FOUR ATTENTIONAL MOTIONS INVOLVED IN THE CONSTRUCTION OF ANGULARITY

Hamilton L. Hardison
Texas State University
HHardison@txstate.edu

Quantifying angularity is critical for the study of K–12 school mathematics and beyond; yet, quantifying angularity is challenging for individuals across these grade levels. Using data from a yearlong teaching experiment with ninth-grade students, I address the role of attentional motion in quantifying angularity and present four such motions involved in the construction of angularity. Additionally, I consider implications of these motions for teaching and research.

Keywords: Geometry and Geometrical and Spatial Thinking, Quantifying Angularity

Angle measure is a critical topic in K–12 school mathematics and beyond. As a few examples, angle measure is used when characterizing shapes, constructing coordinate systems, and investigating trigonometric relationships. However, individuals' challenges with angle measure have been documented through multiple studies with students and (prospective) teachers across the educational spectrum (Akkoc, 2008; Baya'a, Daher, & Mahagna, 2017; Clements, Battista, Sarama, & Swaminathan, 1996; Crompton, 2017; Devichi & Munier, 2013; ; Fi, 2003; Keiser, 2000; 2004; Lehrer, Jenkins, & Osana, 1998; Matos, 1999; Owens, 1996; Topçu, Kertil, Akkoc, Kamil, & Osman, 2006). Yet, relatively few researchers have elaborated the mental operations necessary for measuring angles, provided empirical support for these hypothesized operations, or documented interventions resulting in productive modifications in students' thinking about angle measure. There have been some notable exceptions, two of which were reports published in roughly the last decade. Thompson (2008) presented a theoretical approach to measuring angles rooted in the principles of quantitative reasoning (Thompson, 1994; 2011) and based on multiplicative comparisons of circular arc lengths. For example, to say that an angle has a measure of n degrees in Thompson's view means that an angle cuts off $\frac{n}{360}$ of any circle's circumference, provided that the circle is centered at the vertex of the angle. Following Thompson's (2008) conceptual analysis, Moore (2013) empirically demonstrated the productivity of an arc-length approach to quantifying angularity with precalculus students.

Although this arc-length approach to quantifying angularity is productive, the complexity of this approach renders it an unlikely starting point for early instruction in quantifying angularity. In state and national curricular standards, angle measure is typically introduced in fourth grade (e.g., National Governors Association Center for Best Practices, & Council of Chief State School Officers, 2010); at this grade level, many students are not yet able to instantiate such multiplicative comparisons (Steffe, 2017), much less generalize them as holding invariant across a class of circles. Put simply, more research is needed to understand how individuals begin to quantify angularity (Smith & Barrett, 2017). To this end, I conducted a teaching experiment (Steffe & Thompson, 2000; Steffe & Ulrich, 2013) with ninth-grade students (see Hardison, 2018). In the present report, I elaborate some of the mental operations involved in beginning to quantify angularity with particular attention to the role of motion in the quantification process. Specifically, the primary purposes of this report are (a) to explicate the role of attentional motions in constructing an awareness of angularity, (b) to describe the nature of these motions, and (c) to consider the implications of these motions for teaching and research.

Theoretical Perspectives

The present study was informed by principles of quantitative reasoning (Thompson, 1994; 2011). A quantity is an individual's conception of a measurable attribute of an object or situation; quantities are mental constructions, and examples of quantities that individuals might construct include length, area, time, speed, and angularity. A quantity consists of three interrelated components, which are also mental constructions: (a) an object or situation, (b) an attribute or quality, and (c) a quantification, which is a set of operations an individual can enact on the attribute (e.g., a measurement process). Although all three components are critical, the second component—the attribute or quality—is of primary importance for the present study.

Taking this perspective on quantity, for an individual to construct a geometric quantity (like angularity), she must first construct an awareness of the geometric attribute. Steffe's (2013) analysis of the construction of length provides insight into how such an awareness is constructed:

> The construction of length involves motion of some kind in that it entails an uninterrupted moment of focused attention bounded by unfocused moments. It might be a sweeping of one's hand through space, walking along a path, scratching a path in the frost on a window with a fingernail, or moving ones [sic] eyes over the trunk of a rather tall tree. (p. 29)

According to Steffe (2013), attentional motion is critical in constructing an awareness of length. To be clear, it is the *attention* of the cognizing subject on the motion that is critical and not simply the motion itself. Steffe described three levels of awareness an individual might construct regarding this motion. If an individual is aware of the duration of the motion from beginning to end as well as the visual records of this motion while in the presence of (what an observer would call) a linear object, then an individual has constructed an *awareness of experiential length*. If the individual internalizes the motion—i.e., can re-present the visual records, the bounded motion, and the trace of this motion in the absence of the perceptual linear object—then an individual has constructed an *awareness of figurative length*. Finally, if an individual interiorizes the motion—i.e., can subject the re-presented experience to further mental operations (e.g., imagine the motion backwards, imagine stopping and re-starting the motion, imagine several successive iterations of the motion, etc.)—the individual has constructed an *awareness of operative length*.

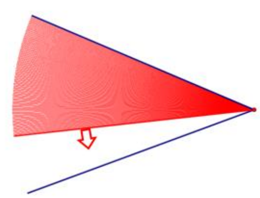

Figure 1: Radial Sweep Through the Interior of an Angle

Generalizing from Steffe's analysis of the construction of length, I consider an awareness of any geometric attribute to involve attentional motion. In particular, the construction of angularity involves attentional motion of some kind through (what an observer would call) the interior of an angular object. Attentional motion through the interior of an angle is critical because,

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

fundamentally, angle measure is a description of the size of the interior of an angle. At the onset of the reported study, I hypothesized an awareness of a radial sweep—the trace of a rotating ray constituting the interior of an angle—would be a critical development in students' construction of angularity (Figure 1). The reason this would be such a critical development in students' quantifications of angularity is because, if the radial sweep were interiorized, then students would be able to subject the interior to further operation and produce, for example, angles three times or three-fourths as open as a given angle. Clements and colleagues (Clements et al., 1996; Clements & Burns, 2000) also argued for the significance of such a motion. However, multiple studies (Clements, et al., 1996; Crompton, 2013; Lehrer, Jenkins, & Osana, 1998; Mitchelmore, 1998; Mitchelmore & White, 1995; 1998; 2000) have shown that students tend *not* to spontaneously insert this rotational motion into non-rotational angle models (e.g., two line segments sharing a common vertex). For this reason, Mitchelmore and White (1998) recommended that angle measure as an amount of turn between two lines be abolished from elementary mathematics curricula. However, failure to insert rotational attentional motion does not imply an individual has not constructed an awareness of angularity. In other words, it is possible that an individual has constructed an awareness of angularity dependent upon some other attentional motion through the interior of an angular object. This issue raises the important questions that will be addressed in this report: what attentional motions might individuals use to constitute the interior of an angular object and what are the implications of these motions?

Methods

The data presented in subsequent sections is drawn from a teaching experiment (Steffe & Thompson, 2000; Steffe & Ulrich, 2013) conducted in the southeastern U.S. with two ninth-grade students, Camille and Kacie, over an academic year. At the time of the study, both students were enrolled in a first-year algebra course, and neither student had taken a dedicated geometry course. The overarching goal of the teaching experiment was to investigate how the students quantified angularity and how these quantifications changed throughout the study (see Hardison, 2018); the author served as teacher-researcher for all teaching sessions. Throughout the study, students engaged in mathematical tasks involving rotational angle models (e.g., rotating laser) and non-rotational angle models (e.g., hinged wooden chopsticks).

Camille and Kacie participated in 14 and 13 video-recorded sessions, respectively, which were conducted approximately once per week outside of their regular classroom instruction; each session was approximately 30 minutes in length. For each student, 2 sessions were initial interview sessions, and 1 session was a final interview session; interview sessions were conducted with each student individually to establish their ways of reasoning at the beginning and end of the teaching experiment. The remaining sessions were teaching sessions wherein the teacher-researcher worked to engender productive changes in students' ways of reasoning in addition to understanding their ways of reasoning; teaching sessions were conducted individually or in pairs. Beyond video-recordings, additional data sources included digitized student work and field notes. The records of students' observable behaviors (e.g., talk, gestures, written responses, etc.) were analyzed in detail during the teaching experiment (on-going analysis) as well as at the conclusion of the teaching experiment (retrospective analysis) via conceptual analysis (Thompson, 2008; von Glasersfeld, 1995). This report focuses on the attentional motions I abstracted from students' observable behaviors at the onset of the teaching experiment, specifically in the initial interview sessions and in the students' first paired teaching session.

Findings

From an analysis of students' activities in the teaching experiment, I abstracted three motions students enacted to account for the interior of angle models. In the subsequent sections, I illustrate each of these motions using data from the teaching experiment.

The First Motion: Radial Sweep

To investigate if students had constructed an awareness of angularity involving a radial sweep (see Figure 1), I posed a rotating laser task to each student during her initial interview. This task was purposefully posed after all other angular tasks in the initial interview to avoid leading students to use a radial sweep on other tasks. The rotating laser task involved a GSP sketch containing the image of a laser pointer. At the click of a button, the laser beam rotated 57° counterclockwise about the endpoint of the ray representing the beam. Initial and terminal positions of the laser pointer are shown in Figure 2. As I introduced the task to each student, I "turned on" the laser by clicking an action button, which showed a red ray emanating from the rightmost end of the laser. I informed the students that I would "turn off" the laser and then click the button to move the laser pointer. I asked the students to keep track of all the places on the screen the beam would have hit if the laser had been "on" as it moved; after the motion stopped, students were asked to shade the portion of the screen the beam would have hit.

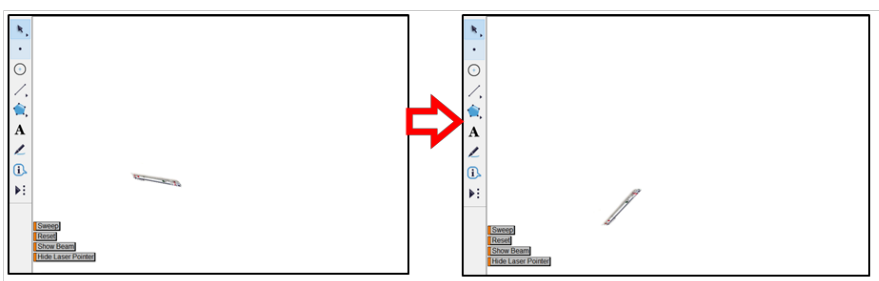

Figure 2: Initial (left) and Terminal (right) Positions of Pointer in the Laser Task

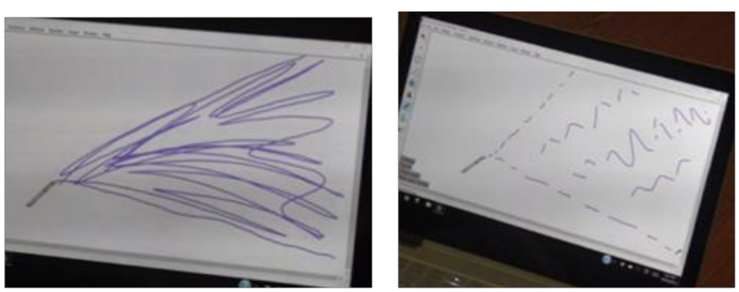

Figure 3: Camille's (left) and Kacie's (right) Shadings on the Rotating Laser Task

Both students shaded the screen accounting for three components: the initial location of the ray, the trace of the ray (i.e., the interior of the angle), and the terminal location of the ray. Camille shaded in the order that the motion occurred—shading the initial ray, then the interior, and finally the terminal ray (Figure 3 left). Kacie shaded in the opposite order (terminal ray, interior, and finally the initial ray), as if re-presenting the experience in reverse (Figure 3 right).

Both students described the boundary of the region as linear and in temporal terms, which indicated the boundedness of the imagined radial sweeping motion. For example, Camille lamented, "I didn't draw it straight but that would be like where it would stop," which indicated

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

her intent to create a linear boundary. Camille's shading of the interior in a counterclockwise motion also indicated that she was reimagining the motion of the radial sweep.

Each student re-presented the uninterrupted motion of the imagined beam, along with its trace, in visualized imagination. Therefore, I consider each student to have demonstrated an awareness of angularity via radial sweep. Camille's activities demonstrated at least an awareness of figurative angularity in that she re-presented the motion of the beam as she had observed it to occur; she had at least internalized the rotational imagery. In contrast, Kacie demonstrated an awareness of operative angularity because she indicated she could imagine reversing the direction of the sweep in re-presentation, which indicated she had interiorized the rotational imagery of the rotating beam.

Considering my hypothesis regarding the importance of a radial sweep, I viewed the students' ability to re-present the rotational motion as a promising base for developing students' angular operations. However, throughout the entirety of the teaching experiment, I saw no evidence that either student ever spontaneously inserted this rotational motion into a non-rotational angle context. In the subsequent sections, I present two other motions, which were not hypothesized, that students inserted into non-rotational angle models on comparison tasks.

The Second Motion: Re-presented Opening

The first angular task posed to each student during the initial interview session was an angular comparison task. In this task, I briefly displayed two pairs of hinged chopsticks (Figure 4) for each student and then covered them with a cloth. I asked each student to draw the chopsticks and describe the similarities or differences they noticed. A portion of Kacie's response is described in the excerpt below where Kacie's actions are described in italicized text.

Figure 4: Obtuse and Acute Chopsticks for the Drawing/Comparison Task

> Kacie: This [obtuse] one is like that [acute] one but it's farther out. [*Holds palms together and fingertips closed (Figure 5 left); then opens fingertips while keeping the bases of her palms pressed together (Figure 5 right)*]. So like they, [*gestures four times as if opening the hinged chopsticks from a closed position in the air*] uh, took it apart, I guess. But, it's still like together. And so, the same with like this [acute] one. It's just like somebody pushed it together [*holds hands open and then closes them some as if closing the obtuse chopsticks to the acute configuration*].

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

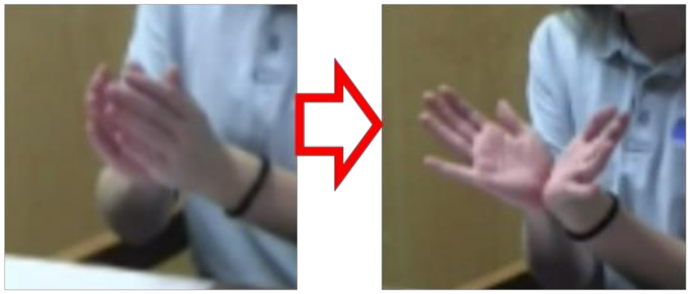

Figure 5: Kacie Imagines Opening Chopsticks from a Closed Position

During the preceding excerpt, Kacie repeatedly opened and closed her hands (Figure 5). From this, I inferred she imagined opening a pair of chopsticks while bounding this motion between two configurations: a closed configuration, which she represented with her hands, and the obtuse configuration she had drawn, which was perceptually available. I use "configurations" rather than "angles" here to emphasize the transformation of a single angle model from one state to another. From her repeated gestures, I infer that Kacie re-instantiated this motion—opening a pair of chopsticks to the obtuse configuration from a closed configuration—in visualized imagination at least four additional times. I interpret Kacie's verbal description, "so like they, uh, took it apart," as additional support for this inference. Kacie's actions indicated she mentally constituted the interior of the obtuse chopsticks by re-presenting the action of opening the chopsticks to the obtuse configuration from a closed configuration. Thus, I consider Kacie to have demonstrated an awareness of angularity via *re-presented opening*.

Some readers may interpret Kacie's re-presented opening of the chopsticks in terms of rotational motion (i.e., a radial sweep); however, I interpret Kacie's actions more conservatively. That Kacie re-presented the opening action does not necessitate that she also held in mind a fixed position for the vertex of an angle model. Additionally, I distinguish an awareness of angularity via re-presented opening from radial sweep as the former involves motion on two distinct rays while the latter involves motion of a single ray through the interior of an angle.

The Third Motion: Segment Sweep

To illustrate a third motion used to constitute the interior of an angle, I describe and analyze Camille's activities on an angular comparison task during the pair's first teaching session. At this point in the session, Camille was explaining why her long chopsticks, which were set to an acute configuration from my perspective, were less open than Kacie's short chopsticks, which were set to an obtuse configuration from my perspective.

To compare the two angle models, Camille held her thumb and index finger together over the vertex of the long chopsticks (Figure 6 left) and then dragged her thumb along one side of the chopstick and her index finger along the other (Figure 6 center and right). As she moved her fingers over the sides of the angle model, her thumb and index finger grew further apart. As she moved her hand, Camille explained, "Mine's starting off really small and getting bigger." Camille made a similar gesture as she referenced Kacie's short chopsticks, "and that one's just like – it's just open really big," moving her hands as if tracing out the short pair of chopsticks.

Figure 6: Camille Drags Her Thumb and Index Fingers Across the Angle Model

Camille's gestures over the long chopsticks suggested a new motion through the interior of the angle. She moved as if sweeping a growing segment, whose endpoints were determined by her thumb and index finger, through the interior of the model (Figure 7). Camille's explanation indicated she implicitly considered the experiential rate at which the segment bounded by her thumb and index fingers was lengthening as she moved her hand away from the vertex. I infer Camille compared the openness of the two pairs of chopsticks by considering which chopsticks would cause her fingers to separate more quickly as she moved her hands away from each vertex. In other words, she was not considering the duration of the motion she enacted as she moved her hands over the chopsticks; instead, she was comparing the intensity of this motion.

Because the Camille enacted these motions sequentially (i.e., one after the other) and not simultaneously, I infer she was able to make the comparison by imagining the segment sweep for one angle model while physically enacting the segment sweep for the other. Therefore, Camille demonstrated at least an awareness of figurative angularity via *segment sweep*.

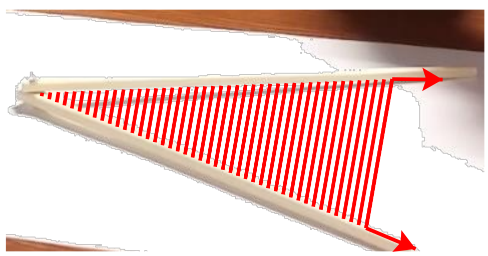

Figure 7: Segment Sweep Through the Interior of an Angle

Discussion

The construction of angularity involves attentional motion of some kind through the interior of (what an observer would call) an angular object. In my analysis of the students' activities, I have presented three such motions: radial sweep, re-presented opening, and segment sweep (Figure 8). A radial sweep involves the rotation a single ray (or segment), whose endpoint is fixed at the vertex of the angle, through an angle's interior. Re-presented opening involves imagining two distinct rays (or segments) opening from a closed position. Radial sweep differs from re-presented opening in that the former motion involves a single rotating ray while the later involves two distinct rays opening from a closed configuration. Segment sweep involves a linear segment, with one endpoint on each side of the angle, moving through the interior of the angle; the length of the segment increases at it is imagined moving away from the vertex. The motions I abstracted from students' activities were likely influenced by the angle models used in my study (i.e., chopsticks and rotating lasers); therefore, future research might investigate whether other contexts lead to other attentional motions.

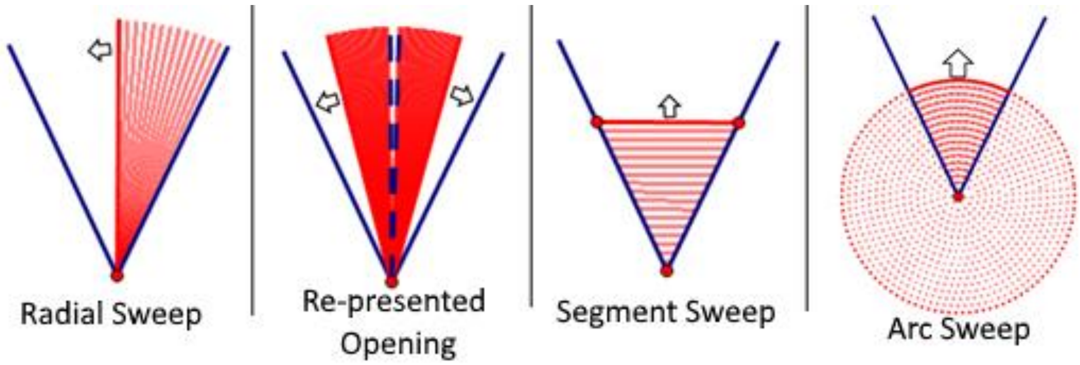

Figure 8: Four Attentional Motions Involved in the Construction of Angularity

In addition to the three motions above, I hypothesize a fourth attentional motion through the interior of an angle—an arc sweep. An arc sweep involves imagining the interior of an angle being swept out by a circular arc bounded by the sides of an angle where the circle containing the arc is centered at the vertex of the angle. Although neither student indicated imagining such an arc sweep, I hypothesize such a motion supports students to develop quantifications of angularity entailing generalized multiplicative comparisons of circular lengths holding across a class of circles (i.e., as advocated by Thompson (2008) and Moore (2013)). Through such a motion, the plane can be conceived as a collection of concentric circles and the angle conceived as a collection of arcs. Future research is needed to investigate this hypothesized attentional motion.

If an individual conceives the interior of an angle model as having an infinite area, all four of these motions can be viewed as being bounded in one sense and unbounded in another. In the case of re-presented opening and radial sweep, the motion is bounded temporally in that the motion has a beginning and an end; the extent of angularity is indicated by the duration of the motion. Re-presented opening and radial sweep are unbounded in a spatial sense, in that the length of the ray(s) moving through the interior of the angle is potentially infinite. In contrast, segment sweep and arc sweep are bounded spatially in that the segment or arc moving through the interior of the angle is always of finite length; however, segment sweep and arc sweep can be thought of as temporally unbounded: the sweeping motion starts at the vertex and continues indefinitely through the interior of the angle. In the case of segment and arc sweep, the extent of angularity is indicated by the rate at which the segment or arc grows as it moves away from the vertex and is, therefore, an intensive quantity (Piaget, 1965) resulting from a coordination of the length of the sweeping entity and this entity's distance from the vertex of the angle. All four of these attentional motions can be productively leveraged to make angular comparisons; yet, I hypothesize radial sweep, re-presented opening, and arc sweep are the most productive motions for developing normative conceptions of angle measure. In contrast, segment sweep may lead to nonnormative conceptions if operations (e.g., iteration) are applied to the sweeping segment.

Further Implications for Teaching and Research

Degrees as a unit of angular measure are often introduced via rotational imagery in terms of a 360-unit angular composite as in the CCSSM. This is a suitable approach for defining degrees if students are considering angles in rotational contexts. However, students' experiences with angles in mathematics classroom often involve non-rotational contexts, such as the interior angles of polygons. Throughout my teaching experiment, students did not spontaneously insert rotational motion into non-rotational angle contexts, which is consistent with the results of previous studies. If students do not spontaneously insert rotational motion into non-rotational angle contexts, introducing units like degrees in non-rotational contexts via rotational imagery is largely inappropriate. However, I do not share the sentiments of Mitchelmore & White (1998), who recommended that angle as turn be removed entirely from the elementary mathematics curriculum. Instead, I suggest that early instruction in angle measure involve both rotational and non-rotational contexts and different approaches for introducing units like degrees are merited depending on the context. Specifically, I hypothesize that emphasizing re-presented opening will be more pedagogically productive in non-rotational angle contexts. After students have interiorized attentional motions and subjected them to further operation (e.g., partitioning and iterating) in different contexts, teachers might work to engender students' recognition of similarities across these two contexts by drawing pictures to represent rotations or inserting rotational imagery into non-rotational angle models, for example.

Future research is needed to determine the prevalence of these attentional motions as students construct an awareness of angularity and the ways in which each motion is related to the other operations constitutive of students' quantifications of angularity. Given that many studies have found students tend to conflate linear attributes (e.g., side length) with angularity, I hypothesize segment sweep may be the most frequently considered motion if students are attending to the interior of non-rotational angle models. In closing, I offer one final hypothesis: generalized quantifications of angularity (i.e., those that are context independent) entail a recognition that different motions through an angle's interior account for the same magnitude of angularity.

References

Akkoc, H. (2008). Pre-service mathematics teachers' concept images of radian. *International Journal of Mathematical Education in Science and Technology, 39*(7), 857–878.

Baya'a, N., Daher, W., & Mahagna S. (2017). The effect of collaborative computerized learning using Geogebra on the development of concept images of the angle among seventh graders. In G. Aldon & J. Trgalová (Eds.), *Proceedings of the 13th International Conference on Technology in Mathematics Teaching* (208–215).

Clements, D. H., Battista, M. T., Sarama, S., & Swaminathan, S. (1996). Development of turn and turn measurement concepts in a computer-based instructional unit. *Educational Studies in Mathematics, 30*(4), 313–337.

Clements, D. H. & Burns, B. A. (2000). Students' development of strategies for turn and angle measure. *Educational Studies in Mathematics, 41*(1), 31–45.

Crompton, H. C. (2017). Using mobile learning to support students' understanding in geometry: A design-based research study. *Educational Technology & Society, 20*(3), 207–219.

Devichi, C., & Munier, V. (2013). About the concept of angle in elementary school: Misconceptions and teaching sequences. *Journal of Mathematical Behavior, 32*, 1–19.

Fi, C. D. (2003). Preservice secondary school mathematics teachers' knowledge of trigonometry: Subject matter content knowledge, pedagogical content knowledge and envisioned pedagogy. Unpublished doctoral dissertation. University of Iowa, Iowa City.

Hardison, H. L. (2018). *Investigating high school students' understandings of angle measure* (Doctoral dissertation, University of Georgia).

Keiser, J. M. (2000). The role of definition. *Mathematics Teaching in the Middle School, 5*(8), 506–511.

Keiser, J. M. (2004). Struggles with developing the concept of angle: comparing sixth-grade students' discourse to the history of the angle concept. *Mathematical Thinking and Learning, 6*(3), 285–306.

Lehrer, R., Jenkins, M., & Osana, H. (1998). Longitudinal study of children's reasoning about space and geometry. In R. Lehrer & D. Chazan (Eds.), *Designing learning environments for developing understanding of geometry and space* (pp. 137–167). Malwah, NJ: Erlbaum.

Mitchelmore, M. C. (1998). Young students' concepts of turning and angle. *Cognition and Instruction, 16*(3), 265–284.

Mitchelmore, M. C., & White, P. (1995). Development of the angle concept by abstraction from situated knowledge. Paper presented at the 1995 Annual Meeting of the American Educational Research Association, San Francisco.

Mitchelmore, M. C., & White, P. (1998). Development of angle concepts: A framework for research. *Mathematics Education Research Journal, 10*(3), 4–27.
Mitchelmore, M. C., & White, P. (2000). Development of angle concepts by progressive abstraction and generalization. *Educational Studies in Mathematics, 41*(3), 209–238.
Moore, K. C. (2013). Making sense by measuring arcs: a teaching experiment in angle measure. *Educational Studies in Mathematics, 83*(2), 225–245.
National Governors Association Center for Best Practices, & Council of Chief State School Officers. (2010). *Common core state standards for mathematics.* Washington, DC: Author.
Owens, K. (1996). Recent research and a critique of theories of early geometry learning: the case of the angle concept. *Nordisk matematikkdidaktikk, 4*(2/3), 85–105.
Piaget, J. (1965). *The child's conception of number.* New York: Norton.
Smith, J. P., III., & Barrett, J. E. (2017). Learning and teaching measurement: Coordinating quantity and number. In J. Cai (Ed.), *Compendium for Research in Mathematics Education* (pp. 355–385). Reston, VA: NCTM.
Steffe, L. P. (2013). On children's construction of quantification. In *Quantitative Reasoning in Mathematics and Science Education: Papers from an International STEM Research Symposium* (pp. 13–41). Laramie, WY: University of Wyoming.
Steffe, L. P. (2017). Psychology in mathematics education: Past, present, and future. In E. Galindo, & J. Newton (Eds.), *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 27–56). Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.
Steffe, L. P. & Olive, J. (2010). *Children's fractional knowledge*. New York: Springer.
Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education* (pp. 267–307), Hillside, NJ: Erlbaum.
Steffe, L. P. & Ulrich, C. (2013). Constructivist teaching experiment. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 102–109). Berlin: Springer.
Thompson, P. W. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 181–234). Albany, NY: SUNY Press.
Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundations of mathematics education. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano & A. Sépulveda (Eds.), *Proceedings of the Annual Meeting of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 45–64). Morélia, Mexico: PME.
Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In S. Chamberlin, L. L. Hatfield & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education: Papers from a planning conference for WISDOM^e* (pp. 35–57). Laramie, WY: University of Wyoming.
Topçu, T., Kertil, M., Akkoc, H., Kamil, Y., & Osman, Ö. (2006). Pre-service and in-service mathematics teachers' concept images of radian. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 5, pp. 281–288). Prague: PME.
von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. Washington, D. C.: Falmer.