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Lacking a Formal Concept of Limit: Advanced Non-Mathematics Students' Personal Concept Definitions

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Abstract

This mixed-methods study examines the conceptual understanding of limit among 22 undergraduate engineering students from two different sections of the same introductory differential equations course. The participants' concepts of limit (concept images and personal concept definitions) were examined using written tasks followed by one-on-one investigative interviews. The findings suggest that, at the end of their mathematics coursework, many engineering students do not consider employing a formal definition of limit to solve limit-based mathematics problems. Furthermore, the findings suggest that most participants' personal concept definitions of limit are both inoperable for solving limit problems and inconsistent with the formal definition of limit.

Keywords: Definition of limit, concept definition, concept image, advanced non-mathematics students, calculus, differential equations

Calculus is a gateway course for studying the higher mathematics necessary for engineering, economics, statistics, and the natural sciences (Zollman, 2014). The concept of limit is the theoretical bedrock of calculus-based mathematics and is essential for advanced mathematical thinking, both in academic and in data-rich work environments. It is, therefore, essential that student learning of limit concepts is researched in advanced courses of calculus-based mathematics.

Success in calculus-based mathematics is associated with a deep understanding of limit. Calculus students who possess a formal and operable concept definition at the center of their conceptual understanding of limit are more likely to be successful in solving limit-based calculus problems (Przenioslo, 2004). Additionally, Roh (2010) states that without understanding the essential components of a formal limit definition, students will struggle with understanding limit-based mathematical concepts.

Beyond an introductory calculus course, and prior to an advanced course in real analysis, limit-based concepts are often developed without a connection to the formal definition of limit. For example, in a second semester calculus course, the standard definition for the volume of a solid of rotation is given as

$$\lim_{n\to\infty}\sum_{i=1}^{n}\pi f^2(x_i)\Delta x = \pi\int_a^b f^2(x)dx.$$

Much attention is given to developing the sum on the left-hand side of the above statement, and rightfully so, but often less attention is given to the equality itself, using a formal δ-ε approach. Without connecting limit-based concepts back to a formal definition of limit, many students of advanced mathematics may not have a deep conceptual definition of limit to study and utilize in advanced calculus-based subjects, such as differential equations.

In most introductory differential equations courses, there are topics that employ components of the formal definition of limit for their development. Such topics include the existence and uniqueness of solutions, and the stability of critical points. Therefore, students of applied mathematics fields such as those majoring in engineering (referred to in this paper as advanced non-mathematics students) would benefit from possessing a rigorous and operable understanding of limit at the end of their academic mathematical studies.

Advanced non-mathematics students seem to be an underrepresented population in the mathematics education literature. When the learning of limit concepts is studied in undergraduate mathematics, prior to real analysis, participants are most often examined in an introductory (first or second semester) calculus class (Bezuidenhout, 2001; Cappetta & Zollman, 2009, 2013; Dubinsky, Cottrill, Nichols, Schwingendorf, Thomas, & Vidakovic, 1996; Juter, 2005; Oehrtman, 2009; Szydlik, 2000; Williams, 1991). However, when studies of advanced mathematics (beyond introductory calculus) are conducted, they often involve learners pursuing a degree in pure mathematics or mathematics education (Cory & Garofalo, 2011; Dawkins, 2012; Moore, 1994; Weber, 2005). There are, however, reasons to be interested in the limit understanding of students, such as engineers, who are (a) not intending on taking an advanced real analysis course, (b) at the end of their mathematics coursework, and (c) specifically intending to use advanced mathematics in their fields of study.

A high level of mathematical competency is undoubtedly necessary among engineering students. Alpers, Demlova, and Mustoe (2013) state, "the mathematical education of engineers is of paramount importance for high quality engineering education" (p. 49). Specifically, engineers engage in cost-precision analysis (Kaddoura, King, & Helal, 2005) and measurement precision analysis (Westerwheel, 2000), both of which contain analogues to the δ-ε relationship found in the formal definition of limit. A formal understanding of limit would deepen an understanding of such engineering topics to recognize, communicate, investigate, evaluate and use advanced mathematical reasoning in their applied fields.

How rigorous are engineering students' understanding of limit at the end of their mathematics coursework? The purpose of this study was to examine the personal concept definitions of limit among 22 engineering students in an introductory differential equations course – the declared terminal class in their mathematics coursework.

Background

It is well established that students of calculus struggle with the concept of limit (Tall & Vinner, 1981; Williams, 1991), and they often have a fragmented and incomplete understanding of the topic (Cornu, 1981). This struggle persists for students who have advanced, in their mathematics coursework, beyond introductory calculus (Moore, 1994; Roh, 2010; Weber, 2005).

In order to study the conceptual understanding of limit among students of calculus-based mathematics, a theoretical framework is required. There are a number of frameworks that have been used to study calculus learning (Dubinsky, et al., 1996; Rasmussen & King, 2000; Tall & Vinner, 1981). Following several others (Cory & Garofalo, 2011; Juter, 2005; Moore, 1994; Przenioslo, 2004), this study makes use of Tall and Vinner's (1981) notion of concept image and concept definition. Tall and Vinner's framework provides an efficient system for examining the use of definitions in solving advanced mathemematics problems.

Tall and Vinner (1981) describe *concept image* as "the total cognitive structure…associated with a concept, which includes all mental pictures, properties, and processes" (p. 152). With regard to the concept of limit, *concept image* refers to all associated theorems, examples, solution methods, metaphors, and processes involved with a student's concept of limit. Subsequently, the authors define a *concept definition* as a collection of words and symbols used to specify a concept. With regard to limit, a student's *personal concept definition* refers to the collection of words and symbols used by the student in one's own explanation of limit, whereas a *formal concept definition* of limit refers to Cauchy's (1821) original verbal description, now translated into symbols as the δ-ε definition.

Additionally, Bills and Tall (1998) describe the notion of *formally operable* as follows: "a (mathematical) definition or theorem is said to be formally operable for a given individual if that individual is able to use it in creating or (meaningfully) reproducing a formal argument" (p. 104). For students of advanced mathematics to truly possess a formal definition within their concept image, they must be able to do more than simply recite the definition. They must be able to employ the definition in order to consciously and formally construct valid mathematical arguments.

Deep understanding of an advanced mathematical concept entails possessing a formal and operable definition of the concept (Edwards & Ward, 2008). Hsu (2013) mentions that most engineering students, who have completed their mathematics coursework, can successfully implement various advanced mathematical procedures, such as solving second-order linear differential equations. Yet, many of these same students struggle with solving advanced mathematics problems that require the employment of formal mathematical definitions. Edwards & Ward (2008) note that their "research indicates that some undergraduates with advanced mathematical training and decent, sometimes excellent, grades do not completely understand the nature and role of mathematical definitions" (p. 227). Thus, one might expect that the formal concept definition of limit has a role in limit-based problem solving for advanced non-mathematics students, and yet, there may be reason to suspect that it is not entirely operable for many, if not most.

This paper describes a study of the conceptual understanding of limit among several differential equations students, who at the time of the study, had declared to be engineering majors and to be in their terminal mathematics course. Specifically, the study seeks to investigate the following three research questions:

1. To what extent are the participants' personal concept definitions of limit consistent with the formal concept definition of limit?
2. What portions of the participants' concept images of limit are evoked when describing the concept of limit and when solving limit-based problems?
3. To what extent are the participants' personal concept definitions of limit operable in solving limit problems?

Method

This study took place during the last four weeks of a standard 16-week semester with participants from two different sections of the same introductory differential equations course. Participation in the study was voluntary. Assessments, both quantitative and qualitative, for the study were performed individually for each participant.

There were a total of 22 participants in the study, all of whom were enrolled at a medium-sized suburban community college in the United States. All participants were pursuing degrees in engineering, and all participants declared differential equations — the course in which they were currently enrolled — to be their last mathematics course. At the time of the study, course grades of the participants ranged from low C to high A using the traditional grading scale. Three of the participants were female, and the other 19 were male.

For each participant, assessment consisted of two components taking place successively in one sitting. The first component was written and comprised of three test items, two of which were short answer format and one of which was multiple choice. Participants were given 45 minutes for the written component. The second component of the assessment immediately followed the first and consisted of a one-on-one follow-up interview for approximately one hour. Participants were allowed to use TI-84 graphing calculators for both components.

The written component of the assessment for this study consisted of three test items (see Appendix A). In accordance with the purpose of this study, these items were selected with the intent of analyzing the participants' personal concept definition of limit. The items gave the participants an opportunity to employ their personal concept definitions both procedurally and conceptually. The second component (see Appendix B) was a follow-up of the students' answers of the first component. It was developed to probe deeper into a student's responses and give the student an opportunity to elaborate on one's thinking. If a student's first component response to a question was incomplete or incorrect, a counter-example was presented for the student to evaluate.

Appendix A

Written Test Items and Follow Up Questions

Answer each item to the best of your ability. Afterwards, I will follow-up with questions to better understand your thinking on the problems you just worked. There are no wrong answers. You will not be graded. I am only interested in how you think about the concepts.

1. In your own words describe the mathematical meaning of limit. Use any means, including symbols, tables, graphs, and/or a verbal explanation.

2. Suppose f is a function. Circle each statement below that must be true if $\lim_{x\to 2} f(x) = 3$. If none of the statements must be true, then circle (e.). No justification is required.

   a. f is continuous at $x = 2$.

b. $f(x)$ is defined at $x = 2$.

c. $f(2) = 3$.

d. For every positive integer n, there exists a positive number δ such that if $0 < |x - 2| < \delta$, then $|f(x) - 3| < \frac{1}{n}$.

e. None of these must be true.

3. Consider the function $f(x) = \sin\left(\frac{1}{x}\right)$. Determine the value of $\lim_{x\to 0} f(x)$, or state that it does not exist. Thoroughly justify your answer using any means, including symbols, tables, graphs, and/or a verbal explanation.

Item 1 asked the participants to describe the meaning of limit. This item was specifically asked first so the participants could reflect on the concept without being influenced by the other test items. This item was designed to examine elements of each participant's concept image of limit, including metaphors, representations of functions, specific examples, and any traces of the formal definition. This item also was included to examine the extent to which students' personal concept definitions of limit were consistent with their responses to the other two test items.

Item 2 was a multiple-choice question adapted from a study (Bezuidenhout, 2001) on introductory calculus students' conceptions of limits and continuity. This item examined how the participants' definitions of limit are related to other portions of their concept images, such the concepts of continuity, domain, function value, and the formal concept definition of limit.

Item 3 asked the participants to determine a specific limit value, or its lack of existence. This item contains a standard introductory example of a function whose limit fails to exist at a point. Correctly answering this item requires referencing components of the formal definition of limit. This item examined the extent to which the participants' personal concept definitions of limit are operable in solving limit problems. It was included to examine the portions of the participants' concept images of limit that are evoked when solving limit problems.

Additionally, item 2 was scored as either correct or incorrect. Item 3 was scored with a 2-point scale as follows: completely correct response with correct work (2 points), partially correct but incomplete response (1 point), and incorrect or no response (0 points). For item 3, a coherent graphical or numerical interpretation – one that is logically consistent with the formal concept definition of limit – was accepted as correct work. Rubrics for all three

assessment items had a qualitative component, whereby solution methods and explicitly evoked portions of participants' concept images were documented.

The principal author of this study conducted all interviews. During the interview, each participant was encouraged to explain one's answers from the written component. Upon completion, the participants were asked one of several follow-up questions (see Appendix B) or may have been shown an example, followed by prompts for further explanation.

Results

There were several notable patterns in the participants' thinking on limit. What follows is an examination of both the quantitative and qualitative results of the assessment. This includes a distribution of participants' scores for items 2 and 3 of the written assessment, statistics, general observations of participant responses, and an examination of specific participant responses. Regarding the statistics, it should be noted that, due to small sample sizes and distribution issues, any statistical inferences from the data would be inappropriate.

Quantitative Results

Below is the distribution of answers to the test items 2 and 3 for the written component of the assessment. Item 1 was not scored quantitatively, and item 2 was scored as either correct or incorrect. Note that the majority of participants were unable to provide a correct answer for either item 2 or 3.

Table 1. *Results of Test Items 2 and 3*

Test Item	*Correct*	*Incomplete*	*Incorrect*	*Total*
Item 2	6	N/A	16	22
Item 3	4	6	12	22

Many participants reasoned (incorrectly) that the limit in item 3 did not exist because the function was discontinuous at $x = 0$. Many of those same participants made logically consistent (though also incorrect) choices for their answers to item 2. Using a proportion comparison, 100% (12 of 12) who gave incorrect responses to item 3, chose option (a) for item 2, if $\lim_{x\to 2} f(x) = 3$ then f must be continuous at x = 2; but only 40% (4 of 10) who gave correct responses to item 3, chose option (a) for item 2.

Again, when using examples almost all participants relied heavily on continuous functions to describe the meaning of limit. This observation appears to be related to the participants' responses to item 2. Using a proportion comparison, 86% (12 of 14) who used a continuous function to describe limit for item 1 chose option (a) for item 2, but only 50% (4 of

8) who did not use a continuous function to describe limit for item 1 chose option (a) for item 2.

Almost all of the participants attempted standard graphing and tabular procedures in their approach to answering item 3, and these approaches continued in their explanations in the interview. Those who seemed to have difficulty were dependent on standard graphing calculator procedures and gave explanations using a finite set of specific values. Participants who scored correct or partially correct but incomplete in their response to item 3 were less or not at all dependent on their graphing calculators and were able to refer to their personal concept definition of limit in their responses. Using a mean comparison, the mean score for item 3, was 1.1 for those who did not restrict themselves to graphical/tabular approach, but only 0.2 for those who did.

Qualitative Results

Few participants were able to use their personal concept definitions to describe specific limit concepts or to solve the assessment items. One participant said that "[definite] integrals must be related to limits since...you know...there are the limits of integration." Many participants could recite a definition or theorem relating to limit, yet few were able to correctly apply them in their solutions. When they used discontinuous functions in their examples, such functions only contained a single discontinuity that was either removable, a jump, or a vertical asymptote.

Appendix B

Interview Protocol

1. In your own words describe the mathematical meaning of limit. Use any means, including symbols, tables, graphs, and/or a verbal explanation.
    a. Can you think of any metaphors or everyday uses of limit that would help you describe this concept?
    b. Have you found it necessary to have a formal definition of limit to excel in calculus? When?
    c. Can you recall other topics in Calculus directly related to limit?
2. Suppose f is a function such that $\lim_{x\to 2} f(x) = 3$. Is it necessary that f is continuous at $x = 2$? Explain why or why not.
    a. Can you recall the definition of continuity as it relates to limit?
    b. Can you think of any metaphors or everyday uses of continuity that would help you describe it?

c. Is it required that the number $x = 2$ be in the domain of f?

d. Explain how you would use a graph to support your conclusions.

e. Explain how you would use table values to support your conclusions.

f. Explain how you would use the definition of limit to support your conclusions.

g. Can you describe the meaning of part (d)?

3. Consider the function $f(x) = \sin\left(\frac{1}{x}\right)$. Determine the value of $\lim_{x\to 0} f(x)$, or state that it does not exist. Thoroughly justify your answer using any means, including symbols, tables, graphs, and/or a verbal explanation.

   a. How did you answer to Item 1 play a role in your answer to this item?

   b. Is it sufficient to check that $f(0)$ is undefined to know that $\lim_{x\to 0} f(x)$ does not exist?

   c. Explain how you would use a graph to support your conclusions.

   d. Explain how you would use table values to support your conclusions.

   e. Explain how you would use the definition of limit to support your conclusions.

The general observations above are here further investigated through an analysis of particular responses given by participants. Three participants, called Mac, Moby, and Layla, represent prevalent trends among all the participants. These three participants were identified through their work as being low achieving, middle achieving, and high achieving students, respectively.

Mac – Low Achieving Student. Mac's responses to the written items and in the interview revealed his difficulty with the subject. His description of limit relied on graphs and evoked the notion of continuity. The following excerpt from his written response to item 1 illustrates this point:

> *Limit is usually a convergence to points in space. This is not necessarily true in cases where the limit goes to infinity. If the limit is established as finite and $f(x)$ is continuous, then for example, at some point on a graph, which is inclusive on an interval (a, b), there should be the same limits as we approach from both sides of the line.*

It is evident that Mac could recall portions of key topics from introductory calculus, but his description was fragmented. From the first two sentences of the passage, it seems Mac held the view of limit as a dynamic process. In the last sentence, he attempted to describe limit by recalling parts of a theorem on one-sided limits. Mac used words such as *graph*, *line*, *point*,

and *visually* frequently throughout the interview, which implied a preference for a graphical description of limit. Mac's personal concept definition of limit is incoherent and largely inoperable, as he was not able to apply it in answering items 2 and 3.

Moby – Middle Achieving Student. Moby's personal concept definition of limit is, at least, partly operable and less dependent on algebraic examples. His reliance on metaphor was evident, and this reliance created potential conflicting factors within his concept image. For example, Moby firmly believed that a limit is unreachable, and he reaffirmed this belief with a metaphor in the opening statement of his written response to the first test item:

> *A limit is a mathematical concept that allows someone to get close to a given number or point but never reaching that point. An example of this is the idea of taking an object and placing it some distance away from a wall. You move the object half the distance to the wall and then half it again. You can do this as many times as you want, and each time you will get closer but you will never reach the wall.*

All of Moby's examples relied on continuous functions. He carefully illustrated his description by using a table and graph of $f(x) = x^2$ to discuss the limit $\lim_{x \to 0} x^2$. When presented with an example of a limit involving a constant function, he said, "Maybe limits are reachable, but most of the time they're not." Regarding item 2 on the written component, he chose statements (a), (b), and (c) as logically following from the statement $\lim_{x \to 2} f(x) = 3$. Even though Moby's concept image possessed Tall and Vinner's (1981) potential conflict factor of *limit as unreachable*, his personal concept definition of limit was not entirely inoperable, as he referred to portions of it, consistently at times, during the interview.

Layla – High Achieving Student. Layla's concept image of limit seemed to contain the most operable and closest to the formal definition among all the participants. Layla had a diverse example space. She was not dependent on tables or graphs, although she did use them in some of her explanations. Layla had a personal concept definition of limit that was closely aligned with the formal concept definition, and her personal concept definition of limit was directly referenced in her problem solving. In the written component, she described limit in terms of arbitrary closeness as follows:

> *A limit is a number to which values of the dependent variable can be made to stay arbitrarily close based on choices of the independent variable.*

Her personal concept definition missed a few key components, such as logical quantifiers, and it contained words whose meanings are not well defined, yet her definition contained elements of the formal definition. When asked to elaborate on her personal concept definition in the interview, she sketched δ-ε strips without actually referring to the δ-ε terminology. She used the *arbitrary*

closeness language from her written response, showing consistency with her concept definition of limit across test items. Regarding Item 3, when asked to elaborate on her justification, Layla referred back to her personal concept definition several times, saying "the values of f can be made to stay … to be –1 or 1 for certain choices of x. They don't stay close to a number." This showed that her personal concept definition was not simply an inoperable collection of words or symbols that she memorized. Layla possessed an operable concept definition of limit. Her personal concept definitions of limit, and its role within her concept image, were an exception among the participants' understanding of limit in this study.

Discussion

This study found participants employed metaphors and continuous functions as examples when describing the concept of limit, and they relied heavily on graphical and tabular methods for solving limit problems. The findings of this study may suggest that, at the end of their mathematics coursework, many engineering students do not employ a formal concept definition of limit for solving limit-based mathematics problems. Furthermore, the findings suggest that most participants' personal concept definitions of limit are both inoperable for solving limit problems and inconsistent with the formal definition of limit. These findings may have implications for the advanced mathematical thinking such students will use in their fields of study.

With one exception, all participants of this study appeared not to possess both an operable and formal concept definition of limit within their concept images of limit. This conclusion is consistent with several other studies on limit understanding, such as those by Williams (1991) and Roh (2010), which indicate that students have difficulty adopting and using a formal concept of limit.

This study indicates that the participants relied on many of the common metaphors characterized in several studies, such as by Oehrtman (2009), to describe their concepts of limit. It is not clear whether such metaphors helped or hindered the participants' ability to solve the limit problems used for this study.

This study also found that, when prompted to describe limit or to solve limit problems, the participants evoked examples of continuous functions. Continuous functions tend to be more easily accessible both conceptually and procedurally for describing limits, than discontinuous functions. This may be because engineering students encounter discontinuous functions less frequently in practice than in a theoretical mathematics classes. When they do encounter such functions, discontinuities may likely be isolated, finite in number, and either removable, a jump, or a vertical asymptote. Thus,

advanced non-mathematics students may approach limit tasks by appealing to concepts of continuity, without appealing to the formal concept definition of limit.

However, Layla's responses provide evidence that engineering students can have a personal concept definition of limit that is operable in solving limit problems and that shows emerging elements of the formal concept definition. Her approaches may offer a model for how engineering students can approach limit-based problem solving in their fields, and how to understand limit concepts in general.

Generalizability of Study and Future Research

This study was observational and investigative in nature, and so there are a number of limitations. The results are based on a small sample of students from two sections of a differential equations course, all at the same college, and so the results do not generalize to other students or institutions. The assessment instrument had only three items and was thus very limited in scope. Additionally, sample sizes and distribution issues render any statistical inferences from the data as impractical. Much of the data collected was qualitative and gathered through interviews, and thus categorizing this data was a subjective endeavor. The wording of the interview questions may have lead some students away from the formal concept definition. Finally, while Layla's gender did not play a role in the decision to include her individual responses, it should be noted that her gender might potentially be a conflating factor for the results.

A controlled longitudinal study across several semesters, involving more participants and a broader assessment instrument is necessary to draw more definitive conclusions on limit concept understanding among the stated population. An experimental study might include examining the effect of regular treatments (of δ-ε lessons) on limit concept understanding imposed on randomly selected engineering students, compared against a control group of engineering students, as they complete their mathematics sequences. Previous research suggests these lessons could be designed to complement the topics that the participants are currently studying, and could be implemented with specified pedagogical practices (Cappetta & Zollman, 2009, 2013; Cory & Garofalo, 2011; Dubinsky, et al., 1996; Oehrtman, 2008). Such practices have been successful in developing an operable personal concept definition of limit that is consistent with the formal concept definition.

Participant responses to assessment item 3 seem to indicate that students who have difficulty with continuity are dependent on standard graphing calculator procedures. This observation would be an area to research in more detail.

Lastly, this study was not designed to draw conclusions on how the participants' beliefs are related to their responses to the test items. However, research by Szydlik (2000) and Williams (1991) indicate that beliefs play a role in students' conceptual understanding of limit. Many advanced non-mathematics students may view the formal concept definition of limit as superfluous. Following Szydlik (2002) a similarly constructed study might reveal the extent to which advanced non-mathematics students' beliefs about themselves (that they are users of mathematical procedure and not developers of mathematical theory) play a role in their willingness to adopt an operable and formal concept definition of limit.

Conclusion

The purpose of this study was to investigate if students' personal concept definitions of limit were operable and consistent with the formal concept definition while solving limit problems. The students were presented both formal and informal representation of limits. This study was looking for traces of formal thinking in students' personal concept definitions. It should be noted that traces of formal thinking can occur in graphical, numeric, algebraic, or oral descriptions, but ultimately, these traces need to be directly connected to the formal δ-ε definition. While this study was only meant to provide a cursory glimpse into students' concepts, it stills identifies a unique and fertile area of mathematics education research that instructors of higher-level mathematics need to consider.

It is imperative that students of applied mathematics fields, including those who are advanced non-mathematics majors, possess a conceptual understanding of key calculus topics. This understanding needs to go beyond purely procedural and metaphorical. Students that have a formal and operable understanding of limit concepts will be able to recognize, communicate, investigate, evaluate, apply, and create advanced mathematical reasoning in their applied fields.

As other researchers have found (Cappetta, 2007; Castillo-Garsow, 2012; Cipra, 1988; Peterson, 1986; Tall, 1990; Vinner, 1983), students in university mathematics courses have a fairly strong procedural knowledge and a fairly weak conceptual understanding of mathematics concepts. Most students use only their strengths, not their weaknesses. They keep a mindset of mathematics that hinders their deeper learning of mathematics. High-ability students, on the other hand, have openness to working with formal mathematical concepts. While these students also begin with their strengths, they venture into less comfortable areas while studying and reflecting on the mathematics. High-ability students have a repertoire of representations they used depending on the context of the mathematics. They can move more easily

between numerical, algebraic, graphical, and application representations (Patel, McCombs, & Zollman, 2014).

To address this problem, there are several recommendations that can occur in the introduction of the limit concept and that should be reinforced in subsequent courses. Students need a learning environment that connects the graphical and numerical representation of the limit concept to the formal concept. This can be done by utilizing such learning activities as the ε-strip (Roh, 2010) or a δ-ε table. Further, students' common metaphors need to be openly addressed in mathematics classes. Students need to see their metaphors as a bridge to formal concepts, not as an endpoint itself.

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