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Computational Technique for Teaching Mathematics (CTTM): visualizing the polynomial´s resultant

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Abstract: We find several applications of the Dynamic System Geogebra – DSG related predominantly to the basic mathematical concepts at the context of the learning and teaching in Brasil. However, all these works were developed in the basic level of Mathematics. On the other hand, we discuss and explore, with DSG´s help, some applications of the polynomial's resultant at the academic level. In this context, we have the opportunity to identify certain limitations of this software. In fact, when we consider the Algebraic Geometry (AG), some analytical methods permit to describe when two plane algebraic curves f and g have a common factor. So, we will show some cases that the CAS Maple is necessary too for providing some conclusive dates. Finally, with this example, we wish to describe/systematize and structure a proposal of a Computation Technique for Teaching Mathematics (CTTM) in the academic level.

Key words: Computational Technique, Visualization, Polynomial's resultant, Geogebra, Mathematics Education.

1. Introduction

In Brazil, we find several works that are within the scope of Mathematcs Education – MA. These works are related to the use of technology in MA. In a specific branch of study, we observe a strong interest about the visualization of mathematical concepts, specially, in the academic level. In our work, we have pointed out the use of two softwares for specific academic content instruction (Alves, 2014a; 2014c; 2014f). Moreover, several limitations and the didactical space of exploration permit identify some implications for the teaching and learning (Alves, 2014a; 2014b; 2014d; 2014e).

In Algebraic Geometry (AG), by a standard way, we seek to find the intersection multiplicities to their common points, in such a way that the Bezout´s classical theorem holds. For example, given two curves indicated by the following equations $f(X,Y)=0$ and $g(X,Y)=0$, what can we talk about the intersection and the common points? We assume that these curves have no common components.

We know a general mathematical methods that involves the one variable selections, we take X, for example, and figurate like a coefficient. Explicitly, we write f and g polynomials in a variable Y, with coefficients in the ring $K[X]$. Finally, we try to conclude if the polynomials $f(x,Y)$ and $g(x,Y)$ (or $f(X,y)$ and $g(X,y)$) have a common factor. Moreover, from this method, we desire to determine the projections, over the axe Ox, related to the points $f\cap g$. Mathematically, we called it by elimination's theory and, we have a powerful tool called the two polynomials´ resultant (Dieudonné, 1974; Fulton, 1969). In the next section, we will indicate some formal elements that we need to establish, en virtue our discussion. For this, the actual technology has a prominent role en virtue to describe several implications to the teaching and learning at the academic level.

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2. The Resultant

We consider a commutative ring A, for example $A = K[X]$ and take the polynomials $f = a_d Y^d + a_{d-1} Y^{d-1} + \cdots + a_0$ and $g = b_e Y^e + b_{e-1} Y^{e-1} + \cdots + b_0$, with condition $d \geq 1$ and $e \geq 1$. We define the resultant of the f, g by the following determinant (see Vainsencher, 2009, p. 22; Tignol, 2002, p. 56):

$$R_{f,g} = \det A_{(d+e)\times(d+e)} = \begin{vmatrix} a_d & a_{d-1} & a_{d-2} & a_{d-3} \cdots a_0 & 0 & 0 & 0 & 0 & \cdots 0 \\ & a_d & a_{d-1} & a_{d-2} \quad \cdots a_0 & 0 & 0 & 0 & 0 & \cdots 0 \\ & & a_d & a_{d-1} \quad \cdots a_0 & 0 & 0 & 0 & 0 & \cdots 0 \\ 0 & & & & a_d & a_{d-1} & a_{d-2} & a_{d-3} & \cdots a_0 \\ b_e & b_{e-1} & b_{e-2} & \cdots \ b_0 & 0 & 0 & 0 & 0 & \cdots 0 \\ 0 & b_e & b_{e-1} & b_{e-2} \ \cdots \ b_0 & 0 & 0 & 0 & 0 & \cdots 0 \\ 0 & 0 & 0 & 0 \ \cdots 0 & b_e & b_{e-1} & b_{e-2} & b_{e-3} & \cdots \ b_0 \end{vmatrix}$$

We show a matrix determinant of order $(d+e)\times(d+e)$, with 'e' lines determined by a´s and 'd' lines determined by the terms b´s. There are some spaces and we write the digit zero. In this definition, we observe some limitations en virtue to explore a mathematical software. In fact, if we take the following polynomials equations $f(X,Y) = X^4 + Y^4 - 1 = 0$ and $G(X,Y) = X^5Y^2 - 4X^3Y^3 + X^2Y^5 - 1 = 0$ in $IR[X,Y]$. These polynomials determine two algebraic plane curves in the IR^2. However, we can visualize it only by a software´s exploration!

Finally, all points in the intersection must verify the following condition indicated below:

$$R_{f,g}(Y) = 0 = 2Y^{28} - 16Y^{27} + 32Y^{26} + 249Y^{24} + 48Y^{23} - 128Y^{22} + 4Y^{21} - 757Y^{20} - 112Y^{19} + 192Y^{18} - 12Y^{17} + 758Y^{16} + 144Y^{15} - 126Y^{14} + 28Y^{13} - 25Y^{12} - 64Y^{11} + 30Y^{10} - 36Y^9 - Y^8 + 16Y^5 + 1 \quad (*)$$

We show in Figures 1 and 2, some date provided by the *CAS Maple*. In Figure 1, we visualize the matrix $A_{9\times9}$ provided by the software and, computing its determinant $\det A_{9\times9}$ we find the analytical equation described in (*). Immediately, we verify and obtain its roots and, and based on Figure 2, we conclude that exists only four numerical real solutions that we indicate in the following numerical set $\{-0.09242096683, -0.5974289870, 0.7211133862, 0.96650662969\}$. Some of these dates can be collected directly from the graph behavior (Figure 2). However, the four numerical values are obtained immediately by the *CAS Maple*.

Even before the abstract character of certain concepts in the Algebraic Geometry, we can identify a view of interpretation that show an intuitive interpretation os some os its concepts. In fact, we explore the analytical, the numerical and geometrical characters. We record that "in a CAS environment, teaching combines two kinds of technique: paper and pencil techniques and intrumented techniques" (Artigue, 2001, p. 9). Moreover, each technique has an epistemic and pragmatic value. In our case, we can combine the pragmatic value in the context of the use the paper and pencil with the use of the CAS that we indicate below. In this case, in accordance with the Artigue (2001a, p. 10), our technique plays a fundamental role in the proof of various theorems in AG.

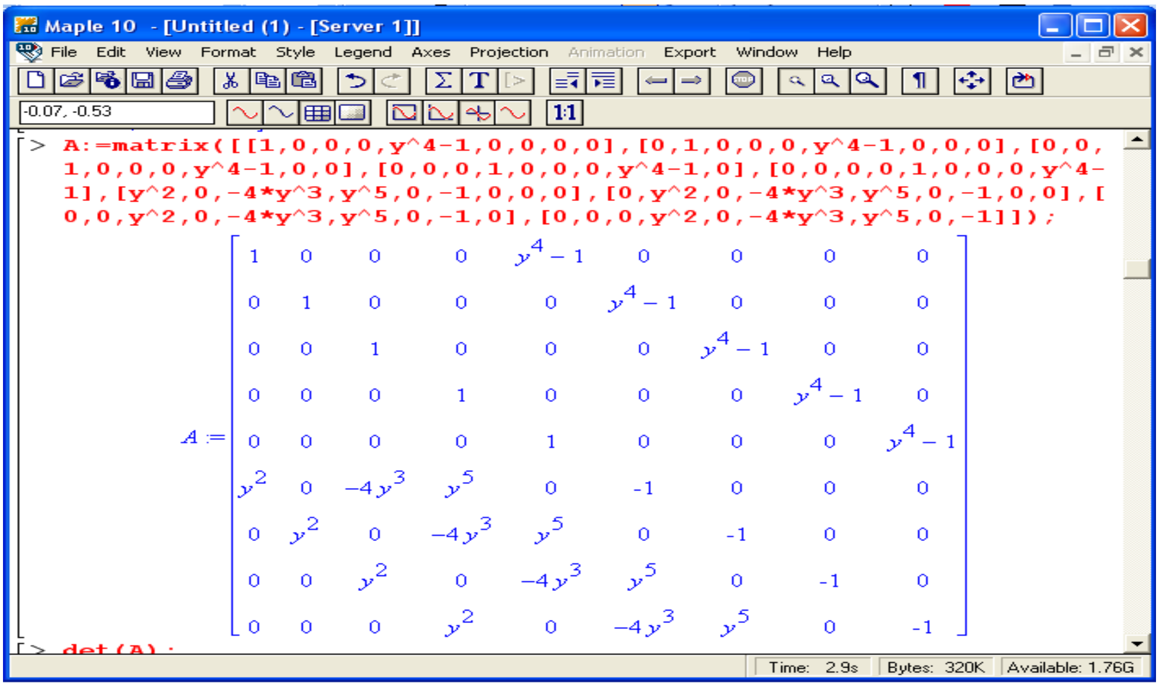

Figure 1. Description of the matrix $A_{9\times9}$ related to the resultant's method

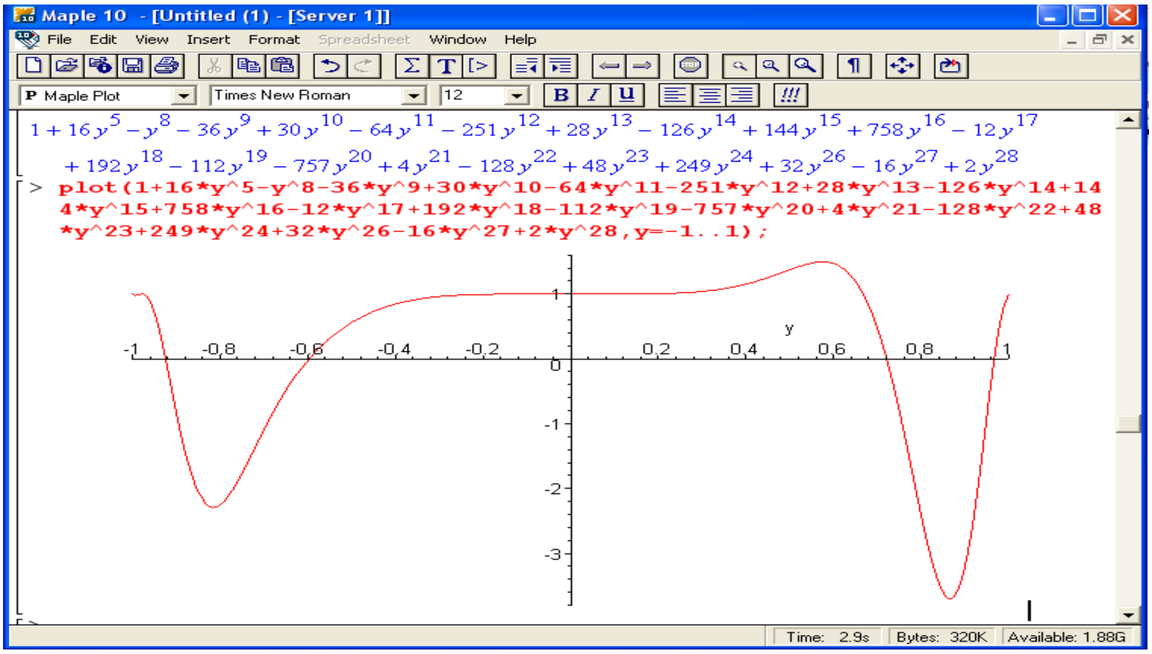

Figure 2. Visualization the four roots related to the intersection of the algebraic curves with the CAS Maple

3. Some theorems in Algebraic Geometry

In AG we have a powerful tool for to analyze the intersection's points related with two algebraic plane curves. In fact, from the specialized literature, we know that $R_{f,g}$ is null if, only if, the two curves f and g have a common component and, in this case, we have $f \cap g$ is finite (Vainsencher, 2009, p. 24). Moreover, when such intersection is finite, we can estimate the points in the plane, by counting its abscises (that is limited, en virtue the degree of the polynomial's resultant). Formmaly, such properties can be summarized in the following theorem.

Theorem 1. (Bezout): Let $f,g \in K[X,Y]$ be polynomials of degrees r and s, respectively. If f and g have no common factor of degree >0, then exists at most $r \times s$ solutions to the system $\begin{cases} f(x,y)=0 \\ g(x,y)=0 \end{cases}$. (see Harris 1977).

We will don´t demonstrate Theorem 1. On the other hand, we find several analytical methods related with this theorem and some elementary proof of this theorem. In fact, let for example the curves $f(X,Y)=X^2-2XY+3X=0$ and $g(X,Y)=Y^2-4X=0$. We can compute $\operatorname{Re}s_Y(X)=\det\begin{pmatrix} -2X & X^2+3 & 0 \\ 0 & -2X & X^2+3 \\ 1 & 3-2Y & -4X \end{pmatrix}=X^2\left(X^2-10X+9\right)$ and $\operatorname{Re}s_X(Y)=\det\begin{pmatrix} 1 & 3-2Y & 0 \\ -4 & Y^2 & 0 \\ 0 & -4 & Y^2 \end{pmatrix}=Y^2\left(Y^2-8Y+12\right)$. This procedure is not precise, en virtue that, we can find some points with the same abscise. However, in this simple case, is easy to conjecture the existence of the intersection's points extracted directly to these little expressions.

Consider now $f(X,Y)=XY^2-Y+X^2+1, g(X,Y)=X^2Y^2+Y-1$. After that, an easy algebraic calculation leads to the following result $\operatorname{Re}s_Y(X)=X^2\left(X^6+2X^4+2X^3+2X^2+3x+1\right)$ and we obtain too $\operatorname{Re}s_X(Y)=(Y-1)\left(X^6+X^5-X^4+2Y^3-3Y^2-1\right)$. Now, the situation appears more complicated, when we consider only the analytical elements.

The classical Bezout´s theorem can be proved by some mathematical properties of resultants. However, we seek to emphasize some visual and others qualitative characters related to this classical method. With this goal, in the next section, we will indicate some presupposes that will conditioning our didatical approach (Alves, 2014f). In accordance with the preliminary example (*), we desire indicate an interesting route for to explore another mathematical software.

4. Computation Technique for Teaching Mathematics - CTTM in the Mathematics Education

Before discuss some aspects related to our mathematical goal, we should indicate certain elements related to what we will called in this paper by Computational Technique for Teaching Mathematics - CT^2M. Practically, we find several mathematical contents that permit a graphical and complex geometric interpretation. In our case, we are interest only in academic contents. On the other hand, in this locus, the mathematical complexity increases considerably (Tall, 2001). So, the first difficulty is to identify a content the permits a graphical and geometric interpretation in IR^2 and IR^3. Second aspect is recognizing that all software manifests some kind of limitation. Since, we have declared our concern related to the polynomial's resultant, we will indicate situations that demand the use of two softwares.

Moreover, we explore and indicate some elements that are to become unworkable when we proceed any kind of analysis without the technology. Soo, in order to understanding and acquire a heuristic approach, we explore, in a complementary perspective of view, the GeoGebra and the CAS Maple.

Finally, in each case, we will point the formal mathematical model on which we establish de discussion.

Without losing the major goal that involves the visualization of some construction provided by the two softwares, in the academic level, we can enphasise the following elements:

(i) particular type of mathematical theory requires the use of technology in order to explore a heuristic interpretation; (ii) certains kinds of graphical-geometric construction that allow the exploitation of the motion and the dynamics of the conceptual objects; (iii) using two kinds of the softwares we can explore the transition to the graphical to geometric representation and vice-versa; (iv) with the use of technology we can promote the apprehension of geometric and graphical properties; (v) its use provides a description of a pragmatic value for the use of CAS and the GeoGebra in the context of AG; (vi) its use provides a description of a epistemic value for the use of CAS and the GeoGebra in the context of AG; (vii) promote the situations of the imbalance between the technical work and the student´s conceptual action and the teacher.

Certainly, we must use and extract a theorical perspective in Mathematics Education en virtue to obtain some kind of systematization in our methodological approach en virtue to predict some elements related an observed practice in research in this branch (Artigue, 2009, p. 3). Moreover, is appropriate to make some comments about the last item (vii). Artigue (1997, p. 166) notes several modifications in the teaching supported by the technology. She observes still the problem about the didactic transposition in the context of technology use. Some of these elementos have to be considered too in the CTTM. In the next section, we will discuss some exemples and preview several didactical possibilities.

5. Some exemples with the Geogebra´s help

In fact, the Dynamic System Geogebra (DSG) has a limited performance for some complex and sophisticated mathematical properties (Alves, 2014a; 2014f). The reader can verify an example and introduce the algebraic expression (*) in the DSG. In Figures 3, 4 and 5, we determine the intersection points related to some plane curves. In Figure 4, on the right side the DSG manifests considerable limitation in the same algebraic task; and on the left side we can understanding and observe the strange behavior of the parameterized curve near at the origin.

We have analyzed the example $f(X,Y)=X^2-2XY+3X=0$ and the curve $g(X,Y)=Y^2-4X=0$. In the case of $\operatorname{Re}s_Y(X)=X^2\left(X^2-10X+9\right)$ and $\operatorname{Re}s_X(Y)=Y^2\left(Y^2-8Y+12\right)$ we can conclude, immediately, that the numbers $0,1,9$ are the roots of $X^2\left(X^2-10X+9\right)=0$. Similarly, we observe that the roots of $Y^2\left(Y^2-8Y+12\right)=0$ are exactly $0,2,6$. However, we can indicate several possibilities, like: $(0,0);(0,2);(0,6);(1,0);(1,2);(1,6);(9,0);(9,1);(9,6)\in \mathrm{IR}\times\mathrm{IR}$.

On the other hand, from Figure 6 (on the right side), we can conjecture that all these ordered pairs are not all desired solutions. In fact, we visualize only tree solutions (see it on the left side). Both in the case of $\operatorname{Re}s_Y(X)$, as like in the case of $\operatorname{Re}s_X(Y)$. So, based in the graphical-behavior and in the formal analytical method, we still observe that only the pairs $(0,0);(1,2);(9,6)$ are the solutions of the system of theses equations $\begin{cases} X^2-2XY+3X=0 \\ Y^2-4X=0 \end{cases}$ (see Figure 5 on the left side). Now, what we can declare about the algebraic system $\begin{cases} XY^2 - Y + X^2 + 1 = 0 \\ X^2Y^2 + Y - 1 = 0 \end{cases}$? We can visualize some solution in $\mathrm{IR} \times \mathrm{IR}$ (see Figure 6)?

With the GeoGebra´s help, we can predict the existence os some solutions of this system. In this way, we can mobilize a pragmatic knowledge. On the other hand, we could only emply the analythical procedures. In this case, we have indicated the following expression: $\mathrm{Re}\,s_Y(X) = X^2\left(X^6 + 2X^4 + 2X^3 + 2X^2 + 3x + 1\right)$ and we obtain too $\mathrm{Re}\,s_X(Y) = (Y-1)\left(X^6 + X^5 - X^4 + 2Y^3 - 3Y^2 - 1\right)$. In the case discussed above, the roots easily identify and anticipate the ordered pairs that correspond to the intersection of this system. However, in this case, the same analytical procedure manifests certain obstacles. Indeed, in this system, for example, we find only the pair ordered solution $(0,1) \in \mathrm{IR} \times \mathrm{IR}$.

In fact, in Figure 5, we visualize (on the left side), tree intersections point. This date agree with the graphical-behavior of the right side, at the same figure. However, in Figure 6, we can conjecture that exist only one solution, we can not declare anything about the quantity of the ordered pairs that satisfy the system above. Is easy observing that $X = 0$ is a double multiplicity point. On the other hand, the $Y = 1$ is another and unique solution, en virtue that $(0,1)$ is solution to this system.

Other example, when we take the polynomials $F(X,Y,Z) = Z^3 - X^2Y$ and $G(X,Y,Z) = Z^5 - X^4Y$ and computing a determinant of order 8 we find that $R_{F,G}(X,Y) = X^{10}Y^3(Y^2 - X^2)$. In Figure 4 we show (on the left side) the graph behavior provided by DSG. Well, directly from the expression $X^{10}Y^3(Y^2 - X^2) = 0$ we can take the following points: $(1,0);(0,1);(1,1);(1,-1) \in \mathrm{IR} \times \mathrm{IR}$ of multiplicities 10, 3, 1 and 1, respectivelly. In Figure 4, we find some seriously limitation of the DSG.

In fact, in the left side (Figure 4), we visualize the graphical behavior related to the plane curve described $X^{10}Y^3(Y^2 - X^2) = 0$. Although, in this case, we can not predict in the real plane, the intersection of the equations $\begin{cases} F(X,Y,Z) = Z^3 - X^2Y = 0 \\ G(X,Y,Z) = Z^5 - X^4Y = 0 \end{cases}$. On the right side, we show a case without problem.

Finally, in Figure 3, we show some example in which some functions of the DSG does not show a good performance, especially those that relate to the calculation of the intersection of objects. We suggest to the reader to investigate and determine, with the DSG´s help, the intersection points of the two or more algebraic curves. In Figure 4, on the left side, the problem is more serious, according to what we see in Figure 4 below. In other works (Alves, 2014g), we have pointed another limitations of the Geogebra when we are in the case of Bezier curves, that constitutes another subject in AG. Once again, we employ in a complementary character the both softwares mentioned here.

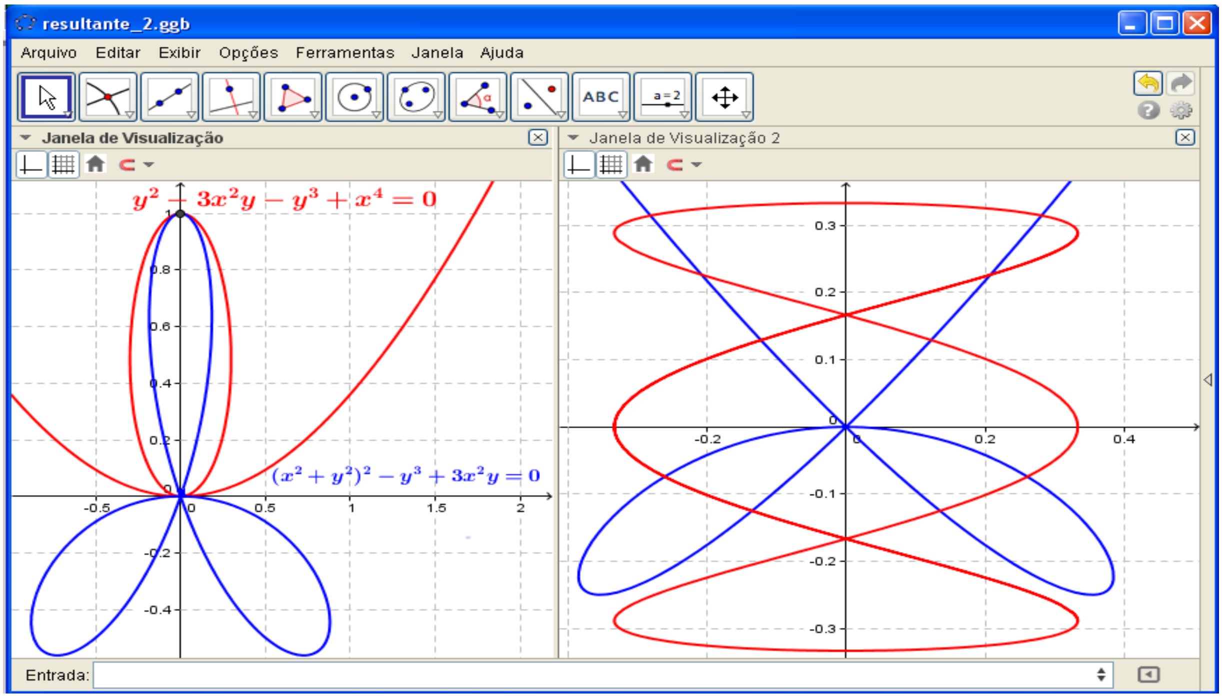

Figure 3. Some limitations examples related to the use of DSG

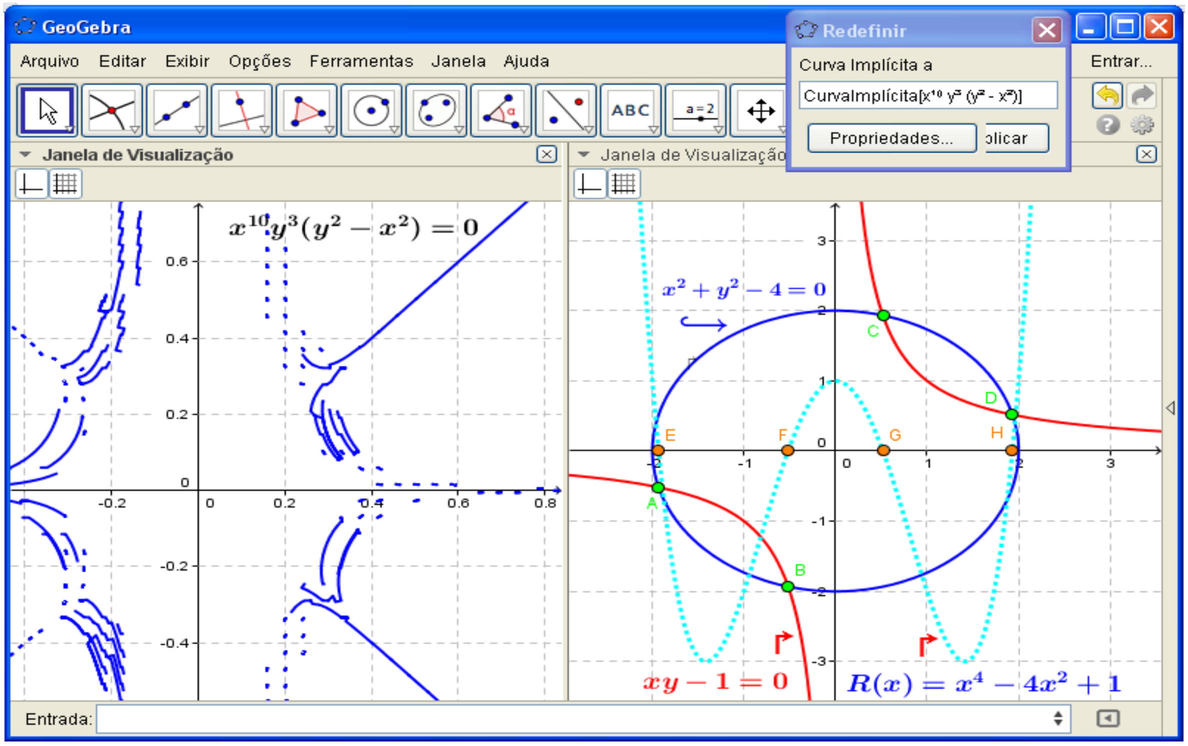

Figure 4. Some limitations examples related to the use of DSG

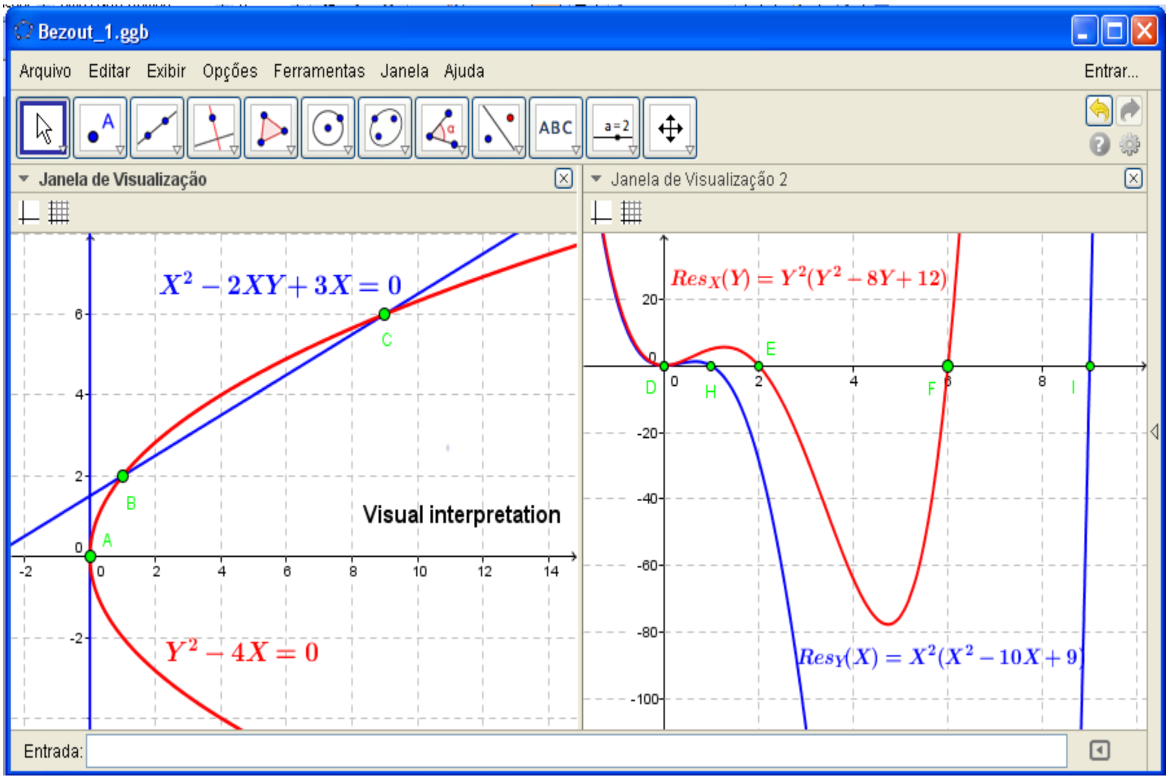

Figure 5. Visualization of the intersection's points related two curves f and g

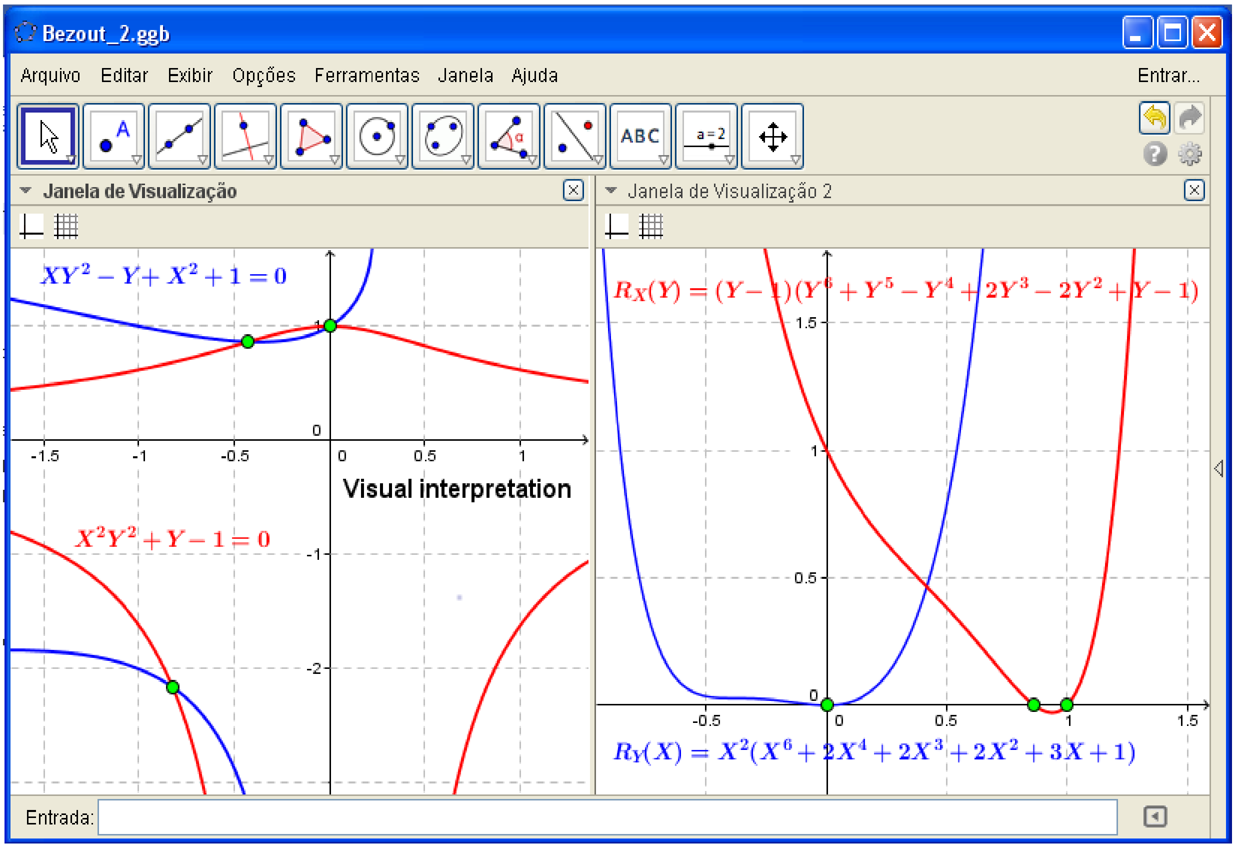

Figure 6. Visualization of the intersection's points related two curves f and g

We finish this section by recalling one way of interpretation of the contents in AG. In fact, we record the following words: "intuitively, the irreducible components of the curve are the "pieces" that are and that are also curved."(Vainsencher, 2009, p. 12). Well, in this quote we observe the role of intuition in order to aggregate an epistemic ans pragmatic value to these contents in AG. These examples can inspire us relatively the academic teaching and learning.

Moreover, we can constate the development of technology involving mathematical tasks that allow us to the visualization of some kind os algebraic curves (Busser, 2014). For example, we can compare Figure 7 with the others that we have constructed by the GeoGebra´s help. From this comparison, we observe several modifications related to the pragmatic and epistemic value. On the other hand, we note the advances of the softwares when we compare Figure 7 with the others figures.

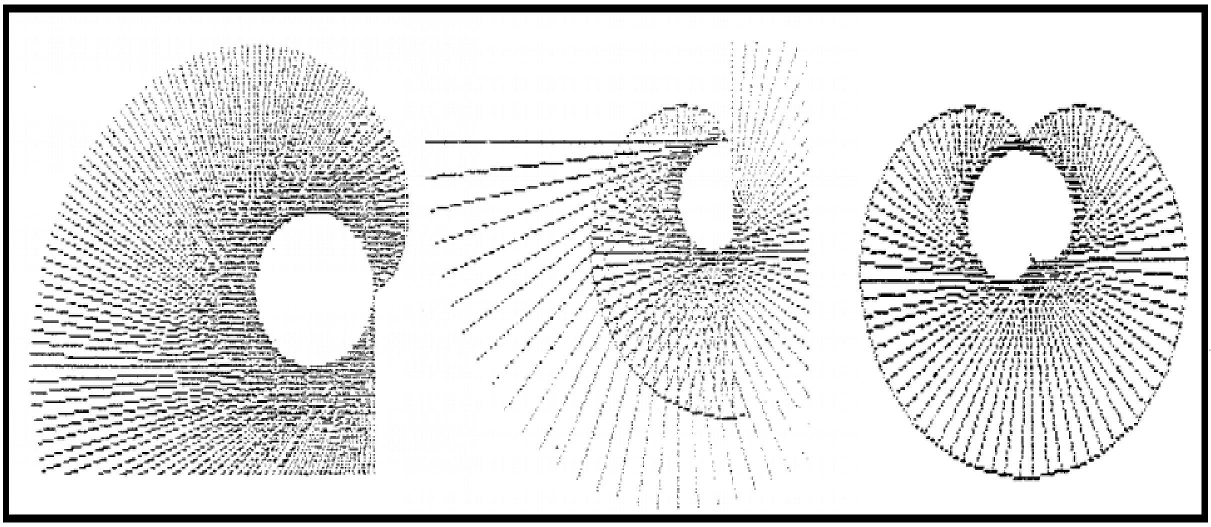

Figure 7. Visualization of the some curves traced by the software *LOGO* (Jarraut, 1986)

3. Conclusion

At the academic level we find several mathematical complex topics involving certain methodological adjustements in line with its epistemological nature (Artigue, 1999). En virtue a complex epistemological nature, is very difficult provided a visual and qualitative interpretation for these conceptual objects, similarly as did the mathematicians in the past (Vaisencher, 2009, p.1-2). In this article, supported by the CTTM, we have indicated a complementary didactical way for use of the DSG and the CAS Maple.

We find some works at the Mathematics Education (Artigue, 2001b; Lavicza, 2005; Robert & Speer, 2001) that indicate several potencialities of the use of the technology for improve the teaching and the learning. Artigue (2001) discusses the epistemic value and the pragmat value in the context of technology approach for teaching and learning. So, our approach, named by Computational Technique for Teaching Mathematics (CTTM) permits the identification of some aspects related to this two kind of cathegories. In fact, the pragmatic value of the GeoGebra´s use faced by some limitations. In addition, the barrries in the computacional calculation when we disregard the CAS Maple, for example.

In this paper, we bring a specific discussion about the polynomial's resultant. Certainly, Algebraic Geometry is the study of zero sets of polynomials. With introductory examples, we showed some situations supported by the visualization and, therefore, the pragmatic value for understanding it. In fact, in certain circumstance, the directly compute of $R_{f,g}$ is very complicated (see *). In same case, the determinant order requires a computational algebraic system like CAS Maple, as we have seen in Figures 1 and 2. We observe that when we explore this standard CAS, we can improve some didactical issues (Winslow, 2003, p. 275).

Historically, we know that the polynomial's resultant has a prominent role in algorithmic algebraic geometry (Chionh; Zhang & Goldman, 1998, p. 1) and there are two matrix representations: the Sylvester Resultant and Bezout Resultant. In this work, we explore the second class of matrix. In certain cases, some authors indicate this as a somewhat accurate method for analyzing the behavior of intersection. On the other hand, we find some cases in which we can explore the DSG´s potentialities en virtue to conjecture and visually predict the solutions. Moreover, we can visualize the graphical behavior of the polynomial associated to the symbol $R_{f,g}(X,Y)$, indicated in this work by the symbols $R_Y(X)$ or $R_X T(Y)$ (see Figures 5 and 6).

In Figure 4 (on the right side), we indicated a nice example supported by the DSG. We can clearly perceive all elements and its conceptual relationship from a visually and analytical point of view. Although, in this case, we have a situation that requires a simple analysis. Indeed, we compare each root of the $R_y(X)$ with each intersection point related to the two algebraic plane curves. On the other hand, with the CAS Maple, we can obtain interesting 'visual algebraic' results that become impossible that to obtain without the actual technology.

Finally, the mathematical research shows that even for the mathematicians is very difficult to understanding all actual information in a specific mathematical branch. Indeed, we remember the words due a Dieudonné (1989, p. 15) when explains that "it is quite difficult to us understanding the point of view of the mathematicians who undertook to tackle topological problems in the second half of the nineteen century. When dealing with the curves, surfaces, and latter, manifolds of arbitrary dimension, with their intersection of their existence when submitted to various conditions, etc, they relied exclusively by 'intuition' […]".

Well, in Mathematics Education, in our methodological approach specifically, we have desired to highlight some elements that admit an access through our mathematical specialized intuition. Without the two softwares, we have only the logical and formal path (Alves, 2014b). Although knowing that mathematical knowledge evolves by progressive accumulations (Dugac, 2003, p. 351) and evolutions (Albuquerque, 1937; Becar, 2007), we cannot disregard that such development is result, too, of the contribution of various mathematical insight. However, when we consider not the experts but our students, we need a systematic way to describe didactical and pedagogical situations to exploit this fundamental cognitive ability. In this way, our CTTM permits some interesting mathematical insights in Brazil.

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