Dividing Fractions Using an Area Model: A Look at In-service Teachers' Understanding

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The paper reports an investigation into how a group of elementary and middle school teachers collectively attempted to solve and understand a fraction division problem using an area model. Solving the word problem required that teachers determine how many two-thirds fit into three-fourths. The teachers struggled to conceptualise fraction division, to meaningfully connect it to the area model, and to interpret the fraction remainder. Developing such understanding was facilitated by allowing sufficient time for group discussion and collective thinking, supported by use of visual representation. During this process, it was important for the teachers to identify an appropriate unit of measure and referent unit, and to make sense of these in relation to each other and to the problem. The importance of connecting concepts to procedures and to comprehending and using other fraction models (linear, set) is noted.

Keywords · fractions · division · area model · word problems · in-service teachers

Introduction

Effective mathematics instruction involves an ability to integrate disciplinary and pedagogical knowledge (e.g., Ball, Thames, & Phelps, 2008). Teachers thus need to have sound mathematics content knowledge in addition to teaching skills (Ball et al., 2008; Wu, 2011). Unfortunately, many prospective and practicing teachers internationally have weak conceptual understanding of division of fractions, which limits their ability to teach the concept effectively (Chinnappan & Desplat, 2012; Fazio & Siegler, 2011; Isik & Kar, 2012; Lin, Becker, Byun, Yang, & Huang, 2013; Luo, Lo, & Leu, 2011; Rizvi & Lawson, 2007). In this paper, we discuss the results of an investigation into how teachers collectively attempted to solve and understand a fraction division problem using an area model.

Review of Related Literature

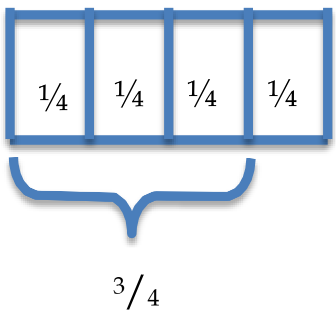
Dividing Fractions Using a Procedural Approach

The invert-and-multiply and common-denominator approaches are the two most common procedures used for dividing fractions (Petit, Laird, & Marsden, 2010). Most adults, including teachers, poorly understand the frequently used invert-and-multiply method (Philipp, 2008; Yimer, 2009). The same is true for students. Sharp and Adams (2002) note, "For many students, using the invert-and-multiply algorithm is an activity completely isolated from concepts and meaning" (p. 336). The common-denominator method is another procedural approach to dividing fractions (see, for example, Cramer, Monson, Whitney, Leavitt, & Wyberg, 2010). The lack of understanding that often accompanies use of procedural methods stems from a tendency to teach these approaches as sets of memorised procedures (cf. Petit et al., 2010; Philipp, 2008).

When teachers learn mathematics superficially and thus do not fully understand underlying concepts, they cannot help students learn mathematics meaningfully (Ball et al., 2008; Philipp, 2008). This understanding is important because teachers should balance teaching algorithmic procedures with engaging the meaning behind them (Li, 2008; Petit et al., 2010).

Aspects of Fraction Division

A number of key actions and factors are involved in performing fraction division. *Unitizing* plays a central role in solving problems involving fractions. According to Lamon (2012), unitizing is a subjective and natural process that involves "constructing mental chunks in terms of which to think about a given quantity" (p. 104). For example, one could think of a case of soda as a case or four six-packs or 24 cans. Allowing and encouraging flexibility in the manner in which items are chunked can benefit problem solvers. Another concept fundamental to understanding fractions is the knowledge that an individual object or a set of objects can be *partitioned*, or divided into equal-sized parts or equal quantities in numerous ways that allow for a unit or fractional part to assume different names (Lamon, 2012; Lee & Orrill, 2009; Luo et al., 2011; Petit et al., 2010). Identical copies of a fractional part can be *iterated*, or repeated, across a designated area or added numerically to achieve a goal, such as covering or totaling a unit of one or a given fractional part, as in Figure 1 (Lee & Orrill, 2009; Son, 2011).



$\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{4}{4}$ or 1

$\frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$

Figure 1. Partitioning a unit of 1 or a fractional part (3/4) into ¼'s and iterating the partitioned parts to total the referent unit.

The idea of a *referent unit*, a concept that is challenging to and rarely addressed by teachers, is important for understanding fraction division. This concept is based on the fact that "the divisor becomes the referent unit for the dividend" (Orrill, de Araujo, & Jacobson, 2010, p. 3). For example, when dividing ¾ by ½, the answer 1 ½ indicates that there are one and one-half ½'s in ¾, the referent unit being ½ rather than one. This involves a multiplicative relationship within the notion of fractions as operators, and it is important for interpreting the quotient. Context, too, plays a role in fraction operations. As Fosnot and Dolk (2002) note, "Different contexts have the potential to generate different models, strategies, and big ideas" (p. 16). Nevertheless, textbooks tend to omit context for procedural methods to fraction division, such as invert-and-multiply, or they include context superficially before quickly moving to symbolic methods (Cramer et al., 2010). Context can both support and constrain problem solvers' ability to make meaning of dividing fractions. In the former case, context can help connect symbolic procedures to realistic situations; in the latter, some contexts may not be sufficiently general to support different models of fraction division and can thus limit flexibility in thinking and even foster confusion (Orrill, et al., 2010). Even with realistic contexts, students tend to ignore the setting and use procedural approaches unless they are encouraged to engage the context through probing questions and rich discussion (Widjaja, Dolk, & Fauzan, 2010). In general, students need to work with problems that are set in varied contexts and involve both partitive and quotitive division models and have different remainders (Petit et al., 2010).

Conceptualising Fraction Division with an Area Model

One model used to conceptualise division of fractions is an area model. Figure 2 shows Lamon's (2012) area model for a division problem that addresses how many 2/3's there are in ¾ (for the problem ¾ ÷ 2/3).

Instruction on Division of Fractions

As noted, teaching fraction division is problematic because both teachers and students have difficulty understanding the method conceptually (Fazio & Siegler, 2011; Isik & Kar, 2012; Luo et al., 2011; Orrill et al., 2010; Petit et al., 2010; Rizvi & Lawson, 2007). Petit et al. (2010) state, "Multiplication and division of fractions are among the most complicated fraction concepts that elementary students encounter…. [They are] consistently a source of confusion for students" (p. 161).

One major difficulty is meaningfully connecting visual representations of fraction division problems with their corresponding symbolic procedures (e.g., Perlwitz, 2005). For example, Perlwitz (2005) asked pre-service teachers to determine how many pillowcases can be cut from a 10-yard-long piece of fabric if each pillowcase requires $^3/_4$ yard of length. Although pre-service teachers could solve the problem using visual representations (e.g., drawings), they had difficulty reconciling their differing pictorial and algorithmic answers. They got an answer of 13 ¼ using a visual representation but 13 $^1/_3$ using an algorithm. The 13 represents the number of ¾-yard pieces obtained from the 10-yard fabric. The accompanying ¼ in one case refers to the length of leftover material, whereas the $^1/_3$ represents $^1/_3$ of a pillowcase that could be made with the leftover material. Perlwitz points out that the students had difficulty making meaning of the fractional parts, which must be interpreted in relation to their appropriate referent units. The correct answer of 13 $^1/_3$ results from determining what portion the leftover ¼ yard of fabric comprises of its referent unit ¾ yards (the amount required to make a whole pillowcase).

Because teaching and learning division of fractions is a significant concern in mathematics education, we studied how in-service teachers solved and understood a fraction division problem. Although both teachers and students struggle with this concept, we believe it is especially critical to gain insight into teacher understanding as a precursor to student learning.

Method

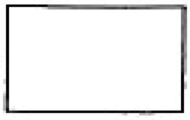
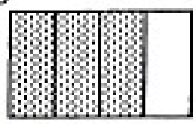
In this study, we investigated how teachers collectively attempted to solve and understand a fraction division problem using an area model, specifically, Lamon's (2012) area model. We chose an area model as an entry point for examining this concept because U.S. teachers, the participants in this study, have been shown to find this model easier than linear or set models in working with

AREA MODEL FOR DIVISION

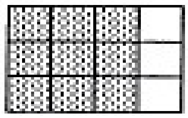
Division may also be interpreted as the composition of two operators and may be modeled using an area model.

- $\frac{3}{4} \div \frac{2}{3}$

The division answers the question: "How many $\frac{2}{3}$ s are there in $\frac{3}{4}$?"

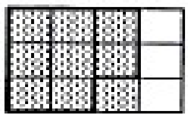
Let  be the unit and shade $\frac{3}{4}$ of it. 

In the horizontal direction, divide the unit into thirds but DO NOT SHADE.



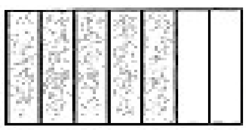
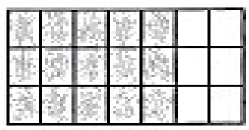
How much is $\frac{2}{3}$ of 1? Notice that it is the area of 8 squares.

So our division question becomes: How many times can we measure an area of $\frac{2}{3}$ (or 8 small squares) out of the shaded area representing $\frac{3}{4}$?

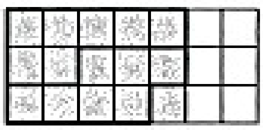


Out of the area $\frac{3}{4}$ we can measure the area $\frac{2}{3}$ once and then we have 1 small square out of the next 8. So the answer is $1\frac{1}{8}$ times.

Notice that the unit area was used to determine the area that corresponds to $\frac{3}{4}$ and $\frac{2}{3}$, but the divisor, $\frac{2}{3}$, became the new unit of measure and the remainder was written as part of that unit.

- $\frac{5}{7} \div \frac{1}{3}$  

$\frac{1}{3}$ is equivalent to 7 small rectangles. So we ask: How many times can we measure $\frac{1}{3}$ (7 small rectangles) out of $\frac{5}{7}$ (the shaded region)?



We can do that twice with 1 rectangle remaining. So we get $2\frac{1}{7}$.

Figure 2. Lamon's (2012) area model for dividing fractions (pp. 200-201).

fractions (Luo et al., 2011), in addition to the fact that an area model provides more room for flexibility and creativity in partitioning objects. The area model involves consideration of a region as a whole in terms of both its length and its width and can thus be partitioned in different directions (e.g., resulting in a 4 x 3 rectangle), whereas a linear model is only partitioned along one dimension, that of length (e.g., resulting in a 4 x 1 rectangle or distance along a number line), and a set model uses discrete objects that may or may not be the same size and shape and which collectively comprise one whole. Within the choice to use an area model, we chose Lamon's rectangular model, given that rectangles are easier to partition equally than other area models, such as circles.

Participants in this study were all twelve teachers who took part in professional development sessions designed to improve practicing teachers' content knowledge. These teachers taught grades 3-9 in a rural school district in a western state in the United States. Their teaching experience spanned 5-15 years, thus including a range from more novice to more veteran status.

The data for this paper are drawn from one four-hour summer session that was devoted to fraction division. Although the teachers worked at tables in groups of four, the data reported here are drawn only from the whole-class discussions that followed the small-group work. Before engaging in fraction division tasks, the teachers were asked to do some reading on fraction computation, including Lamon's (2012) chapter on division and multiplication. During their work, the teachers had access to materials such as pattern blocks, chart paper, and coloured transparencies that teachers could overlay on each other.

The professional development session was videotaped with participant comments transcribed. The data were categorised into themes based on teachers' efforts to make sense of dividing fractions. These themes were constructed using Corbin and Strauss's (2008) constant comparative method, whereby new themes were added as they were identified and data were sorted into these themes. During multiple reviews of the data, categories were added, combined, deleted, and renamed until the themes accurately reflected the data. Chart paper on which the teachers collectively shared work during their whole-group discussions served as an additional data source to support this investigation.

Findings and Discussion

The study results are organised into major themes that appeared in the teachers' efforts to understand fraction division. These include: difficulty conceptualising division of fractions; making sense of the divisor and dividend; visualising the multiplicative relationship in fraction division; measuring to find an answer; limitations of the visual illustration.

Difficulty Conceptualising Division of Fractions

The teachers in this study found visualizing and understanding the area model difficult. The two professors facilitating the professional development struggled to determine how to lead the conversation in a way that might enable the teachers to make sense of fraction division through cooperative problem solving. Based on previous experience where they noted teachers having difficulty seeing the referent unit, they chose to focus teachers' attention on unitizing. The teachers were asked to think independently about Lamon's (2012) area model for division before discussing it with the group, specifically, to determine what is involved in the process of solving a fraction division problem. In other words, the orientation became: "What are you actually doing when dividing fractions?"

While pondering this directive individually, the teachers had difficulty understanding the model. Meaningfully connecting the model to division of fractions was not obvious. The following conversations illustrate this point.

> Mary: This division, I was trying to make sense. I don't see it! Invert and multiply! (Raises hand to express frustration.)

Mary's response indicates that she attempted to understand the area model by herself, but it did not make sense to her. She indicated that she had difficulty seeing how the problem-solving strategy using an area model represented division of fractions. Her comment "Invert and multiply!" showed her frustration in linking the two methods. Mary's response implies that when she cannot comprehend a mathematical idea, she reverts to a procedural approach.

Donna also indicated that she could not "digest" the task. She made the following comment about dividing fractions using Lamon's (2012) area model:

> In doing this, I was confused. The pictures threw me off because I had learned the old algorithm. It did not make sense to me…. When you are dividing the fraction the unit does not stay the same and I realised that, and it was always confusing to me why when you divide you are coming up with bigger numbers in the end…. I don't understand that yet, but at least I have a start to where I am beginning to get it.

Like Mary, Donna had difficulty visualising division of fractions. However, she made a greater effort to grasp the area model, whereas Mary just gave up in frustration. Donna indicated that her knowledge of the algorithm hindered her from interpreting the area model. However, she seemed to understand the meaning and role of the referent unit. She also observed that the answer sometimes gets bigger when dividing two fractions and determined that this had something to do with the size of the divisor and dividend. However, at this point the problem-solving process was still not conceptually clear to her. She was able to partially understand the model, recognising that units and change in units was involved. Comments made by these two teachers illustrate that simply using the area model is not sufficient to make sense of fraction division. These responses also confirmed our conjecture that teachers have difficulty understanding representations of fraction division.

Next, we posed the following problem:

I have ¾ cups of sugar. My cake recipe needs $^2/_3$ cups of sugar. How many cakes can I bake? We asked the teachers to think about the problem individually and then talk to others about how they could solve the problem by meaningfully applying the area model. (The problem involved dividing ¾ by $^2/_3$.)

Donald volunteered to share his thinking. He explained his strategy to the whole group rather quickly by drawing on chart paper. Because the others were a bit "lost," we asked Donald to explain his thinking again. This time we requested that the other teachers pose specific questions if anything was not clear to them. We also asked them to add further explanation and engage in joint sensemaking of the process involved in dividing fractions. We asked Donald to again record his thinking on chart paper to facilitate this process. The goal of simply understanding how Donald solved the problem had shifted to thinking about the problem in the broader context of discerning what happens when one divides two fractions.

Making Sense of the Divisor and Dividend

The teachers examined the meaning of the divisor and dividend in relation to the original unit of one. They also explored the multiplicative relationship between the divisor and dividend. The following conversation illustrates the teachers' thinking about the divisor and dividend in terms of part-whole units. Donald drew a diagram on the chart paper and the discussion involved interpreting the drawing and the problem context.

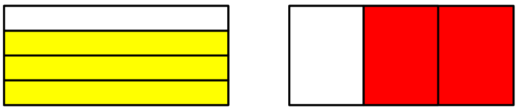


Figure 3. ¾ and 2/3 of the original whole unit.

Gina: The original unit, I am wondering too, since we are finding $^2/_3$ of the whole unit, should we put, "How many $^2/_3$ are there in ¾ of the original unit to lead us to there?" (See Figure 3.)

Gina: Or you would say, "How many $^2/_3$ cups are there in ¾ cup?"

Troy: But I still don't think we changed any units yet. We are looking at the whole unit. We are looking at what does $^2/_3$ of the whole look like?

Mary: Okay.

Gina: Can't we write $^2/_3$ cups? Would that help? Would that make any difference?

Donald: It could be miles. It could be feet. So now what we want to do here is to see what $^2/_3$ of our unit looks like. (See Figure 3.)

Gina: You want to look at $^1/_3$ and $^2/_3$. At this point it does not matter it is shaded. It does not matter that ¾ is shaded. We just want to see $^2/_3$ of one whole.

Mary: Oh, okay! I see it!

Donald: We want to see $^2/_3$ of the whole!

The teachers discussed the meaning of the divisor and dividend in relation to the whole unit of one. (In this paper, we will refer to the original unit of one as the "original unit.") The teachers identified the divisor and dividend as part-whole relationships with the original unit. They concluded that the ¾ and $^2/_3$ needed to refer to the same size original unit of one. Mathematically thinking about the dividend and the divisor in relation to the unit of one is important. This problem would not make sense if different-sized units, such as a square and a rectangle of differing area, were used to determine the ¾ and $^2/_3$.

The teachers started to think about the relationship between the divisor and dividend in terms of measurement. This is illustrated in the language used: "How many $^2/_3$ are there in ¾ of the original unit to lead us there?" This statement implies that in order to figure out the answer, one has to identify "how many $^2/_3$ make up the ¾ unit?" The teachers pointed to the $^2/_3$ and the ¾ drawing when making these comments. During this process, they demonstrated that $^2/_3$ was the unit of measure that needed to be overlaid on top of the ¾. Their comments illustrate that visualising this relationship is different than simply looking at a part-whole relationship. They were considering the measurement relationship, which is multiplicative in nature. In other words, "How many multiples of the divisor, or unit of measure, make up the dividend, or amount to be measured?"

Even though the problem context was given, the teachers determined that giving the unit an appropriate label, such as "$^2/_3$ *cup*," was not important. Donald pointed out that this model would work with any kind of unit. Therefore, it is likely that the teachers were trying to generalise what happens to the divisor and dividend during the problem-solving process as opposed to simply thinking about the specific problem context. In other words, they were trying to create a general approach for solving a fraction-division problem that made sense to them.

Visualizing the Multiplicative Relationship in Fraction Division

Once teachers had determined that the problem required thinking about "How many $^2/_3$'s are in ¾?" they engaged in the physical action of overlaying the $^2/_3$ on top of the ¾. Donald drew and shaded ¾. Then he drew $^2/_3$ on top of it to indicate how much of the original unit of one the unit of measure (divisor) covered. He drew a crosshatch design on the chart paper to fully extend all horizontal and vertical lines from side to side such that ¾ had been partitioned horizontally and $^2/_3$ had been partitioned vertically. These markings, when extended, divided the original unit of one into twelve sections. (See Figure 4.)

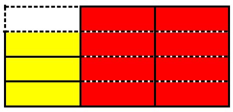


Figure 4. Finding $^2/_3$ of ¾ of the original unit.

Mary: You have to move the crosshatch to the bottom, which is what they did not do in the book. Which would have it made much clearer or rather than just.,,

Dawn: Or you could just use two colours like that.

Maria: Mmm hmm.

The drawing allowed the teachers to visualise what a $^2/_3$ unit of measure looked like in relation to ¾ of the original unit of one. They could see the parts that overlapped and did not overlap. They also saw how they could divide the original unit of one into smaller parts that could make the unit of measure (divisor) cover the part to be measured (dividend).

Mary's comments indicate that seeing $^2/_3$ and watching it be physically moved onto ¾ was an important part of being able to visualise the multiplicative relationship. Use of different colours also helped. This is because "seeing" the multiplicative relationship in a drawing is not readily apparent. It is important to note that the overlay shows that both the $^2/_3$ and ¾ refer to the unit of one. Troy's comments indicate that visualising the unit of one is important.

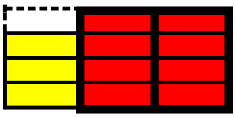


Figure 5. Examining the partitioned unit of measure.

Troy: I think because you started with $^2/_3$ and the numerator being 1, it just made sense because you see whenever you stay into fourths. Your next answer, instead of changing into eighths or whatever, it changes into…that was the part that threw me into a loop.

Donald: Now we know what $^2/_3$ of this whole looks like. How many blocks do we have in the $^2/_3$? There are eight. (See Figure 5.)

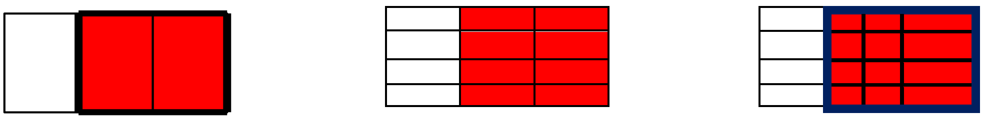
Troy: All right.

Troy was able to visualise $^{2}/_{3}$ of the whole unit that was overlaid on top of the 3/4. He could even see the dividend that was divided into fourths. However, he found it confusing to understand what Don meant by the eight blocks. It was only when Don asked how many blocks were in the shaded $^{2}/_{3}$ that he was able to visualise the unit of measure, or divisor, as being partitioned into eight equal pieces (see Figure 5).

Once the unit of measure ($^{2}/_{3}$) was placed on top of the amount to be measured (¾), the teachers were able to visualise the multiplicative relationship. They thought about how many $^{2}/_{3}$'s are in ¾ in relation to the original unit. When they overlaid the two-thirds on top of the three-fourths in the drawing, they were able to see that the unit of one could be partitioned into twelve equal pieces. Each partitioned piece is one-twelfth of the original unit. Mathematically, the teachers were informally finding the least common denominator for the two fractions.

In order to physically measure how many times the unit of measure (divisor) fits in the amount to be measured (dividend), one has to decompose the unit of measure into smaller equal- sized units so it will fit evenly inside the amount to be measured. The teachers flexibly viewed the unit of measure as an entity that could be decomposed and recomposed as long as they kept the original total area intact.

Troy noticed that the unit of measure was partitioned into eight equal pieces once Donald clarified what he wanted Troy to note about the unit of measure. In doing so, the two viewed each piece of the unit of measure as representing one-eighth of that unit and treated the unit of measure as one unit partitioned into eight equal pieces. One-eighth of the unit of measure represented one-twelfth of the original unit. The diagrams in Figure 6 show how the unit of measure was viewed differently during the problem-solving process:



2/3 of the original unit of 1 | 8/12 of one original unit | Unit of measure partitioned into eighths

Figure 6. Unit of measure re-unitized during the problem-solving process.

The first diagram in Figure 6 illustrates two-thirds of an original unit. Then it was reconfigured to represent eight-twelfths of the original unit even though the teachers never mentioned the term eight-twelfths. The unit of measure was further viewed as a single unit partitioned into eight equal pieces. Therefore, each piece represented one-eighth of the unit of measure. Finally, it was necessary to convert the unit of measure into pieces that could be de- composed and moved into a different shape while maintaining the same total area. In other words, two-thirds was re-unitized to represent eight-twelfths. The teachers did not symbolically represent the twelfths. Rather, they were able to see that the original unit could be partitioned into twelve equal pieces. The goal in doing this was to fulfil the act of measuring. The teachers needed to break up the unit of measure to make it physically fit into the amount to be measured. Therefore, they viewed the unit of measure as one unit partitioned into eight pieces. They considered each piece of the unit of measure as one-eighth. They needed to do this to determine how many blocks fit inside the measured amount (dividend). Further, they needed to know that the unit of measure was re-unitized as eight blocks so it could be decomposed to fit into the shaded area of ¾ (the dividend) for measurement purposes.

Measuring To Find an Answer

The teachers' discussion below reflects their collective transformation in understanding as they implemented a measurement method that involved decomposing the unit of measure and determining how many times it would cover the amount to be measured.

Donald: So now, how do you want to show these 8 blocks fit into that ¾?

Mary: Two that are in this side. Just crosshatch on the other side.

Donald: That is probably the easiest way, right? We want to take these areas in the blank areas (pointing) and put it down here.

Donald: We were working with this area right here (pointing to the shaded ¾ area). What we want to do is to take these guys here (pointing to the two one-eighth pieces located inside the original unit but outside the amount to be measured, or the dividend). That is what we have done is to take these guys here and move them down here.

Maria: Would you draw an arrow to show that

please? Donald: Sure (draws arrow). (See Figure 7.)

…

Troy: Now it makes more sense to me. "How many $^2/_3$'s are in there?" Now is where the sentence on the bottom part actually fits in because now we are looking at ¾ being our unit or however we want to phrase it. Now we are actually looking at ¾ being our unit, because we are looking at how many $^2/_3$'s are in that ¾ that we have!

Donald: Yeah, this is where the ¾ becomes our unit (of reference).

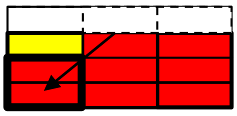


Figure 7. Fitting the unit of measure into the referent unit.

Once the unit of measure was partitioned so it would fit inside the amount to be measured, Donald extended his drawing to illustrate the physical action of moving parts of the decomposed unit of measure to fit inside the amount to be measured, as illustrated in Figure 7. This physical act involved thinking about measurement multiplicatively. In other words, the teachers were not looking at just a part-whole relationship. Rather, they examined how much of the unit of measure would fit inside the amount to be measured. It is important to note that this process also involved visualising the dividend, or amount to be measured, as being made up of the nine blocks of the original unit partitioned into twelfths. This involved visualising the referent unit in a different way, as illustrated in Figure 8.

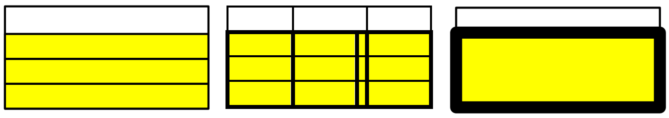


Figure 8. Shifting view of amount to be measured during the task.

First, the ¾ was considered a part-whole concept (three of four equal parts) in relation to the original unit. Then the original unit was partitioned into twelve equal squares, with the ¾ (dividend) now seen as representing nine of twelve squares comprising the decomposed original unit. Finally, the dividend (¾) was considered a single unit once it was viewed as a total amount to be measured based on the task context.

Limitations of the Visual Illustration

One teacher commented that students might look at the visual illustration of the referent dividend and think the answer should be 1 1/9 because the drawing of the dividend is divided into nine blocks. Therefore, visually it might seem that eight blocks make up one unit and the leftover block might be considered one-ninth because it is one of nine blocks left over from the amount to be measured, or dividend (revisit Figure 7). The following discussion took place:

> Donald: Yeah, and this thing right here (pointing to the 1/8 square in the referent unit—return to Figure 7)…the little square that is leftover to me is a big issue! Because we said that 2/3 of this thing was these 8 squares. Ok, also this square right here is the eighth. I think this is going to be an issue with a student because if you look at this there are 9 squares. Why isn't it 1/8 instead of 1/9?
>
> Troy: The unit is in
>
> eighths. Donald: Exactly.

The discussion indicated that it was important for the teachers to actively think about the unit of measure as being made up of eight squares. So, the size of each block represents one-eighth of the unit of measure. The unit of measure had to be re-unitized during the problem-solving process as illustrated in Figure 9.

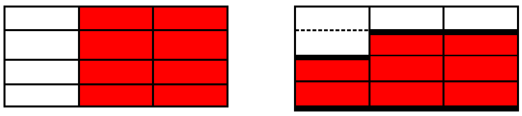


Figure 9. Re-unitizing the unit of measure from 2/3 to 8/8.

Closing Comments

Cognitively, the problem used in this research involves determining how many 2/3's fit into 3/4. The teachers re-conceptualised the question into how many of eight blocks of the unit of measure (2/3) fit inside the nine-block amount to be measured (3/4 of the original unit). Each section of the unit of measure thus equals one-eighth of that unit, or one-twelfth of the original unit, and each section of the amount to be measured represents one-ninth of that total area and also one- twelfth of the original unit. Understanding the area model used in this problem required that participants make sense of both fraction division using the model and the fractional part of the remainder. Both of these can present challenges, especially the latter. Cramer et al. (2010) discuss the importance of recognising that fractional parts can have different names and that "understanding the role of the unit is critical when using pictures to divide fractions" (p. 341). Indeed, the teachers' struggles interpreting the remainder in fraction division in this study mirror that found by Perlwitz (2005), noted earlier. Making sense of a remainder in division can be challenging with whole numbers but appears to be even more so when dividing fractions. Accordingly, teachers should have much experience solving and making meaning of division problems involving different fraction pairs set in varying contexts, with one important goal being to conceptually grasp the fact that a leftover fractional part must be compared to the divisor as the referent unit to figure the fractional remainder that appears as part of a non-whole-number answer. This taxing concept should be thoroughly examined mathematically and in meaningful

contexts, beginning with easier problems such as ½ divided by $^1/_4$. Discussion of the appropriate referent unit and why it is not the original whole, or one, should be explicit.

This study shows that Lamon's (2012) area model can be a useful tool for making sense of fraction division. However, simply examining or constructing the drawings did not help teachers meaningfully connect the area model with fraction division on their own. Specific discussions needed to take place that attended to unitizing and, accordingly, partitioning. This process is as much cognitive as physical, involving active reflection supported by discourse and visual representation. It involves development of negotiated understandings because not everyone sees the same thing when they view a model. When it comes to dividing fractions, Cramer et al. (2010) note the importance of "using language to facilitate the translation from the picture to symbols" (p. 346). The findings from this study thus resonate with Chamberlin's (2009) research in which teachers reported small-group collaboration and use of visuals as being important factors that supported their own mathematics learning during a professional development course.

Although the teachers in this research collectively made progress in their understanding of the meaning of dividing fractions through exploration of an area model, they need more time to generalise and crystalise their thinking, especially–as noted–with regard to presenting fractional parts of quotients and understanding how they are derived (which involves an understanding of the referent unit). They also need time to connect pictorial models to symbolic procedures. Although the area model used in this study can link to the common-denominator procedure of dividing fractions, it is more hard-pressed to illustrate the invert-and-multiply procedure. It is important that meaning be made of models rather than simply learning to construct them appropriately and identifying and reporting an answer, which differs little from symbolic procedural approaches. This is a risk to keep in mind when using an area model for dividing fractions. It must not become an exercise that problem solvers learn to execute without conceptual understanding of the steps and the outcome.

In research conducted with preservice teachers in Taiwan and the United States, U.S. participants showed significant differences among area, set, and linear models in fundamental fraction knowledge, including meanings of fraction operations (Luo et al., 2011). U.S. participants found the area model easiest and were most troubled by the number line, a linear model. Because the Taiwanese participants significantly outperformed the U.S. participants in fundamental fraction knowledge and they showed no significant differences in performance in relation to the three types of fraction models, a reasonable speculation is that developing facility with area, set, and linear models might improve U.S. preservice teachers' fraction knowledge. More research is needed with varied models to help teachers—and, ultimately, students—understand division of fractions conceptually and to meaningfully connect various pictorial models to the efficient procedural methods that are part of the important repertoire of strategies both teachers and students must possess in mathematics.

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