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How Concrete is Concrete?

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Abstract

If we want to make something concrete in mathematics education, we are inclined introduce, what we call, 'manipulatives', in the form of tactile objects or visual representations. If we want to make something concrete in a everyday-life conversation, we look for an example. In the former, we try to make a concrete model of our own, abstract, knowledge; in the latter, we try to find an example that the others will be familiar with. This article first looks at the tension between these two different ways of making things concrete. Next another role of manipulatives, will be discussed, namely that of means for scaffolding and communication. In this role, manipulatives may function as means of support in a process that aims at helping students to build on their own thinking while constructing more sophisticated mathematics.

Key words: Conceret Learning Materials, School Math, Common Sense, Scaffolding, Communication

Abstrak

Jika kita ingin membuat sesuatu yang konkret dalam dunia pendidikan matematika, kita cenderung memperkenalkan, apa yang kita sebut, alat peraga, dalam bentuk representasi visual. Jika kita ingin membuat sesuatu yang konkret dalam percakapan kehidupan sehari-hari, kita mencari sebuah contoh konkret. Pertama, kami mencoba untuk membuat model konkret sendiri, abstrak, pengetahuan; dan kemudian, kami mencoba untuk menemukan contoh sehingga setiap orang akan familiar dengannya. Di awal artikel ini akan di fokuskan pada dua cara yang berbeda dalam membuat hal-hal yang konkret. Selanjutnya, peran lain dari manipulatives, akan didiskusikan, yaitu sarana untuk perancah (scaffolding) dan komunikasi. Pada bagian ini, alat peraga dapat berfungsi sebagai sarana penunjang dalam proses yang bertujuan membantu siswa dalam membangun pemikiran mereka sendiri, setelah itu, kemudian membangun matematika yang lebih canggih.

Kata kunci: Bahan belajar konkret, matematika sekolah, akal sehat, perancah, komunikasi

Introduction

Mathematics is abstract, and not easy to access by students. In education we often try to accommodate students by introducing tactile or visual models of the abstract mathematics we want them to learn. The idea then is to make the abstract mathematics concrete. However, if we want to make something concrete in an everyday-life discussion, we give an example that the others will be familiar with. So there is an interesting contrast between the way we make something concrete in everyday-life, and the way we do this in mathematics education. We may elaborate this distinction further by observing that in mathematics education, we use so-called manipulatives—either in the form of tactile objects or as visual representations--to help students to make connections with what *we* know. While, when giving an example that the others will be familiar with, in a conversation, we try to make a connection with what *they* know. The argument I want to make in this article is that our common way of making things concrete for the students does not work, and that we had better try to follow the other way of making things concrete by trying to connect to what the students know. I will substantiate this argument with some examples.

Comparing fractions

The first example is taken from a teaching experiment in grade 6, where the students we told about a bakery that would cut banquet bars (a sort of large cookies) to order. In this context, the students were given paper strips of a given length to enact the cutting process. They would for instance be asked to cut the banquet bars into eight equal pieces, or six, or ten, and so forth.

After that they we asked to use similar strips to compare 1/3 and 2/6. The students solved this problem by comparing the lengths of pieces produced in the two different divisions—either dividing by three or dividing by six. In doing so they came to the conclusion that 1/3 was <u>not</u> equal to 2/6. The reason for this surprising result was that the way they cut the strips was not very precise. Had they been given ready-made fraction bars, they would have come to the correct conclusion that 1/3 = 2/6. But what would that conclusion have been worth if it was only based on the--for the perspective of the students--arbitrary lengths of the pieces? With ready-made fraction bars, the students will 'see' that 1/3 equal 2/6, but they would just as easily believe that 1/3 dies not equal 2/6. In other words, the tactile representations do not support an insightful solution. We may argue that this is because they are not required to think, all they have to do is to believe that what they see in a given instance is a universal truth. We may argue that hat is no mathematics. From a mathematical point of view, we would want the students to reason that 1/3 *has to be* equal to 2/6. Moreover, this form of reasoning was well within the roam of possibilities for those students. For, when dividing banquet bars in various ways in the earlier activities, they use reasoning as a strategy. When dividing a banquet bar into 6 pieces, for instance, they used the following strategies:

- Either first divide the strip into 3 pieces, and divide each into halves,
- Divide the strip in two halves first, and divide each of those halves into three pieces.

On the basis of this, we may assume that these students would have been able to reason that 1/3 has to be 2/6. So instead of offering students concrete materials as a means of showing mathematical knowledge, we might capitalize on what they know. For many students it appeared rather natural to divide a bar into three parts first and each piece into halves next, to get six equal pieces. Thus on a practical level, they already knew that 1/6 is half of 1/3. According to the idea that we would have to connect with what the students are already familiar with, we would want to try to build on this practical knowledge to help students to reason how 1/6 relates to 1/3.
With this example in mind, we may denote the two ways in which 'concrete' can be understood as either “material concrete”, or as “common-sense concrete”. (See also Gravemeijer & Nelissen, 2007).

Another way to look at the same divide is to distinguish between an observer’s point of view and an actor’s point of view. The former relates to how we—as experts—see a problem. The latter concerns the way the students see the situation. As observers, we see the mathematics in the concrete models that are used. We see the relation between 1/3 and 2/6 in the paper cuttings, or in the ready-made fraction material. For the students, who do not bring our mathematical knowledge to the table, these are just blocks of various sizes. While trying to take an actor's point of view, we have to look at tactile and visual models from the perspective of the student, and ask ourselves:

- What does it signify for them?
- Is what is being represented the student’s own knowledge, or someone else’s knowledge?

When we ignore the students' point of view, we run the risk of disconnecting mathematics the students learn from their common sense. As a consequence, they may start to treat school mathematics and everyday-life reality as two disjunct worlds.

School math

We may illustrate this with an interview with a first-grader, called Auburn, conducted by Cobb (1989). The interview starts with some addition tasks that are presented as numerical expressions:

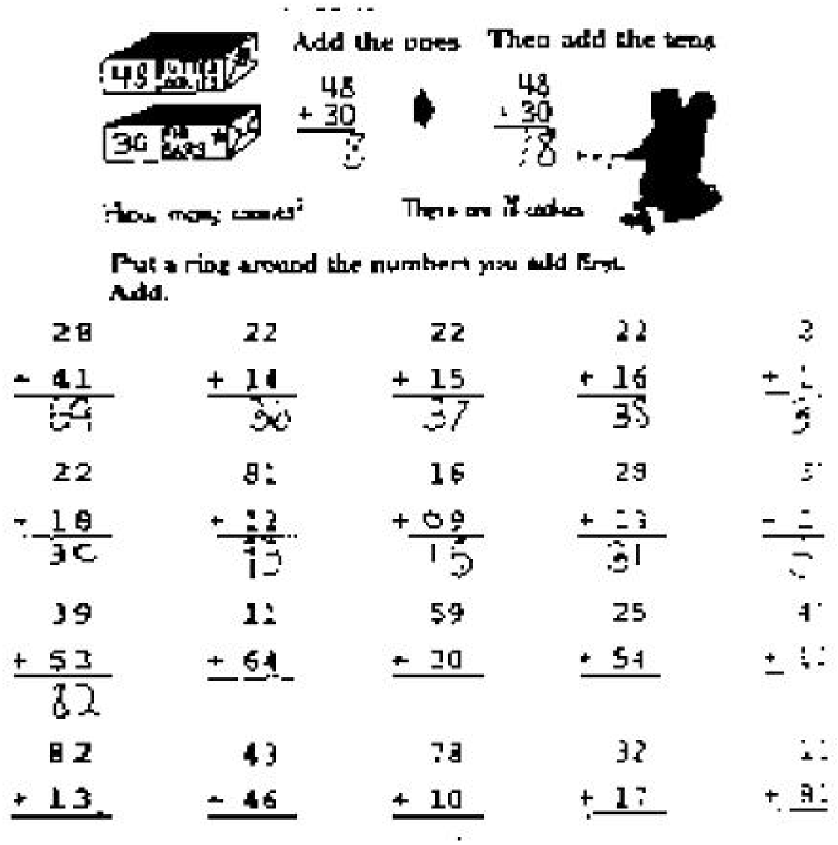
16 + 9 =

28 + 13 =

37 + 24 =

39 + 53 =

Auburn solves the first task, ‘16 + 9’, by counting on, and she arrives at the answer, ‘16 + 9 = 25’. Later, Auburn has to fill out a worksheet that contains the same task, now written in a column format.



Auburn solves this problem in the following manner:

16

<u> 9 +</u>

15

This then constitutes the starting point for the following exchange between the interviewer (I), and Auburn (A).

I : Is that correct that there are two answers?

A : ?

I : Which do you think is the best?

A : 25

I : Why?

A : I don't know.

I : If we had 16 cookies and another 9 added, would we have 15 altogether?

A : No.

I : Why not?

A : If you count them altogether you would get 25.

I : But this (15) is sometimes correct?
Or is it always wrong?

A : It is always correct.

For us this answer may be highly surprising, but for Auburn, the mathematics of the worksheets seems belong to a different world, a world that appears to be disconnected from the world of everyday-life experience. One of the consequences is that Auburn will not be inclined to use everyday-life knowledge to make sense of ‘school-math’ problems. For her mathematics has its own set of arbitrary rules that you just have to accept on the authority of teachers and textbooks.

Knowledge gap

We may conclude from the above that using tack tile or visual materials to make mathematics causes severe problems. The large difference between the abstract knowledge of the teachers and the experiential knowledge of the students causes a mismatch. Teachers and textbook authors (miss)take their own more abstract mathematical knowledge for an objective body of knowledge with which the students can make connections. However, the gap between the knowledge of the teachers and the knowledge of the students is too big to make this work. Manipulatives cannot bridge this gap, because, what those instructional materials signify is in the eye of the beholder. Experts who know the mathematics, see the mathematics, novices don’t. A way to overcome this problem is to shift towards a form of instruction that offers opportunities for the students to construct their own mathematical knowledge. In relation to this, Freudenthal (1987) offers the guideline, "Mathematics should start and stay within common sense”. He connects this with his idea of reality, which he defines as, “What common sense experiences as real”. He points out that what’s common sense for a layman is different from what’s common sense for a mathematician. The mathematician’s common sense will be on a higher mathematical level. A student may for instance reason that two odd numbers will add up to an even number on basis of concrete examples. For a mathematician, an algebraic approach will be common sense. Notating the first odd number as, 2m+1, and the second odd number as, 2n+1, and concluding that 2m+1 + 2n+1 = 2m+2n+2, which is even. From this perspective, learning than can be seen as expanding one's common sense, which corresponds with a growth of what constitutes mathematical reality for the learner. We may illustrate this difference in what is common sense for novices and experts with another example.

Common sense

For us as adults, “1+1=2”, is a matter of common sense, but this may be very different for young children. At a certain age, young children do not understand the question: “How much is 4+4?” Even though they may very well understand, that 4 apples and 4 apples equals 8 apples. The explanation for this phenomenon is that, for them, number is still tied to countable objects, like in “four apples.” At a higher level: 4 will be associated with various number relations, such as:

$4 = 2 + 2 = 3 + 1 = 5 - 1 = 8 : 2$, etc.

At this higher level, numbers have become mathematical objects that derive their meaning form a network of number relations (c.f. Van Hiele, 1973). When an elementary-school teacher is talking about numbers, he or she may very well be talking about mathematical objects that do not exist for students. So here again our everyday-life notion of teaching as helping students in making connections with new knowledge proves to be inadequate. How can students, for whom a number is a sort of adjective, make connections with numbers as mathematical objects?

In reflection, we may conclude that trying to make abstract mathematics concrete by representing the mathematics with tactile or visual models, is highly problematic. Such an approach presumes that learning can be seen as making connections between the internal knowledge of the student and some external knowledge that has to be acquired. This does not fit mathematics education, since the abstract mathematical knowledge they have to acquire does not yet exist for them. In this respect, teachers and students live in two worlds, the mathematical world of the mathematics of the teacher, and the world of everyday life of the students. The only way to bridge this gap is by trying to connect to what the students know, and helping the students to construct mathematics in a bottom-up manner.

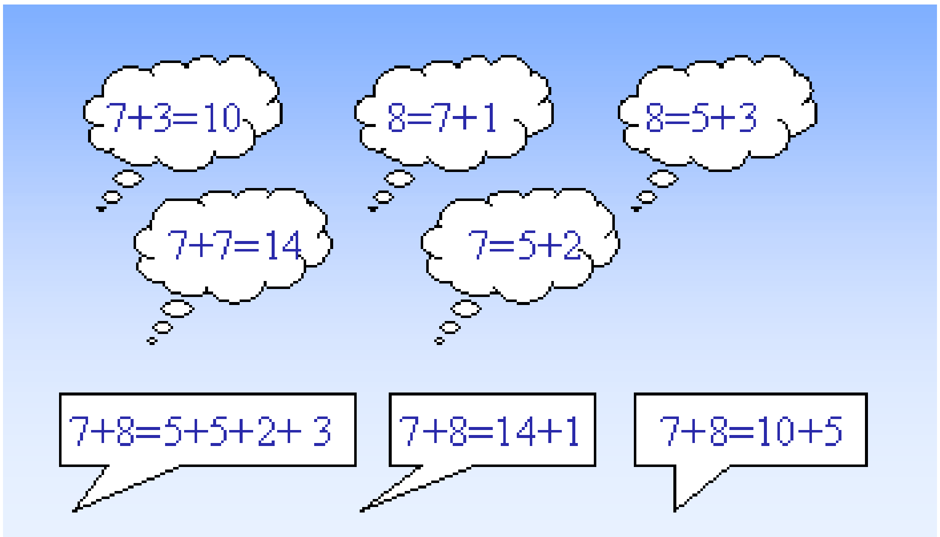
One might, of course, counter that experience shows that (at least some) people appear to have learned mathematics in spite of this problem. We may reason, however, that their actual learning process may have been very different from the presumed process of making connections. We may conjecture that what those mathematics learners really did was construct their personal theories about the alien body of knowledge that was presented to them. Theories they revised and adjusted on the basis of experiences and feedback.

This kind of learning has serious drawbacks, however. In the first place, it is very difficult. The process is prone to produce misconceptions that one has to overcome. The second drawback is the inherent uncertainty, the learner is always guessing about whether he or she has guessed the mathematics right. Knowledge and understanding is always preliminary in such cases; until the next contradiction, which will show that one's latest conjecture of what the body of knowledge entails is still off. A very likely consequence is math anxiety. Moreover, this lack of certainty, and always being dependent on the authority of teachers and textbooks, is in contradiction with the very nature of mathematics. Even if one develops some proficiency in this manner, we may ask ourselves if it is mathematics what has been learned.

Bottom-up, connecting with what students know

The alternative is to help students to construct mathematical knowledge in a bottom-up manner connecting with what the students are familiar with. In case of early number, the goal will be to help students in developing a network of number relations. A way to do so is by activities that involve structuring quantities. Here we will focus on helping students in coming to see that the same number relations hold for various contexts. In addition, we will have to support students in reasoning about number relations. Important steps here are (1) construing resultative counting as a curtailment of counting individual objects, and (2) construing 'counting on' and 'counting back' as extensions of resulative counting. On the basis of these two insights, students can establish the correctness of the number relations they find by generalizing over various contexts.

When building a framework of number relations for addition and subtraction up to 20, we may start by looking at the informal strategies that students invent by themselves. Research shows that proficient students develop strategies that make use of the doubles, and fives and ten as points of reference, such as 7+6=14-1, or 7+6=7+3+3=10+3, or 7+6=5+2+5+1=10+3. Mark that the goal of this bottom-up approach is to foster the flexible use of number relations, not to teach strategies. In our view, student knowledge of number relations forms the basis for what--from an observer's point of view--looks like the application of strategies. We would argue that what the students do is combining pieces of knowledge (number facts that are ready to hand to them) to derive new number facts.

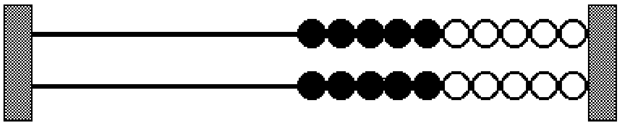


Scaffolding

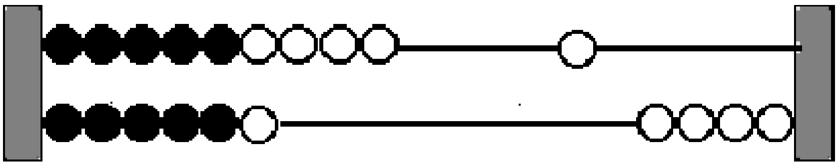
We started this article by observing that, in mathematics education, we often try to accommodate students by introducing tactile or visual models of the abstract mathematics the students have to learn. In the above we discussed the problems that come with trying to make the abstract mathematics concrete in this manner. That does not mean, however, that tactile and visual models cannot play a role. Also in the alternative both-up approach, we are advocating, such so-called manipulatives may offer a valuable means of support. Their role, however, is very different. Instead of function as means of showing the mathematical knowledge of the mathematics educators, manipulatives may be used to help students to express their own thinking.

The use of manipulatives will then be cast in terms of scaffolding & communicating. We may take the so-called arithmetic rack as an example (Treffers, 1990).

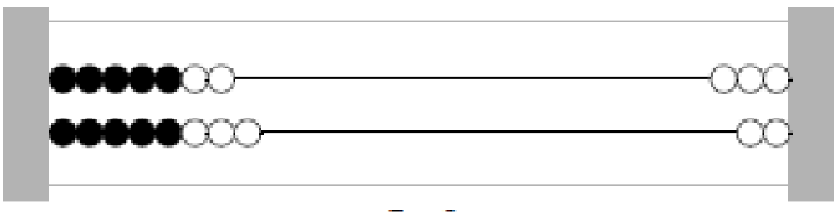
The so-called arithmetic rack may be used as a means of scaffolding & communicating. The arithmetic rack consists of two bars with five dark and five white beads on each bar.



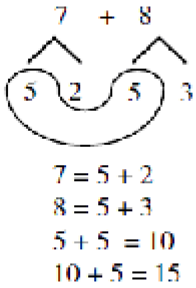
Students can visualize numbers on the rack by shifting beads to the left, while the beads on the left represent the number.



The structure of the colored beads on the rack can support the students' arithmetical reasoning. When adding 7 and 8, for instance. Capitalizing on their prerequisite knowledge students may realize that 7=5+2 and 8=5+3, and visualize that on the arithmetic rack.



Or they may realize that 5+5=10, or 7+7=14, or 8+8=16. As a next step, we may ask students to anticipate how to solve a given problem, thinking of how they might use the rack. An important activity then becomes, notating. Students are asked to invent ways of symbolizing to describe their reasoning.



Over time the students may become so proficient that they will not need visual scaffolding anymore. Their thinking may become so automated even that they do not consciously have to execute the intermediate steps, constructing the answers has become automated—some relations may even have become memorized facts.

Pitfalls

Summarizing, we may conclude that tactile and visual models can support learning processes that start with situations that are concrete in the sense of familiar to the students. Note, however, that this approach is not without risks. One of the most evident pitfalls is that the students may just count beads on the rack, or start to read off number relation from what they see on the rack. Mark that this would be quite similar to the risks we discussed earlier in relation to the ready-made fraction bars. Instead, we would argue that the more basic number relations have to be seen as a prerequisite. Before introducing the arithmetic rack, students have to become familiar with basic number relations—such as 5+2=7, or 5+3=8 and so forth. When these relations have become part of their “common sense”, they can use these relations to fluently place 7 or 8 beads, for instance, on the rack. Then the students can start to focus on arithmetical reasoning. When having to add 7+8, they might even anticipate using "5+5=10", before putting 7 and 8 on the rack. In such cases the rack may function as a means of scaffolding, it may help the students to keep track of the pieces they have to combine in a clever way; in this case 5+5=10, and 2+3=5, resulting in 10+5=15, as the answer to the original problem of 7+8.

Conclusion

"Making things concrete", may be elaborated as either making concrete what we know or hooking up with what the students know. We may call the first 'material concrete', and the second 'common sense concrete'. We pointed to the problems with the former, and elaborated the latter as a fruitful alternative. In relation to this we argued that, it helps to make a distinction between an actor's point of view and an observer's point of view. We tend to look at mathematics from an observer's point of view; implicitly bringing in all the mathematical knowledge we have. From such an observer's perspective many things may seem logical for us that are not so self-evident for the students. From the perspective of the students who have to solve the problems, or interpret the models we present to them, but do not have a similar mathematical background these same things may be incomprehensible. In this sense, it can be very valuable to try to imagine the actor's point of view of the student, and try to look through his or her eyes. In this manner, we can start with what is common sense for the students. From this point onwards, we may try to follow Freudenthal's adagio that 'mathematics should start and stay within common sense', by trying to foster the growth of what is common sense for the students. In such an approach, tactile and visual models will not be used to make the students "see" the abstract mathematics, instead, material and visual representations may be used by the students as means of scaffolding and communicating their own ideas.

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