Investigations in Mathematics Learning

Fall Edition 2015, Volume 8, Number 1

Calculus Students' Understanding of Area and Volume Units

Allison Dorko
Oregon State University
dorkoa@onid.oregonstate.edu

Natasha Speer
University of Maine

Abstract

Units of measure are critical in many scientific fields. While instructors often note that students struggle with units, little research has been conducted about the nature and extent of these difficulties or why they exist. We investigated calculus students' unit use in area and volume computations. Seventy-three percent of students gave incorrect units for at least one task. The most common error was the misappropriation of length units in area and volume computations. Analyses of interview data indicate that some students think that the unit of the computation should be the same as the unit specified in the task statement. Findings also suggest that some students have difficulties correctly indicating the units for computations that involve the quantity π. We discuss students' correct and incorrect use of unit in relation to their understanding of area and volume as arrays, as well as in terms of Sherin's Symbolic Forms.

Keywords: undergraduate students, units, area, calculus

Introduction

Units of measurement are important in many disciplines including those in science, technology, engineering and mathematics (STEM). Values are often meaningless without units and there are often multiple options for the unit in which something is measured (e.g., meters, yards), creating potential ambiguities if quantities are reported without units. In addition, understanding units can be useful for problem solving. For example, knowing what units the answer must be in can help one determine which quantities to combine to obtain the answer. In addition, as Rowland (2006) noted, “Units can also be used to check the validity of equations: while an equation can be wrong if the units are right, an equation can’t be right if the units are wrong” (p. 553).

Many future scientists and engineers get their foundational understanding of key ideas of measuring quantities (e.g., rates, rates of change) in undergraduate calculus courses. Success in any STEM discipline requires both the knowledge to obtain quantitative answers and the knowledge of how those quantities relate to the physical world, including the units in which such quantities are measured. Findings from research on the teaching and learning of calculus indicate that applied problems as well as those involving modeling of quantities are quite challenging for students (e.g., Engelke, 2008; Martin, 2000; Orton, 1983).

Despite the importance of understanding units of measure and anecdotal evidence from instructors that undergraduate students struggle with units, there is little research about the nature and extent of these difficulties. The few studies that exist provide evidence that units pose difficulties for undergraduate students in differential equations, physics, and chemistry (Redish, 1997; Rowland, 2006; Rowland & Jovanoski, 2004; Saitta, Gittings, & Geiger, 2011). Additionally, researchers have found that elementary school students have difficulty with measurement in general and the units of area and volume in particular (Battista & Clements, 1998a; Curry & Mitchelmore, 2006; Lehrer, 2003; Lehrer, Jacobson, Thoyre, Kemeny, Strom, Horvath, Gance, & Koehler, 1998; Lehrer, Jenkins, & Osana, 1998).

The exploratory investigation reported here focused on calculus students’ understanding of the units of area and volume computations. Understanding what area and volume are and being able to carry out computations for them are important in the study of calculus. Topics such as the definition of the definite integral and the volumes of solids of rotation are built on ideas of area and volume. Understanding students’ thinking about units is one window into the understanding of these key issues and, to our knowledge, this is the first exploration of unit understanding for university calculus students. We were curious if the unit issues found in elementary school students persisted through their mathematical careers; given the findings about trouble with units in physics and differential equations, it seemed logical that they might. Knowing more about calculus students’ understanding of and difficulties with units of measurement can enhance the education field in two important ways. First, topics involving units are known to be difficult, but what is not clear is the extent to which the units aspect is causing the topics to be difficult. Second, armed with an understanding of the nature of those difficulties, instruction can be designed to help students develop more robust knowledge in this area in conjunction with learning the calculus content.

To begin an investigation of how calculus students understand units, we gave calculus students basic area and volume computational tasks with the goal of answering the following research questions:

1. How do calculus students answer area and volume computational tasks?
2. How do calculus students think about the units in area and volume computational problems?
3. How do calculus students' difficulties with units compare to elementary school students' difficulties with units?

Analyses of written survey and interview data revealed that calculus students struggle with the units of area and volume computations; that these difficulties are related to students' difficulties with understanding arrays and dimensionality; and that some calculus students' difficulties are similar to those documented in elementary school students.

Student Understanding of Units

Investigating what students understand about units in mathematical contexts is of interest to science and mathematics educators alike. Units are an important part of many scientific calculations and findings from research indicate that students' understanding of units affects their learning of some physics concepts (Rowland, 2006; Van Heuvelen & Maloney, 1999). While units may be less emphasized in mathematics, there are topics in which they play a critical role (e.g., mathematical modeling of physical phenomena, "word problems" of various sorts, measurement of one-, two-, and three-dimensional objects). In this section, we give examples of the importance of units in physics, chemistry, and mathematics, then review literature about elementary school students' understanding of units.

Units in Physics and Chemistry

In disciplines with applied mathematics such as chemistry, physics, and engineering, the variables used in equations relate to physical quantities in the real world. Most of these quantities possess dimensions and these dimensions are an essential part of their nature. Density, for instance, is the amount of matter in a unit volume and thus thinking about density involves thinking about the units involved. Equations used for computation or modeling must both make mathematical sense and be *dimensionally consistent*. For instance, Newton's Second Law *($F=ma$)* is dimensionally consistent because the product of the units of mass and acceleration is kgm/s^2, equivalent to Newtons (a unit of force). The Ideal Gas Law, $PV = nRT$, is an example of an equation in chemistry which must be dimensionally consistent. In chemistry, units are particularly important in stoichiometry and dimensional analysis.

Converting between units is especially problematic; in particular, "although unit conversions are taught in a variety of subjects over several grade levels, many students have not mastered this topic by the time they enter college" (Saitta, Gittings, & Geiger, 2011, p. 910).

We did not locate any physics education articles explicitly stating that students struggle with units, but several researchers write that students do not make use of implying units' importance, difficulty, or both. Redish (1997) noted students may interpret symbols as pure numbers rather than as standing for physical quantities; students are also likely to substitute numbers into an equation right away. The implication is that students may miss opportunities to use the units of the quantity as a 'clue' in the equation (e.g., figuring out what other quantities might be needed). Van Heuvelen and Maloney (1999) proposed a way of using units to help physics students understand concepts. The researchers suggested that problems be written *backwards*; that is, rather than give students a physical situation and have them provide an equation to model it, the students are given an equation with quantities and their units and students create physical situations to match. They give the example of proposing to students the equation N-(60kg)(9.8 m/s^2)=0 , from which students could reason that kg indicates mass and m/s^2 indicates acceleration due to gravity, and so on. The students would then create scenarios, such as a 60kg person standing on a flat surface (Van Heuvelen & Maloney, 1999, p. 252-253). A strength of these problems is that "the units become more meaningful since they become the key to determining what quantities are involved… consequently the units become useful sources of information" (Van Heuvelen & Maloney, 1999, p. 254). Redish's (2007) note that students, to their detriment, often try to work with only pure numbers and Van Heuvelen and Maloney's (1997) emphasis on units as useful in decoding equations indicate the importance of units to the physics world, and their usefulness to students as conceptual tools.

Units in Mathematics

The abstract nature of mathematics means that key issues in the field have to do with values and their relationship with one another. That variables sometimes model quantities in the physical world is not always the core issue; however, there are a number of topics for which units are an important part of a concept. One example is applications of integration, in which the units of the integral depend on the units of the integrable function and the units of the dependent variable. For instance, if *v(t)* is a velocity function with units distance/time, $\int_a^b v(t)dt$ is the total distance travelled with a distance unit. If *a(t)* is an acceleration function with units distance/time^2, $\int_a^b a(t)dt$ is velocity with units distance/time. Rasslan and Tall (1997) studied understanding of definite integration in above-average high school students and found the majority of students had "difficulty interpreting problems, calculating areas, and with definite integrals in wider contexts" (Rasslan & Tall, 1997, p. 8).

Units are also crucial in modeling problems in differential equations. Rowland (2006) and Rowland and Jovanoski (2004) found that few first-year undergraduate engineering students checked that equations were dimensionally consistent when constructing a first-order differential equation to model physical situations. Some of these students also believed that the units on one side of an equation did not have to be the same as the units on the other side of the equation. In a later study, Rowland (2006) asked students to provide units for and interpret the terms in a differential equation modeling the amount of drug in a patient's system as a function of time. Students had trouble identifying which terms had units of rate (i.e., some quantity per unit of time) and which terms represented an amount (e.g., milligrams of drug). Students also had difficulty determining the units of k, a proportionality factor, when given an equation for the rate of change of velocity with respect to time, $dv/dt = -kv^2$.

Area and volume understanding has been studied in depth in elementary school students. However, we found no research about calculus students' understanding of unit. In the remainder of this section, we present findings from research about elementary school students' understanding of unit that informed our research design and interpretation of our findings.

Elementary School Students' Understanding of Units

Units are difficult for elementary school students and these difficulties seem to stem from trouble understanding area and volume. Specifically, elementary school students have trouble with arrays (Battista & Clements, 1998) and they frequently misappropriate units of length for units of other spatial computations such as area, volume, and angle (Lehrer, 2003).

Area and volume computations rely on the idea of arrays of unit squares and unit cubes, a representation troublesome for elementary school students (Battista & Clements, 1998; Curry & Outhred, 2005). Researchers have found that students have trouble visualizing and using the unit structure of an array (Battista & Clements, 1998; Curry & Outhred, 2005). Battista and Clements (1998) found that only 23% of third graders and 63% of fifth graders could determine the number of unit cubes in a 3x4x5 array of cubes.

Students frequently misappropriate units of length measure for area, volume, and angle measurements (Lehrer, 2003). Researchers have replicated this finding with different tasks and concluded that some elementary school students do not see the need for a unit of cover in area measurements (Iszák, 2003; Fuys, Geddes, & Tischler, 1998; Lehrer, Jenkins, & Osana, 1998). Rather, they add the lengths together to get the "area." For example, some children measure the area of a square by measuring one side of the square, moving the ruler a short distance parallel to the side of the square, and measuring the lengths of the two sides perpendicular to the ruler with each successive horizontal movement (Lehrer, 2003). Children also often believe length units are used with measures of angle (Lehrer, Jenkins, & Osana, 1998). Misappropriating length units for other spatial measures seems indicative of trouble with dimensionality (e.g., length is one-dimensional but area is two-dimensional).

These findings from different disciplines and age groups suggest that units are important and difficult, but the reasons such difficulties exist are not entirely clear. Our study was designed to contribute knowledge about older students' unit use, as well as why units are difficult.

Research Design

To begin to develop a base of research findings on undergraduate students' understanding of units, the present analysis focused on calculus students' understanding of units in computational area and volume problems and compared it to findings about elementary school students. We conducted this study within a cognitivist framework (Byrnes, 2000). We gave students mathematical tasks and analyzed the reasoning underlying their answers. This is consistent with the cognitivist orientation toward focusing on the cognitive events, like a student's conceptual understanding of a question, that cause behaviors (Byrnes, 2000).

Data Collection

This research is part of a larger study about calculus students' understanding of area and volume in non-calculus contexts (Dorko & Speer, under review; 2014a; 2014b; 2013). The data analyzed here are from written surveys completed by 198 differential calculus students and 20 clinical interviews with a subset of those students. All students were enrolled in differential calculus at a large public northeastern university and student volunteer participants were recruited. The university offers a single track of calculus for all majors and therefore participants included those pursuing majors in physical sciences, engineering, biological sciences and education, as well as other disciplines. The data collection had two phases: first, students completed the written tasks and data were analyzed using a Grounded Theory inspired approach (Glaser & Strauss, 1967). Clinical interviews (Hunting, 1997) were used to investigate patterns from the written data; that is, interview subjects were selected because their answers on written tasks represented an emergent category. This methodology allowed for a quantitative analysis of a large number of written responses and a qualitative analysis of student thinking about those written responses.

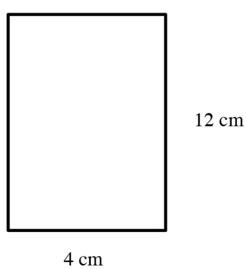
Research Instrument

The written survey and interview tasks consisted of diagrams of figures with dimensions labeled that included units. Students were directed to compute the area or volume of a figure and to show their work. Two of the tasks are shown in Figure 1; the others are included in Appendix A.

We created some of these tasks based on those used by other researchers. Area-of-rectangle tasks have been used by a number of researchers (e.g., Battista, Clements, Arnoff, Battista, & Borrow, C.V.A.; 1998; Lehrer et al., 1998; Lehrer, Jenkins, & Osana, 1998). The rectangular prism was included to mirror the tasks used by Battista and Clements (1998). The area of the circle task was inspired by Lehrer, Jenkins, and Osana (1998), in which the researchers found that elementary school students had difficulties envisioning an array for a shape with a curved boundary. The first iteration of the survey, given to three sections of calculus students, contained the following tasks:

- Area of rectangle
- Area of circle
- Volume of rectangular prism
- Volume of cylinder

What is the area of the rectangle? Explain how you found it.



What is the area of the circle? Explain how you found it.

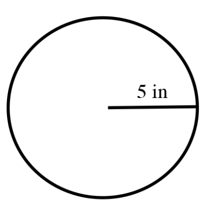


Figure 1. Area of rectangle and area of circle tasks.

When we analyzed these responses, we found many students had written formulae such as A = LW, V = LWH, A = πr^2 , and V = $\pi r^2 h$. We suspected these might be memorized and as the larger study was about students' understanding of area and volume, we wanted to create a task involving a formula which students were unlikely to have memorized. Thus we added a fifth task before giving the survey to additional students. This task was a triangular prism, shown in Figure 2.

The clinical interviews used all five written tasks. Interviewees were a subset of the 198 students who had completed the written tasks, and were asked to do the problems again while "thinking aloud." We asked questions to probe understanding, like "Why is that measured in inches?" If a student's response during the interview did not match what the student had written earlier on the written survey, the researcher showed the student the earlier work and asked about the discrepancy. Interviews were audio recorded and transcribed.

Find the volume of the right triangular prism.

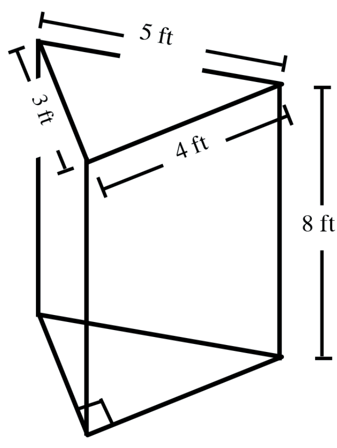


Figure 2. Volume of a right triangular prism task.

Data Analysis

Both written and interview data were analyzed using an approach inspired by Grounded Theory (Glaser & Strauss, 1967). This entailed first looking for patterns in a portion of the data and using patterns to form categories, followed by forming category descriptions and criteria. Those criteria were then used to code all the data, testing the criteria until new categories ceased to emerge. The researchers' departure from Grounded Theory was that literature was accessed prior to coding and findings from these studies informed coding.

For instance, we knew that elementary school students often misappropriated length units for other spatial measures, and we looked for the occurrence of that in our data in addition to our search for emergent categories.

The three parts of responses to a task were the magnitude, the unit, and any work shown. For the present article, researchers coded only the unit written without regard to whether the magnitude was correct. Correctness of magnitude is an additional issue that was part of a larger study (Dorko & Speer, under review; 2014a; 2014b; 2013) but is not reported on here.

The first data analysis yielded three categories: correct units, incorrect units, and no units. This formed our first, and broadest, layer of analysis. The second layer was to look at answers with incorrect units and see if patterns arose there; that is, what were the incorrect units students were writing and why might they write these? This second layer had two sections: misappropriation of length units and issues specific to circles and cylinders. We discuss these two layers of coding in the following subsections.

Correct units, incorrect units, no units. We first coded answers as having correct units, incorrect units, and no units. Students had a variety of correct and incorrect responses (e.g., a student might use correct units for the area problems, but no units for the volume problems) and so we chose to code each response as independent of the others. Any squared unit for an area problem (e.g., cm^2, $units^2$) or cubic unit for a volume problem (e.g. ft^3 , $units^3$) was coded as Correct Unit. Any number without a unit (e.g., 48, 72 π) was coded as No Unit. Any incorrect unit (e.g., 48 cm, 200 cm^2, 480 ft^4) was coded as Incorrect Unit. All responses fit into exactly one of these categories.

Incorrect units: Misappropriation of length units. One of the findings from research with elementary school students was that students may see length units as adequate for other spatial measures. We coded our data to see if this issue persisted in undergraduate students. We coded any answer with a one-dimensional unit (e.g., cm, ft, in, unit) as Misappropriation of Length Units and all other answers as 'other' (it was not relevant here to classify the 'other' category further).

Incorrect units: Circle and cylinder. Findings from analyzing interview data (discussed later) took us back to responses to the circle and cylinder tasks. The finding that spurred this was that some students seemed to think π was a unit. Looking at only the responses to the area of circle and volume of cylinder task, we coded all responses that ended in π and had no unit as "π with no unit." For instance, 25π was coded as "π with no unit" but 25π in^2 was coded as "other." We only broke this "other" into further categories for ten students (those who had π with no unit for both the circle and cylinder). This was done to see if those responses had anything else in common. The categories we used (see Table 4) were 'other tasks units correct,' 'other tasks units incorrect,' and 'no units.' The criteria for these labels were the same as in the first layer of coding (see *Data Analysis*).

Interview data. Recall that we selected interviewees by finding students who had made specific mistakes on the surveys (e.g., length units, circle and cylinder units incorrect). We also included some students who had all of the units correct, as comparing the thinking used by successful students can provide insights into difficulties encountered by others, as well as indicate what might be emphasized in instruction. We took transcript excerpts in which students discussed a particular issue we had identified, then used Grounded Theory to look for patterns. For example, we identified interviewees who had all units correct and looked for patterns in their reasoning. Details of how transcript excerpts were coded are presented in conjunction with the results.

Results

In this section, we present the results of coding, giving the various frequencies for students' unit use. The section is separated into Correct, Incorrect, and No Units; Misappropriation of Length Units, and Issues Specific to the Circle and Cylinder.

Correct, Incorrect, and No Units

Our first research question was "How do calculus students answer area and volume computational tasks?" We answered this question by sorting the data in terms of correct, incorrect, and no units, and then further sorting the Incorrect Units category (see subsequent section). Table 1 shows the results of the first round of coding that was used to determine what percentage of students used correct units, incorrect units, and no units on the tasks.

Table 1. *Correct, Incorrect, and No Units*

Task	Area of rectangle	Area of circle	Volume of rectangular prism	Volume of cylinder prism	Volume of triangular
n total	197	197	197	197	128
n responses	195	195	196	182	107
Correct unit	159	113	165	125	77
% responses	81.54	57.95	84.18	68.68	71.96
Incorrect Unit	26	27	23	23	19
% responses	13.33	13.85	11.73	12.64	17.76
No unit	10	55	8	34	11
% responses	5.13	28.21	4.08	18.68	10.28

Percentages in Table 1 are the percent of the total responses (not all students answered all questions). For instance, 84.18% of students had the correct unit for the volume of the rectangular prism task; 81.54% had the correct unit for the area of the rectangle task; and so on.

Note that the area of the circle and volume of the rectangular prism tasks had the highest percent of 'correct unit' responses (81.54% and 84.18%, respectively). In contrast, area of the circle and volume of the cylinder tasks had the lowest percentages of 'correct unit' responses (57.95% and 68.68%, respectively). The 'no unit' category is also much higher for the circle and cylinder than for other shapes. We used interview data to examine some of these discrepancies; the thinking of students who had correct units is discussed below, and the circle issues are considered later.

We found that interviewees who had correct units talked about at least one of the following: dimensionality, arrays, or rules of exponents. For instance, Steven, Rose, and Isaac explained how they decided what unit was correct as follows:

Steven: Area is a square [unit]. Every time we multiply one dimension by the next, we're multiplying centimeters by themselves. We change from a linear to an area, then area to volume. It's recognizable that volume is cubic, as opposed to area, which is squared. [The volume of the rectangular prism] is the length times the width of the base times the height. It's like the area formula only now we have a vertical dimension so we have 200 little centimeter boxes inside.

Rose: Area is a two dimensional measurement of the amount of space that um something occupies. But it's two dimensional where a length, like each side, would be one dimensional, I believe. And then volume would be three dimensional, so it's cubed.

Isaac: If we think about this [5 cm x 10 cm face] in terms of an area – you have 50 boxes [draws a 5 x 10 array of squares on the face]. You know that you have four of these, fifty times four. So you can think of it as having four layers of this area because the difference between volume and area is just adding another dimension … so we can think of it as four sheets of 50 squares.

Steven thought about dimensions and rules of exponents as he identified a correct unit. Rose also thought about dimensions, though she did not explicitly connect dimensionality of the figure to dimensionality of the unit. Isaac, who determined that the volume of the box was 200 cm^3, thought about arrays, and used the idea of sheets of squares (we suspect he meant 'sheets of cubes') to reason that the unit should be cubic centimeters. These three students' thinking is representative of others who had all the units correct. Therefore, we can conclude that ideas of dimensionality, arrays, and rules of exponents are helpful for students in understanding the units of a spatial computation.

We were also curious how many students had correct answers to all tasks. We took data that had been coded for ‘correct units,’ ‘incorrect units,’ and ‘no units’ and looked for students whose answers were all marked ‘correct units.’ There were 45 (26.63%) such students of 169 who answered all the tasks. In the next section, we use interview data to shed light on some of the thinking of students who had incorrect units.

Misappropriation of Length Units

One of elementary school students’ most common difficulties with units is the misappropriation of length units for other spatial measure; that is, writing a linear unit for an area or volume problem (Lehrer, 2003). Some calculus students have this same issue, as shown in Table 2.

Table 2. *Misappropriation of Length Units*

	Area of rectangular	Area of circle circle	Volume of rectanglar prism	Volume of cylinder	Volume of cylinder prism
n total	197	197	197	197	128
n responses	195	195	196	182	107
Length Units	29	25	18	13	16
% responses	14.87	12.82	9.14	7.14	15.00

Though only two students used length units on all tasks, a number of students used length units on one or more tasks. Interviewees who used length units for all the tasks told us that the unit of the computation was [to paraphrase] ‘the same as the unit you’re told in the problem.’ Rae and Alex wrote “cm” with their answers. Here, they respond to the interviewer asking them to “tell me about the centimeters.”

Rae: That’s the thing that goes along with the sides. Whatever the side is in, that’s what the answer is in.

Alex: We were given a unit. The numbers are centimeters.

Jolie wrote length units for all of the tasks except the triangular prism, in which she wrote ‘60’. The interviewer directed her back to the area of the rectangle task:

Interviewer: On this rectangle question, you wrote 48 centimeters. Does this question have anything like that [points to the ‘cm’]?

Jolie: Well, it’s feet, but I don’t know if it’s cubed because it’s a triangle. I know it wouldn’t be just feet.

Interviewer: Do you remember volumes of other things being cubed?

Jolie: I don't remember.

We can suggest several possible explanations for the occurrence of students writing a length unit for an area or volume computation. One is that the students believe length to be an adequate measure of area or volume, as is the case with some elementary school students (Lehrer, 2003). Another explanation is that the students do not believe the answer is a length, but do believe that the answer is given in the units the problem gave them. That is, the students may understand volume as the amount of three-dimensional space something takes up, without a corresponding understanding that it is measured in cubic units. Students' initial learning experiences with units may predispose them to this sort of thinking, in which they believe the correct unit is whatever is given in the initial problem. For instance, consider an elementary school arithmetic problem in which a teacher reminds students to "include the units" in arithmetic problems like 6 apples plus 6 apples. The unit corresponding to the answer (12 apples) is the same unit as in the problem statement. Students may incorrectly generalize this to other problems that require units, such as the tasks shown here. A third explanation related to this one, which we explore later, is that students have a Symbolic Form (Sherin, 2001) for problems involving units; that is, students believe that the answer to a spatial computation looks something like Figure 3,

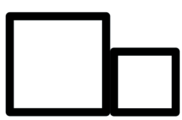


Figure 3. *Symbol pattern for "measurement" symbolic form.*

In a symbol pattern, the larger square is filled with a magnitude and the smaller square is filled with a unit.

While we are not sure which of these explanations (if any) are correct, the common theme is that the misappropriation of length units seems to stem from not recognizing the connections between units, arrays, and dimensionality. In all of these cases, it seems that the underlying issue is that students do not see the connections between units, arrays, and dimensionality.

Issues Specific to the Circle and the Cylinder

As noted earlier, the initial coding of answers (see Table 1) revealed patterns in the types of tasks that seemed most problematic for students. In particular, students did not include units with their answers to the area of a circle and volume of a cylinder tasks at higher rates than for the other tasks. Table 3 shows the results of coding circle and cylinder task data for "π and no unit."

(Note that these figures are the same as in the 'no unit' category of Table 1).

Table 3. *Issues specific to the circle and cylinder*

	Area of circle	Volume of cylinder
n total	197	197
n responses	195	182
π with no unit	55	34
% responses	28.21	18.68

This table is to be interpreted as follows: 55 of 195 students (28.21%) had an answer like 25π (magnitudes may have differed) for the area of the circle. Thirty-four students (not a subset of the 55) of 182 (18.68%) had an answer like 72π for the volume of the cylinder. These students' unit choices for the other tasks and their discussion of these tasks in the interview give us reason to believe that something about the presence of the π creates issues with the units of the computation.

We find it useful to consider the responses from the students who fell into the "π and no unit" category for both the circle and the cylinder. There were 10 such students (see Table 4). These students gave answers such as 25π for the area of the circle and 72π for the volume of the cylinder, indicating that they could carry out measurement computations and knew those computations involve π. Their answers, however, did not include units for either task. We wondered if these students' answers to the other tasks would shed any light as to why they did not give units for the circle and cylinder tasks. Their responses to all tasks are shown in Table 4. Only one student, Ian, was in the group of students given the triangular prism task; the 'not applicable' mark in the volume of the triangular prism column indicates that triangular prism task was not part of the student's set of questions (as opposed to a blank answer).

Table 4 shows that of the 10 students who did not include units with the circle and cylinder task answers, 8 of them did include units with their other answers. While some of the units are incorrect, the fact that the students included some sort of unit on the other tasks seems a compelling piece of evidence that something about circle and cylinder area/volume computations causes students to have issues with the units.

Table 4. π *as and no unit for both circle and cylinder tasks*

Student	Area of Circle	Area of Rectangle	Volume of Rectangular Prism	Volume of Cylinder	Volume of Triangular Prism	Code
Amy	48 cm^2	25π	200 cm^3	144π	n/a	Other tasks units correct
Bob	48 cm^2	25π	200 cm^3	72 π	n/a	Other tasks units correct

Cory	48 cm^2	25π	200 cm^3	$72\ \pi$	n/a	Other tasks units correct
Deb	48 cm^2	25π	200 cm^3	$72\ \pi$	n/a	Other tasks units correct
Eric	48 cm^2	25π	200 cm^3	$72\ \pi$	n/a	Other tasks units correct
Fritz	48	25π	200	72π	n/a	Other tasks no units
Gillian	60	10π	200	48π	n/a	Other tasks no units
Henry	48 cm	25π	200 cm	24π	n/a	Other tasks units incorrect
Ian	48 cm	25π	200 cm^3	556π	48 ft^2	Other tasks units incorrect
James	48 cm	25π	200 cm^3	48π	n/a	Other tasks units incorrect

One might argue that a student "forgot" to write the units, was being "careless," or perhaps did not think units were important. We think that these students' responses are evidence to the contrary. That is, it seems unlikely to us that a student who includes units on two problems would neglect to put units with the other two problems out of sheer forgetfulness. Moreover, of these ten students, half had the units for the other tasks correct. Again, we find it unlikely that students who give correct units with other tasks would "forget" units for the circle and cylinder tasks.

Interview excerpts helped us understand why circles and cylinders might be tricky. Students talked about the presence of π affecting their unit use. For instance, Amy and Bob said that they did "forget" the unit, but it was because of the π involved:

Amy: I probably didn't even think of it because I was using pi, so I left pi in it and I didn't think to label it. But I labeled all the rest of them. That's really weird. Well I know pi is an actual value, but I guess I would … I don't know. It probably just slipped my mind because I was using pi to represent a number rather than saying 3.14 and I probably just forgot to put a label on it. I probably have a tendency to do that with circles because you really only use pi with circles and it kind of doesn't have a label on it. And I guess it makes sense that I would use it consistently with circles. You can multiply it out but I tend to leave pi as pi.

Bob: I think it's because I forgot [the units for the circle problem]. Either that or the pi threw me off and then I forgot. Pi doesn't have a unit. I think I forgot because of the unitless pi.

Three students had the units for at least one of the other tasks incorrect.

Having an incorrect unit for one of the other tasks, in addition to not putting a unit on the circle or cylinder, seems indicative of general issues with units; that these students included units with neither the circle or nor the cylinder seems indicative of something about the circle contributing to these unit issues. The final two students in this subset who gave no units for any of the tasks are less telling in terms of some sort of circle issue. In a way, they are the opposite of the students who had the other tasks' units correct: these students may have thought units were not needed, and thus their answers for the circle and cylinder tasks may not be indicative of a "π and no unit" issue. While our data does not fully explain what is happening with the circle tasks, the findings do suggest that π being related to unit difficulties warrants further investigation.

Discussion

Our first research question was "What percent of differential calculus students write correct units for area and volume computational tasks? What is the thinking and reasoning of these students?" The percentage of students who wrote correct units differed by shape. The highest percentage was for the volume of the rectangular prism (84.18%), followed by the area of the rectangle (81.54%), then the volume of the triangular prism (71.96%) and the volume of the cylinder (68.68%), and finally 57.95% for the area of the circle task. Another way to answer this question is that only 26.6% of the students wrote the correct units for all tasks. We did not locate percentages for elementary school students' unit use that we could compare to these percentages for calculus students' unit use. However, our results are somewhat similar to what Rowland (2006) found with undergraduate differential equations students. Rowland (2006) found that only 13% of students gave the correct units for a constant term in a differential equation, though roughly 80% of students could give correct units for dD/dt (see Rosken, 2006, p. 555). We would hope that all students know the correct units for area and volume by the time they enter college mathematics. However, our findings indicate that they do not. We speculate this may cause difficulty in differential equations and science classes as well as in application problems in calculus. Moreover, not understanding units prevents students from using them to gain information about equations and how to manipulate them; this is a skill that is useful in physics (Redish, 1997) and other sciences.

Interview data analysis revealed that students who wrote correct units could explain area and volume as arrays and/or could explain dimensions of planar figures and solids, and connect this knowledge to the shapes' units. In contrast, students who struggled with units did not seem to possess knowledge of arrays or dimensionality.

The link between correct units and understanding area and volume in terms of arrays and dimensionality is similar to what other researchers found true of elementary school students (e.g., Battista & Clements, 1998; Battista et al., 1998; Lehrer, 2003). For instance, Lehrer (2003) found some elementary school students treat length as a space-filling attribute, indicating that they did not understand dimensionality or tiling (that is, creating an array). While none of our interviewees measured the area of a rectangle as Lehrer's (2003) students did, their consensus that the computation retained the original unit of measure is indicative of a lack of understanding of dimensionality and/or arrays. Battista and Clements (1998) found that not all fifth graders understand volumes of rectangular prisms in terms of arrays, and findings from this study indicate that this may also be true of some undergraduates. Battista and Clements found that only 63% of fifth graders could use an array model to find the volume of a rectangular prism. While we do not have a figure for how many of our subjects could find volume using an array, findings do suggest it is less than 100% of students, and a related study (Dorko & Speer, under review; 2014a, 2014b; 2013) details some of the issues students have with finding volume and the potential ramifications.

The second research question was "What percent of differential calculus students write incorrect units for area and volume computational tasks? What is the thinking and reasoning of these students?" We direct the reader to Table 1 for the per-task percentages, but the 26.6% of students who had all of the units corrects leaves 73.4% of students who had at least one unit incorrect or missing. Interviewees gave several reasons for these incorrect or missing units. Students who used length units cited "the unit of the initial measure is the unit of the computation," which is reasoning indicating that they may have not understood arrays or dimensionality. Students who had incorrect units for the circle and cylinder tasks talked about the presence of the π causing difficulties. We suspect geometry and trigonometry instruction may play a role. For instance, it is not uncommon to tell geometry students to "leave your answer in terms of/in units of π." Additionally, when students learn about radians, we express angles using π and no other unit (e.g., 3π/2, sin(π/4). While the intent is that students grasp the idea of radians, the meaning that students may take away is that numbers with π need no other unit. Further research is needed about student understanding of π to clarify this issue.

Symbolic Forms

Sherin's (2001) theory of symbolic forms may explain some of the results of this study, most notably the π and incorrect unit findings. Developed to explain how students understand and construct equations in physics, the symbolic forms framework hypothesizes that students have conceptual schema with which they associate certain symbol patterns in equations. Sherin (2001) developed a list of these symbolic forms, noting that it is not comprehensive and hypothesizing mathematics-specific symbolic forms also exist for mathematics equations. Izsák (2000) explored the existence of a few symbolic forms in students' modeling of physical situations with algebra. It is important to note that in Sherin and Izsák's work, the symbolic form is connected to understandings that students have about the situation they are modeling. That is, a symbolic form and symbol pattern are not something that arise from rote memorization or how something is "supposed to" look: rather, the form and symbol pattern have meaning to the student. For instance, students who understand upwards acceleration and acceleration due to gravity as competing forces might write an equation with a □ - □ symbol pattern because that specific pattern represents a "competing terms" symbolic form. Similarly, an exponential decay situation might trigger a "dying away" symbolic form, expressed in an exponential symbol pattern such as e^{-x}.

Neither Sherin's nor Izsák's accounts of forms include forms regarding units, but our data lead us to believe that (a) such forms exist and (b) they may explain students' unit use. We might say that students understand an area or volume calculation to be measuring something, and because it is a measurement, it has a unit. Certainly Steven, Rose, and Isaac, understood that volume was a measurement of three-dimensional space. Rae, Alex, and Jolie, who wrote length units seemed to understand that the computation did need a unit. The evidence for this is that they included the unit given to them in the problem. For instance, Alex's comment "we were given a unit. The numbers are centimeters" suggests that she does indeed associate a measurement with the computation. We propose that there may be a *measurement* symbolic form and an associated symbol pattern as shown in Figure 3 discussed previously. We think that students' early experiences with units likely contribute to the development of this form; for instance, in our 'apples' example earlier, the answer of 12 apples has the form

$$\boxed{\mathbf{12}}\ \boxed{\textbf{apples}}$$

Figure 4. *Symbol pattern for units in early addition problems*

We find it quite possible that students generalize a symbol pattern as shown in Figure 3 for problems with units, and thus come to see other unit problems as fitting the 'magnitude, unit' pattern.

Some students may have a more nuanced symbol pattern for the measurement form. For instance, Steven, Rose, and Isaac expressed an understanding of the unit's exponent as connected to unit squares and cubes and their dimensionality. Steven explicitly stated that "volume is cubic, opposed to area, which is squared." These students' symbol patterns for measurement may look like Figure 4, where the biggest box is for a magnitude, the medium box is for the unit, and the smallest box is for an exponent.

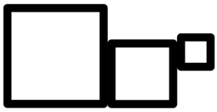


Figure 5. *Nuanced symbol pattern for "measurement" symbolic form.*

These forms may also explain some students' tendency to include π, but not a unit, for circle and cylinder computations and units on other problems. If "measurement" cues the symbol pattern in Figure 3, π may fill the second box, and students may think the symbol pattern is satisfied. They may then proceed to a different task (say, the volume of the rectangular prism), activate the same measurement schema, and in this case include a unit to satisfy the symbol pattern. In short, if, as seems reasonable, these two forms exist, it would explain students' unit use and the sometimes contradictory behavior of including units on some problems but not others, or including an incorrect unit. From an instructional viewpoint, this would suggest good news that students do indeed connect area and volume with the measurement concept. We discuss further instructional implications next.

Implications for Instruction

Successful students tended to think about arrays, dimensionality, or both. We suggest that instructors of elementary school mathematics introduce area and volume with arrays of unit squares and unit cubes, respectively, and that students gain experience with arrays for a variety of different shapes (more than just rectangular prisms) throughout their learning. This is important because findings from research indicate that elementary school students sometimes have difficulty with tiling to form an array (e.g., Lehrer, 2003); in addition, data from a related study (Dorko & Speer, under review; 2014a; 2014b; 2013) indicate that some interviewees could describe an array for the rectangular prism but had difficulty with the arrays of other shapes. With younger students, we suggest emphasizing that area is a two-dimensional measure and can be tiled with unit squares, so it has square units and that volume is a three-dimensional measure and can be tiled with unit cubes, resulting in cubic units. We think this would be useful because it fosters ideas about dimensionality that can be made more formal. That is, once students have learned about the rules of exponents, we suggest helping students see that, for instance, in measuring area, 4 cm x 5 cm = 20 cm^2 because the centimeters are multiplied and rules of exponents dictate that cm^1 X cm^1 = cm^2. This can be linked back to the unit square and unit cube: a unit square is, for instance, 1 in^2 because it is 1 in X 1 in = 1 in^2 (and similarly for the cube). We think that having students write the unit with the computation (e.g., 4 cm x 5 cm = 20 cm^2, not 4 x 5 = 20 cm^2) is a good way to reinforce the connections between dimensionality, rules of exponents, and units.

These suggestions are based on how successful students thought about the units of the computation. One of the most common errors in this study was that students misappropriated length units for the units of other spatial measures, and their reasoning was that "the units of the initial measure are the units of the computation." We think that the above suggestions, particularly writing the units with all steps of the computation, would reduce the chance of students giving length units for area and volume. Per the finding that some students seem to view π as a unit or as unit-like, we suggest that instructors clarify what π is and that the phrase 'in units of π / in terms of π' does not mean that π is a unit like, say, a square inch. Further research about student understanding of π is needed before more specific instructional implications can be offered.

Our final suggestion for instruction is based on physics problems that direct students' attention to units. Van Heuvelen and Maloney (1998) suggest that creating problems that force students to make sense of and pay attention to units aids in understanding. We gave an example of one of their problems for physics earlier. A way to adapt this for area and volume might be problems like

$$16\ m^3 = \pi(2m)^2(4m)$$

and

$$51\ in^2 = (17in)(3in)$$

and asking them what shape the equation represents (e.g., the first of these examples could be a cylinder with radius 2m and height 4 m, and the later could be a rectangle with length 17 inches and width 3 inches).

Suggestions for Further Research

Findings indicate that some calculus students struggle with the units of area and volume in non-calculus contexts. One area for further research is how these struggles interact with students' learning of calculus. For instance, related rate and optimization problems often involve area and volume and their units; it is possible that students' issues with the units for area and volume negatively impact their learning of calculus. This has been the case, for instance, with student understanding of function (Carlson, 1998) and variable (Trigueros & Ursini, 2003). Additionally, given the known difficulties students have with integration (Orton, 1983), it could be fruitful to study how students understand the units of integration (for instance, the units from the integral of a velocity or acceleration function).

These findings demonstrate that some students have developed solid understandings of these ideas while others have not but the data do not provide insights into how students came to have an accurate understanding. It would be productive to examine how students transition from having difficulties with these unit ideas to having a robust understanding of units and dimensions. Teaching interviews or teaching experiment studies could yield insights into the mechanisms by which students can develop normative understandings of one-, two- and three-dimensional units of measurement.

Because a finding from this study is that students seem to attach a different interpretation to π than mathematicians do (that is, π is a unit or is unit-like), a second suggestion for further research is to learn more about student understanding of π. Finally, more research about student understanding of unit in physics, chemistry, and engineering could be useful in helping students succeed in these disciplines.

References

Battista, M. T., & Clements, D. H. (1998). Students' understandings of three-dimensional cube arrays: Findings from a research and curriculum development project. In R. Lehrer & D. Chazan (Eds.), *Designing learning environments for developing understanding of geometry and space* (pp. 227-248). Mahwah, NJ: Erlbaum.

Battista, M.T, Clements, D.H., Arnoff, J., Battista, K., & Borrow, C.V.A. (1998). Students' spatial structuring of 2D arrays of squares. *Journal for Research in Mathematics Education, 29,* 503-532.

Bohren, C.F. (2004). Dimensional analysis, falling bodies, and the fine art of *not* solving differential equations. *American Journal of Physics*, *72*, 534-537.

Byrnes, J. B. (2000). *Cognitive development and learning in instructional contexts* (2nd Edition). Pearson Allyn & Bacon.

Carlson, M. (1998). A Cross-Sectional Investigation of the Development of the Function Concept. In E. Dubinsky, A. H. Schoenfeld, & J. Kaput (Eds.), *Research in Collegiate Mathematics Education* (Vol. 1, pp. 114–162). Washington, DC: American Mathematical Society.

Curry, M., & Outhred, L. (2005). Conceptual understanding of spatial measurement. In P. Clarkson et al. (Eds.), *Building connections: Theory, research and practice (Proceedings of the 27th MERGA conference)*, Sydney: MERGA.

Curry, M., Mitchelmore, M., & Outhred, L. (2006) Development of children's learning of length, area, and volume measurement principles. In H. Novotná, M. Krákáta, & N. Stehliková (Eds.) *Proceedings of the 30th conference of the International Group for the Psychology of Mathematics Education Charles University* (Vol. 2, p. 377-384). Prague: PME.

Dorko, A., Speer, N. (2013). Calculus students' understanding of volume. In S. Brown, G., Karakok, K., Roh, & Oehrtman (Eds.). *Proceedings of the 17th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 190-204). Denver, CO: University of Northern Colorado.

Dorko, A., & Speer, N. (2014a). Calculus students' understanding of volume. *Investigations in Mathematics Learning, 6*(2), 48-68.

Dorko, A., & Speer, N. (2014b). Calculus students' understanding of units. In T. Fukawa-Connolly, G. Karakok, K. Keene, & M. Zandie (Eds.), *Proceedings of the 17th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 584-592). Denver, CO: University of Northern Colorado.

Dorko, A., & Speer, N. (under review, 2015) Deepening students' understanding of area and volume by focusing on units and arrays.

Engelke, N. (2008). Developing the solution process for related rate problems using computer simulations. Contributed report, *Eleventh Annual SIGMAA on RUME Conference*, San Diego, CA, Feb. 29, 2008.

Fuys, D., Geddes, D., & Tischler, R. (1988). The van Hiele model of thinking in geometry among adolescents. *Journal for Research in Mathematics Education Monograph* 3. Reston: NCTM 1988.

Glaser, B., & Strauss, A. (1967). *The Discovery of Grounded Theory*. Chicago: Aldine.

Hunting, R. (1997). Clinical interview methods in mathematics education research and practice. *Journal of Mathematical Behavior, 16*(2), 145-165.

Izsák, A. (2005). You have to count the squares: Applying knowledge in pieces to learning rectangular area. *Journal of Learning Sciences, 14,* 361-403.

Izsák, A. (2000). Inscribing the winch: Mechanisms by which students develop knowledge structures for representing the physical world with algebra. *The Journal of the Learning Sciences*, *9*(1), 31-74.

Lehrer, R. (2003). Developing understanding of measurement. In J. Kilpatrick, W. G. Martin, & D. E. Schifter (Eds.), *A research companion to principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Lehrer, R., Jacobson, C., Thoyre, G., Kemeny, V., Strom, D., Horvath, J., Gance, S., & Koehler, M. (1998a). Developing understanding of space and geometry in the primary grades. In R. Lehrer & D. Chazan (Eds.), *Designing learning environments for developing understanding of geometry and space* (pp. 169-200). Mahwah, NJ: Erlbaum.

Lehrer, R., Jenkins, M., & Osana, H. (1998b). Longitudinal study of children's reasoning about space and geometry. In R. Lehrer & D. Chazan (Eds.), *Designing learning environments for developing understanding of geometry and space* (pp. 137-167). Mahwah, NJ: Erlbaum.

Martin, T. (2000). Calculus students' ability to solve related rates problems. *Mathematics Education Research Journal*, *12*(2) 74-91.

Orton, A. (1983). Students' understanding of integration. *Educational Studies in Mathematics, 14,* 1-18.

Rasslan, S., & Tall, D. (2002). Definitions and images for the definite integral concept. In A. D. Cockburn & E. Nardi (Eds.) *Proceedings of the 26th Conference PME*, Norwich, 4, 89-96.

Redish, E.F. (1997). Student difficulties with math in the context of physics. Available online at: http://www.physics.umd.edu/perg/papers/redish/talks/math

Rowland, D.R. (2006). Student difficulties with units in differential equations in modelling contexts. *International Journal of Mathematics, Education, Science, and Technology. 37*(5), 553-558.

Rowland, D.R. & Jovanoski, Z. (2004). Student interpretation of the terms in first-order ordinary differential equations in modeling contexts. *International Journal of Mathematics, Education, Science, and Technology. 35*(4), 503-516.

Sherin, B. (2001). How students understand physics equations. *Cognition and instruction*, *19*(4), 479-541.

Trigueros, M. & Ursini, S. (2003). First year undergraduates' difficulties in working with different uses of variable. *CBMS Issues in Mathematics Education 12*, 1-29.

Saitta, E. K. H., Gittings, M.J., & Geiger, C. (2011). Learning dimensional analysis through collaboratively working with manipulatives. *Journal of Chemical Education, 88*(7), 910 – 915.

Van Huevelen, A. & Maloney, D.P. (1999). Playing physics jeopardy. *American Journal of Physics, 67*(3)*,* 252-256.

Appendix A: Volume Computation Tasks