Acta Didactica Napocensia

Volume 5, Number 1, 2012

INVESTIGATING AND COMMUNICATING TECHNOLOGY MATHEMATICS PROBLEM SOLVING EXPERIENCE OF TWO PRESERVICE TEACHERS

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Abstract: In this paper, I report on preservice teachers' reflections and perceptions on their problem-solving process in a technological context. The purpose of the study was to to investigate how preservice teachers experience working individually in a dynamic geometry environment and how these experiences affect their own mathematical activity when integrating content (nonroutine problems) and context (technology environment). Careful analysis of participants' perceptions regarding their thinking while engaged in problem solving, provided an opportunity to explore how they explain the emergence of problem solving when working in a dynamic geometry environment. The two participants communicated their experience both through the lenses of themselves as problem solvers and as future mathematics educators. Moreover, the results of the study indicated that problem solving in a technology environment does not necessarily allow focus on decision-making, reflection, and problem solving processes as reported by previous research.

Key words: dynamic geometry system, problem solving, reflection

1. Introduction

"For many people the appropriate use of technology has significant identity with mathematics problem solving. This view emphasizes the importance of technology as a tool for mathematics problem solving" (J. W. Wilson, Fernandez, & Hadaway, 1993, p. 69). In the Principles and Standards for School Mathematics (NCTM, 2000), the authors emphasize the importance of technology in teaching and learning mathematics. Specifically, the standards advocate use of dynamic geometry technology to teach geometry at all levels of mathematics education, emphasizing that the dynamic interactive nature of this technology helps in engaging students in meaningful mathematical activities and promotes deeper understanding of concepts.

In the past two decades a new genre of educational software has become prominent as a classroom tool to support the teaching and learning of geometry. This type of software has been labeled dynamic geometry software or system (DGS) or dynamic geometry environments, a term coined by Nick Jackiw (Olive & Makar, 2010), describing in its name the main features of the software: direct manipulation of geometric figures possible via a pointing device, for instance, by dragging parts of the figure (Hoyles & Noss, 2003). Various dynamic geometry software packages, such as Geometer's Sketchpad (GSP), Cinderella, Cabri, and GeoGebra are available on discs for installation on computers as well as online for download that can be used in classrooms. Geometer Explorer is an APP now available for the iPad that will examine Geometer's Sketchpad files. DGS such as GSP uses a few primitive building objects such as points, lines, line segments, rays, circles, and polygons, and the interface of a DGS creates an opportunity to transform a mathematics classroom into an environment of investigation of interesting phenomena where students engage in observing, manipulating, predicting, conjecturing, testing, and developing explanations for observed phenomena (Fey, Hollenbeck, & Wray, 2010; NCTM, 2000, 2005). For instance, once objects are constructed, the dynamic feature of the software allows for the objects to be dragged via any of their constituent parts while maintaining their constructed properties and underlying geometric relationships, in contrast to a

Received 17 March 2012.

paper-and-pen representation. This feature creates a kind of feedback to the user and an opportunity for a perturbation that was not available before the introduction of DGS (Olive & Makar, 2010). Olive and Makar (2010) provide an in-depth discussion through constructivist lenses on the potential for DGS to transform the way mathematics is learned.

As this type of software allows for direct construction, manipulation, and measurement of geometric figures, it became increasingly popular in mathematics education; particularly it found a role in the geometry curriculum and made a contribution to problem-solving strategies (Fey et al., 2010; Olive & Makar, 2010; J. W. Wilson et al., 1993). Goldenberg et al. (1988) argue that providing the opportunities and dynamic tools for students' explorations promotes the habits of mind that constitute true mathematical power. According to Goldenberg et al. (1988) and Cuoco, Goldenberg and Mark (1996), habits of mind are the ways of thinking that are there to allow students to develop a myriad of approaches and strategies that can be applicable in situations varying from challenges in school to those in life. Some of the habits of mind they discuss include but are not limited to looking for patterns, exploring, communicating, argumenting, conjecturing, refuting, and generalizing. The habits are there to enlighten students about the creation of mathematics, and most importantly to help them learn the way mathematicians think about mathematics. Consequently, students' engagement in these habits helps develop and increase their ability to determining on their own how to think mathematically; deciding what information is needed, choosing a particular strategy, testing their conjectures, and examining what is learned and how it can be applied to a different problem-solving situation. On the down side, users may tend to abuse the power of DGSs. Instead of appreciation for the structure of the system, the system becomes a tool in the hands of the user; the user uses the system because he or she wants to get the problem done which takes away the cognitive load of mathematical thinking (Hoyles & Noss, 2003; Olive & Makar, 2010). Thus, new and emerging technologies continually transform the mathematics classroom and redefine ways mathematics can be taught (Fey et al., 2010). The research (e.g., Artigue, 2002; Hollebrands, 2007; Lagrange, 2003; Thompson, 2009) attests to the fact that DGSs provide the user a well-tuned system within which different mathematical concepts and mathematical problems may be explored (Hoyles & Noss, 2003).

On the other side, much attention has been paid to the concept of reflection in the literature in the last century. Dewey (1933) is acknowledged as a main originator of the concept of reflection who considered it to be a particular form of problem solving, thinking to solve a problem that involved a certain ordering of ideas linking each with its predecessors. However, reflection can also be defined as the conscious consideration of personal experiences often in the interests of establishing relations between ideas or actions (Hiebert, 1992). In mathematics learning, according to Siegel (1981, as cited in Wheatley, 1992), reflection is characterized by distancing oneself from the action of doing mathematics. According to Wheatley (1992), it is one thing to solve a problem and it is quite another to treat one's action as an object of reflection. He emphasizes that is not enough for students to complete tasks, but that they must be encouraged to reflect on their activity (exploring why they acted as they did, and in doing develop sets of questions and ideas about their activities and practice). A major value in solving problems occurs when students step back and reflect on how they actually solved the problem and how the particular set of strategies they used were suboptimal and might be improved.

2. Purpose Statement

Many studies on the teaching and learning of geometry have emphasized the benefits of using dynamic environments (DeVilliers, 1999; Fey et al., 2010; Hollebrands, 2007; Jones, 2000; Laborde, 2000) pointing in an important direction focused on understanding the impact of working in dynamic geometry environments, such as the GSP, on student mathematical problem solving and learning in mathematics. However, no study addressed students' perception on the importance of problem solving in a dynamic geometry environment on their problem solving process, and learning through such experience as a result of reflective activities. By reflecting on the experience, students can consolidate their knowledge, and develop their ability to solve problems. Providing reflection on one's own knowledge monitoring and other regulatory skills embedded in problem-solving learning activities can enhance both students' awareness of their own abilities, and adequate selection of strategies and, possibly, students' performance on learning tasks. Moreover, it could give them an opportunity not only to reflect on their experience as learners, but also as future educators. Such research must be continued and extended if we are to obtain convincing evidence concerning students' mathematical achievement with dynamic technology tools. In light of these considerations, the following questions were central to this study:

- Which elements of reflection can be identified in preservice teacher communication of the experience when problem solving using technology?
- How do preservice teachers perceive the importance of problem solving in a dynamic geometry environment when faced with nonroutine geometry problems?

The paper offers terms of describing and explaining what and how learners appreciate and make out of technology problem solving endeavor.

3. Methods

For this study, a qualitative research design was used. In qualitative research, the researcher is the primary tool to conduct the research (Patton, 2002). Qualitative research study is reflexive as data collection, data analysis, development and modification of theory, and modification of research questions occur simultaneously each influencing all of the others (Patton, 2002). Hence, the focal point of qualitative research is on the research process as a whole, allowing the researcher to understand, organize and report on the phenomena under investigation rather than focusing on a specific outcome (Patton, 2002).

For the main study, I used a purposeful sampling strategy was as a way of collecting rich and deep data from the research participants. The participants were two preservice teachers, Wes and Aurora (pseudonyms), each serving as a unique case, from the mathematics education program at a large southeastern university in the United States. Given that qualitative research is highly interactive, and that the researcher is constantly in the process of questioning and refining the research tools (Patton, 2001), the basic criteria for participation in this study were based on the four following conditions: (1) participants wanted to be a significant part of investigating and communicating the experience of technology in mathematical problem solving (TMPS), (2) participants had previously experienced investigating mathematics in a dynamic geometry environment and are comfortable working with it, (3) participants were comfortable with me (and I with them) discussing not only their mathematics but also their experiences and feelings while engaged in TMPS, and (4) participants have taken college geometry course where GSP was used. Based on research and personal experience, and after consulting with my major professor, I determined several people that would be ideal, that worked well individually, and were reflective thinkers who articulated their thinking well. Not only had they been used to working in a dynamic geometry environment, but they had substantial mathematical background, and we had established a rapport where they felt comfortable interacting with me on a variety of levels.

Data collection procedures

Data collection methods for this study consisted of the following: interviews, audio/video taping of the sessions as the participants were engaged in solving problems, and my observations of the sessions to investigate preservice teachers' thought processes during problem solving in a dynamic geometry environment. The first phase of formal data collection started by using the interview protocol that was intended to elicit the participant's background, and prior technology learning experiences and perceptions. At the core of the present study, the next stage of data collection concentrated on the participants' involvement in solving of three mathematical problems in a dynamic geometry environment (see next section for the mathematical problems). Data collection occurred in a one-to-one setting between the participant and me, as in the preliminary interview and a practice think-aloud session. Next, the participants were asked to solve the given problem. After receiving the problem, the participants began by reading the problem aloud or in silence. They used as much time as they needed in solving each problem. The individual interviews took place shortly after the participants finished solving each problem. We stayed in the same room and talked about the participant's problem-solving session that intended to elicit the participant's experience about using technology solving the particular task. The field notes included the descriptions of questions, reactions, and behaviors that occurred during data collection that were then used during a retrospective interview. The same procedures were used for the following two mathematical tasks. At the end of each appointment, we discussed the date and time for the next problem-solving session.

Mathematical problems

The mathematical problem solving tasks included three nonroutine problems selected and modified from a variety of sources, including mathematical journals, textbooks and web sites. The problems that met the following criteria were considered for study inclusion: (1) The problems are nonroutine; that is, the participants are faced with an unfamiliar problem situation without an apparent solution path. These problems require problem solvers to use information and strategies in unfamiliar ways; that is, they demand strategy flexibility; thinking flexibility, such as logical thinking; abstract thinking; and transfer of mathematical knowledge to unfamiliar situations, as well as extension of previous knowledge and concepts (Schoenfeld, 1992); (2) "The problem should be well chosen, not too difficult and not too easy, natural and interesting, and some time should be allowed for natural and interesting presentation" (Pólya, 1945/1973, p. 6); (3) The problems provide students with opportunities to engage in metacognitive activity, (4) The problems cover mathematical content in geometry, and solutions should not require mathematical concepts and skills that participants have not learned in their college geometry course in order to allow them to engage in problem-solving profitably; and (5) The problems should challenge participants to experiment, conjecture, and prove, if possible, and should invite different strategies and extending existing knowledge and problems to new problems.

The first task (see Figure 1) used in this study was The Longest Segment Problem, adapted from the web material (Cut-the Knot). Problem 1 was a construction problem.

Given two intersecting circles. Draw a line through one of the intersection points, say, A. That line also intersects circles in exactly two points, say, B and C. What choice of the point B results in the segment BC such that the segment BC is the longest?

a. Formulate and prove your conjecture.
b. Find the construction for a point B such that the length of BC is the longest.

Justify your answers as best as you can.

Figure 1. *The Longest Segment Problem*

A second problem, The Airport Problem, was taken from Understanding Geometry for a Changing World (Craine & Rubenstein, 2009), one of the NCTM Yearbooks, and adapted in this study. Problem 2 was an applied problem.

Three towns, Athens, Bogart and Columbus, are equally distant from each other and connected by straight roads. An airport will be constructed such that the sum of its distances to the roads is as small as possible.

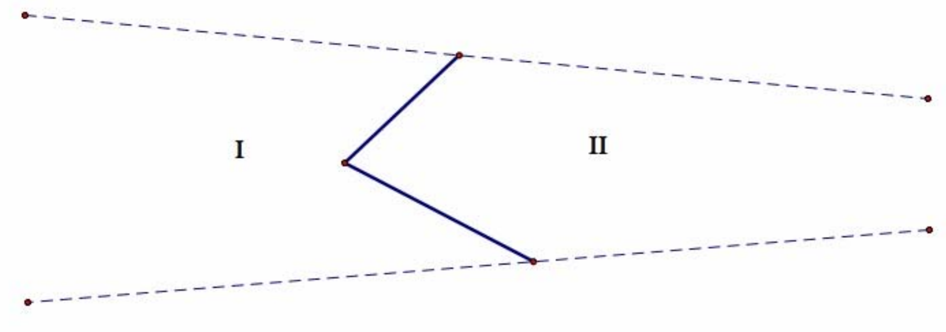
a. What are possible locations for the airport?
b. What is the best location for the airport?
c. Give a geometric interpretation for the sum of the distances of the optimal point to the sides of the triangle.

Justify your answers as best as you can.

Figure 2. *The Airport Problem*

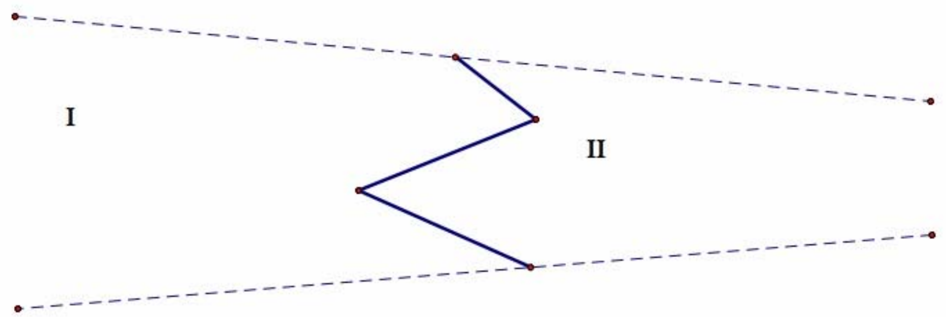
A third problem, The Land Boundary Problem, was taken from the book Euclidean and Transformational Geometry: A Deductive Inquiry (Libeskind, 2008), but it was also available from the TIMSS video study–Japan. Problem 3 was an exploration problem.

The boundary between two farmers' land is bent, and they would both like to straighten it out, but each wants to keep the same amount of land. Solve their problem for them.



Justify your answers as best as you can.

What if the common border has three segments?



Justify your answers as best as you can.

Figure 3. *The Land Boundary Problem*

Data analysis

For the purpose of this study, two stages of analysis, the within-case analysis and the cross-case analysis, as suggested by Merriam (1998) were conducted using inductive analysis. Inductive analysis contends "the patterns, themes, and categories come from the data rather than being imposed on them prior to data collection and analysis" (Patton, 2002, p. 306). When using inductive analysis, I focused on creating codes and categories from the data, developing or enhancing theory during the act of analysis and the use of constant comparative method during analysis of the data (Charmaz, 2006). Using constant comparative method all data from the problem solving sessions, the interview sessions, hard copies and GSP sketches of participants' solutions, and researcher's field notes were categorized that consisted of comparing and generating categories, integrating categories, and delimiting the theory to help illuminate common themes across cases and within cases. The cross-case analysis was used to create a sound theory offering general explanations of perspectives on the experience of using technology that comply each problem for both participants.

4. Results

Elements of reflection

The first research question focused on identifying reflective elements when communicating technology problem solving experience. They reflected both on their experience as learners of mathematics, and as teachers of mathematics. When talking about being successful in a problem solving session, Wes attributed his success to his knowledge of geometry, whilst Aurora attributed it to solving the problem using affordances of the tool,

> In the Land Boundary problem, I struggled for a little while with what would actually allow me to make the fence straight and keep the areas of the farmers' land the same. Once I finally solved the problem, I felt very successful because I had never seen anything like it. I think technology was a large part of why I was successful. It would have been a tough problem to solve without use of GSP because of the movement of triangles (i.e. ones with the same base and height, but different shapes). I liked that even though I reached a few stopping points, I ultimately worked through those issues and arrived at a solution (although it took me a little while to decide if there was more than one solution). It was rewarding to be able to work through a problem that was difficult at first. I did not really know how to start the problem and had to do some experimentation with where the point might lie first. That gave me some ideas about the solution, but also made me concentrate too much on the ending point rather than the work which would allow me to arrive at that answer.

However, Wes and Aurora attributed their unsuccess, to an overreliance on technology and an incorrect problem solving strategy, respectively. Nevertheless, Wes concluded, "I was very disappointed in myself; however, I used this experience to my advantage. When solving problems, I now consider regularly assessing my progress and understanding of the problem. For example, I may ask myself: How will this help me solve the problem?"

With respect to improving their problem solving, Wes concluded that he "should use technology when it is appropriate to do so." That is, he should devise a plan in order to develop a better understanding of a problem, which he said was "crucial" before proceeding the use of GSP. Aurora similarly said that using technology made her "focus on the answer, rather than the process" making her not focus on the big picture. She concluded the she should "focus more on the big picture" in order to choose "effective means in solving a problem."

Both participants reflected on this experience and its influence on their future problem solving. Aurora concluded by emphasizing the latter,

> In my teaching, I think I will encourage technology in problem solving more. We give students long word problems sometimes, which are more easily interpreted if students can use a modeling program or helpful tool. Additionally, I think I will be more apt to encourage different styles of approaching a problem. Before, I would have probably encouraged students to learn an algorithm for working through problems. However, after this study, I think I will allow students to work through the problem in their own way and provide helpful suggestions only if they are stuck.

Lastly, and most importantly both participants changed their stand on using technology when problem solving. Wes nicely explained,

> My stand on using technology changed significantly. Prior to participating in this study, I relied heavily upon technology. However, this experience has given me an opportunity to reflect upon my use of technology. I now see technology as a tool rather than a crutch. My goal is to use technology appropriately. In the past, I have used technology immediately. I now take the time to organize my thoughts. To do so, I usually record my thoughts on paper and create a preliminary sketch.

Aurora, however, provided with a different answer that focused more on her future instruction,

> I think that I will be more apt to use technology in the future. I have always found GSP to be very helpful in working through novel problems. This study reinforced that belief and allowed me the freedom to explore the problems and make mistakes. I feel that the process of exploration needs to be fluid; you can make a mistake and quickly get back to the last point that you knew you had correct. This is essential and technology allows students the ability to be wrong and easily correct it. I do not feel that paper/pencil allows this ease in correction.

Both Wes and Aurora exbihited specific reflective behaviors during problem-solving sessions. Both participants, after unproductive engagement in particular activities in exploration, or a planning-implementation-verification episode, engaged in a behavior they referred to as “taking a step back,” or “going back.” This concept included a set of actions, such as thinking over and investigating their actions, chosen strategies, entire problem, that provided an important and instructional phase in their problem solving process. Data from this study confirmed that reflective activities in a problem-solving environment have the power to enhance the learning benefit of the exercises; it gave the opportunity to review previous actions and decisions before proceeding to a next stage. Consistent with the previous literature (e.g., Zimmerman, 2002) reflexive thinking is the foundation of metacognitive awareness that provides not only a better understanding of what the learner or problem solver knows, but also a way of improving metacognitive strategies, because then one can examine how he performed on a specific learning task. Hence, reflective behavior of “taking a step back” was a precondition to promote students’ metacognitive awareness and monitoring skills.

Perceptions

The second question sought to explain participants’ perceptions to the importance of GSP to their problem solving when faced with nonroutine geometry problems. Both participants expressed a belief that GSP was important and useful tool during problem solving centering around these qualities: problem solving activities and processes, visualization, speed, and accuracy.

Wes and Aurora felt strongly about GSP helping them during various stages of problem solving, most notably during exploration. Wes said that GSP gave him the opportunity to explore problems with relative ease; he was able to manipulate the figure and monitor the change that helped him gather relevant information he then used working through problem solving space. He also added that using GSP helped him assss his actions and conjectures and deciding whether to refine, revise or abandon a particular perspective. Aurora held a similar perception adding that exploring the problem aided better understanding the problem, accessing knowledge and strategies she then considered if they were relevant or not to the problem as well as aided organizing that knowledge in moving successfully towards a solution. Verifying that the answer was an appropriate solution and examining the path to obtain it using GSP’s Measurement Tool was important for Aurora and was used also to double-check her actions.

During the final interview Aurora stated “The ease of discovery provides an excellent resource.” She viewed GSP helpful in allowing her to outline the solution and strategy and test them cycling back making the process of problem solving fluid and less discouraging. Though both participants perceived many attributes of GSP helpful during their problem solving, they viewed it as hindrance as well. For instance, Wes stated that technology can be addictive to the extent of not planning appropriately, assessing and monitoring his actions, while Aurora said that using dynamic features of the software was detrimental to quality of her reasoning and outlined plans.

Both participants perceived the importance of being able to represent not only the problem, but an idea with GSP. Making a representation of the problem was held important in developing an understanding of the problem, examining relationships between conditions and the goals of the problem as well as considering and selecting a choice of perspective. Although imagining was exhibited for both participants, sometimes their ideas seemed to go beyond their mental visualizing abilities. The ability of being able to make a screen representation of it with GSP was paramount in the problem solving process and helped avoid false actions.

There was no session where both participants did not mention they were able to work through a problem quickly and accurately. For instance, Aurora often emphasized that she could quickly make a representation of a problem, use the measurement tool to enable her to quickly arrive at a solution. This was used to direct her thinking on how to actually construct it. Wes explained that the construction process was expedited by elimination of redundant steps helping him to stay organized and focused. Both participants became aware with time, however, that moving quickly through the problem hindered them as they relied on GSP to the extent of not taking into consideration why they were doing so. Thus, management of different resources is crucial for effective usage of a technology tool allowing then the problem solver to focus on problem solving processes and strategies, decision-making, and reflection. Relying on GSP with respect to accuracy was mostly specific for Wes. On numerous occasions he explained that he cannot draw accurately on paper-and-pen, and that having accurate representation of the problem helped his problem solving process and focusing on relevant objects on the representation by using appearance features of the GSP.

There were several admissions that a solution to a problem might have not been possible or successful without the use of GSP. For instance, Aurora on several occasions elaborated that it would have required a lot of hand constructions, which would be time consuming and would detract her from the purpose of the problem. In addition, she felt she would have become more easily confused about a direction to take. Wes had similar perception, but added that not having GSP in some instances might have aided to accessing greater problem specific knowledge instead of doing a construction on GSP or paper.

Hence, both participants seemed to perceive that for all three problems, problem solving without the use of GSP would have been time consuming, detracting them from the process. Instead GSP allowed them to stay organized and focused. Aurora believed GSP was very helpful in working through novel problems, and used it as a “crutch.” Wes, on the other hand, perceived it as an incredible “tool,” an additional resource for working through novel problems.

5. Conclusions

In this article I focused on discussing preservice teachers’ reflections and percpetions on their technology problem solving experience. Through data analysis of the participant-researcher communication several categories were identified: reflecting on their experience as problem solvers (e.g., suggesting and elaborating improvements in the problem solving process, and in their use of technology), and reflecting on their experience as future mathematics teachers. Even though it was intended that participants reflect on their problem solving, it was astonishing they brought perspectives of a future educator indicating meaningless of such discussion, and activities for a preparation for the teaching profession.

Participants in this study reflected on their problem-solving practices in a technology environment, and how they might implement problem-solving in their future classroom; both participants became aware of their problem solving and what actions improved or hindered their own problem solving activity. Taking into consideration the influence of an increasingly global and technological society on teaching practices, teachers need to become aware of the pedagogical and cognitive implications of technology and be able to take advantage of technology as a powerful and engaging teaching tool. The opportunity to experience genuine problem solving, reflect on their metacognitive behaviors that are consistent with the use of the GSP and identify the possible effects they have on mathematical problem solving, teaching, and learning is powerful. The implementation of dynamic geometry environments in the classroom affects not only pedagogy, but student cognition as well (Goldenberg, 2000). Students working with this type of technology might show ways of thinking about mathematics that are otherwise difficult to observe. Hence, preservice teachers before becoming inservice teachers and taking those responsibilities on themselves should have experience in genuine technology problem solving as well as opportunities to discuss curricular, pedagogical, and learning issues with respect to that mission in variety of contexts.

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