

# DISTINGUISHING SCHEMES AND TASKS IN CHILDREN'S DEVELOPMENT OF MULTIPLICATIVE REASONING

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*We present a synthesis of findings from constructivist teaching experiments regarding six schemes children construct for reasoning multiplicatively and tasks to promote them. We provide a task-generating platform game, depictions of each scheme, and supporting tasks. Tasks must be distinguished from children's thinking, and learning situations must be organized to (a) build on children's available schemes, (b) promote the next scheme in the sequence, and (c) link to intended mathematical concepts.*

**Keywords:** Constructivism; Multiplicative reasoning; Scheme; Task

Distinción de esquemas y tareas en el desarrollo del razonamiento multiplicativo de los niños

*Presentamos una síntesis de hallazgos de experimentos de enseñanza constructivistas en relación con seis esquemas que los niños construyen para razonar multiplicativamente y tareas para promoverlos. Proveemos una plataforma de juego generadora de tareas, descripciones de cada esquema y tareas para apoyarlos. Las tareas deben distinguirse del pensamiento de los niños, y las situaciones de aprendizaje deben organizarse para que (a) se basen en los esquemas que los niños tienen disponibles, (b) promuevan el siguiente esquema en la secuencia y (c) se relacionen con los conceptos matemáticos pretendidos.*

**Términos clave:** Constructivismo; Esquema; Razonamiento multiplicativo; Tarea

In this paper we propose a developmental framework that makes distinctions and links among schemes—conceptual structures and operations children construct and use for reasoning in multiplicative situations. We provide a set of tasks to promote construction of such schemes. Elaborating on Steffe et al.'s seminal

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work (see Steffe, 1992; Steffe & Cobb, 1998; Steffe, von Glaserfeld, Richards, & Cobb, 1983), this framework synthesizes findings of our teaching experiments with over 20 children who have disabilities or difficulties in mathematics. This empirically grounded framework contributes to articulating and promoting multiplicative reasoning—a key developmental understanding (Simon, 2006) that presents a formidable conceptual leap from additive reasoning for students and teachers (Harel & Confrey, 1994; Simon & Blume, 1994). In place of pedagogies that focus primarily on multiplication procedures, our framework can inform teaching for and studying of children's conceptual understandings. Such understandings provide a basis not only for promoting multiplication and division concepts and procedures but also for reasoning in place-value number systems (Chandler & Kamii, 2009), and in fractional, proportional, and algebraic situations (Thompson & Saldanha, 2003; Xin, 2008).

We contrast our stance on children's cognitive change and teaching that promotes it with the Cognitively Guided Instruction (CGI) approach (Carpenter, Franke, Jacobs, Fennema, & Empson, 1998). CGI grew out of research on children's solutions to addition and subtraction tasks. By asserting that "children's solution processes directly modeled the action or relationships described in the problem" (Carpenter, Hiebert, & Moser, 1983, p. 55), CGI researchers seemed to equate children's cognitive processes with tasks. In contrast, we argue for explicitly distinguishing between task features as adults conceive them and schemes adults can infer on the basis of children's actions and language when solving tasks. Consider a joint task such as, "We had 7 toys and got 4 more; how many toys we then had in all?" A child may solve such a task by counting-all 1s (1–2–3–…10–11), by counting-on (7; 8–9–10–11), or by using a through-the strategy ( $7+3=10$ ;  $10+1=11$ ). The latter two indicate the child understands number as a composite unit, hence preparedness for multiplicative reasoning, whereas the first does not (Steffe & von Glaserfeld, 1985). We concur with CGI's premise of the need to use children's ways of thinking in teaching. However, we disagree that the structure of a task as seen by an adult determines, in and of itself, the way a child makes sense of and acts to solve it. The next section presents the conceptual framework that underlies our synthesis.

## CONCEPTUAL FRAMEWORK

Our framework builds on the core notion of scheme—a psychological construct for inferring into the mental realms of thinking and learning. Drawing on Piaget's (1971, 1985) work, von Glaserfeld (1995) depicted scheme as a tripartite mental structure: a situation (recognition template) that sets one's goal, an activity triggered to accomplish that goal, and a result expected to follow the activity. Tzur et al. (Tzur & Lambert, 2011; Tzur & Simon, 2004) further distinguished effect from goal and result, asserting that effect can more precisely pertain to

both anticipated and actually noticed outcomes of a mental activity on/with certain objects. As a person's mind runs activities and regulates them by the goal triggered by a situation, novel effects can be noticed, differentiated from anticipated ones, and related anew to the activity. An *activity-effect relationship* (AER) is therefore conceived of as a conception—a dyadic relation that constitutes a scheme's second and third parts. Existing or newly noticed AERs can be linked to a given scheme's situation, transferred to, and linked with other situations (Tzur, Xin, Si, Woodward, & Jin, 2009).

A mathematical task pertains to a pedagogical tool used to promote student learning, that is, advancing from current to intended schemes (Watson & Sullivan, 2008). Typically, a task consists of depictions of relationships among quantities, some given and some unknown, including a question for figuring out the latter (Watson & Mason, 2007). In recent years, tasks have become a primary tool through which to foster mathematics learning, as opposed to a way of applying taught concepts after learning takes place (National Council of Teachers of Mathematics, 2000; Watson & Mason, 1998). In our constructivist framework, to solve a task, a child has to (a) assimilate it into the situation part of an existing scheme, (b) identify the quantities (mental objects) involved, (c) set a goal compatible with the question, (d) initiate mental activities on those quantities that (in the child's mind) correspond to the depicted relationships, and (e) constantly compare the actual effects of the activity to the goal to determine conclusion of the activity.

A key construct for distinguishing multiplicative from additive reasoning, which pertains to the mental object one operates upon, is number as a composite unit (CU) (Steffe, 1992). To reason additively requires students to operate with number as a composite unit. Children establish this in situations that trigger a goal of determining the amount of 1s in a collection of items and the activity of counting, which involves iterating the unit of one to compose larger units (e.g.,  $1+1+1=3$ ). Gradually, the nested nature of the resulting, composed quantity becomes explicit (e.g.,  $[1+1+1]+1=4$ ;  $+1=5$ , etc.). When number is conceived of as a composite unit, children can anticipate decomposing units into nested sub-units (Steffe & von Glaserfeld, 1985). For example, a child can think of  $11-7=?$  as  $7+?=11$ , that is, a composite unit of 11 (whole) of which she knows one part (7) and can find the other. This part-to-whole decomposition highlights a key aspect of additive reasoning, namely, that the referent unit is preserved (Schwartz, 1991):  $11 \text{ apples} - 7 \text{ apples} = 4 \text{ apples}$ .

Learning to reason multiplicatively requires a major conceptual shift—a coordination of operations on composite units (Behr, Harel, Post, & Lesh, 1994). Consider placing 2 apples into each of 3 baskets; 2 is one composite unit (apples per basket) and 3 is another (baskets). Multiplicative reasoning entails distributing one unit over items of another (2 apples per basket) and finding the total (goal) via a coordinated counting activity: 1 (basket) is  $1-2$  (apples), 2 (baskets)

are 3–4 (apples), 3 (baskets) are 5–6 (apples). Coordinated counting entails deliberately keeping track of composite units while accruing the total of 1s based on the distributed composite unit (2 apples-per-basket). As this example indicates, in multiplicative reasoning the referent unit is transformed (Schwartz, 1991) via the coordinated distribution, and the product has to be conceptualized as a unit of units of units (Steffe, 1992): here, 6 apples is a unit composed of 3 units (baskets) of 2 units (apples per basket). The simultaneous count of two composite units and the resulting unit transformation constitute a key, initial conceptual advance from additive reasoning.

## FROM ADDITIVE TO MULTIPLICATIVE SCHEMES

We first describe tasks we used to promote students' construction of multiplicative schemes—revolving around a platform game called *Please Go and Bring for Me* (PGBM). Then, a six-scheme developmental framework is presented. This order helps to delineate teaching that can foster construction of the schemes while clearly separating between instructional tasks and children's thinking.

### Tasks for Fostering Multiplicative Schemes

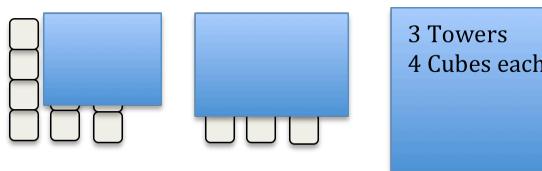
PGBM is an example of a task-generating platform game. It fosters multiplicative reasoning by engaging children in tasks conducive to carrying out and reflecting on coordinated counting activities. The basic form is played in pairs. Partners switch roles in each turn—one playing a *sender* and the other a *bringer*. The sender begins by asking the bringer to produce, one at a time, towers composed of the same number of cubes. Once the bringer has produced the needed amount of same-size towers (e.g., 5 towers, 3 cubes each; denoted  $5T_3$ ), the sender asks her four questions: (1) How many towers did you bring? (2) How many cubes are in each tower? (3) How many cubes are there in all? (4) How did you figure it out? Questions 1 and 2 orient student reflections on the composite units involved—to distinguish activities of producing/counting a compilation of composite units from counting 1s to produce each composite unit. Questions 3 and 4 foster coordinated counting of composite units (e.g., raising one finger per tower) while accruing the total of cubes (e.g., 3–6–9–12–15) based on the size of the distributed composite unit (e.g., 3 cubes per tower).

First, we promote students' facility in playing PGBM with tangible objects (cubes and towers). Making these objects available can support the child's operation on the corresponding mental objects—units of one (1s) and/or composite units. The reason is that tangible objects continually prompt the child's unit-generating operations of unitizing (separating into single items) and uniting (organizing into larger units) (see Steffe & von Glaserfeld, 1985).

When students seem facile with operating on composite units based on tangible objects, we follow a Chinese practice of teaching with variations (Gu, Huang, & Marton, 2006; Jin & Tzur, 2011a) to foster abstraction of coordinated

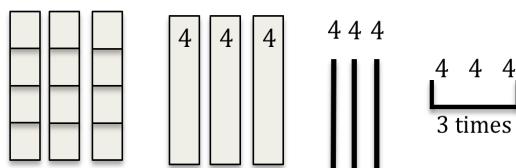
counting. Variation (1) supports students' shift from operating on tangible objects to figural objects—in which a substitute item stands for real objects the students attempt to quantify. For example, to keep track of tangible towers that are covered, the child may substitute the invisible towers by raising her fingers, or jotting tally marks on a paper, while uttering numbers for each. Variation (2) supports students' shifts from operating on figural objects, to abstractly symbolized objects and to mental objects.

In Variation (1) partners produce a given set, say  $3T_4$ , cover the towers (Figure 1), then answer the four questions.



*Figure 1.* Covered towers

Initially, we let children use spontaneous ways of keeping track of composite units and 1s (e.g., count on fingers, tally marks, etc.). Later, we guide them to sketch towers in a gradually more abstract manner. They begin with tower diagrams comprising of single cubes. Then, they sketch tower diagrams with a numeral indicating the tower's size. Then, a line-with-number represents each tower, which gives way to representing the tower by just a number (Figure 2). Using these diagrams fosters a shift in the child's attention (Mason, 2008), from attending to 1s that constitute a composite unit to the numerical value that symbolizes the effect of how each composite unit could have been produced.



*Figure 2.* Tower modeling

In Variation (2) partners are asked to pretend as if they were producing towers, but not to actually do so. As in Variation (1), we guide students to sketch increasingly abstract diagrams, beginning with figural objects and progressing to abstractly symbolized 1s and composite units. When a student can anticipate the structure of the 1s and the composite units, this suggests she or he can operate on composite units as mental objects. Like in the Singapore approach (Ng & Lee, 2009), these variations foster students' advancement from acting on composite units as tangible objects, to tangible replacing the invisible, to mental objects.

Within Variations (1) and (2) we use different amounts of towers and cubes to support students' productive participation. Initially, we purposely direct children to use familiar numbers (2, 5, or 10 cubes per tower) and small compilations

of composite units (up to 6 towers). These constraints support children's focus on operations they recurrently use instead of on calculations involved. Then, when children seem to be facile with coordinated-counting, we guide them to use more difficult numbers (towers of 3-4 cubes, and later of 6, 7, 8, or 9 cubes) and larger compilations (up to 12 towers). When students operate on cubes/towers as figural objects, we introduce similar tasks in other contexts (e.g., How many cookies are in 5 bags, if each bag has 3 cookies?). In doing so, we further promote students' use of coordinated-counting to figure out the total of 1s (e.g., cubes, cookies) across situations constituted by a number of same-size composite units (e.g., towers, bags of cookies).

Building on Xin's (2008) work, we gradually introduce children to a single symbolic structure that ties both multiplication and division. We begin with: *Cubes in each tower × Number of towers = Total of cubes* (Figure 3). As they solve tasks in different contexts, we maintain the structure, replacing cubes and towers with items and groups, respectively: *Items in each group × Number of groups = Total of items*.

$$\frac{\text{Cubes in each tower}}{\text{Number of towers}} + \frac{\text{Number of towers}}{\text{Cubes total}} = \frac{\text{Cubes total}}{\text{Cubes total}}$$

*Figure 3. Equation modeling*

After students solve tasks in different contexts, we introduce unit rate and composite unit to the structure: *Unit rate × Number of composite units = Total of 1s* (adapted from Xin, 2008; Xin, Wiles, & Lin, 2008). This symbolic structure supports students' determination of the needed computation (multiplication, or division, or different operations). In a multiplication situation, the total of 1s is unknown. In a division situation, either the number of composite units or the number of 1s per composite unit is unknown. In other situations, a more complex, multi-step operation may be needed (see the third and fourth schemes in Table 1).

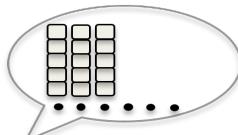
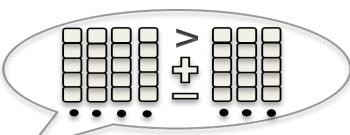
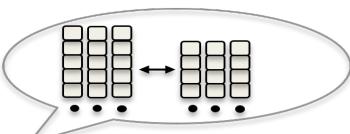
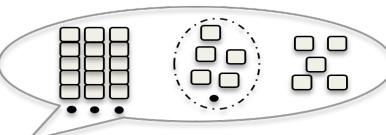
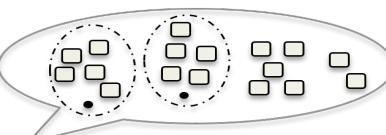
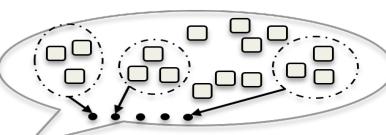
### A Six-scheme Developmental Framework

This section describes each of six schemes that, combined, constitute the framework we propose for promoting children's development of multiplicative reasoning with whole numbers (Table 1). For each scheme, we indicate what the scheme involves, provide a sample task linked to the scheme, explicate goals, activities, and results that constitute the scheme, and articulate mathematics that the established scheme supports.

The first scheme a child may construct is termed *multiplicative double counting* (MDC) (Steffe, 1992; Steffe & Cobb, 1998; Woodward, Kenney, Zhang, Guebert, Cetintas, Tzur, et al., 2009). It involves recognizing a given number of composite units, each consisting of the same number of 1s. Typical tasks include

Variations (1) and (2) of the PGBM platform game. The child's goal is to figure out the total of 1s in this compilation of composite units, and the activity is simultaneous, coordinated (double) counting of composite units and 1s that constitute each composite unit. When established, MDC includes a child's anticipation that a total number of items (say, 24 cookies) is a composite unit constituted of another composite unit (4 bags), each of which a composite unit itself (6 cookies). This scheme provides a basis for the strategic use of known facts to derive unknown ones (e.g., "7  $\times$  5 is like 5 towers of 7, and I know it includes 35 cubes; so 7  $\times$  6 is as if I brought one more unit of 7, hence it is the same as  $35 + 7 = 42$ ").

**Table 1**  
*Six-scheme Developmental Framework*

| Scheme                                   | Anticipatory mental structure   | Constitutive operations  |
|--|---|--|
| Multiplicative Double Counting (MDC)     |    | Coordinated-count of 1s and composite units (distributing items of one composite unit across items of another).        |
| Same-Unit Coordination (SUC)             |   | Additive operations on two sub-compilations of composite units (comparing, adding, subtracting).                       |
| Unit Differentiation and Selection (UDS) |  | Recognizing quantitative differences and similarities between sub-compilations; coordinated-count of difference in 1s. |
| Mixed-Unit Coordination (MUC)            |  | Coordinating multiplicative (segmenting) and additive operations (on composite units) within a global compilation.     |
| Quotitive Division (QD)                  |  | Segmenting a given number of 1s into a compilation of given-size composite units.                                      |
| Partitive Division (PD)                  |  | Partitioning a given number of 1s into a compilation of a given-number of composite units.                             |

The second scheme is termed *Same Unit Coordination* (SUC). It involves operating additively on composite units without losing sight of each composite unit being, simultaneously, both a unit in and of itself and composed of 1s. Typical tasks linked to this scheme involve two sub-compilations of composite units and a question to figure out sums of or differences between the sub-compilations. SUC tasks may ask: "You brought  $7T_5$  and then I brought  $4T_5$ ; How many towers do we have in all?" or "You brought  $7T_5$ ; I brought a few more; Together, you and I have  $11T_5$ ; How many towers did I bring?" The child's goal is to figure out the sum or difference of composite units (not of 1s), and the activity may be any of those a child has constructed for operating additively on 1s (counting-all, counting-on, through-ten, fact retrieval, etc.). Like with units composed of 1s, the key in this scheme is the child's conception of the embedded (nesting) nature of composite unit sub-compilation within a larger, global compilation (e.g., a global compilation consisting of 11 units of 10 can be decomposed into 7 units of 10 and 4 units of 10). When established, the SUC scheme provides a basis for operating on specific composite units such as 10s, 100s, and 1000s in a place-value system, with contexts including distance, weight, money, etc. (Fuson, Smith, & Lo Cicero, 1997; Fuson, Wearne, Hiebert, Murray, Human, Olivier, et al., 1997).

The third scheme is termed *Unit Differentiation and Selection* (UDS) (McClintock, Tzur, Xin, & Si, 2011). It involves explicitly distinguishing operations on composite units from operations on 1s, and operating multiplicatively on the difference of 1s between two sub-compilations of composite units. Typical tasks include, "You have  $7T_5$  and I have  $4T_5$ ; How are our collections similar? Different? How many more cubes do you have?" (Note: sub-compilations may differ in number of composite units, or in unit rate, or in both.) The child's goal is to specify the similarities and differences, and to figure out the difference in 1s between the two sub-compilations. The child's activity can include (a) operating multiplicatively on each sub-compilation to find its total of 1s and then find the difference (*Total-First* strategy) or (b) finding the difference in composite units and then multiplying it by the unit rate (*Difference-First* strategy). We promote children's use and coordination of both strategies. When established, the UDS scheme includes a situation recognized as two sub-compilations of composite units that can be similar or different with respect to quantities that constitute each sub-compilation. The UDS scheme provides a conceptual basis for making sense of and using the distributive property of multiplication over addition [e.g.,  $7 \times 5 + 4 \times 5 = 5 \times (7 + 5)$ ] and for solving algebraic equations such as  $7x + 4x = 55$ .

The fourth scheme is termed *Mixed-Unit Coordination* (MUC) (Tzur et al., 2009). After the UDS scheme has enabled distinguishing composite units from 1s, the MUC scheme involves operating on 1s to answer questions about composite units in two sub-compilations. Typical tasks include, "You have  $7T_5$ ; I'll give you 10 more cubes; if you put these 10 cubes in towers of five cubes each,

how many towers would you have in all?" (Note: The question can be, "How many cubes would you have in all?") The child's goal is to figure out the number of composite units (or of 1s) in a global compilation that combines both given quantities (sub-compilations). To this end, the child's activity includes selection and coordination of the unit rate (e.g., 5) from the given sub-compilation with a segmenting operation (Steffe, 1992) on the given number of 1s to yield the additional number of composite units (2 towers) in the other sub-compilation, and then adding this newly found number of composite units to the initially given sub-compilation ( $2 + 7 = 9$  towers). The MUC scheme includes a situation recognized as one sub-compilation of composite units and another, potential sub-compilation that is initially composed of 1s. The MUC scheme supports the segmenting of a composite unit of 1s based on a given unit rate, which is a precursor to partitioning a given quantity of 1s as required for division. It is also a critical conceptual foundation for operating on different quantities, such as tens and ones (e.g., If you have 7 bags with 10 marbles each and seventeen more marbles, how many marbles do you have in all?).

The fifth scheme is termed *quotitive division* (QD). It involves operating on a given composite unit of 1s (say, 28 cubes) in anticipation of the count of iterations of a sub-composite unit (e.g., towers of four). That is, the child anticipates the effect of a segmenting activity on the given total. Typical tasks include, "You have 28 cubes; pretend you will (or actually) take them back to the box in towers of 4 cubes each. How many towers are brought back to the box?" The child's goal is to figure out how many sub-composite units constitute the given total, and the activity is segmenting of the total via MDC regulated for stoppage when accruing and given totals are equal (e.g., 1-tower-is-4-cubes, 2-is-8, ..., 7-is-28). When established, a QD scheme reverses MDC. The QD scheme provides a basis for conceiving of division as an inverse operation to multiplication, and thus for using fact families of the latter to solve division problems in which the total and the size of each group is given. While playing a game in which children posed PGBM tasks, with conditions specified about the fit between the given totality and sub-composite units (e.g., you need to give me a total and a number of cubes in each tower so when I run out of cubes there will still be 2 cubes left), we could foster in children a conceptual prerequisite for division with remainders.

The sixth scheme is termed *partitive division* (PD). Similar to the QD scheme, the PD scheme involves recognizing a situation with a given totality of 1s. However, the other aspect of the situation a child must recognize is that a given number of sub-composite units requires accomplishing the goal of figuring out the equal-size of each. A typical task would be "You want to put 28 cubes in 4 equal towers. How many cubes will you have in each tower?" Initially, children may accomplish the goal through the activity of distributing all given 1s to each group one after another. We consider this an important precursor for partitive division, but not yet the scheme itself. The child is yet to construct an anticipated

conception of distribution that operates on composite units (not merely on 1s). Introducing prompts and constraints to the child's activity (e.g., "Do you think there would be more than one cube in each tower? Will 3 cubes work? Why?"), children with whom we worked began to anticipate that each round of distribution of 1s would yield a composite unit. They then could reorganize their coordinated-counting activity to figure out the end result (unit rate) without carrying out the distribution—the essence of the PD scheme. The PD scheme provides a basis for seeing division as a twofold (QD/PD) inverse of multiplication, and thus for making sense of and solving corresponding algebraic operations with equations.

## DISCUSSION

In this paper, we proposed a developmental framework of six schemes that underlie children's learning to reason multiplicatively with whole numbers. Table 1 juxtaposes these schemes in terms of the anticipatory, goal directed ways of operating children seem to use for making sense of and reasoning through or about tasks they solve. This framework can guide both the assessment of students' available conceptions and the setting of corresponding teacher goals for students' learning. To both ends, we included tasks and playful activities (e.g., the PGBM game) that may be linked to each scheme. Thus, this paper can support fostering children's multiplicative reasoning via adaptive teaching (Steffe, 1990), a conception-based pedagogical approach (Jin & Tzur, 2011a; Simon, Tzur, Heinz, Kinzel, & Smith, 2000) that reactivates (daily) children's prior knowledge as a necessary step to promote transforming this knowledge into the intended mathematics.

### Theoretical Contributions

The developmental framework of schemes and tasks presented in this paper makes two main contributions. First, it contributes a blueprint of anticipatory structures that can help specify and implement hypothetical learning trajectories (Simon, 1995) for children. This blueprint is grounded in empirical studies of students with learning disabilities or difficulties in mathematics as well as their normal achieving peers, conducted as case studies of individuals/pairs (McClintock et al., 2011; Tzur, Xin, Si, Kenney, & Guebert, 2010; Tzur et al., 2009; Woodward et al., 2009; Xin, Tzur, Si, Zhang, Hord, Luo, et al., 2009) and as whole-class teaching experiments (forthcoming). An example of how this blueprint supports curriculum design can be found in the software that we have been developing as part of the activities of the *Nurturing Multiplicative Reasoning in Students with Learning Disabilities in a Computerized Conceptual-Modeling Environment* (NMRSD) (Xin, Tzur, & Si, 2008). This software engenders students' learning via solving problem situations (individually or in pairs)

adapted to their available, evolving conceptions<sup>1</sup>. An important way in which this blueprint can contribute to practice is the distinction of SUC, UDS, and MUC. These three critical conceptualizations can serve in altering current curricula, which commonly teach division directly after multiplication.

We suggest that structures in this blueprint can be traced back to children's conceptions of number sequences (Steffe, 1992). The first structure (MDC) marks the child's conceptual leap from additive to multiplicative reasoning. Here, the child coordinates (operates on) units of one and composite units into a single compilation of composite units that is understood dynamically (i.e., the child understands the number of composite units as potentially being increased or decreased). In this sense, it seems that MDC is rooted in and is indicative of the Explicitly Nested Number Sequence (Steffe, 1992). The second structure (SUC) transforms the single compilation of MDC into a global compilation consisting of two or more sub-compilations. Within this global structure, the child can operate on the composite units as entities in and of themselves without losing sight of the 1s that constitute each composite unit, sub-compilation, and the global compilation. The third structure (UDS) further transforms the SUC global compilation by orienting the child's attention to explicitly distinguishing among the units she operates on, 1s or composite. In UDS, the child essentially coordinates SUC and MDC to figure out differences in 1s between two sub-compilations. It thus seems that UDS and SUC are rooted in and indicative of a transition to the Generalized Number Sequence (Olive, 2003; Steffe & Olive, 2010). The fourth structure (MUC) further transforms UDS by coordinating multiplicative and additive operations within a global compilation. This coordination is required because one sub-compilation is given as composite units and the other as 1s. The child needs to select and impose the unit rate given for the composite units in order to multiplicatively segment the number of 1s, which can support transition to the last two, divisional schemes. It seems that MUC (a) is rooted in and indicative of the Generalized Number Sequence and (b) provides a conceptual foundation for the three upper strategies that Fuson et al. (1997) identified in children's solution of problems involving 10s and 1s. In the fifth (QD) and sixth (PD) schemes, the child reverses MDC while segmenting a given number of 1s into a single compilation of composite units. When the total of 1s is not a multiple of the given unit rate (QD) or of the given number of composite units (PD), the child may also bring forth, use, and reverse MUC. This is to say that these multiplicative structures and students' conceptions of number sequences seem reflexively related. Said differently, students' conceptualization of number sequences seems likely to inform and/or constrain children's development of these structures.

The second contribution of our framework is in instantiating a fundamental constructivist principle, namely, the child's thinking and the task are not equiva-

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<sup>1</sup> While writing this paper, the beta version is being programmed; it would be tested and refined in 2013.

lent. In depicting and using the six schemes, we do not focus on task characteristics or the child's behaviors when successfully (or not) solving a task per se. Rather, we focus on the invisible and thus necessarily inferred ways of operating—situations and goals the child sets, activities she initiates toward these goals, and effects she notices to follow the activity—that may be engendered by the task and underlie how the child solves it. From this perspective, studying transformations in schemes can be done via design and use of task sequences that may foster, but do not determine, children spontaneous and/or prompted thought processes. Said differently, task design can reflexively be guided by and provide guidance to conceptual analysis of scheme components to increase the likelihood of promoting, and detecting, particular transformations in children's reasoning.

### Practical Implications

For teaching and teacher education, the explicit distinction between children's ways of reasoning when using each scheme and the type of tasks they solve implies the need to pay close attention to (a) units upon which a child operates, (b) the extent to which such operations are spontaneous or prompted, and (c) numbers used in a task (e.g., avoid MDC tasks with  $3T_3$ , or MUC tasks with  $5T_7 + 14$  cubes). Such attention supports using bridging tasks (Jin & Tzur, 2011b) that deliberately reactivate those schemes as a means to foster construction of more advanced schemes. For example, two 4th graders with whom we worked solved a bridging task, "Pretend you have  $9T_3$ ; Together you and I have  $14T_3$ ; How many towers of 3 cubes each do I have?" by counting-up on their fingers ("9, 10–11–12–13–14; so that's  $5T_3$ "). But when asked to solve a task with  $19T_3$  and  $24T_3$ , which from an adult's perspective seemed structurally similar, they had no idea how to proceed. One of them could later solve it after drawing diagrams of the first compilation, whereas the other child could only do so after actually producing all the towers. The work of these children illustrates that even if a child's work with smaller numbers may be spontaneous, solving same-structure tasks with larger numbers may require prompting or be beyond the child's current capacity. In another teaching episode, one of our team members engaged students who had not yet constructed MDC as an anticipatory structure in solving SUC tasks while operating on tangible objects. Although the students could obtain answers to SUC tasks, their reasoning seemed limited to counting perceptual singletons. By promoting previous schemes in the sequence through successfully solving related tasks, students not only could complete tasks successfully, but more importantly, could engage in reasoning that supported their construction of more advanced schemes. To this end, our six-scheme blueprint can provide a basis for designing platform tasks and variations in those tasks (Gu et al., 2006; Jin & Tzur, 2011a) to engender apt reasoning in multiplicative situations while addressing gradations and individual differences in children's thinking.

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## REFERENCES

- Behr, M. J., Harel, G., Post, T. R., & Lesh, R. A. (1994). Units of quantity: a conceptual basis common to additive and multiplicative structures. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 121-176). Albany, NY: State University of New York.
- Carpenter, T. P., Franke, M. L., Jacobs, V. R., Fennema, E., & Empson, S. B. (1998). A longitudinal study of invention and understanding in children's multidigit addition and subtraction. *Journal for Research in Mathematics Education*, 29(1), 3-20. doi: 10.2307/749715
- Carpenter, T. P., Hiebert, J., & Moser, J. M. (1983). The effect of instruction on children's solutions of addition and subtraction word problems. *Educational Studies in Mathematics*, 14(1), 55-72. doi: 10.1007/BF00704702
- Chandler, C. C., & Kamii, C. (2009). Giving change when payment is made with a dime: the difficulty of tens and ones. *Journal for Research in Mathematics Education*, 40(2), 97-118.
- Fuson, K. C., Smith, S. T., & Lo Cicero, A. M. (1997). Supporting latino first graders' ten-structured thinking in urban classrooms. *Journal for Research in Mathematics Education*, 28(6), 738-766.
- Fuson, K. C., Wearne, D., Hiebert, J. C., Murray, H. G., Human, P. G., Olivier, A. I., et al. (1997). Children's conceptual structures for multidigit numbers and methods of multidigit addition and subtraction. *Journal for Research in Mathematics Education*, 28(2), 130-162.
- Gu, L., Huang, R., & Marton, F. (2006). Teaching with variation: a Chinese way of promoting effective mathematics learning. In L. Fan, N.-Y. Wong, J. Cai, & L. Shiqi (Eds.), *How Chinese learn mathematics: perspectives from insiders* (pp. 309-347). Singapore: World Scientific Publishing.
- Harel, G., & Confrey, J. (Eds.). (1994). *The development of multiplicative reasoning in the learning of mathematics*. Albany, NY: State University of New York.
- Jin, X., & Tzur, R. (2011a, January). *Progressive incorporation of new into known: a perspective on and practice of mathematics learning and teaching in China*. Paper presented at the Annual Conference of the Association of Mathematics Teacher Educators, Irvine, CA.

- Jin, X., & Tzur, R. (2011b, April). '*Bridging': an assimilation- and ZPD-enhancing practice in Chinese pedagogy*'. Paper presented at the The 91st Annual Meeting of the National Council of Teachers of Mathematics (Research Pre-Session), Indianapolis, IN.
- Mason, J. (2008). Being mathematical with and in front of learners: attention, awareness, and attitude as sources of differences between teacher educators, teachers, and learners. In B. Jaworski & T. Wood (Eds.), *The international handbook of mathematics teacher education: the mathematics teacher educator as a developing professional* (Vol. 4, pp. 31-55). Rotterdam, The Netherlands: Sense Publishers.
- McClintock, E., Tzur, R., Xin, Y. P., & Si, L. (2011). Engendering multiplicative reasoning in students with learning disabilities in mathematics: Sam's computer-assisted transition to anticipatory unit differentiation-and-selection. In L. R. Wiest & T. D. Lamberg (Eds.), *Proceedings of the 33rd annual meeting of the North American chapter of the International Group for the Psychology of Mathematics Education* (pp. 164-172). Reno, NV: University of Nevada.
- National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: Author.
- Ng, S., & Lee, K. (2009). The model method: Singapore children's tool for representing and solving algebraic word problems. *Journal for Research in Mathematics Education*, 40(3), 282-313.
- Olive, J. (2003). Nathan's strategies for simplifying and adding fractions in third grade. In N. A. Pateman, B. J. Dougherty, & J. Zilliox (Eds.), *Proceedings of the 27th annual meeting of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 421-428). Honolulu, HI: University of Hawaii at Manoa.
- Piaget, J. (1971). *Biology and knowledge* (B. Walsh, Trans.). Chicago, IL: The University of Chicago (Original work published 1965).
- Piaget, J. (1985). *The equilibration of cognitive structures: the central problem of intellectual development* (T. Brown & K. J. Thampy, Trans.). Chicago, IL: The University of Chicago (Original work published 1965).
- Schwartz, J. L. (1991). Intensive quantity and referent transforming arithmetic operations. In J. Hiebert & M. J. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 41-52). Reston, VA: National Council of Teachers of Mathematics.
- Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, 26(2), 114-145.
- Simon, M. A. (2006). Key developmental understandings in matheamtics: a direction for investigating and establishing learning goals. *Mathematical Thinking and Learning*, 8(4), 359-371. doi: 10.1207/s15327833mtl0804\_1

- Simon, M. A., & Blume, G. W. (1994). Building and understanding multiplicative relationships: a study of prospective elementary teachers. *Journal for Research in Mathematics Education*, 25(5), 472-494.
- Simon, M. A., Tzur, R., Heinz, K., Kinzel, M., & Smith, M. S. (2000). Characterizing a perspective underlying the practice of mathematics teachers in transition. *Journal for Research in Mathematics Education*, 31(5), 579-601.
- Steffe, L. P. (1990). Adaptive mathematics teaching. In T. J. Cooney & C. R. Hirsch (Eds.), *Teaching and learning mathematics in the 1990s* (pp. 41-51). Reston, VA: National Council of Teachers of Mathematics.
- Steffe, L. P. (1992). Schemes of action and operation involving composite units. *Learning and Individual Differences*, 4(3), 259-309. doi: 10.1016/1041-6080(92)90005-Y
- Steffe, L. P., & Cobb, P. (1998). Multiplicative and divisional schemes. *Focus on Learning Problems in Mathematics*, 20(1), 45-62.
- Steffe, L. P., & Olive, J. (2010). *Children's fractional knowledge*. New York, NY: Springer.
- Steffe, L. P., & von Glaserfeld, E. (1985). Helping children to conceive of number. *Recherches en Didactique des Mathématiques*, 6(2-3), 269-303.
- Steffe, L. P., von Glaserfeld, E., Richards, J., & Cobb, P. (1983). *Children's Counting Types*. New York, NY: Praeger.
- Thompson, P. W., & Saldanha, L. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick & G. Martin (Eds.), *Research companion to the NCTM Standards* (pp. 95-113). Washington, DC: National Council of Teachers of Mathematics.
- Tzur, R., & Lambert, M. A. (2011). Intermediate participatory stages as zone of proximal development correlate in constructing counting-on: a plausible conceptual source for children's transitory 'regress' to counting-all. *Journal for Research in Mathematics Education*, 42(5), 418-450.
- Tzur, R., & Simon, M. A. (2004). Distinguishing two stages of mathematics conceptual learning. *International Journal of Science and Mathematics Education*, 2(2), 287-304. doi: 10.1007/s10763-004-7479-4
- Tzur, R., Xin, Y. P., Si, L., Kenney, R., & Guebert, A. (2010, April & May). *Students with learning disability in math are left behind in multiplicative reasoning? Number as abstract composite unit is a likely "culprit"*. Paper presented at the American Educational Research Association, Denver, CO.
- Tzur, R., Xin, Y. P., Si, L., Woodward, J., & Jin, X. (2009). Promoting transition from participatory to anticipatory stage: Chad's case of multiplicative mixed-unit coordination (MMUC). In M. Tzekaki, M. Kaldrimidou, & H. Sakonidis (Eds.), *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 5, pp. 249-256). Thessaloniki, Greece: PME.

- von Glaserfeld, E. (1995). *Radical constructivism: a way of knowing and learning*. Washington, DC: Falmer.
- Watson, A., & Mason, J. (1998). *Questions and prompts for mathematical thinking*. Derby, United Kingdom: Association of Teachers of Mathematics.
- Watson, A., & Mason, J. (2007). Taken-as-shared: a review of common assumptions about mathematical tasks in teacher education. *Journal of Mathematics Teacher Education*, 10(4-6), 205-215. doi: 10.1007/s10857-007-9059-3
- Watson, A., & Sullivan, P. (2008). Teachers learning about tasks and lessons. In D. Tirosh & T. Wood (Eds.), *The international handbook of mathematics teacher education: tools and processes in mathematics teacher education* (Vol. 2, pp. 109-134). Rotterdam, The Netherlands: Sense Publishers.
- Woodward, J., Kenney, R., Zhang, D., Guebert, A., Cetintas, S., Tzur, R., et al. (2009). Conceptually based task design: Megan's progress to the anticipatory stage of multiplicative double counting (MDC). In S. L. Swars, D. W. Stinson, & S. Lemons-Smith (Eds.), *Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 5, pp. 1378-1385). Atlanta, GA: Georgia State University.
- Xin, Y. P. (2008). The effect of schema-based instruction in solving word problems: an emphasis on pre-algebraic conceptualization of multiplicative relations. *Journal for Research in Mathematics Education*, 39(5), 526-551.
- Xin, Y. P., Wiles, B., & Lin, Y-Y. (2008). Teaching conceptual model based word problem story grammar to enhance mathematics problem solving. *Journal of Special Education*, 42(3), 163-178. doi: 10.1177/0022466907312895
- Xin, Y. P., Tzur, R., & Si, L. (2008, January). *Nurturing multiplicative reasoning in students with learning disabilities in a computerized conceptual-modeling environment*. Grant proposal submitted by Purdue University. Arlington, VA: National Science Foundation.
- Xin, Y. P., Tzur, R., Si, L., Zhang, D., Hord, C., Luo, W., et al. (2009). Interweaving tasks and conceptions to promote multiplicative reasoning in students with learning disabilities in mathematics. In S. L. Swars, D. W. Stinson, & S. Lemons-Smith (Eds.), *Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 5, pp. 354-362). Atlanta, GA: Georgia State University.

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