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Analogical Reasoning in Geometry Education

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Abstract: The analogical reasoning isn't use only in mathematics but also in everyday life. In this article we approach the analogical reasoning in Geometry Education. The novelty of this article is a classification of geometrical analogies by reasoning type and their exemplification. Our classification includes: analogies for understanding and setting a geometrical concept, analogical concepts, analogical theorems and properties, analogies inside a problem, solving problems through analogy with a basic theorem or with a general method and mathematical results formulated by observing analogies.

Key words: Mathematics, Geometry, analogical reasoning

1. Introduction

In everyday life we often do transfers because "the transfer is the demarche through which knowledge and skills acquired are mobilized and used in a new situation, more or less different from that in which they were learned" (Voiculescu, 2013, p. 67 cited. Scallon, 2004, p. 109). It can be considered that "similarities and analogies are the transfer basis" (Voiculescu, 2013, p. 68). Polya (1965, p. 57) consider that "*Analogy* is a kind of similarity. Similar objects or things concur in certain aspects, analogical objects or things concur through certain analogue relationships of their component parts". An *analogical reasoning* "is any type of thinking that relies upon an analogy. An *analogical argument* is an explicit representation of a form of analogical reasoning that cites accepted similarities between two systems to support the conclusion that some further similarity exists." (http://plato.stanford.edu/entries/reasoning-analogy/).

The analogical reasoning isn't use only in mathematics but also in everyday life. According to Richland, Holyoak and Stigler (2004, p. 37) "empirical researchers across disciplines have argued that analogical reasoning may be central to learning of abstract concepts (e.g., Brown & Kane, 1988; Gentner, Holyoak, & Kokinov, 2001), procedures (Goswami, 1992; Ross, 1987), novel mathematics (Bassok, 2001; Novick & Holyoak, 1991; Ross, 1987), and to the ability to transfer representations across contexts (Novick, 1988; Reed, Dempster, & Ettinger, 1985)". At these aspects we can add that analogical reasoning develop competencies as: the skill of find known similar aspects in new situations, the skill to apply known things in a new situation, the skill of generalization. In this context the familiarization of using analogy as a specific method in mathematics has a lot of benefits not only mathematics but also in the real life activities.

In our approach we start from the great Romanian mathematician Solomon Marcus statement (1987, p. 91): "Analogical reasoning is one of the most powerful tools of mathematical thinking. Unfortunately it is used in a very low measure in education. But any analogies economy must be compensated by an additional memory effort". In this context the mathematics teacher role is to determine students to identify and use the analogue reasoning as much as possible in various contexts. In this article we approach the analogical reasoning in Geometry education. The novelty of this article is a classification of geometrical analogies by reasoning type and their exemplification. Our classification includes: analogies for understanding and setting a geometrical concept, analogical concepts, analogical theorems and properties, analogies inside a problem, solving problems through analogy with a basic theorem or with a general method and mathematical results formulated by observing analogies.

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2. The analogical reasoning in learning mathematics

For students most often Mathematics seems to be a collection of disparate concepts and formulas. Mathematics is however a complex gear in which each concept is connected more or less visible with other mathematical concepts or with other sciences as well as with everyday life elements. How can we approach this complex of concepts? Even if we can't give a complete answer to this question, we know for sure that "the problem is not to transmit a finalized science but to acquire a way of thinking" (Revuz, 1970, p. 58). In this context the analogical reasoning brings an important contribution in mathematical thinking. On the one hand the analogies are between everyday life elements and Mathematics, and on the other hand analogies aim mathematical content elements that will determine a whole vision of Mathematics.

In Figure 1 (processing after Magdaş, 1999, p. 62) we represent the way of thinking by using the analogical reasoning in real life situations or problems. Let's assume we have to solve a problem P. Through recognition we identify an analogue problem, marked as BP (basic problem), solved previously. Solving steps *1, 2, …, n* of the BP through analogy are transformed in the analogue steps 1, 2, …, n for solving the problem P. But sometimes we need to add new steps *1, 2, …, k* for solving the problem P. All these steps together will give us the solution for problem P.

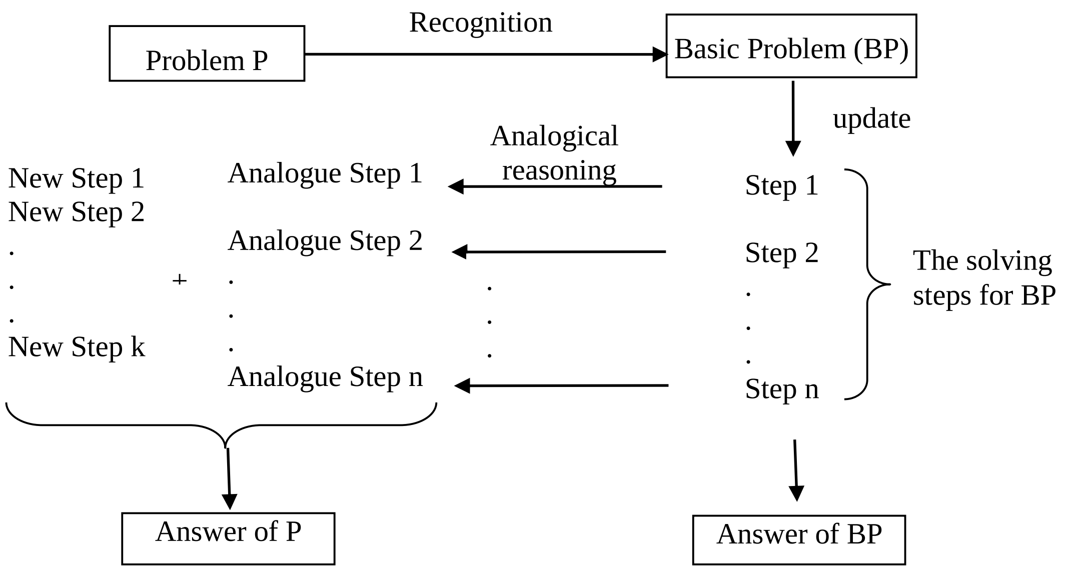


Figure 1. *Scheme of an analogical reasoning to solve problems*

The scheme of Figure 1 can be adapted for introduction of similar concepts. Thus, in Figure 2 we represented the thinking way for introducing a concept C'. In this case the analogue concept is based on the concept C which is defined by the properties *P1, P2, ..., Pn*. Through analogy these properties are transposed to the new situation as the properties *P1', P2', ..., Pn'*. These new properties define the new concept C'.

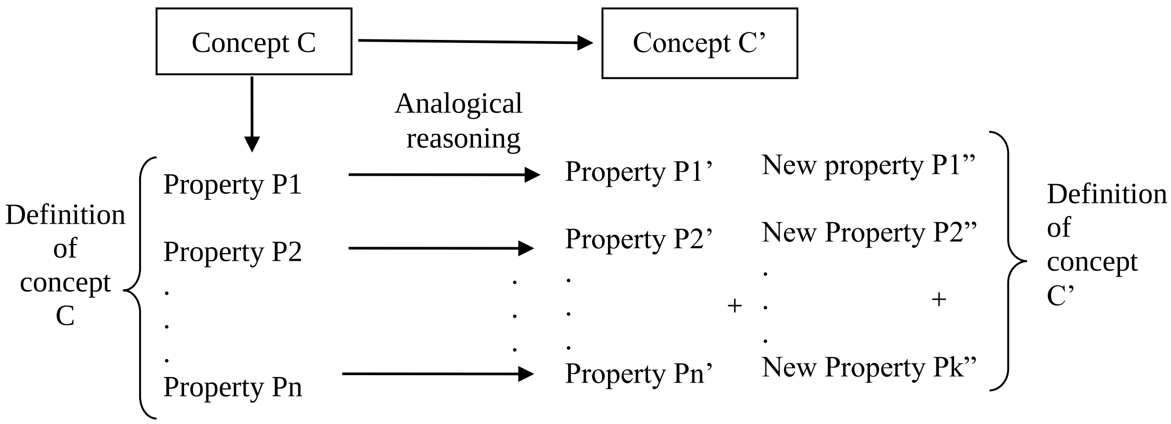


Figure 2. *Scheme of an analogical reasoning between concepts*

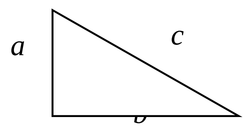
3. Types of analogical reasoning in Geometry

a. Analogies for understanding and setting a geometrical concept

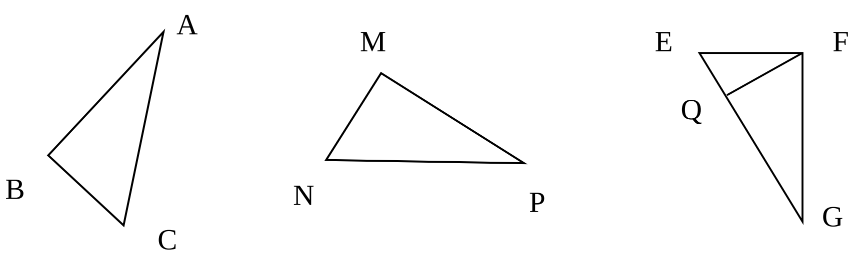
For understanding a mathematical concept it is necessary to make analogies between verbal expression, definition symbols (written expression) and visual representation (material models, pictures, drawings etc.). Changing notation is a "a good way for testing the mathematical understanding level, in other words to recognize mathematical objects and situations independently by the notation used to express them" (Marcus, 1987, p. 44).

Next we illustrate analogies to be made for Pythagora's Theorem:

- Verbal expression: *In a right angle triangle the square of the hypotenuse is equal to the sum of the squares of other two sides.*
- Visual representation:



- Written expression (Pythagorean equation) : $a^2 + b^2 = c^2$
- Connections between the figures and Pythagorean equation have to be made in various context as:



$AB^2 + BC^2 = AC^2$

$MN^2 + MP^2 = NP^2$

in the right triangle EQF: $EQ^2 + QF^2 = EF^2$

in the right triangle GQF: $GQ^2 + QF^2 = GF^2$

in the right triangle EFG: $EF^2 + FG^2 = EG^2$

b. Analogical concepts

Starting from primary school students identify similar concepts. An example of analogical reasoning occurs between standard prefixes for the SI units of measure and standard units for length, mass and volume. SI is an abbreviation from French of *Le Système International d'Unités (International System of Units*. In this case prefixes used are the same for all measures, as well as how to convert from one to another multiplying or dividing by powers of 10. In Figure 3 we have exemplified this analogue reasoning between standard prefixes and standard units for length.

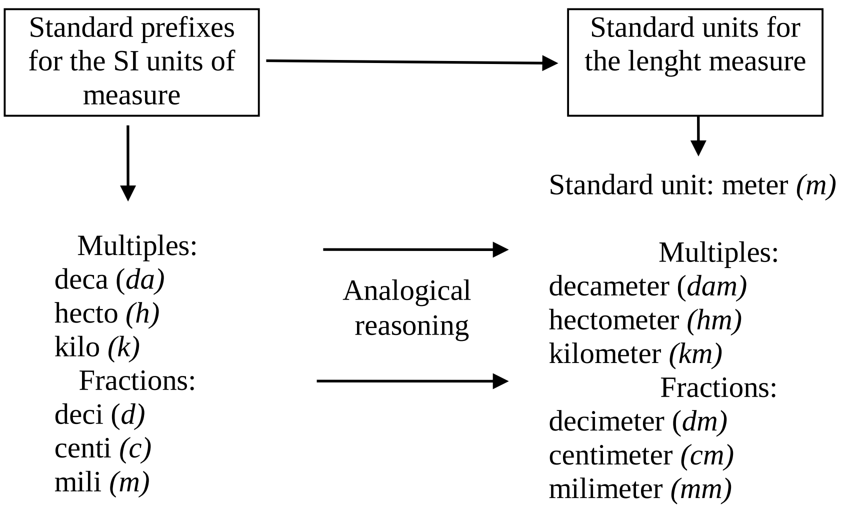


Figure 3. *The analogical reasoning between standard prefixes for the SI units of measure and standard units for lenght*

Most analogical concepts are obtained by generalizations. "Dating back to Babylonian times, analogical mappings have been made between area and volume, line and plane, length and area, shape and solid, triangle and pyramid, trapezoid and frustum, parallelogram and parallelepiped and so on." (Pease, Guhe, Smail, p. 2). In table 1 we present few analogical concepts.

Table 1. *Analogical concepts*

Basic concept	Analogue concept
Angle	Diedral angle
Angle bisector	Dihedral angle bisector
Triangle	Tetrahedron
Square	Cube
Rectangle	Rectangular cuboid
Prism	Cilinder
Pyramid	Cone
Pyramidal frustum	Conical frustum
Middle line in a triangle	Middle line in a trapezoid
Vectors in two dimensions	- Vectors in three dimensions - Complex number - Points in a Coordinate system in two or three dimensions

c. Analogical theorems and properties

Analogical theorems appear after introducing analogical concepts. Usually because of the properties of a concept we can assume that analogue concept has similar properties. These similar properties could be true or false. In case of being true the demonstration could be or not an analogical one. In Table 2 we present few analogical theorems and properties.

Table 2. *Analogical theorems and properties*

Basic theorem/ property	Analogue theorem/ property
Congruence cases for arbirary triangles	Congruence cases for right angled triangles
Mid segment of a triangle properties	Mid segment of a trapezoid properties

Leg Theorem	Altitude of the Hypotenuse Theorem
Pytagora's Theorem (for a right angle triangle)	- Law of cosines (for an arbitrary triangle) - Diagonal lenght of a rectangular cuboid - Descartes theorem (for an tetrahedron in which three concurent edges are perpendicular to each other)
Angle bisector theorem (in a triangle)	Dihedral angle bisector theorem (in a tetrahedron)
Menelaus and Ceva's theorem (in plane)	Menelaus and Ceva's theorem (in space)
Formulas for surface area and volume of: - prism - pyramid - pyramidal frustum	Formulas for surface area and volume of: - cilinder - cone - conical frustum
Vectors addition and subtraction in two dimensions	- Vectors addition and subtraction in three dimensions - Complex numbers addition and subtraction
Vector magnitude in two dimensions	- Vector magnitude in three dimensions - Complex number module - Distance between two points in a coordinate system in two or three dimensions

In Figure 4 we exemplified analogy between Pythagoras's theorem and Law of cosines. The demonstration proofs are not analogues. Although Pythagoras's theorem has over 100 demonstrations the usual way of proof use Leg Theorem. But for Law of cosines proof it is used twice Pythagoras's theorem, as illustrated in Figure 4. In this case the Law of cosines formula is similar to that of Pythagoras's theorem, which makes it easy to remember.

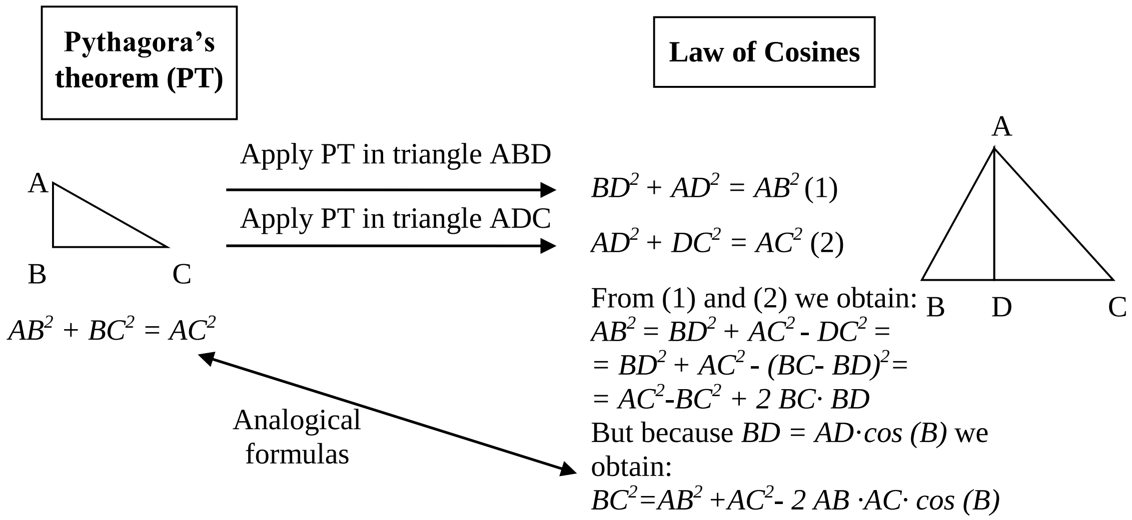


Figure 4. *The analogical reasoning between Pytagora's Theorem and Law of cosine*

d. Use analogical reasoning inside of a geometric problem

Sometimes, especially in Geometry, we use the analogical reasoning inside a problem. To illustrate this situation we propose the next theorem: *The altitudes of a triangle are concurrent.*

Proof. *Let's consider the triangle ABC, A' , B', C' the foots of the perpendicular lines starting from A, B, C respectively on the BC, AC, AB and DEF the triangle obtained by constructing the parallel lines to the sides of triangle ABC through the vertices (see figure 5).*

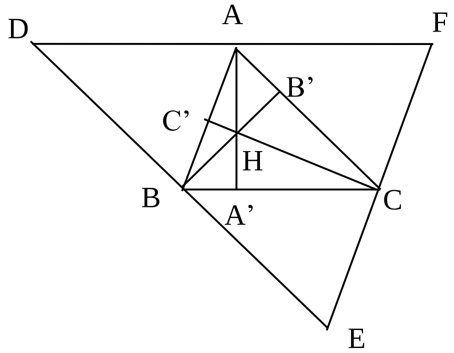


Figure 5. *The concurrent of the altitudes of a triangle*

Through construction ABCF and BCAD are parallelograms, therefore BC = AF = AD. Because BC || DF and AA' ⊥ BC then AA' ⊥ DF. So AA' is the perpendicular bisector of the side [DF].

Using an analogical reasoning BB' and CC' are the perpendicular bisectors of the sides [DE], respectively [EF].

Thus the altitudes of the triangle ABC are the perpendicular bisectors of the triangle DEF and using the property of concurrent of the perpendicular bisectors of a triangle we can conclude that the altitudes AA', BB' and CC' are concurrent.

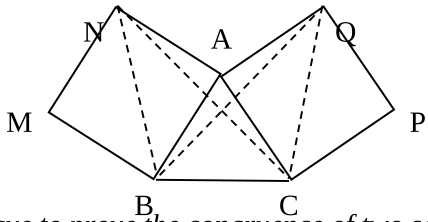
In this proof we observe the using of a analogical reasoning in two situations: to prove that BB' and CC' are the perpendicular bisectors of the sides (DE), respectively (EF). In the science of Mathematics this kind of reasoning are often used but for the children this could be confused. That's why we suggest to write down the whole proof. This can be done by using a table in which the students can see the analogies. The first column will be filled frontal and the students fill in the other columns.

	FRONTAL ACTIVITY:	**STUDENTS ACTIVITY:** (by using analogical reasoning)	
I want to prove…	*AA' is the perpendicular bisector of the side [DF]*	BB' *is the perpendicular bisector of the side [DE]*	*CC' is the perpendicular bisector of the side [EF]*
STEP 1	*ABCF and BCAD are parallelograms therefore BC = AF = AD (1)*	*ACEB and ACBD are parallelograms therefore AC = DB = BE (3)*	*ABEC and ABCF are parallelograms therefore AB = EC = CF (5)*
STEP 2	*Because BC || DF and AA' ⊥ BC then AA' ⊥ DF (2)*	*Because AC || DE and BB' ⊥ AC then BB' ⊥ DE (4)*	*Because BA || EF and CC' ⊥ AB then CC' ⊥ EF (6)*
STEP 3	*From (1) and (2) result that: AA' is the perpendicular bisector of the side [DF]*	*From (3) and (4) result that: BB' is the perpendicular bisector of the side [DE]*	*From (5) and (6) result that: CC' is the perpendicular bisector of the side [EF]*

e. Solve geometrical problems through analogy with a basic theorem

In this category we include analogues reasoning made between a basic theorem and a chapter problems. For a given problem by analyzing conclusion students identify a basic theorem that could use for solving it. Then transfer and apply the basic theorem into the new context doing an analogical reasoning between the theorem hypothesis and conclusion and new conditions of its application. Examples are: prove the congruence of two segment lines by using congruent triangles cases, prove the relationships between segments line ratio using the triangles similarity cases or Thales' theorem, compute segments length using metric theorems, finding the distance from a point to a line by using the Three Perpendicular Theorem etc.

To illustrate this situation we propose the next problem: *On the exterior of an equilateral triangle ABC are constructed two squares ABMN and ACPQ. Prove that NC = BQ.*



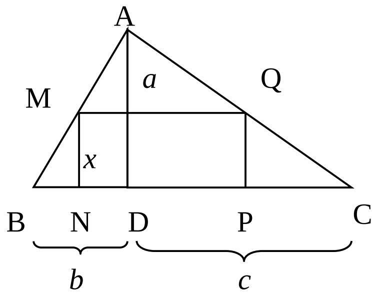
Proof. Because we have to prove the congruence of two segment lines, we identify the basic theorem as one of the congruent triangle cases. We look after two triangles where [NC] and [BQ] are sides, respectively. For this we consider triangles CAN and ABQ. In these triangles we have: $m(< NAC) = m(< QAC) = 150°$, *AC= AB (as sides of an equilateral triangle) and AN=AQ (as sides equals with the sides of the equilateral triangle ABC). Thus we are in the SAS (Side-Angle-Side) case, then triangle CAN is congruent with triangle BAQ. Hence NC = BQ.*

f. Solve geometrical problems through analogy with a general method

In Mathematics there are general solving methods applicable in many mathematical fields. For a given problem by analyzing conclusion students identify a general method used previous that could use for solving it. Then transfer and apply the general method into the new context doing an analogical reasoning between the method and new conditions of its application. The most common general methods are: Reductio ad absurdum and mathematical induction. There are also other less known general methods that can be transferred at a variety range of problems through analogy. We can mention: expressing a quantity (areas, volumes, sums etc.) in two ways, analytical approach of a synthetic geometry problem or vice versa. Sometimes general methods used can exceed the Geometry domain. For example: solve a geometrical problem through algebraic methods, volume or surface area for solids of revolution using definite integrals.

To illustrate this situation we propose the next problem: *In the triangle ABC we consider four points M and Q situated on the sides [AB] and [AC] respectively, N and P situated on the side [BC] so that MNPQ is a rectangle. Find the position of the segment [MQ] for which the area of rectangle MNPQ is maximal.*

Solution.



Because we want to find the maximal area a good idea is to write this area as an algebraic expression. For this we introduce next notations: AD = a, BD = b, DC = c and MN = x. We want to find out NP in term of x. Because triangles BMN and BAD are similar (according to the Fundamental Similarity Theorem) we have that: $\frac{x}{a} = \frac{BN}{b}$, *therefore* $BN = \frac{xb}{a}$. *Then* $ND = b - \frac{xb}{a} = \frac{b(a-x)}{a}$. *By an analogical reasoning we have that* $DP = \frac{c(a-x)}{a}$. *Therefor* $NP = \frac{(b+c)(a-x)}{a}$. *The area of MNPQ is* $MN \cdot NP = \frac{(b+c)x(a-x)}{a}$. *This area is maximal if* $x(a-x) = -x^2 + ax$ *is maximal. We have a quadratic function which is maximal for* $x = \frac{a}{2}$. *Thus the conclusion is that the area of MNPQ is maximal if MQ is the triangle ABC middle line.*

g. Mathematical results formulated by observing analogies

The observation of some mathematical analogical results for particular isolated cases can determine the extension of study to other situations, and in case of obtaining similar results these may lead at more general results. Thus appear some theories more general, conjectures or even theorems. A conjecture is an assertions that is likely to be true but has not been formally proven. Some of that conjectures becomes theorems if they can be formally proven. These results can overcome the geometry framework.

Counting faces, vertices and edges of different polyhedrons, Euler observed a relationship among them. This become an important theorem namely the Euler's formula : *For any polyhedron that doesn't intersect itself, the number of faces plus number of vertices minus the number of edges always equals 2.* This can be written as $F + V - E = 2$.

Another example that integrates a whole class of formulas is due to integral calculus. All volume formulas for solids of revolution can be unified as: $V = \pi \int_a^b f^2(x)dx$.

The number Pi (π) was introduced because of the observation that the ratio between any circle circumference and its diameter seemed to be approximately 3. The four colors' problem appear when a geographer from Edinburgh city observed that for any map are enough four different colors to color it. Poincare's conjecture can also be included in this category, although it looks like it was demonstrated in 2003 by Russian mathematician Perelman who published the demonstration in several Internet articles, thus becoming theorem.

Conclusions

For drawing conclusions we can start from the premise that not all students will become mathematicians but all of them will need Mathematics in life. And we refer both at the basic concepts of mathematics but especially at the mathematical way of thinking. One of the mathematical thinking attributes is the analogical reasoning.

Taking into account the considerations presented in this article we suggest teachers:

- To development constantly an analogical reasoning through mathematics lessons;
- To highlight the links between concepts, theorems, properties and similar problems;
- To realize review and synthesis activities that allow students to see analogies between various fields of mathematics topics or between mathematics and other sciences or real life;
- To put students to make generalizations through analogy;
- To show students that not all analogies prove to be correct.

In this article we approach the analogical reasoning in Geometry education. But analogies are also applicable in other areas of mathematics, between mathematics and other disciplines or even real life. For the future we intend to analyze other aspects of reasoning through analogy.

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