



WHERE ARE THE QUADRATIC'S COMPLEX ROOTS ?

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Abstract: A picture worth more than a thousand words – in mathematics too. Many students fail in learning mathematics because, in some cases, teachers do not offer the necessary visualization. Nowadays technology overcomes this problem: computer aided instruction is one of the most efficient methods in teaching mathematics.

In this article we try to discuss some methods of visualizing complex roots, with examples. The first method presented is the auxiliary polynomial method, followed by the tangent line method. One of the most interesting but less known method is Lill's circle. Another method discussed in the article is Domain Coloring via the Color Wheel.

Key words: quadratic equations; complex roots; visualizing roots; Mathematics Education

1. Introduction

According to Arcavi's definition "visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings".[1]

Visualization is very important for every teacher, especially for mathematics one, because they must often present abstract concepts and symbols which are difficult to understand for students. Unfortunately many of the students fail in learning mathematics because teachers do not pay attention to the basic concepts and use formulas without explaining where they came from.

In order to overcome these difficulties, teachers should use different methods. Nowadays technology is able to provide us the help that we need: computer aided instruction is one of the most efficient methods in teaching mathematics, very effective when it comes to visualize abstract concepts.

In this article we try to discuss some methods of visualizing complex roots, with examples created by us, using Matlab.

When teaching complex roots, teachers do not deal with identifying complex roots graphically and usually use the quadratic formula to solve a quadratic equation. The question is: where are the complex roots located and how can they be visualised?

2. Preliminaries

The roots of the general quadratic equation $y = ax^2 + bx + c$ ($a, b, c \in \mathbb{R}$) are known to occur in the following sets: real and distinct (the discriminant is greater than zero); real and coincident (the discriminant is equal to zero); complex conjugate pair (the discriminant is less than zero. In this case, a simple x - y plot of the quadratic equation does not reveal the location of the complex conjugate roots.)

3. The auxiliary polynomial method , Example:

We have a quadratic function with complex roots $f(x) = x^2 - 2x + 5$ or $f(x) = (x - 1)^2 + 4$. The minimum point (vertex) of the graph is at $V(1,4)$, see Figure 1, so it doesn't cross the x -axis. Because it lies entirely above the x -axis, we know it has no real roots. The discriminant is:

$$\Delta = b^2 - 4ac = 4 - 20 = -16$$

and the roots are:

$$x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} = 1 \pm 2i$$

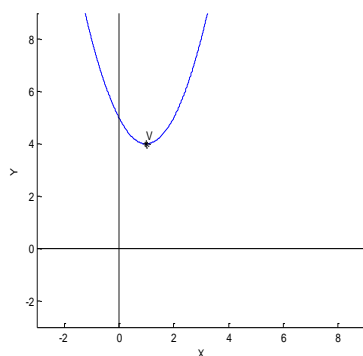


Figure 1. $f(x) = x^2 - 2x + 5$ and vertex at $V(1, 4)$

We reflect the quadratic horizontally over the vertex, in Figure 2, we do this by transforming $f(x) = (x - 1)^2 + 4$ into $f(x) = -(x - 1)^2 + 4$, and find the x-intercepts of this new graph $R(-1, 0)$ and $P(3, 0)$.

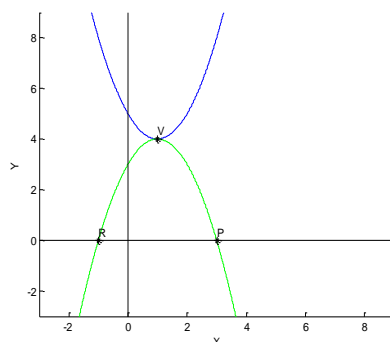


Figure 2. $f(x) = (x - 1)^2 + 4$ (blue) and $f(x) = -(x - 1)^2 + 4$ (green)

If we rotate the RP segment about his midpoint, then the R and P points will describe a circle, with centre $A(1, 0)$ and radius of 2, see Figure 3, 4. In order to graphically see the complex roots of the original function, we need to rotate the reflected image 90 degrees to place the quadratic into the Argand plane.

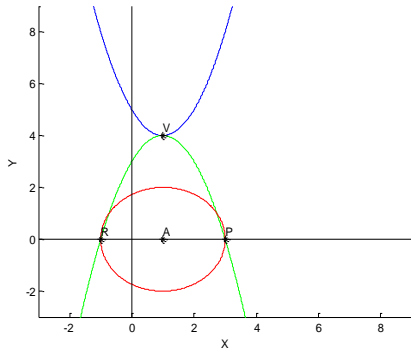


Figure 3.

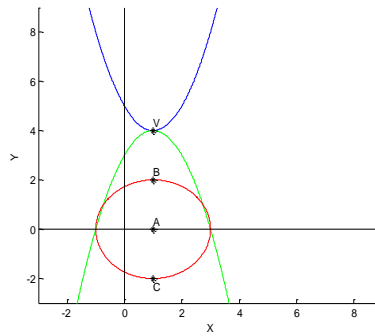


Figure 4.

The points $B(1, 2i)$ and $C(1, -2i)$ on the diagram are a representation of the complex roots. The complex roots of the initial equation are therefore given by $x = 1 \pm 2i$. [2], [13].

4. General case (auxiliary polynomial method)

Let $y = ax^2 + bx + c$ represent any parabola. First we complete the square.

$$y = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a}$$

$$y = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

Its vertical reflection over its vertex is $= -a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$, which expands to $y = -ax^2 - bx + c - \frac{b^2}{4a}$. Its complex roots will be at

$$x = \frac{b}{-2a} \pm \frac{\sqrt{b^2 - 4(-a)\left(c - \frac{b^2}{4a}\right)}}{-2a}$$

which is

$$x = \frac{b}{-2a} \pm \frac{\sqrt{4ac - b^2}}{2a}.$$

Since $(-b^2 + 4ac)$ is negative, these complex roots of the flipped quadratic can be written as:

$$x = -\frac{b}{2a} \pm \frac{i\sqrt{b^2 - 4ac}}{2a}, \quad (1)$$

and real roots of the quadratic we started with

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}. \quad (2)$$

They are the same except for the i in the complex roots. So we can think of (2) as representing a circle of radius $\frac{\sqrt{b^2-4ac}}{2a}$, centred at $(-\frac{b}{2a}, 0)$. Therefore multiplying $\frac{\sqrt{b^2-4ac}}{2a}$ by i has the effect of rotating the point $(\frac{\sqrt{b^2-4ac}}{2a}, 0)$ 90 degrees anti-clockwise around the point $(-\frac{b}{2a}, 0)$. [11]

5. The tangent line method

Let $(x) = ax^2 + bx + c$, then its derivative is $f'(x) = 2ax + b$. Let $\alpha = -\frac{b}{2a}$, $\beta = \frac{\sqrt{4ac-b^2}}{2a}$ and we consider two points from the graph

$$S_2(\alpha + \beta, 2\gamma), \quad S_1(\alpha - \beta, 2\gamma).$$

$f'(\alpha + \beta) = 2a(\alpha + \beta) + b = 2a\left(-\frac{b}{2a} + \frac{\sqrt{4ac-b^2}}{2a}\right) + b = 2a\frac{\sqrt{4ac-b^2}}{2a} = 2a\beta$, this means that the slope of the tangent line to the parabola at S_2 is $2a\beta$.

The tangent line in S_2 to the parabola which contains the point $(\alpha, 0)$ is:

$$y_2 = 2a\beta x - 2a\alpha\beta, \quad \text{and in } S_1: y_1 = -2a\beta x + 2a\alpha\beta, \quad \text{see Figure 6.}$$

$r_1 = \frac{x_{S_2} + x_{S_1}}{2} = \alpha$, $r_2 = \frac{x_{S_2} - x_{S_1}}{2} = \beta$, $x = r_1 \pm ir_2$ are the complex roots of the quadratic. For further information see [2].

Example: Let $f(x) = x^2 - 2x + 5$, see Figure 5., $f'(x) = 2x - 2$, $\alpha = 1, \beta = 2$, $f'(\alpha + \beta) = 4$, $y_2 = 4x - 4, y_1 = -4x + 4$, $S_2(3,8)$, $S_1(-1,8)$. The roots are :

$$x = r_1 \pm ir_2 = 1 \pm 2i, \quad \text{where } r_1 = \frac{x_{S_2} + x_{S_1}}{2} = \alpha = 1, \quad r_2 = \frac{x_{S_2} - x_{S_1}}{2} = \beta = 2$$

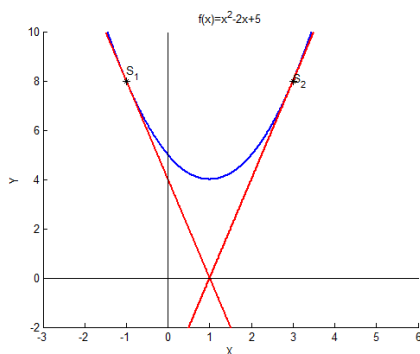


Figure 5. $f(x) = x^2 - 2x + 5$

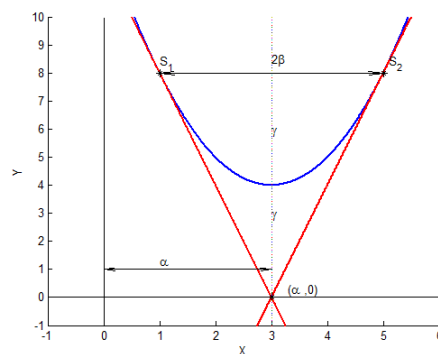


Figure 6.

6. Lill's circle

With this method we can find both real and imaginary solutions, and will work for equations of any degree. We consider a general quadratic equation:

$$ax^2 + bx + c = 0.$$

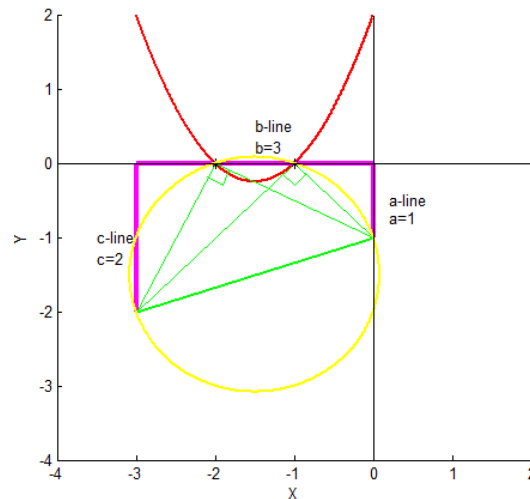


Figure 7. $y = x^2 + 3x + 2$

If $a \neq 1$, we divide the equation by leading coefficient. From the starting point $(0, -1)$ we lay out counter-clockwise perpendicular segments with lengths equal to the a, b and c coefficients. If we have negative coefficient we go upwards on the vertical path, and downwards if the coefficient is positive. On the horizontal path we go left if the coefficient is positive, and right in the case of negative one.

When the quadratic path is finished we draw a line from the starting point to the finishing point, and we draw a circle that has this diameter. The solutions of the equation are the points where this circle intersects the x-axis.

Quadratic equations with complex solutions:

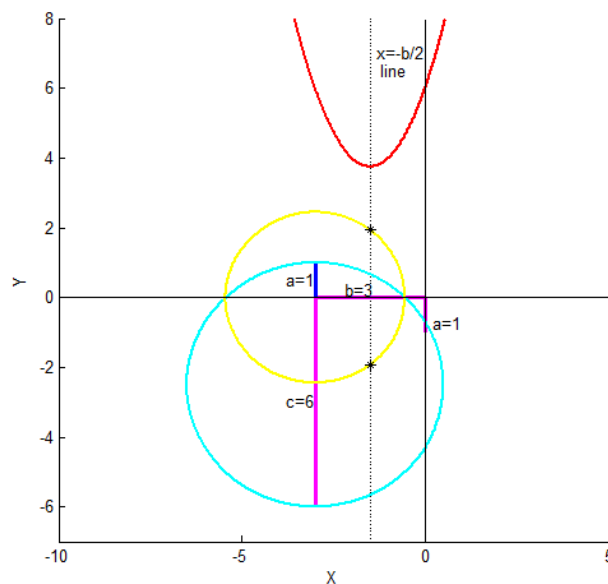


Figure 8. $y = x^2 + 3x + 6$

First of all, we construct the a -line, b -line and c -line like in the previous case. Then we translate the a -line in order to add to the c -line. We draw the blue circle using the $c + a$ lines as diameter. The center of the yellow circle will be the point where the b -line and c -line intersect, and the radius will be given by the intersection of the b -line and the blue circle. Where the yellow circle intersects $x = -b/2$ line, there we find the two complex roots. [5]

7.Surface plot

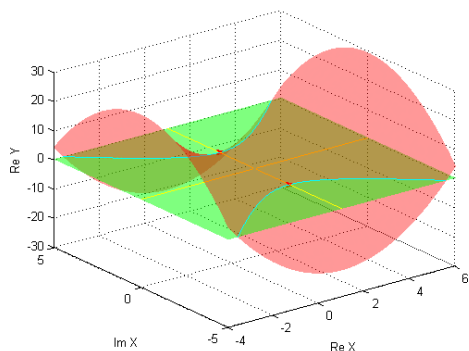


Figure 9. Surface A (red) and Re Y=0 (green)

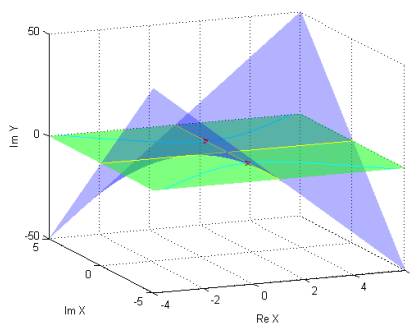


Figure 10. Surface B (blue) and Im Y=0 (green)

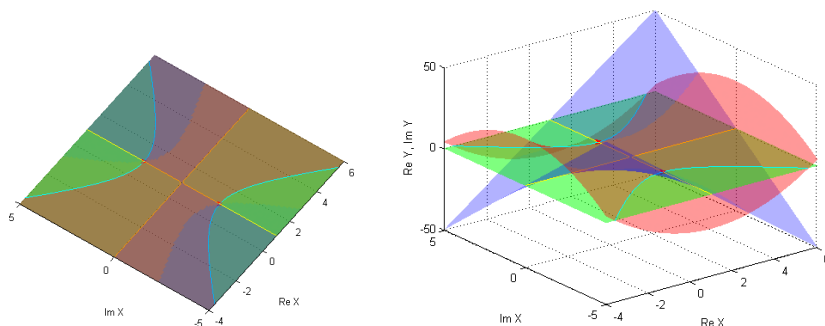


Figure 11. $f(x) = x^2 - 2x + 5$, complex roots at $1+2i, 1-2i$

“The reader is now asked to leave the comfort zone of two-dimensional plots and extend his/her thinking into the third dimension.” said N.Bardell in [3].

We consider $y = ax^2 + bx + c$ and $X = G + iH$ a complex value. If we substitute X in the equation the result is

$$\begin{aligned}
 y &= a(G + iH)^2 + b(G + iH) + c \\
 y &= aG^2 + 2aGHi - aH^2 + bG + bHi + c \\
 y &= (aG^2 - aH^2 + bG + c) + i(2aGH + bH) \\
 y &= A + iB \\
 A &= aG^2 - aH^2 + bG + c, B=2aGH + bH \text{ where} \\
 \text{Re } X &= G, \text{Im } X = H, \text{Re } y = A, \text{Im } y = B
 \end{aligned}$$

Then we plot the surfaces. The surface A has the appearance of a saddle while surface B has the appearance of a twisted plane. Both are hyperbolic paraboloid. The complex roots are where $A = B = 0$ or $\text{Re}(y) = \text{Im}(y) = 0$. This is illustrated in Figure7: we have two points where surfaces A and B intersect each other and on the same time intersect a horizontal plane positioned at zero altitude. The parabola in the x - y plane is actually a section of a more general three-dimensional hyperbolic paraboloid surface.

8. Quadratic equation with complex coefficients

When it comes to finding zeros, the complex function $f(z)$ has an advantage: there are two equations (real and imaginary parts of f) for two unknowns (real and imaginary parts of z).

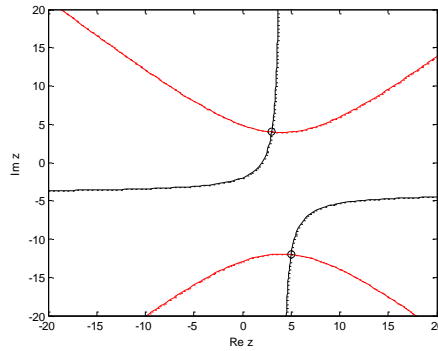


Figure 12. . Contour of real (red) and imaginary (black) part of the function

$$f(z) = z^2 - 8(1 - i)z + 63 - 16i$$

Matlab code :

```
x = linspace(-20,20);
y = x;
[x,y] = meshgrid(x,y);
z = x + i*y;
f = z.^2-8*(1-i)*z+63-16*i;
contour(x,y,real(f),[0 0], 'r');
hold on;
contour(x,y,imag(f),[0 0], 'k');
plot(5-12i, 'kO');
plot(3+4i, 'kO');
hold off
xlabel('Re z'); ylabel('Im z');
```

Example:

Regarding $f(z) = z^2 - 8(1 - i)z + 63 - 16i$, $\Delta = -252 - 64i$, the complex roots are : $z_1 = 5 - 12i$, $z_2 = 3 + 4i$. If we substitute $z = x + iy$ in the equation the result is:

$$(x + iy)^2 - 8(1 - i)(x + iy) + 63 - 16i = 0$$

After expanding the equation:

$$(x^2 - y^2 - 8x - 8y + 63) + i(2xy - 8x - 8y - 16) = 0$$

separating the real and imaginary parts, and setting each equal to zero, we have two real equations: $x^2 - y^2 - 8x - 8y + 63 = 0$, $2xy - 8x - 8y - 16 = 0$.

Then we plot y depending on x and x with respect to y (y as a function of x and x as a function of y), see Figure 12. The complex roots are at the intersection points of the two graphs.[4],[7]

9. Modulus surface

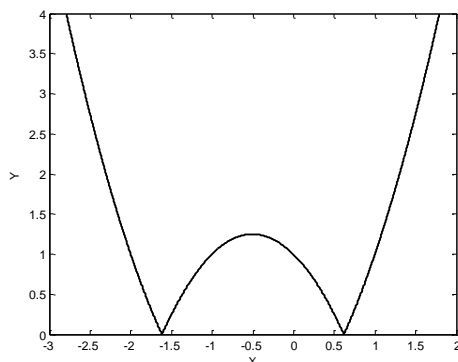


Figure 13. $f(x) = |x^2 + x - 1|$

If $w = x + iy$, $w \in \mathbb{C}$, $x, y \in \mathbb{R}$ then the modulus is defined as $|w| = \sqrt{x^2 + y^2} \geq 0$. The *modulus surface* of the function $f(z) = az^2 + bz + c$ is the surface defined by three real variables x, y and $|f(z)|$. The graph always lies on or above the Argand plane. $|f(z)| = 0$ if and only if $f(z) = 0$, so the modulus surface touches the complex plane in those particular points where the function is zero. Complex roots are found where the modulus surface has minimum points touching the xy -plane. The quadratic $f(z) = z^2 - 2z + 5$ has the modulus surface as seen in Figure 14, where the complex roots of the polynomial are $1 \pm 2i$ and the surface will have exactly those two points that touch the xy -plane. This method can be also used with complex coefficient polynomials and with polynomials that have higher degree. This method enabled us to see how the zeros of a polynomial moved when a single coefficient is changed (surfaces became easy to draw with the use of computer graphics). This method is presented in [10] and [2].

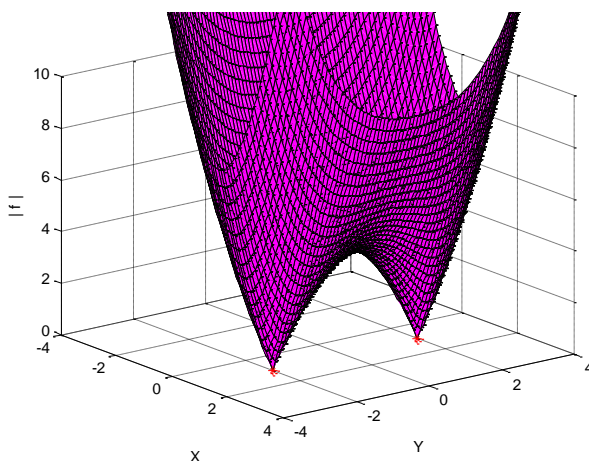


Figure 14. Modulus surface of $f(x) = x^2 - 2x + 5$

10. Newton's method on the complex plane.

Newton's method is an efficient way to approximate solutions to equations of the form $f(x) = 0$. Newton's method involves choosing an initial point x_0 and then, finding a sequence of numbers x_0, x_1, x_2, \dots that converge to a solution.

Select a point x_0 , arbitrarily close to the function's root. Consider the tangent line to the function f at the point $(x_0, f(x_0))$ then we can compute the x -intercept of this tangent line at $x = x_1$. Repeating this algorithm generates a sequence of x values x_0, x_1, x_2, \dots by the rule

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, f'(x_n) \neq 0.$$

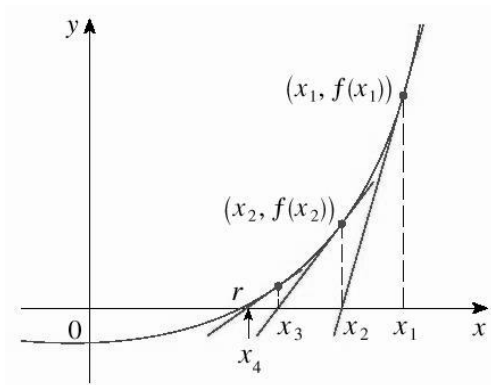


Figure 15. Newton's method

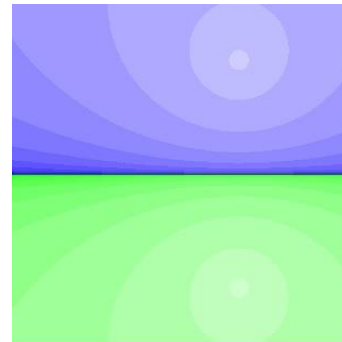


Figure 16. Basin of attraction

Consider $f(x) = x^2 - 2x + 5$, it has two solutions at $x = 1 + 2i$ and $x = 1 - 2i$. If we choose x_0 on the real axis then the iterates behave chaotically. If the chosen x_0 is not on the real axis, Newton's method converges.

$x_0 = 0.5000 + 0.5000i$	$x_0 = -0.5000 - 0.5000i$
$x_1 = 2.7500 + 2.2500i$	$x_1 = 1.4500 - 0.6500i$
$x_2 = 1.4442 + 1.6788i$	$x_2 = -0.2150 - 2.4050i$
$x_3 = 0.9275 + 1.9528i$	$x_3 = 0.7272 - 1.8650i$
$x_4 = 1.0017 + 1.9992i$	$x_4 = 1.0172 - 1.9824i$
$x_5 = 1.0000 + 2.0000i$	$x_5 = 0.9998 - 2.0000i$
	$x_6 = 1.0000 - 2.0000i$

The *basin of attraction* of root r is the set of all x_0 numbers, such that Newton's method starting at x_0 converges to r . Since the real axis is the perpendicular bisector, any initial value chosen on the real axis will not converge. For example, if we pick any complex point in the green regions, when it is used as the initial point in the Newton Method, it will converge to the root $1 - 2i$. Every starting value that converges to $1 - 2i$ is in the basin of attraction of its own.

11. Color Representations (Domain Coloring via the Color Wheel)

With color wheel method we graphically represent complex functions. The color wheel method assigns a color to the point of complex plane. The domain coloring technique gives a continuous and bijective map from the complex plane to the color wheel: 0 maps to white, 1 maps to red, -1 maps to cyan, and the remaining sixth roots of unity map to yellow, green, blue, violet and infinity maps to black. Phase is encoded as hue, while lightness corresponds to the modulus.

The behavior and the basic properties (zeros and their multiplicities, periodicity, symmetry and range) of the function can be rapidly overviewed with this method. [8][14]

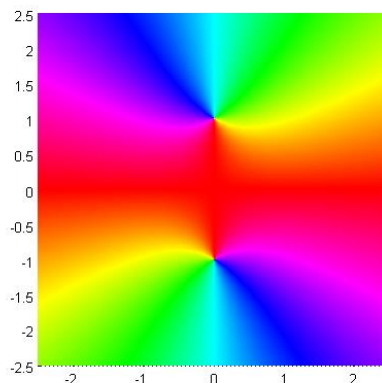


Figure 17. $f(x) = x^2 + 1$, complex roots at i , $-i$

Matlab code:

```
x = linspace(-2.5,2.5,500);
y = linspace(-2.5,2.5,500);
z=complex(ones(500,1)*x,y'*ones(1,500));
f = z.^2+1;
ff = surf(real(z),imag(z),0*z,angle(-f));
set(ff,'EdgeColor','none');
colormap hsv(500)
view(0,90)
axis equal
xlim([-2.5 2.5]);ylim([-2.5 2.5]);
```

Conclusion

Visualising make teaching of mathematics much more interesting because we are able to tell the whole graphical story, not just part of it. We can present a complete analysis to the connection between the roots of a quadratic equation and the graph of its related quadratic function thanks to computer-aided mathematical visualization, obtaining great results. Of course, just as Arcavi [1] warns us, there are some problems which cannot be solved with this method.

Although the effectiveness of the visualization cannot be measured, the researchers of the topic agree about it's usefulness. The "Möbius Transformations Revealed" a short film created at the University of Minnesota by Douglas Arnold and Jonathan Rogness gathered, until now, more than two million viewers on Youtube, proving that science can go viral if properly presented.[12]A Turkish study got the same result: they determined that the use of visualization method at the beginning of the class attracted students' attention to the subject and relieved classroom boredom.[9]

Unfortunately the majority of the teachers from Eastern-Europe, especially from Romania, still rarely use computers for visualizing mathematical examples however it would help students to develop much deeper understanding in such abstractions as similarity, self-similarity. Some drawings, as we have seen examples above, cannot be made during a usual class on the blackboard with chalk, but can be easily drawn by a computer. All these techniques increase our ability to understand complex function behavior.

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