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Thomson's Theorem of Electrostatics: Its Applications and Mathematical Verification

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ABSTRACT

A 100 years-old formula that was given by J. J. Thomson [1] recently found numerous applications in computational electrostatics and electromagnetics. Thomson himself never gave a proof for the formula; but a proof based on Differential Geometry was suggested by Jackson [2] and later published by Pappas [3]. Unfortunately, Differential Geometry, being a specialized branch of mathematics, is normally inaccessible to the majority of engineers and engineering students. This paper provides for the first time a proof that greatly reduces the dependence on Differential Geometry (the reader only needs to be aware of the concept of "curvature"). The emerging applications of the theorem in computational field problems are also discussed here.

Keywords: Thomson's theorem, electromagnetic fields, computational field problems

INTRODUCTION

In 1891, J.J. Thomson, the discoverer of the electron, stated without a proof a formula [1] that relates the vertical potential gradient (or electric field intensity) at any point on the surface of a charged conductor to the mean curvature of that surface. The formula remained unrecognized and unused for nearly 100 years, until the emerging new techniques of the rapid numerical solution of electrostatic and electromagnetic field problems finally put Thomson's formula in the spotlight [4, 5, 6, 7, 8]. The formula was also found to be extremely valuable in a similar application, namely, numerical geodesy (the study of the geoid, or the equipotential surface of the earth) [9]. (Note that Joseph John Thomson's theorem is often confused with another theorem of electrostatics that was published in 1848 by Sir William Thomson. That earlier theorem discusses the minimum energy principle in the distribution of a system of charges.)

While Thomson didn't originally provide a proof for his theorem, there seem to be hardly any proof for it in the scientific literature of the past 100 years either. The few attempts to prove the theorem that have in the past appeared in the literature show that the theorem is in fact quite challenging to prove (except for trivial geometrical configurations). The first attempt to prove the theorem was apparently the one suggested (but not published in a detailed manner) by J.D. Jackson in his popular book *Classical Electrodynamics* [2]. The solution is based on Gauss's theorem of differential geometry (note that this is not the same as Gauss's theorem of electrostatics!). Unfortunately, differential geometry, being a specialized branch of mathematics, is normally inaccessible to the majority of engineers and engineering students. In fact, Jackson's proof would be accessible only to mathematicians and specialized physicists. A second attempt to prove Thomson's theorem (aimed at providing a simpler proof for the theorem) was published by Estevez and Bhuiyan in 1985 [10]. Those authors, however, treated the problem using an incorrect mathematical approach (power series) and essentially failed to give a valid proof for the formula. Finally, a proof was published by Pappas in 1986 [3]. The Pappas proof, however, was essentially a full development of Jackson's differential geometric idea. From 1986 to the present, the scientific literature was once again silent about the origin of Thomson's theorem. It therefore appears that the only valid proof for the theorem that currently exists in the literature is the proof based on differential geometry.

The purpose of this paper is to demonstrate, for the first time since 1891, that Thomson's theorem can be in fact proved in a straightforward manner directly from the fundamental laws of electrostatics, with only minimal dependence on differential geometry (the reader only needs to be aware of the concept of "curvature"). Since such a proof does not exist in the scientific literature, it will be of important educational value for science and engineering students; particularly students of electromagnetism. This paper is organized as follows: Section 2 gives an introduction to Thomson's theorem and its applications in the numerical solution of field problems. Specific suggestions for incorporating the material in a typical undergraduate course in electromagnetism are given. Section 3 gives the new proof of the theorem.

THOMSON'S FORMULA AND ITS APPLICATIONS

In a small footnote written in 1891 [1], J.J. Thomson gave the following formula

$$\frac{\partial |E|}{\partial z} = -|E|\left(\frac{1}{R_1}+\frac{1}{R_2}\right) \tag{1}$$

This formula relates the normal derivative of the magnitude of the electric field intensity $\vec{E}$ to the so called "principal radii of curvature", R_1 and R_2, of an equipotential surface to which $\vec{E}$ is perpendicular. Here, z is the normal direction to the surface at the point under con sideration and X, Y are oriented along the "principal directions" (the directions of maximum and minimum curvature) on the surface (see Figure 1). As shown in the Figure, the electric field $\vec{E}$ is directed along the Z axis. (Note that no prior knowledge of differential geometry is required to understand the derivations in this paper. Differential geometry says that the maximum and the minimum curvatures at any point on a surface will occur at orthogonal directions. See [12] for proof.)

Thomson's theorem and its applications are currently being taught by the author as part of the Electromagnetics I course at the University of West Florida. The simplest way to understand the theorem and visualize its application is to write software for curvilinear squares field mapping. Since the electric field intensity at a point adjacent to an equipotential surface is directly related to the curvature of the surface (or curve, in 2 dimensions), then a simple curvilinear squares field mapper can be written in a language such as C++ or Matlab and used to trace and visualize the electrostatic field throughout the domain of a given problem. Such a software application was developed by the author and is currently being used by the students at the University of West Florida. An outline of the software algorithm is given in the Appendix. More details can be found in [6].

Thomson's theorem is usually taught to the students following the introduction of Laplace's equation, since the theorem can essentially be regarded as a "visual Laplace solver". The student is typically asked to write Matlab code to solve a simple boundary value problem by using the finite-difference (mesh) technique, where potential values are assigned to the nodes in the mesh. The student is then asked to run the code that is based on Thomson's theorem, where a curvilinear squares field map is actually traced step by step on the screen. The student finally compares the graphical field map with the "numerical" map that was obtained and verifies that the two maps

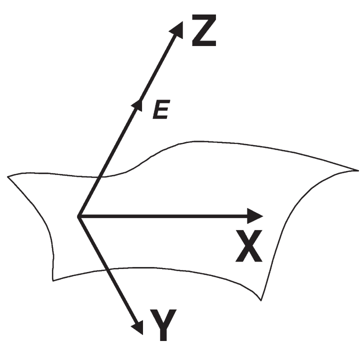

Figure 1. An orthogonal coordinate system at a point on an equipotential surface.

correlate. Examples of the output of the graphical Matlab code are shown in Figure 2. Figure 2(a) shows a plot of the potential function $V = X^2 - Y^2$, which satisfies Laplace's equation. Figure 2(b) is a plot of the equipotentials that exist between two electrodes, where one of the electrodes is a sphere and the other is a plane (the electric field lines have been suppressed for clarity).

In a survey that is conducted each semester, the students are asked about their experience with the visual Matlab tool. The majority of the students typically say that the tool is very important and that it enhances their visualization of electrostatic field problems. The majority of the students also say that

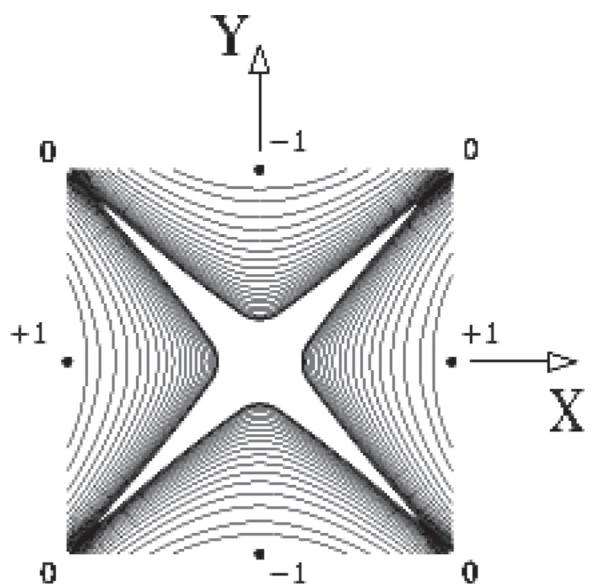

Figure 2(a). Plot of the potential function V=X²−Y².

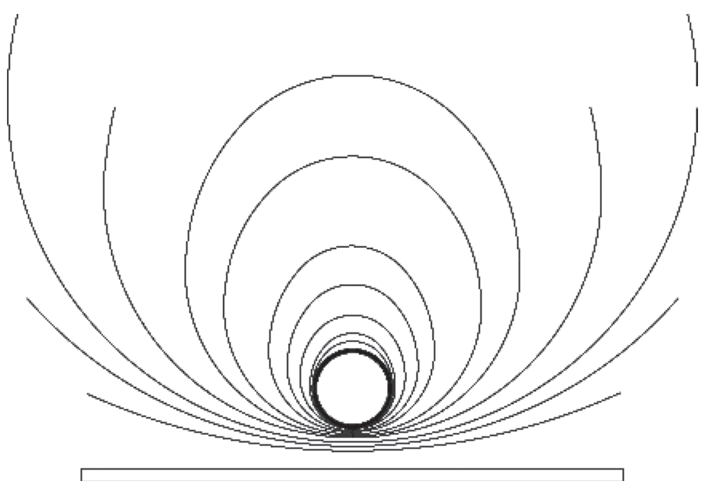

Figure 2(b). Solution of Laplace's equation between a spherical and a planar electrodes.

the study of Thomson's theorem contributed significantly to their overall understanding of the subject of electromagnetics in general and of Laplace's equation in particular. Student test scores do in fact show marked improvement when this graphical tool is used. It is to be noted, however, that other tools do exist for the graphical mapping of field problems and that the real importance of Thomson's theorem lies in its applications in the rapid solution of those problems (see ref. [6] for a discussion of those applications).

PROOF OF THOMSON'S FORMULA

We shall now proceed to prove Equation (1) from the very basic principles of electrostatics. We start by noting that the potential $U(x, y, z)$ is a constant on the equipotential surface. Furthermore, the field intensity $\vec{E}$ is given by the gradient

$$\vec{E} = -\nabla U, \tag{2}$$

according to the theory of electrostatics [11]. In Figure 1, we shall assume that the direction cosines of the vector $\vec{E}$ are δ_x, δ_y, and δ_z (shorthand notation for $\cos\theta_x$, $\cos\theta_y$ and $\cos\theta_z$, or the cosines of the angles formed between the vector and the principal axes). Since $\vec{E}$ is directed along the Z axis, then $\delta_z = \cos 0 = 1$ and $\delta_x = \delta_y = \cos 90° = 0$. We now write an expression for the components of $\vec{E}$ in terms of its direction cosines:

$$E_i = |E|\delta_i \tag{3}$$

where $i = x, y, z,$ and where $|E|$ is the magnitude of the vector. Even though we realize that the components $E_x = E_y = 0$, we are interested in the partial derivatives of those components, which do not vanish. We now take the partial derivative of each term in Eq. (3) with respect to the coordinate i, obtaining

$$\frac{\partial E_i}{\partial i} = |E|\frac{\partial \delta_i}{\partial i} + \delta_i \frac{\partial |E|}{\partial i} \tag{4}$$

Now we write this equation explicitly for each of the coordinates, x, y and z, noting that $\delta_z = 1$ and $\delta_x = \delta_y = 0$. We have:

$$\begin{aligned} \frac{\partial E_x}{\partial x} &= |E|\frac{\partial \delta_x}{\partial x}, \\ \frac{\partial E_y}{\partial y} &= |E|\frac{\partial \delta_y}{\partial y}, \\ \frac{\partial E_z}{\partial z} &= |E|\frac{\partial \delta_z}{\partial z} + \frac{\partial |E|}{\partial z} \end{aligned} \tag{5}$$

The potential U satisfies Laplace's equation [11], that is,

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0 \tag{6}$$

But from Equation (2) we note that

$$\begin{aligned}\nabla U &= \frac{\partial U}{\partial x}\hat{x}, \frac{\partial U}{\partial y}\hat{y}, \frac{\partial U}{\partial z}\hat{z} \\ &= -E_x\hat{x}, -E_y\hat{y}, -E_z\hat{z}\end{aligned} \tag{7}$$

where $\hat{x}$, $\hat{y}$, $\hat{z}$ are unit vectors along the coordinate system axes. Hence Laplace's equation can be written as

$$\nabla^2 U = 0 = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \tag{8}$$

(notice that this is the same as $div\,\vec{E} = 0$). Substituting in the last equation from the three partial derivatives in Eq. (5) yields

$$0 = |E|\left(\frac{\partial \delta_x}{\partial x} + \frac{\partial \delta_y}{\partial y} + \frac{\partial \delta_z}{\partial z}\right) + \frac{\partial |E|}{\partial z} \tag{9}$$

Now we note that

$$|E| = \sqrt{E_x^{\ 2} + E_y^{\ 2} + E_z^{\ 2}}, \tag{10}$$

from which we have

$$\begin{aligned}\frac{\partial |E|}{\partial z} &= \frac{1}{2|E|}\left(2E_x\frac{\partial E_x}{\partial z} + 2E_y\frac{\partial E_y}{\partial_z} + 2E_z\frac{\partial E_z}{\partial z}\right) \\ &= \delta_x\frac{\partial E_x}{\partial z} + \delta_y\frac{\partial E_y}{\partial z} + \delta_z\frac{\partial E_z}{\partial z} \\ &= \frac{\partial E_z}{\partial z}\end{aligned} \tag{11}$$

(note that $\delta_z = 1$ and $\delta_x = \delta_y = 0$). From Eq. (11) and the last of Eqs. (5) we must conclude that

$$|E|\frac{\partial \delta_z}{\partial z} = 0, \tag{12}$$

And hence $\partial \delta z/\partial z = 0$ (this result, as a matter of fact, should have been an obvious result). Eq. (9) now reduces to

$$\frac{\partial |E|}{\partial z} = -|E|\left(\frac{\partial \delta_x}{\partial x} + \frac{\partial \delta_y}{\partial y}\right) \tag{13}$$

The rate of change of a direction cosine, such as δ_x or δ_y, is related to the concept of "curvature" of a curve by a well known relationship. This relationship can be derived in a simple manner without invoking differential geometry, as follows:

In Figure 3, let the arbitrary curve shown be a curve embedded in the surface under consideration and directed along one of the principal directions. At the specific point where the electric field vector $\vec{E}$ is considered, the X axis is tangent to the curve and R is the radius of a circle that is also tangent to the curve. In theory, if we consider R to be the "radius of curvature" of the curve at the point under consideration, then the circle is also known as the "osculating" circle. For an infinitesimal deviation of the vector $\vec{E}$ from the vertical direction, the cosine of the angle θ_x can be determined from the small triangle shown in the Figure. That is, for a small deviation from the vertical, the cosine of the angle θ_x, formed between X and the electric field vector $\vec{E}$, is approximately equal to x/R, where R is the radius of the osculating circle that defines the "radius of curvature" of the curve. We have

$$\delta_x = \cos\theta_x \approx \frac{x}{R} \tag{14}$$

Hence,

$$\frac{\partial \delta_x}{\partial x} = \frac{\partial \cos\theta_x}{\partial x} \approx \frac{1}{R} \tag{15}$$

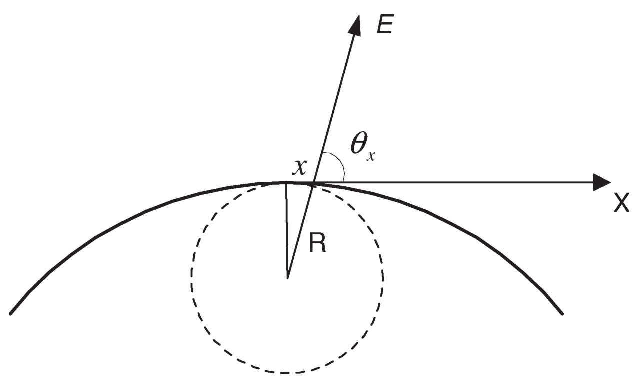

Figure 3. An arbitrary curve and its tangent direction, X, at a point under consideration.

The rate of change of the direction cosine with respect to X is therefore equal to the reciprocal of the radius of curvature along that principal direction. Since we have two principal directions on the surface, X and Y, then Eq. (13) can be written as follows:

$$\frac{\partial |E|}{\partial z} = -|E|\left(\frac{1}{R_1} + \frac{1}{R_2}\right), \tag{16}$$

where R_1 and R_2 are the so called "principal radii" of curvature of the surface at the point under consideration. This concludes the proof of Thomson's formula. The reader who is interested in learning more about the concept of curvature and about differential geometry in general can refer to ref. [12]. It is to be noted that when the formula is applied to the case of geodesy (the study of the earth's equipotential surface), we merely replace the electric field intensity $\vec{E}$ by the earth's acceleration of gravity $\vec{g}$. The problem is otherwise mathematically identical.

The proof given in this paper does not exist in the scientific literature, and it is hoped that it will be a valuable reference for the researchers who are currently using Thomson's theorem as well as to the future generations of students.

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APPENDIX

Here we give the outline of a two-dimensional software algorithm for field mapping based on Thomson's theorem. The reader who is interested in obtaining more details about the actual code should consult [6], as such details are beyond the intended scope of this paper. It is assumed that both Dirichlet and Neumann boundary conditions are given for the problem under consideration (if the Neumann boundary conditions are not given, see [6] for details on obtaining such boundary conditions).

We shall use the simple problem $V = X^2 - Y^2$ of Figure 2(a) as an example to illustrate the field mapping algorithm. The first step in the algorithm is to determine the points or the equipotentials on the boundary of the problem where the potential is a local maximum or minimum. In Figure 4, the local maxima and minima on the boundary are the four points where $V = +1, -1, +1, -1$, as shown.

Our objective now is to determine the equipotential curves V_2, V_3, etc., shown in the figure, starting from any of the indicated four points on the boundary. By using a linearized Taylor series at every point on the equipotential curve V_2, we have

$$V_2 \approx V_1 + \frac{\partial V_1}{\partial s}\Delta s \tag{17}$$

where $V_1 = +1$ is the potential at the starting point, and s is the distance along the electric field line (see figure). But since

$$|E| = -\frac{\partial V}{\partial s}, \tag{18}$$

where the potential is decreasing, thus, in Fig. 4,

$$\Delta s = \frac{V_1 - V_2}{|E|} \tag{19}$$

Hence, by selecting a suitable voltage V_2 (such that the linear approximation holds), and by knowledge of the magnitude of the electric field intensity, a point can be obtained inside the domain that lies on the equipotential curve V_2. The magnitude of the electric field intensity is assumed to be known on the boundary and can be calculated from the boundary conditions. It will be given by:

$$|E|\sqrt{(\partial V/\partial x)^2 + (\partial V/\partial y)^2} \tag{20}$$

Now, given one support point on the equipotential curve V_2, plus two points on the boundary, a polynomial of degree 2 can be fitted to represent that equipotential curve (see any standard text

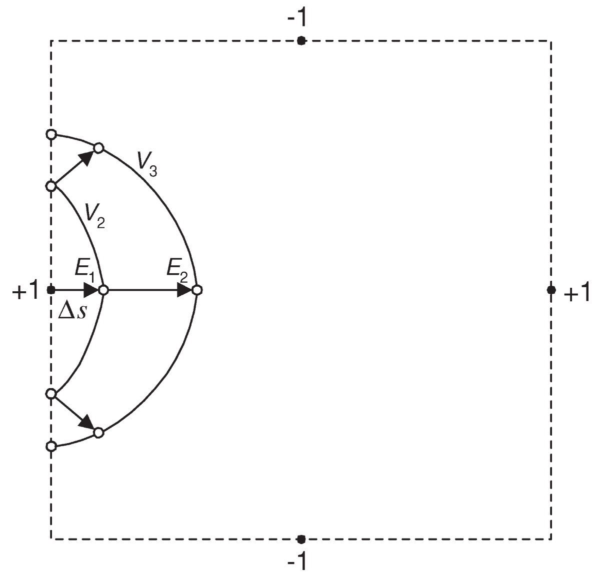

Figure 4. The field mapping.

on numerical analysis, such as [13]). To proceed further inside the domain, we must now obtain the electric field intensity at 3 points on the equipotential V_2 (see Fig. 4) in order to compute a new set of distances $\Delta s_i = (V_2 - V_3)/|E_i|$, and hence determine the location of the equipotential curve V_3. The field intensity at the two points on the boundary are known from the boundary conditions. At the point inside the domain, however, the field intensity can be computed from the linearized approximation

$$|E_2| \approx |E_1| + \frac{\partial |E_1|}{\partial s}\Delta s, \tag{21}$$

where $|E_2|$ is the magnitude of the field near the equipotential V_2, as shown. The quantity $\partial|E_1|/\partial s$ must now be calculated from Thomson's formula in two dimensions, that is,

$$\frac{\partial |E_1|}{\partial s} = -\frac{1}{R}|E_1|, \tag{22}$$

where $1/R$ is the reciprocal of the radius of curvature of the equipotential curve V_2 at the point inside the domain. The quantity $1/R$ is also commonly known as the "curvature" of the curve, and is usually given the symbol K. Now, if the curve will be mathematically represented by a polynomial of the form $y = f(x)$, then the curvature K can be found from a formula that is well known in numerical analysis [13]:

$$k = \frac{y^{11}}{(1 + y'^2)^{3/2}} \tag{23}$$

where $y' = f'(x)$ and $y'' = f''(x)$.

Having a polynomial representation for the equipotential curve V_2, the first and second derivatives of the function $y = f(x)$ must then be computed at all the points on the curve that lie inside the domain (one point, in this example). The most convenient method of interpolation to represent the function $y = f(x)$ under this scheme will be the interpolation by cubic splines [13], since the method of interpolation by cubic splines features some readily available formulas for computing the required derivatives.

Tracing additional equipotentials beyond V_3 will be carried out by simply repeating the above steps. We have given hereinabove an outline for a field mapping algorithm that is based on Thomson's theorem. More details, including a breakdown of the actual working code, can be found in [6].