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The Wonderful Gift of Mathematics

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This theoretical analysis addresses "the unreasonable effectiveness of mathematics in the natural sciences" (Wigner, 1960). For example, how is it that ellipses—constructed by Greek geometers nearly two millennia before Kepler used them—provide such precise models for describing planetary motion? Historical models of gravity and planetary motion frame arguments on the nature of mathematics, particularly the characterization of mathematics as the objectification of action. This characterization helps explain the effectiveness of mathematics while distinguishing mathematics from other disciplines.

"How can it be that mathematics, a product of human thought, independent of experience, is so admirably adapted to the objects of reality?" Albert Einstein (p. 464, in Kline, 1959)

By 200 BC, Greek mathematicians had constructed conic sections and investigated many of their properties. Beyond lines and circles, these constructions included parabolas, ellipses, and hyperbolas, which had no practical application at the time (Kline, 1959). How is it, then, that conic sections provided models for projectile and planetary motion two millennia later, and with uncanny precision? This is just one example of the "unreasonable effectiveness of mathematics in the natural sciences" (Wigner, 1960), which has marveled scientists for centuries. Wigner attributed this unreasonable effectiveness to the *empirical law of epistemology*: "The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve" (p. 8).

In this paper, I argue that, although mathematics is a product of human thought, it is not independent of experience. The main purpose of the paper is to characterize mathematics

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as a human activity distinct from (other) languages and sciences. In particular, people create mathematics through the objectification of action so that actions at one level become objects for us to act upon at another level. This characterization helps explain why mathematics is so effective in modeling our experiences in the world, and at predicting new phenomena. Several examples are provided to demonstrate the effectiveness of objectifying action, but the primary example pertains to modeling planetary motion. The paper sequences arguments in the following way:

1. Tracing the history of mathematical models for planetary motion demonstrates that, despite perpetual (and sometimes radical) revision of underlying physical principles, the mathematics used to model and extend these principles never comes into question.

2. The consistency of mathematics in the face of paradigm shifts in science can be explained in terms of mental actions, which produce mathematical objects.

3. Research in mathematics education buttresses this claim, which in turn, has clear implications for mathematics education.

Elliptical Orbits: A Brief History

On July 4th, 2012, two independent teams of physicists announced they had found empirical evidence for the Higgs boson—a subatomic particle whose existence had been predicted by British scientist Peter Higgs (Cho, 2012). Up to that point, the Higgs boson was only a theoretical particle—a logical-mathematical consequence of the Standard Model of physics. The fact that a sequence of mathematical computations could predict the existence of a particle with precisely described properties is remarkable (some might say unreasonable), but similar examples abound throughout the history of science. As we will see, mathematical models have even predicted the existence of very large objects, such as

Neptune (Darling, 2006). In a sense, Neptune was invented before it was discovered, and that invention began two millennia ago, in ancient Greece.

Plato (427-347 BC) established three simple rules for geometric construction: A straightedge can be used to construct a line passing through two given points, a compass can be used to construct a circle centered at one given point and passing through another given point, and the intersections of lines/circles define new points. Greek geometers—most notably, Apollonius—demonstrated that, by following these rules, they could construct parabolas, ellipses, and hyperbolas. For example, an ellipse can be constructed as the set of all points whose sum of distances from two given points is fixed. The construction, following Plato's rules, is illustrated in Figure 1.

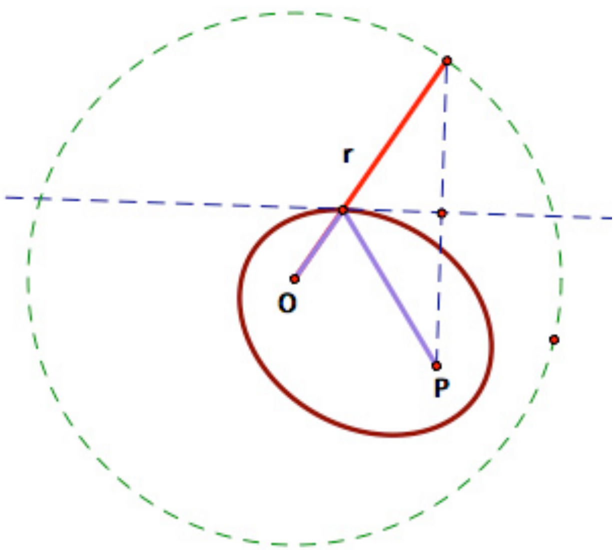

Figure 1. Construction of an Ellipse.

Given points O and P, and radius r, consider the circle centered at O with radius r. Starting from O, we need to determine where we can cut radius r so that the remaining length is equal to the distance from the cut point to P. This will ensure that that sum of distances from the cut point to O and P is r, thus giving us a point on the ellipse. We can find the appropriate point by taking the perpendicular bisector of the base of what-we-know-must-be an isosceles triangle. This construction relies on the fact that the third point on an isosceles triangle lies on the perpendicular bisector of its base. We can generate the ellipse as the locus of all such points, as the direction of the radius is changed.

Before this geometric construction, Greek experiences with ellipses were limited to slicing cones (note that the Greek word *konos* refers to actual pine cones) and observing their cross sections, or to the perception of a circle when viewed from an oblique angle (when looking at a circular object from askew, the planar projection appears elliptical) (Darling, 2006). After the construction, Greek geometers could investigate properties of the ellipse involving their focal points. In addition to the focal point property used in the construction, these include the property that a ray of light emanating from one focal point will reflect off of the ellipse in the direction of the second focal point. However, over the hundreds of years that the ancient Greeks investigated the ellipse, they never considered that they might be riding on one.

Galileo was not the first to suggest a heliocentric view of the heavens (nor was Copernicus), but until his time (1564-1642) people generally accepted that the Earth stood at the center of the universe (Darling, 2006). Greek philosophy and Christian theology perpetuated this view even in the face of strange observations, especially on occasions of apparent backward motion (retrograde) of a planet. Ironically, it was Apollonius—the principle contributor to the study of conic sections—who first suggested that these anomalies could be explained by epicycles, thus delaying the need for elliptical paths. The use of epicycles (circular paths centered on the circular orbits of planets around the Earth) persisted until Tycho Brahe's detailed observations of Mars, and Kepler's use of those observations, led to a new mathematical model for planetary motion—a model too simple to deny.

By 1614, Kepler had guessed at the elliptical orbit of Earth based on empirical data on the position of the Sun and Mars at various times (Darling, 2006). He soon developed this guess into what we now call Kepler's first law: Planets move in elliptical orbits, with the Sun at one focal point. Only later (in 1687) did Newton demonstrate that elliptical orbits were a mathematical consequence of the inverse square law for gravitational force. Once we accept the inverse square law—that the force of gravity between two objects (say the Sun and the Earth) reduces in proportion to the square of the distance between their centers of mass—the elliptical orbits of planets can be calculated with great precision. Figure 2 illustrates how Newton argued that the latter mathematically follows from the former.

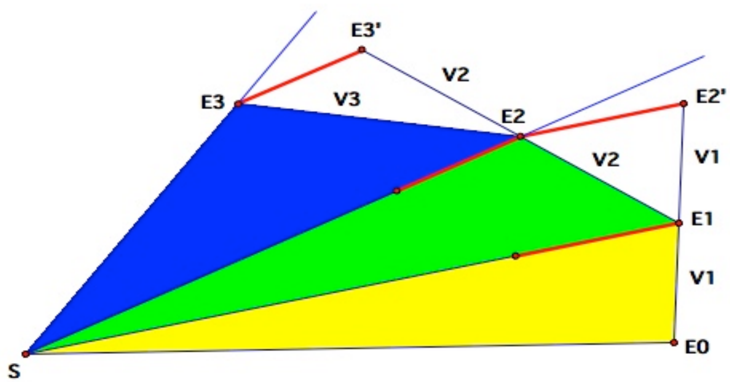

Figure 2. The inverse square law and the ellipse.

Newton's approach depended heavily on Kepler's second law, which states that the radii of planets (from the Sun) sweep out equal areas in equal time. Because this property was already known for circular orbits, Kepler actually produced this law before his first law. The second law can be derived for any orbit in which there is only one attracting force, regardless of whether that force obeys the inverse square law. This can be seen in Figure 2 by considering the movement of Earth from E0 to E1 and then from E1 to E2.

If V1 represents the first movement and if this velocity were to persist for the next time interval, Earth would move from E1 to E2'. However, the inward force toward the Sun (S) causes Earth to fall by some distance, represented by the thick segment from E1, which is also represented, in parallel, from E2' to E2. Thus, Earth travels from E1 to E2 instead. Kepler's second law follows from the fact that the areas of bottom triangle (S-E0-E1) and middle triangle (S-E1-E2) are equivalent. This is easiest to see when considering the common base S-E1, from which the two triangles also have common heights (namely, the component of V1 perpendicular to the base).

From there, Newton argued that (the magnitudes of) changes in velocity (represented by E2'-E2 and E3'-E3) are constant with respect to the internal angles (at S), rather than time. This is where the inverse square law becomes important. Briefly, Kepler's second law shows that any change in time is proportional to area swept out by the orbit ($\frac{1}{2} r^2 \sin\theta$), so that the product of $1/r^2$ and this change in time is a function of θ. Because this product represents the change in velocity (acceleration multiplied by elapsed time), the change in velocity is constant for equal angles. As these angles become smaller, the changes in velocity sweep out a circle (see Figure 3). Finally, from the circular map of velocities, we geometrically derive the elliptical orbit: Find the curve for which the given velocities (v0, v1, v2, and v3) correspond to the given directions of Earth's position relative to the Sun (along e0, e1, e2, and e3). Note that, as θ becomes smaller, the secants in the velocity map become tangent to the circle. But these changes in velocity should be pointed inward toward the Sun (S). Thus, the velocity map is rotated 90 degrees from the position map, and solutions should resemble the construction in Figure 1. A more thorough explanation can be found in *Feynman's Lost Lecture* (Goodstein & Goodstein, 1996).

In the history of the ellipse, we have a geometric object, constructed in pursuit of intellectual curiosity, later used to precisely model planetary motion, which, still later, was mathematically justified based on a physical principle. This example would provide a beautiful testimony to the simplicity of the universe and its language of mathematics, but the story does not end there.

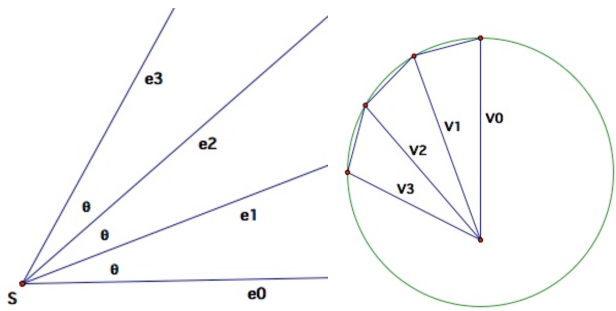

Figure 3. Mapping the Velocities at various points in Earth's Orbit.

In the middle of the 19th century, astronomers were perplexed by the motion of Uranus, which was the newest of the seven known planets, along with Mercury, Venus, Earth, Mars, Jupiter, and Saturn (Darling, 2006). They were already aware that, due to the fact that the Sun was not the only source of gravitational attraction (the three body problem), the planets did not, in fact, move in perfectly elliptical orbits. Since Newton's work, the inverse square law and its mathematical consequences had become primary, taking the place of Kepler's ellipses, which were a consequence of this law when there were only two bodies to be considered. Thus, the simple truth of elliptical motion was broken almost as soon as it was posited. Based on Newton's laws and the deviation of Uranus' path (from that of an ellipse), physicists hypothesized the existence of an eighth planet whose gravitational attraction would explain the deviation. In fact, they mathematically calculated the path and location of that planet, which in 1846 was observed in precisely the place where mathematical calculations determined it should be.

We might say that Neptune's discovery was a mathematical one, rather than an astronomical one. Neptune provided another testimony to the mathematical truth and universal simplicity, this time regarding Newton's inverse square law. However, the simplicity of Newton's law barely outlasted that of elliptical orbits. When astronomers noted similar deviations in the planetary motion of Mercury, they posited a similar solution—the existence of a third body, the planet "Vulcan," orbiting close to the Sun (Darling, 2006). Alas, no such planet exists. Rather, the inverse square law ultimately took the blame, and the simple exponent in Newton's $1/r^2$ relation was replaced by the crude value of 2.00000016. A few decades later, Newtonian physics would be replaced altogether by Einstein's general theory of relativity.

The simplicity of Newton's physics, and the precision of mathematical models based on his laws, cannot be explained by a fundamental accord with the universe. General relativity superseded Newtonian physics because it provided still greater precision while explaining phenomena that Newton could not. In particular, Newton admitted that his theory lacked an explanation for how gravity between two masses communicated itself. General relativity has contributed to a partial explanation in the form of another uncanny prediction—the existence of the Higgs boson. Despite the precision of this prediction, should we be surprised when the underlying model, yet again, falls short? Or when the next predication fails, should we instead blame the mathematical calculations that generate the prediction?

Finding Reason

Mathematicians, scientists, and philosophers have offered various explanations for the effectiveness of mathematics in predicting physical phenomena. Computer scientist Richard Hamming (1980) suggested four "partial explanations"; among these, "we select the mathematics to fit the situation" (p. 89). This was the case for Kepler who selected an ellipse as a curve of best fit for the data Brahe had provided. This curve was available to him because the Greeks had constructed it with three parameters (two focal points and a radius). Thus, Hamming goes a long way in explaining the uncanny precision of mathematical models. However, the unreasonableness goes still further when we consider that the Sun lies at one of the focal points and when we consider Newton's contribution.

Newton demonstrated that Kepler's model was not just a good fit; elliptical orbits were a mathematical consequence of

Newton's inverse square law for the Sun's gravitational force with each planet. On the other hand, because gravitational forces between planets also affect their orbits, the inverse square law simultaneously implies that the orbits are not actually elliptical. Here we find a mathematical model for gravity that worked so well it predicted the existence and position of a new planet. However, we cannot attribute this unreasonable effectiveness to a perfect harmony between mathematics and the universe because, as precise as it was, the inverse square law failed too under the weight of Einstein's general relativity. Neither can we attribute it to the selection of mathematics to fit the situation because Galileo had developed the gravitational laws from observations of objects falling near the surface of the Earth. Only later did Newton demonstrate that these laws generalized and extended to the planets. What in the nature of mathematics enables it to generalize models in such a way, beyond physical experience?

Contrary to Einstein's premise, the story of the ellipse teaches us that physical experience does play a role in mathematical development. In fact, it plays two key roles: one based on physical action, and one due to evolution. Hemming (1980) downplayed the role of evolution due to the relatively short history of mathematics, but others (e.g., Suppes, 2011) have extended the roots of mathematics much further back into our evolutionary history. For example, recent studies in neuroscience indicate that several species of animals have innate abilities to estimate magnitudes, presumably as a consequence of evolution and their need to keep track of offspring, predators, and prey (Dehaene, 1997). Additionally, it is now well established that humans are born with an ability to immediately recognize quantities up to 4 (i.e., subitizing; Revkin, Piazza, Izard, Cohen, & Dehaene, 2008). Such innate abilities demonstrate a primary connection between human mathematical ability and the world in which we have evolved. In the next section, we consider a characterization of mathematics that explains how these worldly roots have kept mathematics grounded, even as mathematics has grown through its various branches.

Mathematics as the Objectification of Action

"*Mathematics is the science of acts without things—and through this, of things that one can define by acts.*" Paul Valéry (1935, p. 811, in von Glasersfeld, 1992).

When a prediction fails, as it did in the case of the imaginary planet Vulcan, physicists question the underlying principles of physics rather than the chain of mathematical arguments that led from those principles to the prediction. Despite Gödel's (1992/1962) incompleteness theorem, numerous paradoxes, and the historically mounting evidence against the Greek philosophy of a fundamental accord with the universe—despite all of this—mathematics remains "queen of the sciences." As humans attempting to predict and control the world we experience, we have no recourse but to rely on mathematical models and mathematical arguments resulting from those models; in other words, we have no more reliable way of reasoning about our experience. Here, I argue that this property rests in the characterization of mathematics as the objectification of action (see Figure 4).

Action ⇉ Object

Figure 4. Mathematics as the Objectification of Action.

The top arrow in Figure 4 represents the actions we perform in the world that we experience and the idea that "knowing an object means acting upon it" (Piaget, 1970, p. 15)—a perspective promoted within constructivist and embodied cognition frameworks (von Glasersfeld, 1992; Campbell, 2010). The bottom arrow represents a process by which those actions become objects themselves, thus subjecting them to further action (top arrow). Piaget (1970) called this process "reflective abstraction," on which several researchers have elaborated: Dubinski through his APOS theory (e.g., Dubinsky & Lewin, 1986; Dubinsky, Elterman, & Gong, 1988); Steffe (1991) in the case of counting schemes; and Simon, Tzur, Heinz, and Kinzel (2004) through their description of activity-effect relations. The underlying idea also closely relates to Sfard's (1994) reification, by which operational thinking becomes structural thinking.

Consider the mathematical object, 8. Although children are born with an ability to subitize numbers up to 4 and to estimate magnitudes in general, they must construct 8 through activity. Steffe (1991) has provided a detailed analysis for how this occurs, through coordinated action. This coordination includes a one-to-one correspondence between pointing acts and word number utterances. As their coordination becomes more internalized, the physical actions of pointing and uttering dissipate. One consequence of this internalization is that when children consider the total of two collections, say 8 and 3, they begin to realize the superfluity of counting to 8 before counting on 3 more. Through such a progression, 8 becomes an object that can be acted upon, as in determining the value of three 8's.

Characterizing mathematics as the objectification of action explains how, on the one hand, mathematics is grounded in experience, and how, on the other, mathematics transcends experience. In particular, although mathematical objects are derived from actions in the world, once they are objectified, they can be acted upon in various ways, including their composition. Sometimes these compositions anticipate physical actions we could perform (e.g., counting three collections of eight objects); other times they form actions that can be carried out only in imagination (e.g., reflecting a cube through a plane of symmetry).

In the case of the ellipse, objectification of action begins with the construction of lines and circles. Whereas Kant had assumed space was an innate concept into which we assimilate the shapes we see, Piaget and Inhelder (1967/1948) demonstrated how children construct both space and shape, largely through haptic experience.

> The abstraction of shape actually involves a complete reconstruction of physical space, made on the basis of the subject's own actions and to that extent, based originally upon a sensori-motor, and ultimately on a mental, representational space determined by the co-ordination of these actions (p. 78).

Similarly, research from an embodied cognition perspective has demonstrated the importance of haptic experience—such as feeling a cube—in constructing geometric concepts (Roth, 2010).

Piaget and Inhelder (1967/1948) referred to lines and circles as "principal perceptual forms" because of the immediacy of their connection to physical experience and their primacy in constructing other shapes. In particular, lines are closely connected to eye movements from one point of fixation to another (saccades), with efficient travel from one point to another, and with the apparent trajectory of light. Circles arise through physical experiences of spinning and swinging objects, including oneself. Indeed, the primacy of lines and circles explains why Plato based his rules of geometric construction on them.

With lines and circles objectified, we can act on them through composition, as Plato prescribed, to construct new shapes. Here, composition refers to the construction of points of intersection from which to construct new lines and circles. The richness of Euclidean geometry (based on Plato's rules) testifies to the utility of objectification; among its products, the construction of the ellipse illustrated in Figure 1. With this construction, the ellipse—previously a product of slicing cones—gained new properties, including focal points and a radius. Kepler could then use the focal points and radius as parameters in finding a curve of fit best for Earth's orbit. The next step was Newton's, to demonstrate that the ellipse was more than a nicely fitting curve; it was a necessary consequence of the inverse square law for gravity and the two-body problem (the Sun and the Earth, neglecting other masses). This, too, relied on the objectification of action.

Newton's argument relied on a chain of equalities for proportional lengths and areas. Hence, his argument was mostly geometric, but included algebraic manipulations as well. The history of mathematics indicates that algebra involves a particular form of objectified action wherein algebraic symbols serve as proxies for mathematical objects, and symbolic manipulations serve as proxies for compositions of these objectified actions. For example, variables were first treated as unknown numbers defined through a sequence of actions (usually additions and proportional scalings) and the numerical result of those actions (Burton, 2007). Eventually they became objects within equations. Still later, Descartes showed how those equations define curves, and in modern algebra, functions, groups, and cohomologies, too, are treated as objects.

Recent research in mathematics education also characterizes the progression of algebra as the objectification of action. For example, work on children's understanding of equal sign (e.g., Knuth, Stephens, McNeil, & Alibali, 2006) implies that children typically think about *x* as the result of computation, as in 5+7=*x*; only later do they begin to think of *x* as an object, through its relation to other objects. In particular, Hackenberg and Lee (2015) have argued that *x* becomes an object for children when they can coordinate its relationship with other units in an equation—even a simple equation such as a*x*=b. Moreover, Tillema and Hackenberg (2011) have demonstrated how children symbolize mathematical objects and manipulate those symbols as a proxy for composing them through actions related to addition and multiplication. Thus, the characterization of mathematics as the objectification of action, with roots in Piaget's structuralism, has support in both the history of mathematics and recent research in mathematics education.

Conclusions

"*Pure logic and pure mathematics are forever capable of transcending experience… But as human action is that of an organism which is part of the physical universe, we understand also why these unlimited operatory combinations so often anticipate the experience, and why when they encounter each other there is harmony between the characteristics of the object and the operations of the subject.*" Jean Piaget (1971/1970, p. 72)

Characterizing mathematics as the objectification of action helps explain the effectiveness of mathematics in the natural sciences, while also distinguishing mathematics from them. Both the history of mathematics and studies of children's mathematical development indicate that physical actions ground mathematics in the world we experience. Moreover, once these actions become objectified, we can act on them (often by composition, as in geometric construction and algebraic manipulation) to construct new actions that generalize and extend into conjectures about the world. This was certainly the case for Newton when he extended gravity into the heavens, but is it reasonable to expect such generalizations to work?

Thanks to Gödel (1982/1962) and various paradoxes that have arisen in the history of mathematics, we know that mathematics does not provide ontological truth (Kline, 1959). Still, when models go awry, we always blame the underlying physical principles and never the long chains of mathematical arguments that led from those principles to the falsified prediction. We saw this play out at least three times in the history of elliptical orbits: circular orbits, elliptical orbits, and inverse square laws were questioned, but not the validity of the geometric constructions and algebraic manipulations. This is because, unlike other sciences, mathematics is about the actions we can perform, either physically or imaginatively, and not observed results of those actions. As a proxy for physical action and for actions on objectified actions, mathematics remains our most reliable way of reasoning.

Galileo (see Kline, 1959) once described mathematics as the language of the universe. Living at a time when he and other scientists were paradigmatically transforming the world with parabolas and ellipses, his view was easily defensible. However, the new shape of planetary motion proved as imperfect in describing the universe as the circles, spheres, and platonic solids that preceded it. If we are to think of mathematics as a language, it is a very human language indeed. Still, mathematics is distinct from all other human languages in that it describes necessary ways of operating—necessary because the operations arise from action and possible actions on those objectified actions.

Of course, in the history of the ellipse there are socio-cultural factors at play too, such as Kepler's study of optics, which made knowledge of ellipses and other conic sections readily available to him (though he was hesitant to use the ellipse because it seemed too simple to have been missed by others) (Darling, 2006). Thus, we can think of mathematics as a culturally-based body of knowledge, but again this body of knowledge is distinct. After all, mathematics is a cultural product that has enabled us to literally escape the world we know and land in new worlds.

Implications for Mathematics Education

Characterizing mathematics as the objectification of action not only helps explain the effectiveness of mathematics in the natural sciences, but also the value of pedagogical practices that leverage students' actions. For example, we know that manipulatives are effective in supporting mathematical development, especially among younger students (Sowell, 1989). This efficacy is easily understood when we consider that young children are constructing their first mathematical objects, and that they do so on the basis of coordinated physical activity (Piaget & Inhelder, 1967). Sometimes educators conflate the roles of manipulatives and visual aids (Moyer, 2001), but to support mathematical development we need to attend to the actions students perform on these figurative materials more so than the formal concepts they might represent.

We might also consider how the role of manipulatives changes for older students who can rely more upon imagined activity and previously constructed objects for constructing new mathematical objects. After all, objectifying action does not end with the internalization and organization of coordinated physical activity; the nature of mathematics allows for level upon level of reorganization. We see this in students' development of multiplication as repeated addition, and exponentiation as repeated multiplication. But for students to generalize these developments to $e*\pi$ (or even $e^{2\pi i}$), for instance, actions of repeated addition (or multiplication) must be reorganized in ways that may not have physical analogs. Successful instructional approaches leverage more abstract actions involving the coordination of covarying quantities (e.g., Abrahamson, Trninic, Gutiérrez, Huth, & Lee, 2011) or mapping one structure to another (e.g., Confrey & Smith, 1995).

When researchers argue for direct instruction, they often rely on theoretical frameworks that do not account for mathematics as a unique domain of knowledge and learning (e.g., Kirschner, Sweller, & Clark, 2006). They focus on figurative aspects of mathematics without recognizing that those figures (e.g., algebraic symbols) serve as proxies for objectified mental actions and that their manipulation serves as a proxy for further action on those objects. The danger in ignoring the role of mental actions and their objectification is that we might teach students to manipulate symbols that have no content, in ways that have no meaning. Effective instruction begins with attention to the mathematical objects students have objectified and the mental actions students have available to act upon them. Here, I have argued that active learning is more than a preferred pedagogy, it constitutes the very nature of mathematics.

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