Key Shifts in Thinking in the Development of Mathematical Reasoning

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This symposium will draw on the evidenced-based learning progressions for multiplicative thinking, algebraic reasoning, geometrical reasoning, and statistical reasoning presented at previous MERGA conferences (see references by symposium authors in the papers that follow). The four papers will consider key shifts in thinking identified within each progression, without which students' progress may be seriously constrained.

Paper 1: *A Disposition to Attend to Relationships: A Key Shift in the Development of Multiplicative Thinking*
[Dianne Siemon]

This paper draws on multiple data sources to better understand the shift from additive to multiplicative thinking, which is crucial to all further participation in school mathematics.

Paper 2: *Key Shifts in Students' Capacity to Generalise: A Fundamental Aspect of Algebraic Reasoning*
[Max Stephens, Lorraine Day, & Marj Horne]

This paper will elaborate five levels of algebraic generalisation and two key understandings based on an analysis of students' responses to RMFII algebraic reasoning tasks.

Paper 3: *Cognitive Flexibility and the Coordination of Multiple Information in Geometry and Measurement*
[Rebecca Seah & Marj Horne]

This paper analyses students' solutions to problems in geometry and measurement situations in order to identify key components needed to nurture reasoning.

Paper 4: *Facilitating the Shift to Higher-order Thinking in Statistics and Probability*
[Rosemary Callingham, Jane Watson, & Greg Oates]

Students have difficulty moving from concrete representations and procedural mathematical statistics to context-based appreciation of data. This paper examines the barriers to this shift to higher-order thinking based on the Statistical Reasoning Learning Progression.

2022. N. Fitzallen, C. Murphy, V. Hatisaru, & N. Maher (Eds.), *Mathematical confluences and journeys* (Proceedings of the 44th Annual Conference of the Mathematics Education Research Group of Australasia, July 3–7), p. 41–57. Launceston: MERGA.

Facilitating the Shift to Higher-order Thinking in Statistics and Probability

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It is increasingly recognised that to be informed citizens and to participate fully in the workforce requires an understanding of statistical data and risk. Such understanding is underpinned by statistical reasoning. It has been shown, however, that students have difficulty moving from concrete representations and procedural mathematical statistics to the context-based appreciation of data drawing on proportional reasoning that is becoming increasingly necessary. Based on the Statistical Reasoning Learning Progression (SRLP), this paper examines the barriers to shifting to higher-order thinking.

Introduction

As statistics and probability began to be acknowledged as a fundamental part of the mathematics curriculum towards the end of the 20th century (Australian Education Council [AEC], 1991), it became important to consider the new challenges for students in mastering this part of the curriculum. Although traditionally the other parts of the mathematics curriculum have claimed to have applications across other school subjects and outside of the classroom, two aspects of statistics and probability add even more potential to the application of the curriculum outside of the mathematics classroom: *uncertainty* and *context* (Callingham et al., 2021). At this point in time, the combination of uncertainty and context is seen starkly in society's experience of the COVID-19 pandemic (Watson & Callingham, 2020). The uncertainty associated with chance events and the confidence associated with decisions in contexts where statistics have been collected, is different from the rest of the mathematics curriculum, which is based on undisputed facts and proved theorems. Further, context is essential to any meaningful data that are collected (Cobb & Moore, 1997), and the entire statistical problem-solving process is based on anticipating, acknowledging, accounting for, and allowing for variability in these data (Bargagliotti et al., 2020). At each step in this process, particular skills and understandings need to be applied and combined to reach the answer to the statistical problem posed.

Students' progress in developing their statistical understanding and reasoning has been described in terms of an 8-zone Statistical Reasoning Learning Progression (SRLP) (Callingham et al., 2019). A question has arisen, however, as to why, as students progress through the middle school years (aged 11 to 16 years), many have difficulty moving to the highest zones in the learning progression but remain around the middle zones, particularly in Zone 4 (Callingham et al., 2019).

Approach

The Statistical Reasoning Learning Progression (SRLP) was developed during the Reframing Mathematical Futures (RMFII) project (Siemon et al., 2018). The SRLP describes an increasingly sophisticated hierarchy in which procedural mathematical statistics, such as calculation of an average or quantifying outcomes from a probability experiment, interact with an understanding of the context of the problem. In Zones 1 and 2, skills are limited to, for

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example, reading a value from a graph or offering an opinion about a context with no reference to data. At the higher levels (Zones 7 and 8), students call on proportional reasoning with data integrated with contextual understanding to make decisions and draw informal statistical inferences. Of particular interest here are the middle levels of the 8-zone hierarchy (See handout).

Students in Years 7 to 10 undertook a series of assessments based on statistical reasoning tasks. The student data reported here are taken from the third round of RMFII assessment, (Callingham et al., 2019) and have not been previously reported. Two tasks are used to exemplify the shifts observed in moving across zones, particularly in respect of the difficulties observed in moving up from of Zone 4: one based on probability (STATS) in the context of the interpretation and implications of winning Tattslotto; and the second based in a statistical context (STWN) with students contrasting two different graphical representations of the same data set to tell a story in the context of how long families have lived in a town. Both tasks are based in social contexts with which students are likely to be familiar. The abbreviated titles were used to identify tasks during the analysis and are used here for consistency. The tasks were marked by teachers based on the rubrics provided. The tasks and rubrics are shown in Figure 1.

Findings and Discussion

Table 1 presents the findings from a sample of 581 students in Years 7 to 9 (aged 13 to 15 years) who undertook at least four statistical reasoning tasks (not just the tasks reported here) during the third round (MR3) of assessment. Student responses were Rasch analysed, and the person measures used to determine the distribution of students across the zones.

Table 1
Number and Proportion of Students across SRLP Zones

	n	Zone 1 (%)	Zone 2 (%)	Zone 3 (%)	Zone 4 (%)	Zone 5 (%)	Zone 6 (%)	Zone 7 (%)	Zone 8 (%)
Yr 7	165	19 (11.52)	19 (11.52))	51 (30.91)	48 (29.09)	20 (12.12)	6 (3.64)	2 (1.21)	0 (0.00)
Yr 8	215	13 (6.05)	16 (7.44)	43 (20.00)	56 (26.05)	41 (19.07)	25 (11.63)	19 (8.84)	2 (0.93)
Yr 9	201	26 (12.94)	29 (14.43)	49 (24.38)	38 (18.91)	38 (18.91)	6 (2.99)	14 (6.97)	1 (0.50)
Total	581	58 (9.98)	64 (11.02)	143 (24.61)	142 (24.44)	99 (17.04)	37 (6.37)	35 (6.02)	3 (0.52)

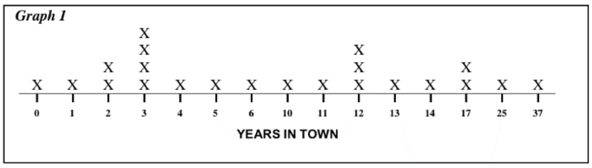
The proportion of students in each zone is very similar to that reported elsewhere (Callingham et al., 2019), and in previous similar studies (Callingham & Watson, 2017). It should be emphasised that this analysis is based on a new and different group of RMFII students, and that the nature of the analysis allows for skewed distributions and is not based on a normal distribution. The very similar patterns shown to previous analyses suggest that the sticking points in the middle zones are not environmental but related to cognitive development.

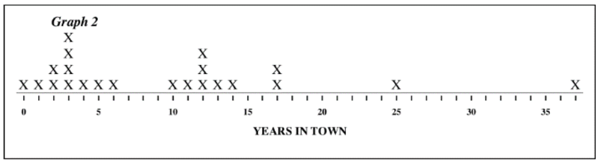
Shifts to Higher Zones

As shown in Figure 1, the rubrics reflect an increasing sophistication and quality of response and their position along the SRLP is based on the Rasch analysis. Across these two different tasks, to reach higher levels of response students need to bring together multiple aspects of reasoning.

STATS	One day Claire won Tattslotto with the numbers 1, 7, 13, 21, 22, 36. So she said she would always play the same group of numbers, because they were lucky. What do you think about this?
Code 1 **Zone 3**	Affirms a belief in being lucky (e.g., *I think it would be lucky I will pick the same number's too; I don't think many numbers are lucky. But I think 4, 7 & 9 are, so I guess I'd agree in a way you can have lucky numbers*).
Code 2 **Zone 4**	Rejects 'luck' (e.g., *There is no such thing as lucky numbers*) or states that numbers were unlikely to occur again, or less likely to occur than other numbers (e.g., *I think she shouldn't go for the group of numbers again because you can't get the same numbers after numbers, you always get different numbers all the time*).
Code 3 **Zone 6**	Implicitly recognises that all combinations of numbers have the same chance of occurring on any draw (e.g., *It was just a stroke of luck because any number could of come up; There is no such thing as a lucky number, things like Tattslotto are picked at random*).
Code 4 **Zone 7**	Explicit recognition that all numbers or combinations of numbers are equally likely, may/may not offer an opinion (e.g., *There is an equal chance for all combinations, but she's already won once, so why keep gambling, why not invest the money, you would get more out of it*).
Code 5 **Zone 8**	Reasoning that recognizes equal chance and interprets Claire's comments relative to context (e.g., *It is a good idea to use the same numbers all the time but there is as much chance as getting any other six numbers*).

A class of students recorded the number of years their families had lived in their town. Here are two graphs that students drew to tell the story.





STWN1: What can you tell by looking at Graph 1?
STWN2: What differences do you notice between Graph 1 and Graph 2?
STWN3: Which graph is better at presenting information and "telling the story"? Explain your answer.

	STWN1
Code1 **Zone 3**	Tautological response (e.g., *The numbers along the bottom tell you how many years; How long people lived in that town).*
Code 2 **Zone 5**	Response refers to one or more specific aspects (e.g., *3 and 12 have the most; 1 family had lived there 37 years, There are 22 kids).*
Code 3 **Zone 8**	Summative or comparative response that reflects some appreciation of information overall (e.g., *They range from all years; Not many families have stayed there for the same time*).
	STWN2
Code 1 **Zone 4**	Incorrect (e.g., *Less people live in the town in Graph 2 than Graph 1; There are more Xs in Graph 2*) or superficial comments related to the appearance of the graph (e.g., *Graph 2 is harder to read because numbers are together, Graph 1 is easier to read because numbers are spread out).*
Code 2 **Zone 5**	Some indication that difference recognised in terms of spread and accuracy (e.g., *Graph 2 goes up in fives and Graph 1 doesn't*).
Code 3 **Zone 7**	Acknowledges that graphs show the same data and describes the difference in terms of the scales used (e.g. *There is no difference from graph 1 to graph 2 except that graph 2 shows the spaces where graph 1 doesn't; graph 2 says all the years between 0 and 37 – while graph 1 only tells the relevant ones*).
	STWN3
Code 1 **Zone 3**	Statistically inappropriate choice (Graph 1) with reasoning that ignores spread (e.g., *Graph 1 because it only has the time it needs*)
Code 2 **Zone 5**	Statistically appropriate choice (Graph 2) with reasoning based on personal preferences (e.g., *Graph 2 because they have set it out better*) or indicates both the same (e.g., *Neither – they tell the same amount of information*).
Code 3 **Zone 7**	Statistically appropriate choice (Graph 2) with reasoning that recognises the importance of seeing all the years (e.g., *Graph 2 because you can see the difference between the years more clearly and the graph is more spaced out; Graph 2 because it has all the years*).

Figure 1. Exemplar items and rubrics.

In the probability item (STATS), it appears that developing the complex concept of random, and the necessity for appreciating the probability of groups of numbers occurring rather than single values, as appropriate for the context of the question, is important in moving responses to the higher zones. The emerging recognition of randomness and the application to groups of numbers is evident in the Code 2 (Zone 4) response but this loses coherence and falls back on individual numbers ("you always get different numbers all the time"). The Code 4 (Zone 7) response, however, is confident about working with groups of numbers but falls back on opinion ("but she's already won once so why keep gambling") to justify the thinking.

The other item (STWN), in its two-part structure (presenting two graphs and requiring a comparison rather than a single description), requires several components of the context, both visual and textual, to be integrated for a higher-level response. Students need to recognise the subtlety of the comparison needed between the graphs and to bring together understanding of the nature of the graphs and the context of the question to reach higher zones. That the lowest levels of the responses (Code 1) appear in Zones 3 and 4 rather than lower down the SRLP indicates that comparing two graphs creates some difficulty for students. The reasoning demonstrated to obtain a Code 1 is procedural, (e.g., Graph 1 is easier to read because numbers are spread out) focussing on aspects of the graph alone, rather than the information each graph conveys. To reach a Zone 7 response, students have to explicitly reason by integrating both the visual appearance of the graph and the nature of the information conveyed (e.g., Graph 2 says all the years between 0 and 37—while Graph 1 only tells the relevant ones).

Conclusion

It appears that coordinating different types of information and bringing together diverse aspects of mathematics and context are critical to shift reasoning to more sophisticated levels of response. This capacity to bring together two or more aspects of knowledge and understanding is important in other areas of mathematics, including in the shift from additive to multiplicative thinking. The inclusion of Statistics and Probability in the Mathematics Curriculum (AEC, 1991) has extended the appreciation of the structure of the multiple understandings required when data and context need to be combined rather than considered separately. It is appreciating this combination that moves reasoning to higher zones.

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