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Investigation into how 8th Grade Students Define Fractals

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Abstract
The analysis of 8th grade students' concept definitions and concept images can provide information about their mental schema of fractals. There is limited research on students' understanding and definitions of fractals. Therefore, this study aimed to investigate the elementary students' definitions of fractals based on concept image and concept definition. The descriptive method was used in this study. The sample under investigation comprised 70 elementary school students in grade 8 from three different regions: the Black Sea, and Central Anatolia and Aegean regions in Turkey. Data were collected by an open-ended questionnaire with two parts. The first part was an examination of the students' written explanations of fractals and the second part was an analysis of the students' fractal drawings. The questionnaire was developed by the researcher based on previous studies in the areas of teaching and learning fractals. Data was categorized by semantic content analysis and analyzed using descriptive and inferential statistical methods. The findings showed that students had problems both in personal concept definitions and the formal definition of fractals. Moreover, students' fractal drawings were more successful than their formal definitions.

Keywords: Definition of fractal • Understanding fractal • Students' concept image • Concept definition • Mental schema

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Recently, fractals began to emerge in mathematics and mathematics education. Studies of the learning and teaching of fractals (e.g., Fraboni & Moller, 2008; Goldenberg, 1991; Kern & Mauk, 1990; Naylor, 1999) often included activities that can be used in the classroom. However, there are few studies (e.g., Bowers, 1991; Bremer, 1997; Hughes, 2003; Karakuş, 2011; Komorek, Duit, Bücker, & Naujack, 2001; Langille, 1996; Murratti & Frame, 2002) relating to how students understand fractals and the kinds of difficulties they face when learning them. One way to determine the students' understanding about fractals is to examine how students define them. Determining the students' concept definitions and concept images can provide information about their mental schema regarding fractals. In this context, students' definitions were focused to fractals.

Definitions in Mathematics Education

Definitions are considered fundamental in mathematics and mathematics education. National Council of Teachers of Mathematics (2000) emphasizes the importance of the students' perception of the roles definitions play, and the usage of conceptual definitions in mathematical studies starting in middle grades. Mathematical definitions have an important role in the concretization of a defined and exact concept, along with an understanding of the concept as powerful (Edwards & Ward, 2008). Tall and Vinner (1981) have dealt with the process of defining concept in students' learning of mathematics. Their model of concept image and concept definition provides the basis for analyzing students' representations of mathematical concept. *Concept image* is defined as, *"to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and process."* (Tall & Vinner, 1981, p. 152). Students' experiences are essential in the formation of a concept image. For example, if a student observes the perimeter of a rectangle increasing, he can surmise that if the perimeter of a rectangle increases then the area always increases also. For such a student this observation is part of his concept image and may cause problems when he encounters a situation where, as the perimeter increases, the area can reduce or remain fixed. *Concept definition* is defined as, *"to be a form of words used to specify*

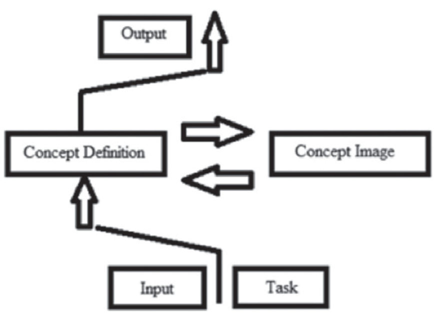


Figure 1. Interplay between definition and image

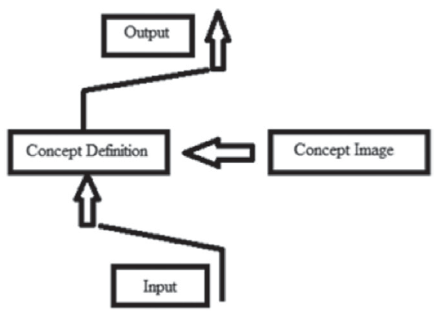


Figure 2. Deduction following intuitive thought

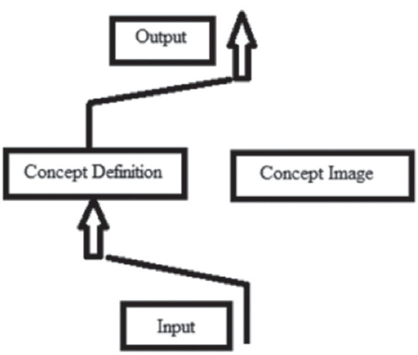


Figure 3. Purely formal deduction

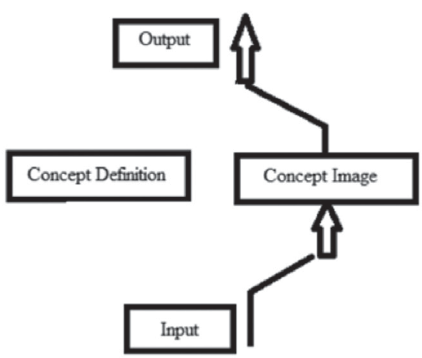


Figure 4. Intuitive response

Figure 1-4: Adapted from Vinner (2002, pp. 71-73).

that concept." (Tall & Vinner, 1981, p. 152). Concept definition can be separated into two parts; formal concept definition and personal concept definition. Formal concept definition, which is an accurate explanation of the concept, is accepted by the mathematical community at large (Tall & Vinner, 1981). However, personal concept definition is the students' personal reconstruction of the definition (Tall & Vinner, 1981). Personal concept definition is part of concept image and, unlike formal definition; it is the student's alternative definition about a concept. Vinner (2002) suggests four situations for the relationship between concept definition and concept image. The focal point of the first three of these (see Figure 1-3) is the concept definition. In these situations, a mathematical task, such as proving a theorem, is completed in a mathematically acceptable way. On the other hand, in the last case (see Figure 4), the concept definition is not consulted during the problem-solving process and is not seen as mathematically acceptable.

For a student, a mathematical definition of the concept depends on what he/she accepts as a definition. In examining the students' understanding of integral, Rösken and Rolka (2007) indicated that students were quite unsuccessful in determining the formal definition of the integral, but their concept image was more dominant in their conceptual learning. This study shows that students' concept images are more effective than their formal concept definitions in conceptual learning. Examples, counterexamples and experiences are very important in the formation of students' personal concept definitions Wilson, (1990). Zaslavsky and Shir (2005) stated that students were often faced with a single definition (by using only one textbook) and did not know any alternative for the learned concept. Therefore, a given example according to an alternative definition was not considered as an example by the student. Hence, alternative definitions, examples and counterexamples play an important role in the formation of the students' personal concept definitions (Tall & Vinner, 1981). Moreover, Wilson (1990) demonstrated that the combination of the natural complexity of definitions and students' lack of knowledge about definitions and examples caused inadequacy in the students' definitions. As such, a student, who provides the formal definition perfectly, is not necessarily showing an understanding of the concept (Edwards & Ward, 2008). In this context, determining the students understanding of a concept, personal concept definitions are very important, especially given that there is not yet an agreed definition in the literature on fractals (Debnath, 2006). Moreover, the infinitive structure of fractals, and the complex formation process, makes them difficult for students to understand (Bowers, 1991; Karakuş, 2011; Langille, 1996; Murratti & Frame, 2002). In this regard, focusing on the students' definitions of fractals can be seen as a way to determine how they understand fractals.

Studies on the Teaching and Learning of Fractals

Studies about teaching and learning fractal geometry have been divided into two subsections: Theory and practice. In the first subsection, (Fraboni & Moller, 2008; Goldenberg, 1991; Kern & Mauk, 1990; Naylor, 1999) activities were frequently developed for the teaching and learning of fractals for teachers to apply in the classroom. Other studies (e.g., Bowers, 1991; Bremer, 1997; Günay & Kabaca, 2013; Hughes, 2003; Karakuş, 2011, 2013; Karakuş & Karataş, 2014; Komorek et al., 2001; Langille, 1996; Murratti & Frame, 2002) focused on how fractals could be integrated into the existing mathematics curriculum, the difficulties faced in the teaching and learning of fractals, and the effect of fractals on changing attitudes towards mathematics. For example, Bowers (1991) determined that students learning fractals have difficulties in three specific areas. The first difficulty arises when learning fractal dimension, the second in determining the scaling factor in the self-similar parts and the third in construction of a fractal. Bowers stated that students have difficulties understanding the process of building a fractal and the definition of the fractal. Similarly, Murratti and Frame (2002) stated that to start teaching fractals by using mathematical definitions can cause many problems with understanding. In particular, they reported that students have difficulties in understanding the formation process and the shape of a fractal. Langille (1996) conducted a study about integration of fractal geometry into the 12^{th} grade mathematics curriculum. He determined that students have difficulties identifying characteristics of fractals. Komorek et al. (2001) determined that students can intuitively identify whether an object is a fractal or not, but they have difficulty defining some of the mathematic characteristics of fractals. However, Bremer (1997) expressed that both elementary and secondary school teachers could learn fractals and have a positive attitude about them. Furthermore, to better understand the process of forming fractals, Hughes (2003) used

both drawing and computer activities in teaching them. Karakuş (2011) determined that pre-service teachers can generally decide whether a given shape is a fractal or not, but they have difficulties in determining the formation process of fractals. The reason stated for this was the particular definition used in the explanation of self-similarity concept. The definition used for deciding about self-similarity, by comparing any part of the object and entire object, was deemed inadequate.

Karakuş (2013) examined elementary and secondary school students' understanding about fractals depending on age. He found that some lack of knowledge and misunderstanding about fractals existed in all (grades 8–10) grades. He also found that, although students recognized the fractals, as the grade level increased the ability to recognize them decreased. Günay and Kabaca (2013) investigated 7th grade students' informal understandings of the concept of fractal in relation to its features. They found that although the students had not formally learned fractals, they could recognize them in an informal way. Moreover, students could distinguish fractals from other patterns according to their characteristics, such as self-similarity and iteration. Karakuş and Karataş (2014) also determined that the secondary school students had misconceptions about fractals. They found that, although students could recognize intuitively a given shape as a fractal, they had misconceptions about how the fractals were formed.

Fractals in the Turkish Mathematics Curriculum

In the Turkish educational system, the teaching of fractals begins with an introduction to fractals at the age of 13-14 years in Grade 8. The Grade 8 mathematics curriculum includes a goal about fractals: "*To build patterns* from *line, polygon and circle models, to draw them and to determine fractals from these patterns*" (Milli Eğitim Bakanlığı [MEB], 2008a). The goal was to build fractal patterns by using figures in Euclid geometry, as well as deciding whether given patterns are fractals or not. In that grade, fractals are learned through drawing activities and finding fractal patterns. When the textbooks were examined, the definition of a fractal is given as, "the *patterns which were built proportionally with the magnification and reduction of a shape*" (Aydın & Beşer, 2008; Cinkol, 2010; MEB, 2008b). The definition emphasizes two important properties of fractals; iteration and self-similarity. Simply put, definitions, examples, and explanations focus on two fractal properties: self-similarity and iteration, in the textbooks and mathematics curriculum. Self-similarity is defined as a part of the whole that closely resembles the whole (Lornell & Westerberg, 1999). Imagine taking a fern and breaking off a piece. That piece looks like the original. The other characteristic of iteration is defined as the same operation being carried out repeatedly, with the output of one iteration being the input for the next one (Peitgen, Jürgens, & Saupe, 1992). In this study, these two features are taken into consideration in the students' definitions.

Challenges of the Study

Examining the students' concept definitions and concept images can provide information about their mental schema regarding fractals. There is limited research on students' understanding and definitions of fractals. Therefore, the present work sought to investigate this gap in the literature and focused on how students define fractals. The main challenges of the study were:

1. Determining the kinds of concept images and concept definitions of fractals that elementary school students have.
2. Determining the relationship between students' concept images and concept definitions regarding fractals.

Method

In order to determine the students' concept image about fractals, descriptive method was used in this study. The purpose of such research was to define what an event is, and describe its components in order to interpret, compare, classify, and analyze (Cohen, Manion, & Morrison, 2007). Moreover, the research methodology of this study was a case study. In a case study, the researcher is primarily focused on understanding a specific individual or situation (Fraenkel, Wallen, & Hyun, 2012). Case study research focuses on individuals' experiences of certain phenomenon and describes the cases in depth.

Participants

The sample under investigation comprised 70 elementary school students (36 boys and 34 girls) in grade 8 that ranged from three different regions: The Black Sea, Central Anatolia and Aegean regions in Turkey. The number of students selected from The Black Sea, Central Anatolia and Aegean regions were 26, 20, and 24, respectively.

The criterion for selecting participants was that they were from different regions, they had some previous knowledge about fractals and that they were from the same socio-economic group. All the participants were from schools in city centers. Furthermore, the selected schools had similar rates of academic success. The participants were selected by use of a purposeful sampling method. Purposeful sampling allows for in-depth research by selecting information-rich cases (Patton, 2002). The purpose of selecting students from different regions was to increase the diversity of data obtained. The three selected elementary schools were from a middle socio-economic level. Students were introduced to fractals for the first time in the fall semester; the study was conducted at the end of the fall semester of the academic year 2010-2011.

Instrument and Data Collection

A way to determine a person's schema for a concept is to ask direct or indirect questions relating to the concept (Vinner, 2002). For that reason, an open-ended questionnaire was used to collect data. The questionnaire was prepared based on interview questions used in Karakuş's (2011) study about pre-service teachers' understandings about fractals. In the first part of the questionnaire, students were asked two questions: *"What is fractal? Can you define it?"* and *"How do you decide whether a shape is a fractal or not?"* The aim of the questions was to reveal the students' concept definitions and concept images for fractals and determine which fractal properties they focus on when defining fractals. The other part of the questionnaire asked the students to draw a shape which is fractal. The aim of this question was to determine students' personal concept definitions about fractals. To examine students' understanding (or personal concept definitions), drawing activities are important tools (Kösa, 2011; Köse, 2008). Moreover, the views of two mathematics teachers were taken in order to ensure the content validity of the open-ended questionnaire. One of the mathematics teachers holds a Masters Degree in Mathematics teaching and has held a position of professional seniority for six years. The other has an Undergraduate Degree and has held a position of professional seniority for twelve years.

Different data collection methods, such as observation, interviews or documents, were used in the case study research. For that reason, the role of the researcher was very important in adhering to the detailed process required for data collection and analysis. Yin (2003, p. 59) defines the abilities of a researcher in a case study as follows:

A case study researcher:

- should be able to ask good questions and interpret the answers
- should be a good listener and not be trapped by his or her own ideologies or preconceptions
- should be adaptive and flexible, so that newly encountered situations can be seen as opportunities, not threats
- must have a firm grasp of the issues being studied
- should be unbiased by preconceived notions

In this study, the researcher was required to record notes and collect the students' opinions without affecting the students' views. For that reason the researcher needed to play an unbiased role in trying to reveal the students' concept images. Moreover, the researcher was not to provide any

Table 1
Categories for Open-ended Questions

Categories	Criteria	Focus of Students Definitions	Examples
Comprehensive definition (CD)	Responses that included all component of the validated response.	Student expressed the characteristics of fractals which are self-similarity, iteration and ratio accurately and correctly.	A shape which is formed by iterated its similar parts which are magnified or decreased with a ratio. (S1)
Partial definition (PD)	Responses that included at least one of the components of validated response, but not all the components.	Student expressed at least two characteristics of fractals, but his/her definition contained minor mistakes.	Magnifying and reducing a shape (S14)
Partial definition with specific misconception (PDSM)	Responses that showed understanding of the concept, but also made a statement, which demonstrated a misunderstanding.	Student expressed at most one characteristic of fractals and his/her definition had errors.	Regular shapes (S42) Iterated shapes (S69)
Specific Misconception (SM)	Responses that included incorrect information	Student did not express any characteristics of a fractal and gave incorrect responses.	Large and small shapes (S54)
No definition (ND)	Contained irrelevant information or left the response blank	Student gave irrelevant information or left the response blank.	I don't know (S59)

education to participants for teaching and learning of fractals during the data collection process. In this study, it was important that researcher maintain an appropriate distance from the participants, so as not to influence their thinking. However, the researcher was also required to maintain a certain level of attentiveness so as not to miss any information.

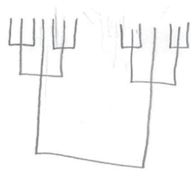
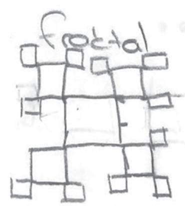
Data Analysis

Data was categorized by semantic content analysis (Miles & Huberman, 1994) and analyzed by using descriptive and inferential statistical methods. In the content analysis, the researcher examined words or concepts within the data, determined the relationship and the meaning of these words and made inferences about the purpose of the research by analyzing the data (Büyüköztürk, Çakmak, Kılıç, Özcan, Karadeniz, & Demirel, 2011). Semantic content analysis is the process of creating categories in order to reveal the main categories and sub-categories dependent on those categories created (Tavşancıl & Aslan, 2001). In this study, categories prepared by Abraham, Williamson, and Westbrook (1994) for analyzing the open-ended questions are presented in Table 1. The use of such classification criteria for student responses provided an opportunity for the researcher to compare students' definitions.

Similarly, students' fractal drawings were categorized by using the studies of Köse (2008); Hoese and Casem (2007), and Göçmençelebi and Tapan (2010) presented in Table 2.

To assure the reliability of the study, suggestions from a well-experienced mathematics teacher were taken into consideration. First, the students' responses and drawings from the open-ended questionnaire

Table 2
Categories for Students' Fractal Drawings

Categories	Criteria	Focus of students drawings	Examples
Comprehensive drawing (CDR)	Drawings reflect all the features of the concept accurately and correctly.	Student defined the generator, applied the iteration rule which is defined in generator in each iteration step, and showed the characteristics of fractals which are self-similarity, iteration and ratio in his/her drawings correctly and accurately.	 (S53)
Partial drawing (PDR)	Drawings reflect the properties of concept in general, but there are incorrect/missing parts or not complete drawings.	Student defined the generator correctly, but the iteration rule defined in the generator was not continued in full or was continued with small mistakes in each iteration step. Student showed the characteristics of fractals which are self-similarity, iteration and ratio in his/her drawings as generally correct, but answers sometimes contained mistakes.	 (S4)
Drawings with misconceptions (DWM)	Drawings that reflect the properties of concept either weakly or too many missing drawings	Student defined the generator incorrectly, improperly built the fractal shape in each iteration step, and drew one characteristic of a fractal at most.	 (S47)
Unsuitable drawings (USD)	Drawings that do not reflect any of the features of the concept	Student drew a shape which does not show any characteristic of fractals or drew non-fractal shapes.	 (S69)
No drawings (Non-D)	No drawings	Student did not do any drawings.	

and categories from Tables 1 and 2 were given to the teacher. The teacher then classified the students' definitions and drawings by using categories. Next, the classifications allocated by the teacher and the researcher were compared. Determining the number of agreements and disagreements, the reliability of the study was calculated by using Miles and Huberman's formula (1994, p. 64). The reliability values of the students' fractal definitions and drawings can be seen in Table 3.

Table 3
The Reliability Values of Students' Fractal Definitions and Drawings

Categories	Reliability values
Definition of fractal	58/(58+12) = 0.83
Drawing fractal	56/(56+14) = 0.80

Because the reliability values were more than 70 in each category, the researcher's classification was able to be deemed reliable. However, to determine any relationship between the students' definitions and the students' drawings, the definition and drawing categories were numerical; students in CD/CDR were given 4 points; PD/PDR were given 3 points; PDSM/DWM were given 2 points; SM/USD were given 1 point and ND/Non-D were given zero points. For example, student coded as S66 was categorized as PDSM and given 2 points for his definition of *"fractal is repeated shape*s." Examining the drawing from the same student (Figure 5), he was categorized as USD and given 1 point.

Figure 5: Fractal drawing of student coded S66.

The researcher therefore concluded that, as the score of a student's definition/drawing approached 4 points, the way he/she defined/drew fractals was considered to be more accurate. In this context, to assess the students' definition/drawing scores, the breakdown shown in Table 4 was used.

In order to determine the relationship between students' definitions and drawings, they were converted to quantitative scores; the Pearson's correlation coefficient was calculated. In order to determine the strength of the relationship, understanding the value of the correlation coefficient is very important (Pallant, 2007). Different authors suggest different interpretations; however, Cohen (1988, pp. 79-81) suggests the following guidelines: if r = .10 to .29 then the relationship is low; if r = .30 to .49, the relationship is medium and if r = .50 to 1.0 the relationship is large. In this study, Cohen's guidelines were used to interpret the relationship.

Table 4
Taken as a Basis for the Interpretation of the Students Score Ranges of Definitions/Drawings

Categories of fractal definition	Categories of fractal drawing	Score range
Comprehensive definition (CD)	Comprehensive drawing (CDR)	3.20 – 4.00
Partial definition (PD)	Partial drawing (PDR)	3.19 – 2.40
Partial definition with specific misconception (PDSM)	Drawings with misconceptions (DWM)	2.39 – 1.60
Specific Misconception (SM)	Unsuitable drawings (USD)	1.59 – 0.80
No definition (ND)	No drawings (Non-D)	0.79 – 0.00

Findings

Students' Concept Images and Concept Definition of Fractals

Table 5 shows the results, with the means and standard deviation scores, for the definitions of fractals obtained by the students' responses.

Table 5
Means and Standard Deviation Scores of Students' Fractal Definitions

	N	Mean	Std. Deviation
Definition of fractal	70	2.34	1.06

Table 5 shows that the fractal definitions of students (X = 2.34; SD = 1.06) fell into the category of PDSM. This means that students did not accurately define the fractals. They gave, at most, one characteristic of a fractal in their definition. Moreover, they had some misconceptions about fractals. The distribution of students' definitions of fractals with respect to categories is given in Table 6.

Table 6 shows that students' fractal definition scores were highest in the category of PDSM (37%) and were followed respectively by PD (30%); CD (14%); SM (13%) and ND (6%). This result showed that more than half of the students had misunderstandings or provided incomplete definitions of fractals. They did not provide exact and/or accurate definitions relating to fractals. Moreover, this result also demonstrates that they did not know exactly the basic characteristics of a fractal: self-similarity and iteration. These

misunderstandings or incomplete definitions were grouped under three categories. The first definition was, *"fractals are only iterated shapes."* This was the most preferred definition allocated by students. The next definition was, *"fractals are regular patterns."* The third definition revealed the students' misconceptions about self-similarity. Some students defined fractals as, *"only reducing shapes"* and others defined them as, *"shapes which are reduced by one within the other."*

Table 6
The Distribution of Students' Definitions of Fractals with Respect to Categories

Categories	*N*	%	Examples
Comprehensive definition (CD)	10	14	• A shape which is formed by iteration of its similar parts which are magnified or decreased with a ratio. (S1) • A shape which is iterated by magnifying or reducing itself continuously with a ratio. (S10) • A pattern which is generated by magnifying or reducing a shape itself continuously. (S27) • A pattern that iterated same shapes which are different sizes. (S38)
Partial definition (PD)	21	30	• A shape which is formed by magnifying or reducing it. (S1, S3, S4, S10) • Shapes that are magnified or reduced with a ratio. (S22)
Partial definition with specific misconception (PDSM)	26	37	• Shapes which are iterated respectively. (S67, S68) • If a shape magnifies or reduces, it can be fractal. (S61) • If a shape is increased the number of its parts, it is a fractal. (S47) • A shape which is reduced with a ratio. (S24)
Specific Misconception (SM)	9	13	• Regular shapes (S42) • Magnified or reduced shapes (S54)
No definition (ND)	4	6	• I don't know (S59) • It is a mathematical subject (S55) • I don't understand this subject (S41)
Total	70	100	

Table 7 shows the results with the means and standard deviation scores for drawing of fractals.

Table 7
Means and Standard Deviation Scores about Drawing of Fractals

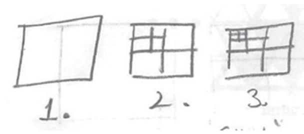
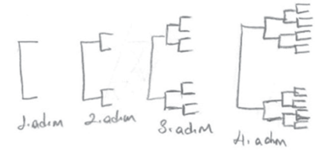
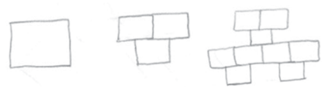
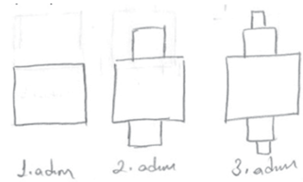
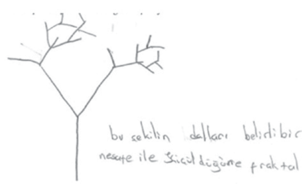
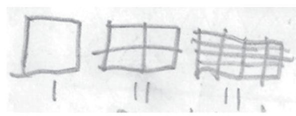
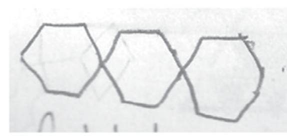
	N	Mean	*Sd*
Drawing fractals	70	3.01	1.17

Table 7 shows the fractal drawings of students (X = 3.01; SD = 1.17) in the category of PDR. This indicated that the students' drawings were generally correct, but some mistakes exist in their drawings, especially in the iteration process. Students defined the generator correctly, but the iteration rule was not continued in full or was continued but with small mistakes in each iteration. The distribution of the students' drawings of fractals with respect to categories is provided in Table 8.

As shown in Table 8, 47% of the students, who were in the category of CDR, drew fractal shapes accurately and in full. 23% of the students, who were in the category of PDR, had some mistakes in their drawings. These students defined the correct generator, but they made some mistakes in the iteration process. For example, student coded S38 defined a generator as: *"reduce a square with a ratio and add two small squares at the top of the original square and one small square at the bottom of the original square."* He implemented the iteration rule in each iteration step, but he did not consider the number and location of the small squares. The shape was not compatible with the generator. 11% of students who were in the category of DWM either defined a generator or continued the iteration process but did not exactly define a generator. Thus, in each iteration step, they drew different patterns and shapes. For example, when student coded S56 drew a fractal tree, she defined a generator as: *"draw a trunk and on the top of the trunk draw two spread branches which were half of the trunk's length."* Yet, after the generator step, she did not apply the iteration rule in each of the iteration steps; she only made some drawings on some of branches. Student coded S47 described a generator in which two small squares occurred as a branch of the original square on both sides, and the two other squares occurred lower in the original square. However, in the following iteration step he continued drawing randomly, without taking the generator in consideration. 19% of students thought of fractals as only patterns and they stated that every pattern they drew was a fractal. Moreover, some of the students drew shapes that were not fractals. For example, student coded S7 divided a square into four equal parts, but this shape is not a fractal, it is only a set of small squares. Similarly, student coded S36 re-drew hexagons side-to-side and stated the shape was a fractal. Thus, misconceptions of drawing a fractal can be classified by four subtitles. These are:

- A generator is described, and then in the iteration steps, the shape is drawn without consideration of the generator.

Table 8
The Distribution of Students' Definitions of Fractals with respect to Categories

Category	*N*	%	Examples	
Comprehensive drawing (CDR)	33	47	 (S3)	 (S44)
Partial drawing (PDR)	16	23	 (S38)	 (S39)
Drawings with misconceptions (DWM)	8	11	 (S56)	 (S47)
Unsuitable drawings (USD)	13	19	 (S7)	 (S36)
No drawings (Non-D)	0	0		
Total	70	100		

- Without describing a generator, the shape is drawn randomly by iterations.
- The shape is drawn without taking care of self-similarity.
- All shape patterns are considered as a fractal.

The Relationship between the Students' Concept Images and Concept Definition of Fractals

The relationship between the students' fractal definitions and fractal drawings is demonstrated in Table 9.

Table 9
The Relationship between the Students' Fractal Definitions and Fractal Drawings

		Drawing	Definition
Definition	r	.345*	1
	p	.003	.
	N	70	70
Drawing	r	1	.345*
	p	.	.003
	N	70	70

*$p < .01$.

According to Table 9, the correlation coefficient between fractal definition and fractal drawing was $r = .345$, with a significant level of .01. This case shows that there is a positive and medium level relationship between students' fractal definitions and drawings.

Discussion and Conclusion

It was determined that students' fractal definitions were in the category of PDSM and students, in providing their definitions, gave more than just one characteristic among three basic characteristics from the curriculum. It was also revealed that there were missing and inaccurate explanations in their definitions. In their fractal definitions, both formal definitions and personal concept definitions were focused, so this finding shows that students had problems with both personal concept definitions and the formal definition of fractals. In this context, it can be stated that students have difficulty in understanding the concept of a fractal. Similarly, in their studies, Bowers (1991), Langille (1996), Komorek et al. (2001), Murratti and Frame (2002), Karakuş (2013), and Karakuş and Karataş (2014) stated that students experienced difficulties defining fractals and specifying their characteristics. The most common inaccurate fractal definition was: *"fractal is an iterated/recursive shape."* Students' experiences in school may be the cause of such a concept image about fractals, given that examples, counterexamples and students' experiences are considerably important in forming their personal concept definitions (Wilson, 1990). Another inaccurate definition was that fractals were defined as only regular patterns. In this definition, although students considered the iteration and pattern rule, they could not demonstrate the difference between pattern and fractal. Similarly, these two definitions show that students could not understand the stage in which fractals occur. In their previous experience, they saw that fractals occur with iteration, but they did not pay attention to how they occur. In support of this, Bowers (1991), Murratti and Frame (2002), Karakuş (2011), and Karakuş and Karataş (2014) stated that students had difficulties in understanding the stage in which fractals occur. Moreover, another reason for providing an inaccurate fractal definition could be inefficiencies in textbooks. Karakuş and Baki (2011) claimed that some explanations and examples in textbooks about fractals are inaccurate or missing.

It was determined that students fractal drawings were at the level of PDR. Nearly half of the students drew fractals correctly and precisely. This finding is in line with the work of Rösken and Rolka (2007). In their studies, Rösken and Rolka stated that students were unsuccessful in defining integral formally. In this study students' missing or inaccurate drawings were classified as follows:

A generator is described and in the iteration steps, the shape is drawn without consideration of the generator.

- Without describing a generator, the shape is drawn randomly by iterations.
- The shape is drawn without taking care of self-similarity.
- All shape patterns are considered as a fractal.

The above classifications for inaccurate drawings reveal that students could not understand the stage in which fractals occur. Since students defined a generator but did not take generator into consideration in each iteration, or did not define a generator at all, they formed shapes by iteration randomly. In this context, these findings are in line with studies done by Bowers (1991), Murratti and Frame (2002), and Karakuş (2011). Similarly, students' inaccurate fractal drawings showed that students have difficulty in understanding self-similarity; one of the basic characteristics of fractals. Bowers (1991), Langille (1996), Komorek et al. (2001), and Karakuş (2011) stated that students had difficulty in understanding self-similarity. Murratti and Frame (2002) indicated that fractals have infinitive structure, but for limiting process its shape can be formed. This could be another reason for mistakes in fractal drawings.

The findings showed that a positive and medium level relationship exists between students' fractal definitions and drawings. So, when the students form a fractal definition in their minds, as Vinner (2002) in Figure 1 stated, concept image and concept definition affect each other.

The results of this study have demonstrated that students generally draw a fractal correctly. However, there are some problems with respect to the definition and formation process of a fractal. For this reason, it is recommended that some changes to the content of the curriculum should be made. For example, students could be introduced to fractals in earlier grades. Moreover, famous fractals like the Sierpinski triangle, Koch curve or fractal dragon should be included in mathematics textbooks in lower grades. This would help students to develop an intuition for fractals.

Students have difficulty in understanding the formation process of a fractal. Having a formal way of forming a fractal, such as initiator-generator-iteration, could be effective in overcoming this difficulty. Because fractals have infinitive structure, it is difficult to generate them with only paper-pencil activities. For that reason, the effect of using computer-based activities on students' understandings of fractals can be examined in future research. Moreover, different studies could be conducted on teaching and learning fractals in higher grades.

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