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Teachers Promoting Student Mathematical Reasoning

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Introduction

The importance of students building arguments to support their solutions to problems and then defending these arguments is undisputed (Alibert & Thomas, 1991; Balacheff, 1991; Ball & Bass, 2003; Francisco & Maher, 2005; Maher, 1995, 2005; Yackel & Hanna, 2003). However, there is a need for more knowledge about the types of classroom communities that promote reasoning and justifications and the teacher's role in the community. In this paper we analyze and discuss specific teacher moves that led to the formation of a community of learners in which students' co-constructed arguments, provided justifications for solutions, and engaged in mathematical reasoning.

Research has shown that certain conditions promote meaningful, mathematical learning. These include a combination of the following: (a) challenged and active students; (b) observant teachers who attend to the developing ideas of students; (c) appropriate, open-ended tasks that invite students to extend their learning as they build and justify solutions; (d) student collaborations that make possible the exchange of ideas; and (e) a setting that respects and welcomes student ideas, conjectures, and alternative ways of working. Under these conditions even young children develop confidence in their ability to solve problems and offer justifications for solutions that take the form of proof (Cobb, 2000; Lampert & Cobb, 2003; Maher & Martino, 1996; Martino & Maher, 1999; Maher 2005, 2009; Yackel & Hanna, 2003). A crucial feature of such communities is the teacher's ability to react responsively, in particular when it comes to facilitating the building of arguments.

Prior research on the value of teacher interventions with elementary school age children has shown the importance of timely questioning in encouraging students to support their solutions (Maher, 2009; Martino & Maher, 1999). This study extends the research on teacher questioning by examining specific *teacher moves* that encouraged students to collaborate, freely share ideas, question each other's ideas and solutions, and build arguments for the solutions posed. Krussel, Edwards, and Springer (2004) define a discourse move as "a deliberate action taken by a teacher to participate in or influence the discourse in the mathematics classroom" (p. 309). For the purposes of this paper, we use the term *teacher moves* to refer to purposeful, verbal interventions made by the teacher after the posing of a task. These deliberate, verbal moves are intended to influence student discussion and reasoning about mathematics by encouraging students to verbalize their ideas, to make these ideas public, and to justify their solutions to their peers.

Theoretical Framework

Guiding our perspective is the notion that in order to build a community that promotes mathematical reasoning, particular conditions need to be in place. In this section we discuss these four interrelated themes: (a) thoughtful teacher interventions; (b) the posing of strategic questions; and (c) the development of a community that supports reasoning and the co-construction of ideas; and (d) the establishment of socio-mathematical norms.

Teacher Interventions

Teacher interventions are a critical component of creating an environment that promotes the sharing of ideas in a learning community. Thoughtful interventions, implemented according to students' developing ideas, allow students to take ownership of their learning and solutions. According to Maher and Martino (1996), by minimizing the teacher's role during initial exploration, students are more likely to engage in mathematical discourse, share representations, co-construct ideas and justifications, and ultimately take a more active role in their own learning. In this way, teachers can facilitate more elegant, clear explanations which lead to detailed, efficient representations and ultimately to further refined arguments by students (Maher, 2009). However, teacher questioning is crucial in drawing out elaborate forms of reasoning and deeper understanding. While our working definition of teacher moves specifically focuses on verbal moves, peripheral interventions such as task design/initiation of tasks and listening play a critical role in establishing a mathematical community and promoting student autonomy.

Tasks

Task selection or design is an intervention that often is planned before a teacher enters the classroom. Many researchers have emphasized the importance of task features in promoting reasoning and understanding (Doerr & English, 2006; Francisco & Maher, 2005; Henningsen & Stein, 1997; Maher, 2002; Maher & Martino, 1996; Stein, Grover, & Henningsen). Challenging, open-ended tasks are open to multiple representations and multiple strategies for solutions (Maher, 2002; Francisco & Maher, 2005). At the same time, mathematical discourse can be promoted as students work together on tasks. Strands of related problems may be later revisited over time (Francisco & Maher, 2005). In addition, tasks that are novel to the students, in that a procedure is not readily available, encourage students to rely on their own mathematical resources. After posing tasks, the teacher encourages students to begin to build their justifications and share ideas. During this phase the teacher engages in observation and careful listening in order to estimate how children are thinking about their solutions. Based on the type of task posed, the teacher initiates specific moves to promote reasoning and understanding. In this paper we will discuss the connection between the kind of task that is posed and teacher moves.

Listening

As they attempt to promote understanding, teachers must practice being skilled and attentive listeners. By listening, teachers are able to recognize if students are constructing their solution from an understanding of the conditions of the problems (as compared with behavior that might suggest that they are parroting the behaviors of others with little understanding), make decisions based on having a meaningful understanding of the ideas and how they are related, and recognize student conceptions that seem plausible as well as student actions that suggest obstacles toward successful problem solving (Martino & Maher, 1999).

Questioning

Teacher questioning plays a crucial role in promoting student understanding, construction of new knowledge, as well as the sharing of ideas (Moyer & Milewicz, 2002). Martino and Maher (1999) stress the importance of teacher questioning in creating an environment that promotes mathematical understanding and problem solving. Sahin and Kum (2008) suggest that teachers should be cognizant of the types of questions they are asking and their purpose of asking these questions. Skillful questioning of student thinking and monitoring of student problem solving can provide teachers with a deeper understanding of the development of student's mathematical ideas and help advance student mathematical growth. Sahin and Kulm (2008) developed a model for looking at teacher questioning. They considered three main types of questions: probing, guiding, and factual.

Probing questions consisted of those that ask students to explain their thinking, offer justifications or proof, and use prior knowledge in attending to the task at hand. These questions, according to Sahin and Kulm (2008), served the role of extending students' conceptual understanding and encouraging them to relate new ideas to prior notions and schemas. Guiding questions were identified as those that sought to guide students' problem solving by asking for solutions, strategies or procedures, and thus scaffolding student understanding of a concept. Guiding questions tended to support students in creating their own heuristics and deriving mathematical concepts. Finally, factual questions were requests for facts or definitions, as well as answers or next steps in a problem. Sahin and Kulm (2008) concluded that the majority of teacher questions are factual, even when using reform-based curricula that includes probing and guiding questions in the teaching guides. They suggest that by asking probing and guiding questions, teachers invite students to share their ideas and justifications with others, and thus create a classroom community that supports the building of ideas and conjectures.

Towers (1998, 2002) studied the process of classroom interactions that leads to the growth of mathematical understanding and identified teacher intervention themes which include the following: showing and telling, leading, shepherding, checking, reinforcing, inviting, clue-giving, managing, enculturing, blocking, modeling, praising, rug-pulling, retreating, and anticipating. According to Towers, teachers traditionally use two types of teaching, each involving multiple interventions: *showing and telling* and *leading.* In the first, teachers usually give information without checking understanding; while with the latter, the teachers ask frequent questions but at a low level. Towers (1998) offers an alternative approach which she calls *shepherding*. This approach involves directing the students "through subtle nudging, coaxing and prompting" (p. 30).

Establishing a Mathematical Community

Teacher moves are crucial in the establishment of mathematical learning communities. Mathematical reasoning and understanding naturally results from the communication that takes place in such communities (Yackel & Cobb, 1996; Forman, 2003). Communities of mathematical inquiry are described by Goos (2004) as classrooms where students learn to talk and work mathematically by participating in mathematical discussions, proposing and defending arguments, and responding to the ideas and conjectures of their peers. The design and posing of thought-provoking tasks leads to such discussions, which in turn lead to a culture of justification and proof. Mathematical discourse improves in quality when teachers and students share the responsibility of communicating about mathematical concepts (McCrone, 2005).

The establishment of a learning community with social norms that promote justification and reasoning promote the development of autonomy in learners. Kamii (1985) characterizes *mathematically autonomous* students as being cognizant of their own mathematical resources and able to call upon and use these resources to make mathematical judgments. Ben-Zvi and Sfard (2007) point out the tension between autonomous learning and collective learning and contend that autonomy must also extend to interaction with others and collective sense-making. The authors expand the notion of autonomy to include the sharing of ideas and strategies that occur as students are working in a learning community. The posing of thoughtful, challenging tasks, the act of listening to students' ideas, and strategic questioning in a community of learners promotes student autonomy.

Ellis' (2011) reported seven categories of generalizing-promoting actions. Each category in Ellis' (2011) framework describes a move that was found to promote the development of reasoning and justification. Ellis (2011) contends that teachers, students, and tasks promote this justification or generalizing as they work collaboratively. Ellis' (2011) seven actions include publically generalizing, encouraging generalizing, encouraging sharing of a generalization or idea, publically sharing a generalization or idea, encouraging justification or clarification, building on an idea or a generalization, and focusing attention to mathematical relationships.

Although it shares several commonalities with our study, Ellis' work is markedly different for several reasons. First, Ellis' study focused on classroom events that encouraged generalization. Our work studies the building of all student justifications. Second, Ellis examined the interactions of a small group of students in the eighth grade students as they worked on complex tasks in algebra. We study the teacher moves that encouraged student explanation and justification while building understanding of fraction concepts. We conduct a more focused analysis of the teacher strategies in a whole-class setting to *create and support* an environment that supported student justification when learning fraction concepts with younger students called for a more focused analysis. Our study also identified ways that teacher moves evolved over the course of the study and as students became more comfortable with providing explanations and justifications for their mathematical ideas. Finally, Ellis did not distinguish between the impact of student interactions and teacher moves that prompted reasoning.

In this paper we focus on particular teacher moves sought to encourage student explanations while eliciting and making public student ideas, strategies, and justifications proposed for problem solutions. We identify two types of teacher interventions similar to Sahin and Kulm's (2008) guiding and probing questions and Tower's (1998, 2002) *shepherding*, which we refer to as eliciting an idea and promoting an explanation that justifies a solution, and include in our design the identification of teacher moves intended to make student ideas public. Focusing on these three types of questions, we traced the forms of reasoning that middle schools students exhibited as they worked on open-ended tasks and that developed in the solutions to middle-grade students' tasks and investigated the following:

1. How does the teacher guide the establishment of the mathematical community of learners?
2. How is the occurrence of specific teacher moves connected to the elicitation of students' reasoning behaviors and autonomy?

Methodology

This research is a component of a larger, ongoing longitudinal study, Informal Mathematics Learning Project[1] (IML), conducted through an after-school partnership between a university and an economically depressed, urban school district. The goal of the project was to study how students from a low-income, urban community build mathematical ideas and engage in mathematical reasoning in an informal after-school program (Maher & Powell, 2002). The data also lent itself to the study of the nature of interventions made by the teacher/researcher (Maher & Powell, 2002).

The current study was built upon two earlier studies, supported by NSF grants, which traced the development of mathematical ideas in children. One of the prior studies focused on fourth graders and the other, a longitudinal study, traced the development of mathematical understanding of a cohort of students from first grade through high school. In both studies, researchers concluded that students were able to use convincing arguments in the development of mathematical ideas (Maher & Martino, 1996; Steenken, 2001; Bulgar, 2002; Powell, 2003; Maher, 2002, 2005; Francisco, 2005; Francisco & Maher, 2005; Reynolds, 2005).

Setting and Participants

The IML program was integrated into an existing after-school program that met twice a week and began when the first cohort of students were in the sixth grade. The students met for approximately 12 two-hour sessions during the first year of the study. During the after-school sessions, students were invited to work collaboratively on open-ended mathematical tasks. The students were placed in heterogeneous groups of four. In each session, problems were posed and students were invited to explore solutions in their groups.

The sessions were organized into cycles which consisted of different tasks; examples include explorations of fractions with Cuisenaire rods, probability

[1] The Informal Mathematics Learning project directed by Carolyn A. Maher, Author B. Powell, and Keith Weber, was supported by a grant from the National Science Foundation (ROLE: REC0309062). The views expressed in this paper are those of the authors and not necessarily those of the funding agency.

explorations with and without a computer program, algebra investigations, and counting problems. The tasks were designed carefully as task choice played an important role in the project's objectives. The researchers worked on the assumption that if the tasks were too simple, then students' schemas would not be enhanced, but, on the other hand, if they were too difficult, the students would not be engaged in finding solutions.

The research program was designed to investigate how mathematical ideas and ways of reasoning developed over time and under certain conditions. Therefore, specific classroom atmospheres were formed to create a learning environment that fostered collective mathematical learning and individual development. These characteristics include: (a) the posing of open-ended, tasks; (b) whole class discussions and small group activities in which the teacher acts as facilitator and/or participant; (c) ample opportunities for students to exhibit their own ways of thinking; (d) encouraging students to defend their ideas and challenge the ideas of others; and (e) sufficient time to explore and build understanding (Bauersfeld, 1995; Martino & Maher, 1996). In this paper, we report on activities of the first cohort of students, 24 sixth-graders, all African American or Latino, during the first cycle of the [illegible][2] This cycle consisted of five, 90 minute sessions.

The Tasks

The explorations used during the first cycle of the study focused on reasoning about fractional relationships. During these sessions, the students worked on fraction tasks in an environment that promoted working collaboratively, sharing ideas, and justifying answers. The dual purposes of these sessions were to help students reconceive fractional concepts and engage in critical thinking and reasoning. The strand of tasks was developed from an earlier research intervention with fourth grade students. In the previous study, it was documented that the students used reasoning to compare fractions, find equivalent fractions and perform operations on fractions after working on the tasks with the Cuisenaire rods (Maher & Martino, 1996, 2000; Steenken, 2001; Bulgar, 2002; Powell, 2003; Maher, 2002, 2005; Francisco, 2005; Francisco & Maher, 2005; Reynolds, 2005).

In the previous study the students had purposely not yet been introduced to fractions.

The students in this study were sixth graders and had been taught fractions; however, they were taught using rules and procedures and they exhibited little conceptual understanding and a fragmented grasp of the

[2] The group of students who volunteered was representative of the overall population of sixth graders of that school. However, the research team deliberately chose not to identify students according to earlier success in school mathematics.

notion of a fraction (Powell, Francisco, & Maher, 2003). Therefore, the first five sessions focused on introducing fractional concepts to students using concrete representations and a focus on relationships and sense-making. Students engaged in naming fractions, finding equivalent fractions, and comparing fractions using models. Through these tasks, they conceptualized the meaning of fractions. Table 1 describes the purpose of each of the five

Session	Purpose	Sample Task
1	Introduction to the Cuisenaire rods. Conceptual meaning of fractions. Naming fractions based on length relationships.	*If I call the orange rod one, what number name would I give to the yellow rod?*
2	Establishing fractional relationships with fraction as number. Promoting the concrete understanding of fractions through visual models.	*Someone told me that the red rod is half as long as the yellow rod. Do you agree?* *If I call the blue rod one, I want each of you to find me a rod that would have a number name one-half.*
3	Establishing the concept of "the whole" when naming fractions. Reinforcing the conceptual meaning of fractions. Connecting fractions to decimals.	*Is 0.3 another name for the light green rod if the orange rod is named one?* *If I call the blue rod one, what number name would I give to the white rod? What name would I give to the red rod?*
4	Naming fractions. Finding equivalent fractions. Simplifying fractions.	*If I called the blue rod one, what number names would I give to the rest of the rods?*
5	Comparing fractions. Operations with fractions.	*Which is bigger, one-half or one-third? By how much?*

Table 1. Purposes and challenges posed during the first 5 sessions

For the sessions analyzed in this study, Cuisenaire rods® were available to the students for use in building their models. A set of Cuisenaire rods, as shown in figure 1, contains colored wooden or plastic rods that increase in length by increments of one centimeter. The rods are given permanent color names. These names, along with their respective lengths, are: white (1 cm); red (2cm); light green (3 cm); purple (4 cm); yellow (5 cm); dark green (6 cm); black (7 cm); brown (8 cm); blue (9 cm); orange (10 cm). Students were encouraged to use the rods to build models to support their justifications. Overhead rods were used to model representations during whole class sharing.

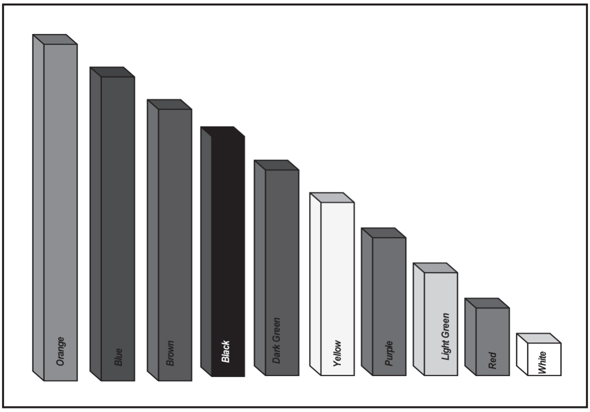

Figure 1. Staircase model of rods.

Groups of students worked on the investigations over five sessions, one and one-half hours in duration, over a three week period. During each session a set of tasks was presented, for example, in Session two, the following problem from the earlier study was presented: *"What number name would you give to the dark green rod if the light green rod is called one? Discuss the answer with your group"* (Maher, 2002). Groups were then provided time to investigate their solutions and make claims. They were invited to collaborate, and they were encouraged to justify and make sense of their solutions first in their small groups and then with the whole class. Once each group had completed the task, they were invited to the overhead projector to share their findings with the larger group. During these whole group discussions, students shared their findings, challenged each other, and often reflected on and revised their solutions.

Data Collection and Analysis

Each session was video-taped with four different camera views. The cameras focused on different groups of students and one of the cameras also captured the presentations at the overhead projector. Video recordings and the transcripts were analyzed using the analytical model outlined by Powell, Francisco, and Maher (2003). The analytical model is used to study the development of mathematical thinking and contains the following seven interacting, non-linear phases: viewing the video data, describing the video data, identifying critical events, transcribing, coding, constructing a story line, and composing a narrative (Powell, Francisco, & Maher, 2003).

After describing the video data, we examined the effects of the teacher[3] moves on learners and their ideas, arguments, and solutions. The following

[3] For all sessions, the teacher, facilitator was a member of the research team.

moves were flagged as critical events: making an idea public, eliciting an idea, and promoting the explanation of an idea. The differences between each group were often subtle and at times overlapped. *Making ideas public* included specific actions focused on encouraging students to listen to and consider the ideas of others. These actions were noted when the teacher reiterated student ideas, asked if students agreed with an explanation (or if students were convinced), and requested that students listen to the important ideas of others. *Eliciting student ideas* included moves geared toward encouraging students to formulate their own ideas and strategies, and in doing so advance student thinking. Finally, *encouraging justifications and/or explanations* consisted of moves that prompted students to give more detailed explanations of their strategies, generalize solutions, and/or make connections.

Sample codes, along with descriptors and examples from the transcripts of the sessions are shown in table 2. The data were also analyzed according to the forms of reasoning students used in negotiating the tasks. The occurrence of these forms of reasoning was analyzed alongside the occurrence of teacher moves and the two sets of data were compared and patterns were noted. The forms of reasoning identified included direct reasoning, reasoning by contradiction, reasoning by cases, and reasoning using upper and reason-

Category	Descriptor	Example
Making ideas public	Specific teacher moves focused on getting students to listen to and consider the ideas of others, such as, reiterating ideas, asking for agreement or if students are convinced, asking students to listen to the important idea	"It's very interesting the way Chanel wrote it, right?" "Let's follow what Dante is saying. Dante says…. Do you all agree?" "Can you convince us?" "I want you all to think about what Malika is saying and try to prove her right or wrong"
Eliciting student thinking and ideas	Specific teacher moves geared toward encouraging students to formulate their own ideas and strategies, and further their thinking.	"Three-ninths, someone said we had another name for the light green rod" "Could you continue?" "Does the blue rod also have two number names?"
Encouraging explanations and/or justifications	Teacher moves that prompt students to go into more detail in explaining their solutions/strategies, generalize solutions, and make connections.	"Do you have a different explanation?" "I'm confused." "I don't understand." "Why is she calling the white rod one-ninth?"

[4] See Mueller (2007) for a detailed analysis of the forms reasoning that emerged during the five sessions.

Table 2: Sample codes, descriptors, and examples

After developing codes, the tapes were transcribed and coded. For validity purposes, cross code-checking was performed by another researcher. Observation notes were used to supplement the transcripts and assist in constructing a story line. In addition, student work was used to make sense of the models that students created and add a written explanation of their thinking.

The occurrence of teacher actions was analyzed by looking for patterns among tasks and forms of reasoning that were connected to the type of teacher actions that was most prevalent in each episode. The numbers and proportions of the different types of teacher actions were noted as they occurred in each session. In addition, the occurrence of these interventions was noted in relation to the task that was the focus of the discussion.

Most tasks were discussed in small groups as well as in a whole class setting. Those tasks which were only investigated in small groups were flagged and the connection between this fact and the occurrence of teacher moves during those tasks was analyzed.

After general patterns were noted, the data set was analyzed further in an attempt to pinpoint anomalies and possible explanations for differences in the occurrence of teacher moves during the investigation of certain mathematical tasks. The forms of reasoning that were elicited during these tasks were analyzed and the nature of the task and the student activity was investigated to determine the root of the patterns and anomalies that were found.

Results and Discussion

Overall Patterns

The proportions of the incidence of the three categories of teacher moves are displayed in Table 3 below. These show clear patterns regarding the frequency of each kind of teacher intervention. In addition, the table shows the total number of teacher moves flagged during each session, which was much greater in the fourth and fifth sessions. During the first two sessions,

Table 3: Teacher moves across the five sessions

Session	Elicit Ideas	Make Ideas Public	Encouraging Explanations	Total (moves per session)
1	13%	65%	20%	63
2	14%	54%	32%	72
3	34%	44%	14%	67
4	37%	42%	19%	124
5	34%	47%	18%	141

a large percentage of the teacher moves were focused on making students' ideas public. During the latter three sessions, a relatively smaller percentage of these moves focused on making ideas public (although they still constituted the bulk of teacher moves within the session), and relatively larger percentages focused on eliciting students' ideas. These findings are discussed and interpreted in the last section of this report.

Eliciting Ideas and Type of Task

As previously stated, *eliciting ideas* was the code given to moves that promoted students' formation of their own ideas and strategies. More of these teacher moves occurred in the last three sessions. A closer look at the tasks that elicited the greatest number of this kind of teacher moves evidences that these tasks were different than the others in their level of difficulty and the forum in which they occurred. For example, during session 4, the students were asked to investigate the following task: *If I call the blue rod one, what number name would I give to the white rod?* While students were working on this task a total of 19 moves were coded as *eliciting ideas*. Three of these tasks were only investigated in small groups. This was unusual, as the remainder of the tasks throughout the five sessions were discussed in small groups as well as in a whole class setting. This may be related to the fact that many of these tasks were more difficult than the ones introduced in the beginning sessions.

We discuss an episode from the fifth session to illustrate our point. The task asked students to compare one half and one third and to find the difference between these two fractions. The students were asked to build a model that justified their solution. This task, too, was more challenging than those involving only one fraction, and the teacher intervened *seventeen* times to elicit students' ideas about the task. The following episode illustrates the ways in which the teacher elicited students' ideas.

Working in small groups, the students began by using the orange rod to represent one. They named the yellow rod one half but then struggled finding a rod to represent one-third. After grappling with the task with their other partners, Ian and Michael created a model to compare the two fractions using the dark green rod as one.

Ian	Alright, I say that this [dark green] is a whole, this [red] will be …
R1	What number name is it?
Ian	One – this will be three [lining up a train of three reds next to the dark green].
R1	The red would be what?
Ian	Red would be a third and the light green would be a half.
R1	Would you say it's bigger?
Michael	We said a half is a lot bigger because…

R1	You said the light green half is bigger than the red, one-third, by how much?
Ian	One cube [holds up a white rod] which would be one-third.
R1	Why is it one-third?
Ian	Because…
R1	What's the white rod?
Ian	A third, because if you take;…
Michael	No, no, that ain't no third
R1	You just told me that red was a third.
Ian	A sixth.
R1	Why a sixth Ian?
Ian	Because if you take six of those [white] it equals to three of those [red] which would be a half and two light greens.

The students struggled with comparing two fractions and finding a "common denominator" to use for comparison. The teacher questions in this narrative served to elicit or *shepherd* (Towers, 1998) students' ideas and thus support them in developing strategies and extending their thinking. The students were not yet comfortable with referring to rods by their number names. In this exchange, the teacher assisted them in attaining familiarity with this language by focusing the students' attention on the use of these number names. She then used short questions to facilitate the students' formation of ideas and strategies. The nature of these tasks may explain why three of the five tasks that elicited this kind of teacher intervention were investigated in a small group setting. These tasks were difficult for the students, and they were provided ample time for exploration, rather than required to quickly provide a justification for a solution that was simpler to derive.

Making Ideas Public and Type of Task

A similar argument to that made in reference to teacher moves intended to elicit ideas can be made for the majority of occurrences of teacher moves that attempted to make ideas public. During completion of difficult tasks, many of which were discussed in the section above, larger numbers of teacher moves intended to encourage students to share their understanding of the task with others. This phenomenon is illustrated in the episode described below.

During session 4, the following task was posed: *What name would you give to the rest of the rods when the blue rod is named one?* In the small group forum, Chanel used the staircase model shown in figure one to incrementally name the remainder of the rods beginning with naming the white rod one-ninth. As she was working, she said the names of all of the rods, "One-ninth, two-ninths, three-ninths, four-ninths, five-ninths, six-ninths, seven-ninths, eight-ninths, nine ninths, ten..– wow, oh, I gotta think about that one, nine-tenths".

The teacher listened as Chanel created her model and asked Dante if he heard her dilemma, stating, “Chanel has an interesting problem that she wants you to hear about. Okay? Tell him about the problem.” Chanel explained and Dante initially correctly named the orange rod ten-ninths but then changed his mind and named it one-tenth. The teacher asked the students to discuss the problem and walked away. A few minutes later the teacher returned to the table and asked the students what they were naming the orange rod. One student named it one-tenth and another one “whole.” The teacher asked that they show the solutions with the rods and again left, asking them to think about the problem.

The students discussed the fact that the numerator of the fraction could not be larger than the denominator. Dante then announced that he overheard another group calling the orange rod ten-ninths. Michael and Chanel insisted that a fraction cannot have a numerator that is larger than the denominator and Dante agreed.

Another teacher then joined the group and asked Chanel to explain what she was working on. Chanel named all of the rods including the orange rod as one-tenth. Rather than correct her, the teacher asked for an explanation and reminded her of what they already knew (the blue rod was named one). She continued to ask a series of questions about what they had already determined about the names of the rods when the blue rod was named one. Finally, Dante named the orange rod ten-ninths. Instead of accepting the solution, the teacher asked Dante to explain and “show” her using the rods. She then asked Dante to convince his partners.

Later in the session, the task was discussed in a whole class forum. Malika and Lorin began the presentations explaining, “Before, we thought that because we knew that the numerator would be larger than the denominator and we thought that the denominator always had to be larger but we found out that that was not true. Because two yellow rods equal 5/9 and 5/9 + 5/9 equals 10/9.” The teacher asked her to repeat her explanation, stressing, “This is such an important thing you’re saying. Could the rest of you listen to this? This is very important. Because some other people told me that they believed that the numerator couldn’t be bigger than the denominator and now they believe something different.”

In the above example, the teacher listened in as Chanel struggled naming the orange rod and then made her idea public by asking her to share her dilemma with Dante. He then walked away and allowed the students time to discuss the problem. Upon returning he asked a series of clarifying questions to elicit the students’ thinking about the relationships and providing them the opportunity to convince themselves of the validity of the relationship. When the second teacher joined the group, she first asked the students to explain their reasoning and then encouraged them to use prior knowledge to construct new understandings. She asked them to explain what they “knew” and had already established during previous tasks. When Dante named the orange rod ten-ninths, rather than accepting his answer, the teacher asked him to justify his solution by suggesting he convince his partners.

During whole class presentations many of the teacher moves were focused on making students' ideas public and encouraging students to revisit ideas. By observing small groups the teacher saw that many students were struggling with the idea that a fraction could be written in the form with the numerator larger than the denominator (as an improper fraction). The goal of this final sharing was to make this representation public and then use the materials to build a model that illustrated the validity or not of the idea.

Encouraging Explanations and Varied Forms of Reasoning

One task, in particular, elicited a relatively large number of teacher moves that were coded as encouraging explanations. This task occurred during the second session. The researcher asked the students, "*If I call the blue rod one, I want each of you to find me a rod that would have a number name one-half*." During the students' investigation and discussion of this task, an unusual variety of forms of reasoning were elicited as well. While attending to this task, students used direct reasoning, contradiction, upper and lower bounds and argument by cases. The level of difficulty and the open-ended nature of the task and justification most likely contributed to the results. However, it is interesting to note the connection between the sheer magnitude of the type of teacher move and the large variety of arguments that were offered during the discussion. This finding suggests that when teachers capitalize on the potential of a well-designed task and encourages students to share their solutions and to justify their solutions to their classmates, varied arguments can be presented and students can learn to discuss, critique, and see the value in others students' ways of thinking.

Conclusions

While studies have been conducted on the types and frequency of questions posed by teachers, these studies have not addressed overall teacher moves that influence the establishment of a learning environment. Furthermore, there is a lack of research addressing the ways in which teacher moves can promote specific student mathematical behaviors and sense-making in the classroom.

After analyzing data from the five after-school sessions, we categorized teacher moves into three main types of moves: (a) those that made students' ideas public, (b) those that brought forth and extended students ideas, and (c) those that encouraged explanations and justifications. Together, these three types of moves were crucial to the establishment of social norms of listening, sharing, and promoting student justifications characterized by various forms of reasoning. Coupled with the posing of open-ended tasks and thoughtful listening, these moves guided the creation of a mathematical community of learners.

The teachers first established social norms by modeling careful listening and consideration of solutions. While students worked in their small groups, they were encouraged to listen to each others' explanations, explain their own solutions, and ask their partners for assistance. Table 3 shows that throughout the five sessions the teachers continuously made students' ideas public although the largest percentage of these moves occurred during the first two sessions as these norms were established.

Examining the effects of the teacher moves on learners and their ideas, arguments, and solutions, showed that certain moves served different purposes. Moves geared toward eliciting students' ideas were often used for tasks that were more challenging for students. For example, in the episode described above, the students had difficulty comparing fractions with unlike denominators. In this situation, the teacher used a series of questions to coax their ideas and encourage them to formulate thoughts and strategies. This type of move often occurred as students were working in small groups and as the teacher attempted to introduce social norms such as persistence in problem solving and the importance of collective solution strategies (Cobb, Yackel, & Wood, 1995).

Moves geared toward encouraging students' explanations and/or justifications occurred more often while students were working on tasks that encouraged them to use varied solution strategies. Such tasks, often posed and discussed in a whole class forum, tended to elicit a large number of moves that made ideas public and elicited explanations. These moves were strategic in creating social norms of explaining and/or justifying solutions, questioning solution strategies, and sharing agreement or disagreement (Cobb, Yackel, & Wood, 1995).

As the students became accustomed to the norms of questioning, probing and careful listening, they questioned and reflected on their own and each others' ideas. Through this process, norms of formulating justifications and convincing each other of their reasonableness were promoted. During the sessions, multiple solutions were encouraged, and therefore students were confident to share different forms of reasoning. Taking ownership of their own solutions and having confidence in their own reasoning allowed them to attain increased mathematical autonomy.

Through specific teacher moves, students were encouraged to take responsibility for their mathematical problem solving and assume roles that might be expected as the teacher's responsibility, such as determining if solutions to a problem were correct, evaluating the reasonableness of arguments, and posing questions. Rather than correcting students' errors, the teachers charged the students with considering the reasonableness of solutions. Students were not praised for correct solutions; rather, all solutions were considered and students were afforded the opportunity to defend and/or modify their arguments. A result was that the learners were comfortable judging their own solutions and those of their peers, and learned that they could determine the validity of a mathematical argument.

The results of this study could be used in preparing future and practicing teachers of elementary mathematics. Teachers often believe that it is impossible to plan effective moves and questions in advance. However, this study suggests that good teacher moves are related to the nature of the task that is posed. By training teachers to appropriately choose and analyze mathematical tasks, they can learn to judge the usefulness of different kinds of teacher interventions as they pertain to different classes of tasks. One way that this can be accomplished is through the introduction of varied tasks in preservice courses, coupled with the viewing of video clips of students' work on this task. Such an approach can train both preservice and inservice teachers to understand how students generally approach different kinds of tasks and the reasoning of students working on these tasks. Examining the facilitation of the process of facilitating small group and whole class problem solving could lead the teachers in a discussion and analysis of the moves used to encourage reasoning, develop a community of learners and promote student autonomy. In addition, students can learn that, with practice and careful analysis of task design, students' approaches to problems can sometimes (although not always) be anticipated, and can learn which teacher moves can effectively move students' mathematical reasoning to a higher level.

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