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Much More Than It's Cooked-Up To Be: Reflections on Doing Math and Teachers' Professional Learning

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ABSTRACT:

The author argues that students' persistent struggles with mathematics suggest a new form of professional development for teachers is needed. The author draws on a model of professional learning in literacy education to propose an analogous model for mathematics education: teachers of mathematics need to produce mathematical ideas, themselves, in order to better support their students in becoming mathematical thinkers. It is not enough to focus singularly on developing teachers' content understanding, however, because mathematical ideas are embedded within their representational forms; therefore, any content-related professional development must also include pedagogical discussions. The author concludes by describing a research-based, high-quality professional development community—the Philadelphia Area Math Teachers' Circle (PAMTC)—in which authentic mathematical inquiry and pedagogical analysis occur hand-in-hand.

Introduction: Believe It or Not, I Do Math for Fun!

"Whaddaya do, go home on the weekends and just do math?"

The Medical Curiosity of Math Teachers

Math teachers get this question a lot. My preteen students asked most often, but I'll bet their parents and my non-math colleagues wondered silently, too. Reading between the lines, I would hear: "He must not have a life! How could anyone teach math and *enjoy* it?" Outwardly, I would laugh off this question and its companion allegations. Inwardly, though, I would bristle at the sentiment and that the words to respond were so elusive.

As a founder of the Philadelphia Area Math Teachers' Circle (PAMTC), I now have better language to use. "Nah," I would begin, brightly. "I do lots of things when I go home. But, yeah, I also *discover math!*" Such an assertion seems outlandish, but—believe it or not—I do, in fact, *do math* for fun. Most people probably imagine that I sit amidst a pile of dusty textbooks and scribble solution after solution to the "homework problems." This could not be further from the truth, however, as I hope to illustrate.

In this article, I argue for reconceiving math teachers' professional development (PD). Note that, throughout, I use the terms "teachers of mathematics" or "math teachers" to refer to K-12 teachers whose responsibilities include mathematics instruction. In making my argument, I describe the common, overly-narrow conceptions of mathematics and teachers' PD, and I present alternative viewpoints that draw on literacy education. I also explain our efforts in the PAMTC, which I believe represents a research-based, high-quality form of professional learning for teachers. I claim that groups such as ours fulfill a pressing need across the U.S.—especially in urban centers, like Philadelphia.

A Dish Served Cold: Challenges of the American Educational Landscape

To put it bluntly, we Americans are bad at math. Decades of research and volumes of anecdotal evidence have proven it. Our K-12 students continue to underperform on national and international tests (NCES, 2013; Provasnik et al., 2012). Adults fare no better. Nearly 60% of two-year college students are placed in remedial math courses (Bailey, Jeong, & Cho, 2010), and compared to other nations, a shockingly high proportion of American adults lack basic numeracy and problem-solving skills (OECD, 2013). Very few of us pursue advanced study of mathematics beyond the required courses in high school or college (Snyder & Dillow, 2013, Tables 179 & 331).

Consider, also, this dismaying 1980's anecdote from Green (2014): the A&W restaurant chain introduced a new hamburger with one-third pound of beef, but it suffered lackluster sales in comparison to McDonald's smaller-sized Quarter Pounder. Even though the A&W burger was the preferred choice on taste tests, and even though both burgers were priced comparably, focus-group research revealed that most people falsely presumed the Quarter Pounder was a better deal. They reasoned, incorrectly, that the "4" implied within the name "Quarter Pounder" meant the McDonald's burger was larger.

Clearly, standard models of teaching haven't worked, and meaningful reform has proven perniciously difficult for generations (Lortie, 2002; Tyack & Cuban, 1995). Improving teachers' PD is seen as a key ingredient for change, but improvements have been difficult to attain (see Yoon et al., 2007; Ball & Cohen, 1999; Cohen & Hill, 2000; Darling-Hammond & McLaughlin, 1995; Desimone et al., 2002; Heck et al., 2008). We therefore need a *different* form of professional development for teachers of mathematics, and teachers, themselves, agree. Most say, in fact, that "much of the professional development available to them is not useful" (Darling-Hammond et al., 2009, p. 5), especially because of the lack of content-oriented PD. Content-oriented PD concentrates on aspects of teaching and learning unique to a particular area, like history or mathematics. The majority of teachers receive fewer than 16 hours (or two days) of content-oriented PD annually, and this amount is decreasing (Darling-Hammond et al., 2009). Research has shown, in contrast, that 50 hours or more is needed (Yoon et al., 2007, p. 14). One-time workshops, the most common forms of PD in American schools, have virtually no effect on teachers or their students (Darling-Hammond et al., 2009; Yoon et al., 2007).

In addition, changes in the educational landscape—including the Common Core State Standards Initiative—aim to support students' development of critical-thinking and problem-solving abilities and place additional demands on teachers. Teachers also say they are unprepared to implement these sorts of changes (EPE Research Center, 2013). One persistent barrier they encounter is time: U.S. teachers spend about 80% of their workdays on teaching, while their counterparts in other, high-performing nations spend about 60% and have the remainder available for collaboration and professional learning (OECD, 2014, Table D4.1).

The Brownie Problem

"What are you giggling at?" my partner, Andrea, asked. We were driving home to Philadelphia from a friend's birthday party on Long Island.

"I just remembered the *Seinfeld* episode I saw, today. Elaine steals a piece of cake from Peterman's fridge. Turns out the cake was a collector's item: it was from the 1937 wedding of King Edward VIII and Wallis Simpson and was worth $29,000. She tries to hide the missing piece by smoothing out the frosting." I laughed, reminded of the cake we inhaled hours earlier.

"Ha! That's interesting," Andrea said, clearly amused. But I also detected a serious tone.

"Why is that?"

"Well, it reminds me of the Brownie Problem from the Math Teachers' Circle you told me about. I've been thinking about it."

"Yeah, me too," I replied. "Let's see if I can remember the problem…."

> At the end of the school term, you bake a tray of brownies to share among your two 7th grade classes. While the brownies are cooling on the counter, some mischievous scamp sneaks into the kitchen, cuts out a rectangular piece, and steals it. The tray of brownies must be split evenly between the two classes, and you have only moments to do so. Is it possible to divide the brownie tray evenly, using a single cut from your knife? Why or why not?

To stay awake on our drive, we chatted about the problem. As I had told Andrea, I wanted to use the problem in an upcoming Philadelphia Area Math Teachers' Circle (PAMTC) workshop, but I wasn't quite sure how to interpret it and I didn't yet grasp possible solutions. So we worked it out together, riffing on each other's questions and ideas.

Different Ingredients in "Doing Mathematics"

The Traditional View: Recipes and Rules

Most American students experience mathematics through models known as direct instruction (Rosenshine & Stevens, 1986) or gradual release (Pearson & Gallagher, 1983). With direct instruction, the teacher sets lesson objectives and then walks students toward these aims; near the end of the lesson, students typically work alone in solving practice problems. In gradual release, the teacher also leads: first, by modeling the steps; then, as the class practices together, by debugging any errors. Finally, as with direct instruction, students complete practice problems on their own. The gradual release model is also known as "I, we, you" because the teaching sequence proceeds from the teacher ("I"), to the class ("we"), to the individual student ("you").

The discipline of mathematics, as interpreted within these forms of instruction, usually consists of a set of rules to follow, or recipes to learn, to arrive at the so-called "right" answers. The teacher's role involves explaining or demonstrating these steps for students. Most people, when pressed, would probably have difficulty conceiving of anything *but* the above models for doing and teaching mathematics. In this article, I portray alternatives to these views, arguing that they are inherently problematic. First, though, I note that these traditional views are so entrenched that, when presented with alternatives, people sometimes dismiss them outright. Teachers employing alternative models are not actually "teaching math"; students of such teachers, the argument goes, are not "learning math"—at least not the math that they need to become "college- or career-ready" (see Rubinkam, 2014).

Problems with the Traditional View

Boaler (2013) argues that this traditional view of math is limiting. She observes that many Americans experience anxiety when faced with mathematical tasks. Being a struggling reader in our society is embarrassing, but it's perfectly acceptable to be math-phobic. Put another way, many Americans feel that they are just not meant to be "math people." Indeed, mathematical ability is largely considered the hallmark of a gifted few whose abilities are innate rather than learned or developed (Burkley et al., 2010). Boaler also explains that decades of research show it does not have to be this way: "when mathematics is opened up and broader math is taught" (2013, para. 3), she argues, students not only become engaged and interested in math, they also do better at it.

Before painting an alternative picture of what it means to *do math* and what Boaler (2013) refers to as "broader math" (para. 3x), I also point out that research has demonstrated schools are contrived and inauthentic arenas for doing mathematics. Studies have shown that students and adults are eminently capable of performing complex calculations, when playing sports or going to marketplaces, but cannot do so when presented with the *exact same* problems on paper tests in classrooms (Herndon, 1971; Lave, 1988; Nunes, Schliemann, & Carraher, 1993). Likewise, studies of word problems in school mathematics show that students often ignore the contexts and simply focus on the numbers and operation keywords, which can lead to nonsensical results (Gerofsky, 2004; Wiliam, 1997). In each case, researchers argued that schools are so disconnected from the real-world that problems posed within them cease to make sense. Additional research has shown that the so-called "traditional" algorithms of school mathematics (e.g., multiplying multi-digit numbers using carrying) actually *obscure* the fundamental properties of number that are key to understanding them (Hiebert et al., 1997). We therefore need mathematical experiences in schools that are meaningful for and understandable to students. I argue that, to achieve this, we need PD experiences that provide the same for teachers.

The Brownie Problem: Part 2

"What are some questions that you have about the Brownie Problem?" I asked Andrea. As she replied, I ticked off her questions from my own mental checklist.

"How big was the piece that was stolen? What shape was it? Where was it cut?" she asked.

"Right! Right!" I replied, excitedly. "I wondered, too—I have no idea! I also wonder how big the tray of brownies is, including how thick it is. And does thickness even matter?"

As we started to ask these questions, we also began to answer them. In other words, we developed a set of assumptions that helped us to interpret the problem and establish boundaries on reasonable approaches.

"Let's start by assuming that the stolen piece is also rectangular and that it's been cut from anywhere in the tray."

"OK, fine. So…."

An Alternative Perspective: Creating New Dishes

The main problem with the traditional view of mathematics and mathematics learning and teaching is that it bears little resemblance to how mathematicians *actually* work. In school, most of us did math the same way that we did cursive or did spelling; that is, we copied and copied the forms and rules established by others. I argue, though, that we received fallacious messages from these practices and that we don't actually know what it means to *do math*.

In contrast, to mathematicians, *doing math* is a creative endeavor. It does not involve following pre-established recipes. Mathematicians are like artists, experimental scientists, critics, politicians, and chefs all rolled into one. They estimate. They mix. They create. They test. When something fails, they scrap their work and start all over again. Often, they work together. And when they do, they create new language, disagree and debate, and build on one another's ideas. In short, they are curious and they pursue lines of inquiry wherever they lead; when doing math, they support and revise their ideas, while also drawing upon the collective wisdom of their predecessors (Lakatos, 1976; Pickering, 1995).

Studying mathematics bears resemblance to studying art or cuisine in one additional, crucial aspect: just as every painting or dish reveals something new each time we view or taste it, mathematical thinkers can discover something new each time they return to seemingly familiar ideas. Ma (1999) noted that teachers in elementary schools in China have a deeper understanding of fundamental mathematical ideas than do their American counterparts. I speculate this is partly because American teachers may be embarrassed to investigate or are simply unaware of the importance of investigating basic ideas of whole number operations, fractions, or decimals. I believe that such reflection would be productive, revealing subtleties either long ago forgotten or otherwise missed.

At its heart, mathematics is both a problem-solving venture and a language. In order to make sense of mathematics, students need to be exposed to problems that are important in their everyday lives, and they need to have opportunities to communicate mathematically—to explain their thinking, to critique others' thinking, and to develop notation and terminology that makes sense to them in the context of solving a problem. This is what Boaler (2013) means by "broader math" and what typifies the work of professional mathematicians (and other critical thinkers!). The Common Core State Standards recognize the importance of active mathematical inquiry, as well, in the eight Standards for Mathematical Practice (NGA Center / CCSSO, 2010). The Practice Standards are easily overlooked but actually represent the foundations of the Common Core reforms and decades of research in math education.

I now explore teachers' professional learning. First, I explain the traditional approaches to supporting teachers and problems with these approaches. As before, I offer alternatives. I also return to our work on the Brownie Problem, as an example of authentic mathematical inquiry and a model for a new vision of teachers' development.

Teachers' Professional Learning: The Old and the New Menus

The Old Menu: Traditional Dishes and Their Problems

As noted above, professional development for American teachers mainly consists of one-time workshops. In most cases, these workshops fail to address learning and teaching within a specific content area, and—with respect to the type, nature, and amount of PD—they rarely align with established best practices (Darling-Hammond et al., 2009). This sort of delivery method, generally speaking, puts teachers in the role of receivers (Darling-Hammond et al., 2009, p. 6). Consequently, the traditional approach to PD replicates problems with traditional approaches to teaching and learning: teachers inevitably become passive, disengaged, and struggle to apply what is being taught to their classrooms. Yoon and her colleagues (2007) explain that the difficulty of translation rests on the highly contextual nature of teaching and learning. In other words, generic PD does not address teachers' questions about *their* classrooms or their students' specific needs.

The New Menu: Alternative, Research-Based Approaches

For these reasons and many more, researchers have argued for a different set of principles to guide PD (Garet et al., 2001; see also Doerr, Goldsmith, & Lewis, 2010). Many of these are general recommendations for improving teachers' learning; in other words, they could apply to teachers in any academic discipline. For example, professional development experiences regarded as high quality are those that provide regular and ongoing opportunities for active learning, for building productive habits of mind, and for collaboration. In this section, though, I argue that PD for teachers of mathematics must also include something *more* and *different*. Math teachers, I believe, face an additional set of challenges that necessitate a novel approach.

First, teachers need to know *much* more mathematics than what they teach. The Conference Board of Mathematical Sciences (CBMS), a blue-ribbon panel of mathematics and mathematics education experts, notes that, across all grades, teachers' mathematical knowledge is often incomplete and superficial (CBMS, 2012). The CBMS recommends, then, that all teachers *continue* deepening and expanding their mathematical understanding while they teach. The panel's recommendations align with the artful vision of mathematics described previously: returning to seemingly basic ideas can reveal complexities previously unnoticed. Research supports regular engagement in mathematical thinking for students (Murayama et al., 2013). Why shouldn't the same principle apply to teachers?

I argue, though, that increasing the quantity of content-oriented PD is not enough, because there simply isn't enough time within the schedule. Requiring teachers to undertake coursework at colleges or universities is not a solution, either: when teachers pursue advanced study, significant improvements in teaching have not occurred (Hanushek & Rivkin, 2006). Graduate coursework, under the pressure of exams and grades, tends to reinforce the outdated models of learning and teaching. Teachers need to understand mathematics, but they also need to understand the models, examples, and representations of mathematics that are helpful for building meaningful and deep connections between ideas (Ball, Thames, & Phelps, 2008). Helping teachers bolster their mathematical knowledge for teaching is, in fact, tied to students' improved achievement (Hill, Rowan, & Ball, 2005; Hill et al., 2007; Jacobs et al., 2007). To accommodate teachers' schedules and to promote higher levels of engagement, I therefore argue for *recreational* professional development in mathematics (much like the playful way Andrea and I worked on the Brownie Problem).

Second, teachers of mathematics require *different* PD opportunities than their peers. This, I argue, is because of the deeply-rooted negative perceptions of mathematics that reside in our culture. Tobias (1978/1995) originally defined math anxiety as the feelings of apprehension that impede our performance on mathematical tasks. Math anxiety also manifests itself when we *avoid* mathematics, perhaps even thinking that it isn't worth our while to explore mathematical ideas. Higher levels of math anxiety translate into lower achievement, and math anxiety can even be transferred from adults to children (see Aronson et al., 1999; Ashcraft & Kirk, 2001; Beilock et al., 2010; Gunderson et al., 2012; Hadley & Dorwand, 2011; Vukovic et al., 2013).

I think that the great majority of us are anxious about math, including many math teachers, themselves (Hembree, 1990). Teachers are, of course, products of our culture; they, too, learned about math from their own in-school experiences, and more likely than not, these experiences were also stressful and uninspiring (Beilock et al., 2010). Why would we expect this chain to break spontaneously? Todays' teachers also face intense pressure to increase students' test scores, to proceed rapidly through the curriculum, and to meet the new standards. These pressures further decrease the chances that K-12 mathematics will be engaging or joyful. Not only do we as a society need cultural retraining to break down unhelpful assumptions about mathematics, but our teachers need retraining, too. Fortunately, research has shown that Boaler's (2013) "broader mathematics" is not incompatible with standardized testing (see Grouws et al., 2013; Tarr et al., 2008). To the contrary, many new state assessments even ask students to demonstrate deep conceptual understanding and an ability to explain their thinking, capacities that would be bolstered by an enthusiasm for mathematics.

Finally, high-quality professional learning opportunities are especially needed in cities like Philadelphia. First, such school districts have few available resources. PD budgets have been slashed, and the available time for engaging in PD is minimal. In Philadelphia, less than one-tenth of 1% of the school budget has been allocated for teachers' PD; shockingly, schools report having less than $200 for supplementary PD for their entire faculty for the entire year (School District of Philadelphia [SDP], 2014; Graham, 2014). Further, mathematics instruction (and instruction in general) is often narrower in urban and under-resourced settings than in suburban and affluent areas (Haberman, 1991; Kohn, 2011). Therefore, schools in under-resourced areas often need additional support, both because of greater need for resources and because of the deeply-entrenched nature of traditional practices.

I now describe a support structure for teachers that has been developed in literacy education before turning to an analogous, high-quality (and relatively low-cost) effort for math teachers. These shift professional learning away from *receiving knowledge* and toward *creating knowledge*, as part of everyday recreational activity.

The Brownie Problem: Part 3

"So, let's assume that the stolen brownie-piece is a rectangle and let's make it really simple," Andrea said.

"What do you mean by 'simple?'" I asked, wondering where she was going.

"Well, imagine that the thief was really greedy and stole half the tray, cutting it right down the middle." I imagined the following picture (see Figure 1a).

"Well, then, it's easy to figure out how to cut the remaining piece in half," I said. "Just cut the brownie in the opposite direction, say lengthwise" (see Figure 1b).

"Right," she said. Mile after mile of the New Jersey Turnpike drifted away, as we problem-solved.

"So, then," I began, "let's move to something a bit more complicated. Instead of taking half, the thief steals one-fourth of the tray—cutting out an entire corner. In that case, you could just make a diagonal cut and you'd create two equal pieces that were the exact same shape, like a trapezoid." (See Figure 1c.)

"But that's not right," she replied. "Imagine a very narrow, skinny rectangle. With a quarter of the tray removed, just like before. If you cut from one interior corner to the opposite corner of the tray, then your two pieces would *not* be the same size. Or shape!" She demonstrated by drawing in the air (Figure 1d).

"Yeah! So cutting along a diagonal, from one corner to another, won't work. Hmm…."

Teachers as Patrons and as Chefs

Teachers' work, we know, consists of oodles of essential tasks: grading homework, planning field trips with colleagues, contacting students' families, and so on. But when do teachers have the opportunity to grapple with the content they teach? To *practice* the content? To grow, as learners (or patrons) and as *producers* of knowledge (or chefs)?

At first, it may seem odd to regard teachers as producers of knowledge instead of transmitters of knowledge. Most teachers, however, are also practitioners of the content they teach. For example, English teachers often read, deeply, for pleasure. Many even write stories, essays, poetry. Art teachers are often artists, themselves, and many music teachers perform in musical groups. These are all avocations, pursued for fun and often within others' company. Scholars have shown that these pursuits are far from frivolous (e.g., Graham & Zwirn, 2010). Atwell (1987) argues:

> When teachers demonstrate interest and excitement in our fields, we invite students to believe that learning is valuable. We answer the question, "Why do we have to do this?" with our own conviction and passion, modeling the power we derive from our knowledge and experience. (p. 48)

In literacy education, scholars have invited teachers to think of themselves as writers—to *become* writers—not only for the purposes of modeling desired behaviors and dispositions, but also because the act of writing is an authentic way to learn about writing (see Graves, 1983; Calkins, 1986; Atwell, 1987/1998). And teachers cannot merely teach; they must be learners, too. In Hall-McEntee's (1998) words, "Teachers retain their vulnerability as learners, sensitive to the complex challenge of putting thoughts into words [by writing]. Writing is a humble act during which the writer is always up against personal inadequacy. Sharing this struggle with students models genuine learning." The National Writing Project and Cochran-Smith and Lytle (2009) also show us the importance of teachers writing about their teaching practice. In other words, for students to become better readers (and writers), they need to write; for teachers to become better teachers of reading and writing, they need to write, too. And since teaching is, inherently, a communicative endeavor, teachers are well served by writing *about* their teaching.

Therefore, while some scholars in literacy education have recognized the interconnectedness of reading texts and writing (or producing) texts and have promoted the use of new pedagogical models, like writing workshops (Atwell, 1987/1998; Calkins, 1986; Graves, 1983), we have no analogue in mathematics education. Modes of teaching in K-12 mathematics, and students' achievement, remain largely stagnant in the U.S. Yet, we need students to become mathematicians—to become creative and curious mathematical thinkers. To get there, I argue, we need teachers to become mathematicians, too.

I hope I am making clear that neither grading homework, nor completing worksheets, is *doing math*. Really doing math is something else, entirely. Indeed, I helped to start the Philadelphia Area Math Teachers' Circle for one simple reason: to provide a *recreational* space for teachers to engage in, and hence deepen their appreciation of, *authentic* mathematical thinking. I detail how we work in the PAMTC, below, noting that the Brownie Problem has thus far illustrated our model.

The Brownie Problem: Part 4

"Oh! I think I have an idea!" exclaimed Andrea. "What if the missing piece was located entirely within one quadrant of the tray? Then, you could…" She elaborated on her approach.

"That would work!" I replied, smiling. "I wonder, though, what would we do—if the missing piece was located elsewhere in the tray." After some thought, both of us realized another possible solution.

"Hm," I began, still unsatisfied. "I wonder if the shape of the brownie tray or the shape of the missing piece matters. Can we find an approach that will work for all types of brownie trays and all types of missing pieces? In other words, for what shapes of brownie trays and missing pieces will our solutions work?" (See Figure 2.)

Days later, long after our trip had passed, we eventually arrived at (and wrote an explanation of) our approach to this extended problem.

A Fun, New Flavor of Professional Learning: Math Teachers Circles (MTCs)

Math Teachers' Circles: The Model and History

The model, which we follow in our workshops in the Philadelphia Area Math Teachers' Circle and which I describe below, was developed by the American Institutes of Mathematics (AIM) in 2006. The AIM is one of eight mathematical sciences research institutes funded by the National Science Foundation that bring mathematicians together to collaborate on solving cutting-edge research problems. The AIM noted that most mathematical discoveries are made by teams of scholars, working together, debating and discussing. They thought, "Why not find a way to not only show K-12 mathematics teachers how professional mathematicians work, but to also provide them with an opportunity to do real mathematical work, themselves?" And, so, Math Teachers' Circles (MTCs) were born (AIM, 2014). Currently, there are about 60 MTCs across the U.S. with a cumulative total of approximately 1,000 regular attendees (Silverstein, 2014).

At MTC workshops, after an initial greeting, a professional mathematician describes a problem that we then tackle by working together. The professional mathematician remains "hands-off," throughout, letting us grapple with possible solutions—of which there are usually many. This approach contrasts with what happens in most American classrooms, wherein teachers often step in to help guide their students through problems toward a single answer and using a single approach (Stigler & Hiebert, 2009).

The types of problems used in MTCs are what Boaler (2014) calls "low-floor, high-ceiling" problems. Such problems are easily accessible to all and, yet, could also be extended to employ more sophisticated approaches. They are markedly different than the artificial problems commonly used in school mathematics, which confound students because of their artificiality (Gerofsky, 2004; Wiliam, 1997). Solving "low-floor, high-ceiling" problems requires probing, investigative work. The Brownie Problem is one such problem; here are several others:

- **The *Futurama* Theorem.** Professor Farnsworth and Amy build a machine that allows them to switch minds. They use the machine, so that Farnsworth can feel youthful in Amy's body, and so that Amy can eat whatever she wants, guilt-free, in Farnsworth's body. After using it, they discover that the machine has one, troublesome flaw: after a pair of minds is switched, the same two minds cannot switch back to their original bodies. What are Farnsworth and Amy to do?
- **The Key Cap Problem.** Rushing to get inside your house, you drop your key ring on the ground. What is the smallest number of colored key caps that you need to distinguish all of your otherwise indistinguishable keys?
- **The Four and Five Problem.** Using the numbers 1, 2, 3, and 4 no more than once and the operations of addition, multiplication, and subtraction, what are all possible results that you could obtain? What's the smallest positive whole number that you *cannot*? What about with 1, 2, 3, 4, and 5?

If you study these problems carefully, you will note that they are unlike traditional math problems. In most, there is no clear-cut statement of a "problem." Instead, a particular situation is presented from which new questions emerge. For instance, with regard to the Key Cap Problem:

- Should we agree that all the keys are physically indistinguishable?
- Is the key ring marked in some way?
- How many keys are on the ring? Does this even matter?

The Key Cap Problem is deceptively simple until the above questions are considered. In order for an actual "problem" to emerge, in the sense of an unknown or unknowable question, we assume that the keys are indistinguishable and their number unknown, and that the ring is unmarked. That said, there are additional variations that are also worth exploring, and in the PAMTC we would mention these before settling on one, agreed-upon problem. Much of our initial work involves simply interrogating what the problems *could* mean. This implies, of course, that our problems rarely have a single correct answer, nor a single solution path. This is one essential aspect of a mathematician's work: precisely identifying a particular question that is worth investigating, defining the important terms, and agreeing upon definitions.

Invariably, we get stuck when solving problems. More often than not, the stickiness involves difficulty articulating or justifying our ideas. When this happens in the PAMTC, we arrive at a teachable moment: this is what our K-12 students experience daily, when they face new mathematical ideas. The essence of mathematical inquiry is not feeling defeated, but rather, recognizing that getting stuck is when actual learning can occur. (The Common Core Practice Standards thereby value "persisting in problem-solving" [MP1].) Getting an answer, so to speak, is less important than developing strategies for how to become unstuck, such as making a table to organize information, working backwards, and trying a simpler case. These strategies invariably allow for progress; we then celebrate the joy in surmounting a block.

Note that in my description of MTC workshops, I have distinguished between "mathematician" and "professional mathematician"; this distinction reflects our belief that, fundamentally, *everyone is a mathematician*. Sure, those who make a living doing mathematical research have more experience than we do—those folks are "professional mathematicians." But we emphasize that pursuing and enjoying mathematics should not be exclusively reserved for professionals, certainly not any more than photography, music, or writing should be. With regard to writing, Atwell (1987) asserts:

> Teachers who write as a way of making their own meanings know about disorder....When we make room for the tentativeness and turbulence of creating written meaning, we and our kids breathe a collective sigh of relief. Writing well isn't a gift God gives to a chosen few. Instead, we provide a powerful demonstration: with enough time to shape and reshape the writing, with topics and audiences we care about and with responses along the way, anyone can write well. (p. 154)

Replacing "writing" with "doing mathematics," Atwell's words are equally apt descriptors of the everyone-is-a-mathematician philosophy underlying MTCs.

Again, doing math involves making discoveries and justifying these discoveries, rather than simply following someone else's prescribed rules without knowing exactly why. Some discoveries might be personal discoveries, but they are still important. Some discoveries may even have relevance beyond our workshops. For example, middle and high school teachers in the Palo Alto, CA MTC recently blurred the lines between professional and amateur mathematicians by publishing research in *The College Mathematics Journal* (Baker et al., 2013). Asked about his MTC team's work, mathematician-facilitator Brian Conrey said, "There are plenty of good, unsolved problems *that anyone can work on*. You don't necessarily need specialized training in mathematical research, so much as a willingness to try something and to be persistent" (Peterson, 2013, italics added).

A New Flavor of MTC: The PAMTC

At each of our workshops in the PAMTC, we draw upon the AIM model, but we differ in our approach in several significant ways. First, we devote a considerable amount of time to making the AIM model transparent to our participants. At the beginning of our workshops, we explain our purposes for coming together: to put ourselves into the position of our students, to become math learners for the next few hours, and to reflect upon our teaching. We announce that we will work together in solving a complex math problem. We stress the importance of productive collaboration and of asking each other questions. We say that it's OK to feel frustrated. After all, problems are (by definition!) intentionally frustrating situations. We note that growth only happens when you learn how to handle and work through challenges. We also say that it's OK to ask for more time to think. We encourage everyone to just try something when encountering an obstacle: make a table, write a list, draw a sketch. We explain that, by participating actively and having fun, you are doing math and building your problem-solving toolkits. And we say that we hope our participants will, in turn, share this understanding and these tools with *their* students, whenever *they* view math in narrow ways or become stuck when solving problems.

Secondly, in the PAMTC, we believe it's not enough to just do math. Teachers should also have opportunities to discuss how to create math-positive environments and foster authentic mathematical experiences in their classrooms. Therefore, beyond simply modeling a more engaging form of mathematical pedagogy—one centered on rich problem-solving experiences, discussions, and debates—we also debrief after each session, to bring the underlying principles to light. Our model for debriefing contains two pedagogical- ingredients: how to make connections to curriculum materials and standards, and how to facilitate classroom inquiry. With regard to making connections to curriculum and standards, we analyze material from commonly used teachers' guides or standards documents, looking at the language and suggested activities, as well as discussing ways to understand and support students' learning. With regard to facilitating classroom inquiry, we look at videotapes of teaching, and we discuss strategies for scaffolding and differentiating problems, promoting student-thinking, and navigating classroom conversations. Some of these strategies are research based, and some emerge organically from our participants.

In general, we aim to support teachers in building the capacity to do problem-solving in their classrooms and to feel more comfortable letting their students do more of the intellectual work. One key challenge involves managing students who are not accustomed to doing so. Authentic problem-solving experiences place teachers in the role of facilitator rather than teller—that is, teachers must engage students in solving problems on their own and in talking with each other, rather than simply explaining what needs to be done. This is often a new role for teachers, and making such a shift requires a lot of deliberate thought and retraining. Instead of giving teachers lists of tips or activities to bring back to their classrooms, we try to build teaching capacities (akin to the aphorism of teaching a person *to fish*, rather than giving *a* fish).

Just Desserts? Research Findings

The American Institutes of Mathematics is currently spearheading a multi-year, NSF-funded study on the impact of Math Teachers' Circles nationwide. The preliminary results, and results from other research, are highly encouraging. In particular, after one year of participation in a Math Teachers' Circle, teachers' confidence increased within their own math classrooms and with deploying inquiry-based (or problem-solving oriented) activities. These self reports were validated by classroom observations, which showed that such classrooms also exhibited a greater frequency of inquiry-based instruction (Marle, Decker, & Khaliqi, 2012). Teachers also reported increased confidence in solving mathematical problems, themselves, and in believing that all of their students are equally capable of thinking as mathematicians (White & Donaldson, 2011). And even during short-term participation in an MTC, teachers' scores have significantly increased on a standard test for measuring mathematical knowledge for teaching (White, 2011; White et al., 2013). Previous studies have associated teachers' mathematical knowledge for teaching with student achievement levels (Hill, Rowan, & Ball, 2005; Hill et al., 2007). Finally, as a consequence of this growing body of support, the CBMS (2012) recently highlighted Math Teachers' Circles as an especially fruitful forum for professional learning.

Further, a key part of the MTC framework and a key ingredient of research-based, high-quality PD is building collegiality (Doerr, Goldsmith, & Lewis, 2010; Garet et al., 2001). This is why MTC workshops meet regularly throughout the school year, in contrast to the usual one-off PD experiences. In the PAMTC, such a community is building, as the majority of our attendees attend several workshops per year and our attendance continues to grow. Teachers rave about our work together, too. They report a deepening appreciation for mathematics as a field of creative inquiry and greater confidence in facilitating open-ended, problem-solving lessons in their classrooms. Most importantly, our participating teachers tell us they are having fun *doing* math!

Conclusion

I've argued that doing mathematics is much more than just grading homework or completing workbook exercises. Instead, through the Brownie Problem, I have illustrated that *really* doing math is like wrestling with a brain-teaser or creating a new dish. Real mathematics involves imagination, questioning, testing, and exploring. Furthermore, authentic, deep-thinking problems permit a variety of interpretations and approaches toward a solution.

We want our students to be able to do *real* math: to become creative problem-solvers and critical thinkers, to be engaged in inquiry, and to have a thorough grasp of concepts. We need them to understand the whys, and not just the whats and hows. To do so, they need to engage in actual mathematical inquiry. Borrowing a sports metaphor, students need to get in the game and play it as it is meant to be played—not just do endless practice drills (or worksheets). They need to get into a real kitchen and invent something new, not just play with an Easy Bake Oven. They need to write, themselves, and not just read what others have written.

The same should be said of our teachers. We want our teachers to be able to inspire our students, but since their own experiences with math were probably uninspiring, they need a different kind of professional development. Professional learning experiences cannot replicate outdated pedagogy, and professional learning—to be truly transformative—must be ongoing and lifelong. As there are very few resources and little time within today's schools (especially in under-supported districts like Philadelphia), professional learning experiences need to be more readily integrated with teachers' out-of-school intellectual and social lives. Most importantly, like writing a poem, creating a painting, or baking a dessert, PD needs to be fun and active. Like being in a book group, or attending a dinner party, it also needs to be a meaningful community experience.

Since 2006, MTCs have challenged the impoverished, bland, common perceptions of mathematics—that solving a math problem simply involves following a prescribed set of steps and practicing these steps in an unthinking way, over and over. For the past three years in Philadelphia, the PAMTC has also supported teachers in connecting their growing mathematical appreciation and understanding to more effective pedagogical practices. It may be a vain, budding notion, but it is nonetheless our hope that when asked what we do in our free time, we can proudly respond, "We discover mathematics"—and our students, their family members, and our colleagues will all want to join us.

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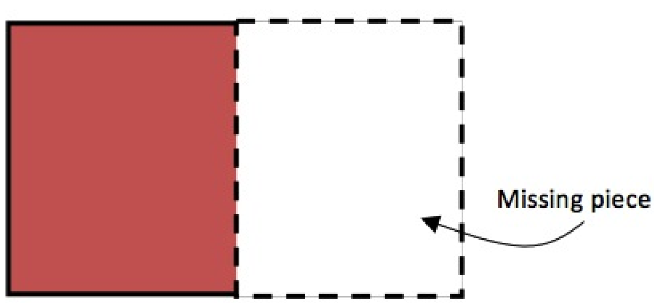

Figure 1a. A brownie tray with half missing

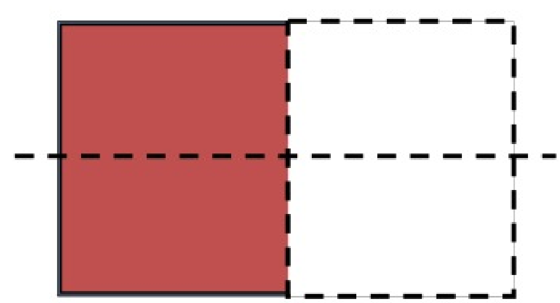

Figure 1b. Cutting the remaining piece in half

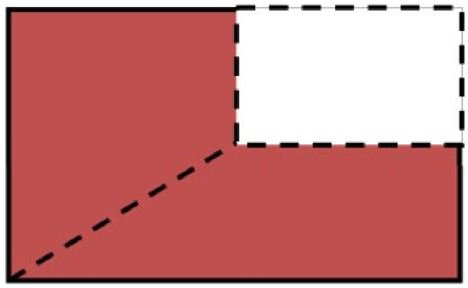

Figure 1c. A fourth of the brownies missing, and the remaining piece cut in half, perhaps?

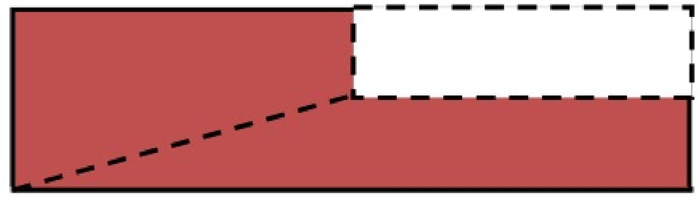

Figure 1d. A long, rectangular tray, and the remaining piece clearly not cut in half

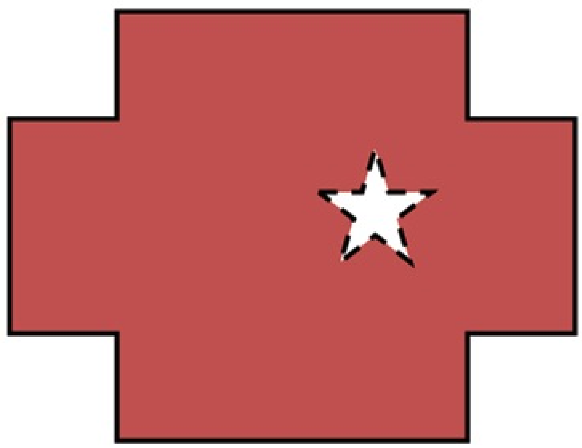

Figure 2. An oddly-shaped brownie tray with an oddly-shaped missing piece

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