TIERING INSTRUCTION ON SPEED FOR MIDDLE SCHOOL STUDENTS

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A design experiment with 18 students in a regular seventh grade math class was conducted to investigate how to differentiate instruction for students' diverse ways of thinking during a 26-day unit on proportional reasoning. The class included students operating with three different multiplicative concepts that have been found to influence rational number knowledge and algebraic reasoning. The researchers and classroom teacher tiered instruction during a 5-day segment of the unit in which students worked on problems involving speed. Students were grouped relatively homogenously by multiplicative concept and experienced different number choices. Students operating at each multiplicative concept demonstrated evidence of learning, but all did not learn the same thing. We view this study as a step in supporting equitable approaches to students' diverse ways of thinking, an aspect of classroom diversity.

Keywords: Rational Numbers, Instructional Activities and Practices, Middle School Education

Today's middle school mathematics classrooms are marked by increasingly diverse ways of thinking (National Center of Educational Statistics, 2016). Traditional responses to such diversity are tracked classes that contribute to opportunity gaps (Flores, 2007) and can result in achievement gaps. Differentiating instruction (DI) is a pedagogical approach to manage classroom diversity in which teachers proactively plan to adapt curricula, teaching methods, and products of learning to address individual students' needs in an effort to maximize learning for all (Tomlinson, 2005). DI is rooted in formative assessment, positions teachers and students as learners together, emphasizes engaging all students in creative thinking, and requires teachers to clarify big ideas for instruction in order to make effective adaptations.

One big idea in middle school mathematics is proportional reasoning (Lesh, Post, & Behr, 1988; Lobato & Ellis, 2010), and a great deal of research has been conducted in this domain (Lamon, 2007). One key finding is that computing to solve problems involving ratios (e.g., by cross-multiplication) does not indicate proportional reasoning and may mask it (Kaput & West, 1994; Lesh et al., 1988). A second finding is that students often double, triple, halve, etc., two quantities together as they learn to solve problems involving ratios (Kaput & West, 1994; Lobato & Ellis, 2010); in other words, students treat the two quantities as a *composed unit* (Lobato & Ellis, 2010) and operate on the quantities multiplicatively. For example, students who are creating mixtures of lemonade with the same flavor as 2 T powder with 3 cups water will double each quantity. A third finding is that operating multiplicatively on a composed unit is still different from thinking of the ratio as a *multiplicative comparison* (Lobato & Ellis, 2010), where students know, for example, that in the recipe the amount of concentrate is always 2/3 the amount of water. In this way of thinking, a person is measuring one quantity with the other, and doing so can lead to ideas about rates (Steffe, Liss, & Lee, 2014; Thompson, 1994).

One context for working on proportional reasoning with students is speed (Lobato & Siebert, 2002). In a teaching experiment with nine 8^{th}-10^{th} grade students, Lobato and Siebert gave a distance value and time value for one character to walk in a computer simulation and asked students to determine a different distance value and time value for another character to walk at the same speed. The researchers found that this problem engaged some students in constructing ratios. For example, ninth grade student Terry explained why traveling 2.5 cm in 1 s was the same as 10 cm in 4 s because he could see the 10 cm-4 s journey as made of up four 2.5 cm-1 s segments, and that each small segment was ¼ of the total journey. Similarly, in a teaching experiment with seven 7th grade students, Ellis (2007) found that such tasks supported students' construction of rate and slope in cycles of generalizing and justifying activity. Across both experiments students showed evidence of one hallmark of the construction of speed as a proportional relationship between distance and time, notably that "partitioning a traveled total distance implies a proportional partition of total time required to travel that distance" and vice-versa (Thompson & Thompson, 1994, p. 283).

The purpose of this paper is to report on this research question: What influences did tiering instruction with speed tasks have on a class of 18 regular seventh grade mathematics students during a unit on proportional reasoning? The report comes from a 5-year project to study DI and relationships between middle school students' rational number knowledge and algebraic reasoning. In the last phase of the project, the research team partnered with middle school mathematics teachers who had participated in a year-long study group to explore differentiation. In fall 2017 a seventh grade mathematics teacher and the project team designed to differentiate during a 26-day unit, and the teacher and first author co-taught the unit. The data for this paper comes from five episodes in which the students experienced tiered instruction focused on exploring speed to support reasoning with ratios. The instruction for these five days was adapted from Lobato's Math Talk project (mathtalk.sdsu.edu) and utilized the geogebra app Races developed by Bowers (https://www.geogebra.org/m/J434Kb54).

Theoretical Frame

In this section we present our definition of DI, our view of mathematical thinking, and a tool we use to understand students' diverse ways of thinking, students' multiplicative concepts.

Definition of DI

Our definition of DI is *proactively tailoring instruction to students' mathematical thinking while developing a cohesive classroom community* (cf. Tomlinson, 2005). For us, "tailoring instruction" requires posing problems that are in harmony with students' thinking and posing challenges at the edge of students' thinking (Hackenberg, 2010). We view a "cohesive classroom community" as students and a teacher who are working together to foster everyone's mathematical learning (Lampert, 2001; Tomlinson, 2005), who regularly talk about their ideas (Sherin, 2002), and who hold diverse points of view that are valued (Bielaczyc, Kapur, & Collins, 2013).

Mathematical Thinking and Interaction

As teacher-researchers we don't have direct access to students' mathematical thinking or points of view. So, we organize our experiences with students' thinking by describing and accounting for it using our constructs: operations, schemes, and concepts. *Operations* are the components of *schemes*, goal-directed ways of operating that involve a situation as conceived of by the learner, activity, and a result that the learner assesses in relation to her goals (von Glasersfeld, 1995). For us, mathematical learning involves a learner making reorganizations, or *accommodations*, in her schemes in on-going interaction in her experiential world.

Indeed, interaction is a core principle of our view of mathematical thinking and learning (Piaget, 1964; Steffe & Thompson, 2000). We find it helpful to think about two non-intersecting domains of interaction (Steffe, 1996): intra-individual interactions of constructs within a person, such as accommodations in schemes, and individual-environment interactions of which social interactions are a major part. Social interaction, such as student-student and student-teacher interactions, can open possibilities for accommodations and make operations and schemes apparent via verbalizations, non-verbal expression, drawn representations, or mathematical notation. Similarly, the construction of particular operations, schemes, and concepts can dramatically influence how a student interacts with others in a classroom (e.g., Hackenberg, Jones, Eker, & Creager, 2017). However, interaction of a particular kind in one domain does not directly cause interaction of a particular kind in the other (Steffe, 1996).

For us, *concepts* arise from re-processing the result of a scheme so that students can use it to structure a situation prior to acting (von Glasersfeld, 1982). Broadly speaking, students enter middle school operating with three different multiplicative concepts that significantly influence rational number knowledge (Norton & Wilkins, 2012; Steffe & Olive, 2010) and algebraic reasoning (Olive & Caglayan, 2008; Tillema, 2014). Transitioning between these three concepts requires substantial accommodations that can take two years (Steffe & Cobb, 1988; Steffe & Olive, 2010). Steffe (2017) estimates that at the start of sixth grade, 30% of students are operating with the first multiplicative concept, 30% with the second, and 40% with the third.

Students' Multiplicative Concepts

We view students' multiplicative concepts in terms of units coordination (Steffe, 1992). *Units* are discrete ones (Ulrich, 2015), lengths, or measurement units. As children progress in their construction of number and quantity, they organize units into larger units, such as composite units (units of units). A *units coordination* entails distributing the units of one composite unit across the units of another composite unit. For example, consider this problem: The length of the balance beam measures 8 skewer lengths. There are 7 toothpick lengths in a skewer length; how many toothpick lengths will measure the beam's length?

Students operating with the first multiplicative concept (MC1 students) solve this problem by counting on by 1s past known skip-counting patterns, tracking the total number of toothpick lengths and skewer lengths. For example, they might know that two 7s is 14. Then they count on by 1s to 21 for the amount of toothpick lengths in 3 skewer lengths. And they keep going. These students think of the result, 56 toothpick lengths, as a unit consisting of 56 units. But they don't see a multiplicative relationship between 1 toothpick length and the 56.

Students operating with the second multiplicative concept (MC2 students) do see a multiplicative relationship: The 56-unit length is 56 times 1 unit. These students also see the 56 toothpick lengths as eight 7s, or 8 units of 7 units of 1, which is three levels of units. However, as they work further, they think of 56 as a unit of 56 units of 1; the three-levels-of-units structure does not remain for them.

Students operating with the third multiplicative concept (MC3 students) can see what MC2 students see, but as they work further, they continue to view the 56 units as 8 units of 7 units of 1. This view is helpful if the number of skewer lengths is not a whole number. For example, if the distance were 8 ¼ skewer lengths, MC3 students are able to reason that to measure the distance in toothpick lengths they need eight 7s and ¼ of 7.

Method, Data Collection, and Data Analysis

To launch the experiment, we observed in two seventh grade pre-algebra classrooms: a participating class taught by Ms. W and a comparison class taught by a different teacher. Following observations, 38 students consented to participate: 18 out of 20 in the participating class, and 20 out of 21 in the comparison class. Before the unit began, we sought to develop initial understanding of students' multiplicative concepts and fractions knowledge and to select focus students: six from the participating class (1-3 operating with each multiplicative concept) and six from the comparison classroom. To gather initial information, we administered three written assessments of students' fraction schemes and operations (Wilkins, Norton, & Boyce, 2013) and multiplicative concepts (Norton, Boyce, Phillips, Anwyll, Ulrich, & Wilkins, 2015).

We used results of the written assessments, as well as our classroom observations, to select 30 students for 40-minute individual interviews prior to the start of the unit. In the interviews we developed deeper understanding of students' thinking, and sometimes we revised our assessment of a student's multiplicative concept. Following the interviews, we had 11 MC1 students (5 in the participating class), 17 MC2 students (9 in the participating class), and 10 MC3 students (4 in the participating class). We selected as participating focus students two MC1 students, three MC2 students, and one MC3 student, as well as six "matched" comparison focus students.

For this paper we have done two analyses. First, we developed second-order models of the six participating focus students; a second-order model is a researcher's constellation of constructs to describe and account for another person's ways and means of operating (Steffe & Olive, 2010). For each student, we repeatedly reviewed video of the student's three interviews; video of the student's activity during the targeted five days of class; and the student's written work, including a quiz on Day 17 and a unit test. To create the models we wrote summaries, interpretations, and conjectures, which we discussed at bi-weekly research meetings with a 6-member research team. We debated and questioned interpretations, coming to consensus through discussion. Each model is a description of the student's operations, schemes, and concepts, with accounts of accommodations that occurred and the individual-environment interactions (e.g., particular small group discussion) that were involved in the accommodations.

Second, we have engaged in close description and analysis of the classroom activity during the targeted five days. We have organized documents of the work of all 18 students across the five days in order to articulate trends and patterns in student ideas and responses.

Findings

Summary of Days 9-13

The unit consisted of three investigations: quantifying orangeyness (Days 1-8), quantifying speed (Days 9-18), and understanding percentages (Days 19-26). By Day 9 we had conducted formative assessment of students' reasoning with ratios during the quantifying orangeyness investigation. We found that MC1 students were not fluidly iterating two quantities as a composed unit (Lobato & Ellis, 2010), while MC2 and MC3 students were. So, we thought that tiering instruction at the start of the quantifying speed investigation would help us target students' current thinking with ratios and could support them to make advances. Students worked in small groups that were relatively homogenous by multiplicative concept from Day 9 to 13.

On Days 9 and 10 students articulated how to measure fastness and how they knew one car was going faster than the other in the Races app (Figure 1). Subsequently they worked on tasks where they were to make the red car go slower than the blue car if both traveled the same distance, and then if both traveled the same amount of time. On Days 11 through 13 students were given a distance value and time value for the blue car, and they explored how to make the red car go the same speed using a different distance value and time value. They were to justify their claims with pictures and explanations. Here we tiered instruction by selecting distance and time values strategically for different thinkers as shown in Table 1.

Now we show how students at each stage worked on the same speed task with a focus on members of the group that made the most progress: a group of three MC1 students in which we focus on Emily, a group of three MC2 students in which we focus on Lisa and Sara, and a group of two MC2 and two MC3 students in which we focus on MC3 student Joanna.

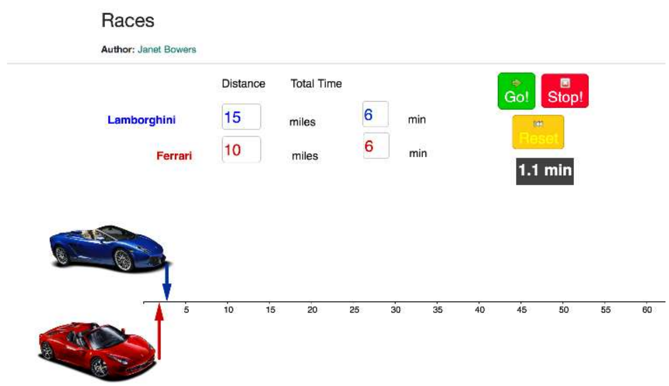

Figure 1: Races App with Blue Car (top) and Red Car (bottom)

Table 1: Students' Multiplicative Concepts and Numbers for Same Speed Task

MC	Same Speed Task (Blue car goes)
1	18 mi in 3 min; yields a whole number unit ratio (6 mi per 1 min)
2	15 mi in 6 min; yields a unit ratio that is a mixed number with ½ (2.5 mi per 1 min)
3	15 mi in 9 min; yields a unit ratio hard to work with as a decimal (5/3 mi per 1 min)

MC1 Students: Emily and Groupmates

When Emily and her two groupmates tried to find a distance and time for the red car to go the same speed as the blue car traveling 18 mi in 3 min, Emily suggested 9 mi in 6 min, 18 mi in 6 min, and 18 mi in 2 min. She seemed to be, primarily, halving or doubling either quantity but not operating on both together.

Then a groupmate suggested 36 mi in 6 min. Emily ran that race and was visibly excited when the cars kept pace with each other. She seemed suddenly subdued when the red car continued traveling after the blue car stopped, but the group concluded that the cars had gone the same speed and that doubling each number "worked."

The first author, Ms. H, asked the group to draw a picture to justify why traveling 36 mi in 6 min was the same speed as traveling 18 mi in 3 min. No one initially had ideas. Emily said, "I know how to tell, but I don't know how to show it." She explained that doubling each number (18 and 3) meant you could then divide each number (36 and 6) by 2, and "it's almost like they're the same number in a way." Ms. H acknowledged this idea but asked them to think about the quantities because that would help them develop stronger ideas about speed.

Ms. H asked if they could draw something to represent each journey. Emily's pictures evolved the most, so we focus on her. First Emily drew a segment to show each journey, identified by labels (Figure 2, left). When asked whether she could show the idea of doubling with the lengths, Emily drew a second picture (Figure 2, right).

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

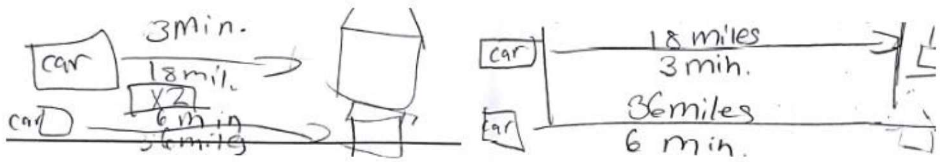

Figure 2: Emily's First Picture (left) and Second Picture (right)

Despite the discussion about doubling and Emily's emphasis on the importance of doubling, the second picture showed lengths about the same size. Ms. H asked whether the journeys were the same size, and Emily said no. She then extended the 36 mi-6 min segment but did not make it exactly twice as long as the 18 mi-3 min segment, in part because she reached the edge of the paper (Figure 3). She identified that there was supposed to be another 18 mi-3 min segment next to the first one, making up the 36 mi-6 min segment. She was about to draw a more exact picture when the period ended. So, Emily went from not knowing how to draw a picture to beginning to show how the 36 mi-6 min journey consisted of traveling the 18 mi-3 min journey twice. She presented this idea to the whole class the next day. In her follow-up interview on a similar question she began by showing two different journeys with equal lengths and then self-corrected to produce a picture showing relative size, which is evidence of learning for her.

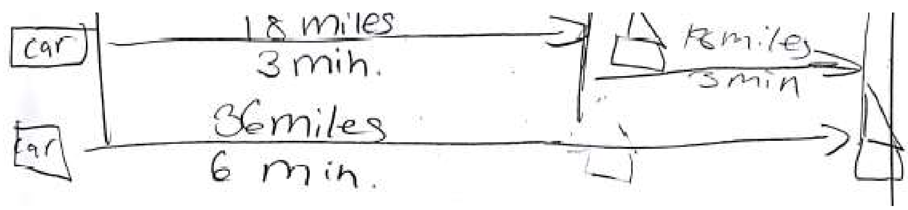

Figure 3: Emily's Third Picture

MC2 Students: Lisa and Sara

When Lisa and Sara tried to find a distance and time for the red car to go the same speed as the blue car traveling 15 mi in 6 min, Lisa suggested 14 mi in 5 min and then 15.1 mi in 6.1 min. When neither worked, both students said it was "impossible!" Ms. H asked them if it was really not possible for two cars to travel the same speed but different distances and times. Sara said: "They probably could, but I can't figure it out." Then she added, "unless you double it." They ran a race where the red car traveled 30 mi in 12 min, and they both seemed excited to find that doubling the quantities produced the same speed. "I figured the system out!" proclaimed Sara. Lisa added that it might be possible to triple both quantities or use other multiples.

Like Emily and her groupmates, Lisa and Sara found it challenging to explain why doubling would work. However, in contrast with Emily, Lisa's first picture showed that the 30-mi distance was twice the length of the 15-mi distance (Figure 4, left). In discussing the picture, Ms. H pointed out that in Lisa's picture it looked like the car traveling 30 mi went a trip of 15 mi in 6 min and then another trip of 15 mi in 6 min (tracing the trips with her pen).

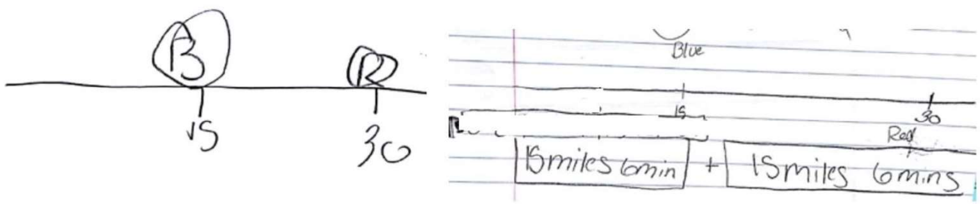

Figure 4: Lisa's First Picture (left) and Second Picture (right)

Lisa agreed and drew another picture (Figure 4, right) that showed the 30 mi-12 min trip as consisting of two "15 miles 6 min" segments added together. In a whole class discussion the next day, Lisa stated the idea that to go the 30 mi-12 min journey "first you'll need to do one 15 miles in 6 minutes and then you'll need to add another 15 miles in 6 minutes." Sara also used this multiple-trip explanation to explain solutions to this and other problems.

When Ms. H asked if they could find smaller distance-time pairs that would produce the same speed, they halved both quantities and indicated that they could continue to halve to find more same speed pairs. We note that Lisa and Sara did not consider dividing the distance and time values by numbers other than 2 without further teacher questioning. Nevertheless, during this instructional segment they went from not knowing how to generate same speeds to using at least whole number multiples and halving to do so, and from not knowing how to justify same speeds to using multiple-trip explanations. They sustained these ways of generating and justifying same speed pairs in their follow-up interviews.

MC3 Student: Joanna

When Joanna's group of four began discussing distances and times for the red car to go the same speed as the blue car travelling 15 mi in 9 min, Joanna quickly suggested 5 mi in 3 min. Her groupmate Mark suggested 16 mi and 10 min, adding one unit to each quantity. Joanna argued that 15 and 9 "reduced" to 5 and 3 but 16 and 10 "reduced" to 8 and 5, so 15 and 9 and 16 and 10 "wouldn't be the same ratio to each other." Using the app to test Mark's suggestion, the group determined that the red car travelling 16 mi in 10 min would actually go slightly slower than the blue car. Then Joanna stated that any numbers "where the miles would reduce to 5 and the minutes would reduce to 3" should work "because they're the same ratio to each other." She suggested 10 and 6 as another pair that would give the same speed.

To justify her claim, Joanna drew a distance line and time line (Figure 5). She partitioned the lines into three equal parts of 5 miles and 3 minutes. Then she used her picture to justify that when the red car travels 5 mi in 3 min, it goes the same speed as the blue car; it just stops earlier. Upon questioning, Joanna elaborated that 5 mi-3 min segment was 1/3 of the blue car's journey. To Joanna, the 15 mi-9 min trip was a unit that could be partitioned into 5 mi-3 min segments, and she saw that any trip made from a multiple of these segments would have to be the same speed as the blue car, a general way of thinking. She created this general way of thinking by determining the smallest whole number pair of numbers that could make the 15 mi-9 min trip.

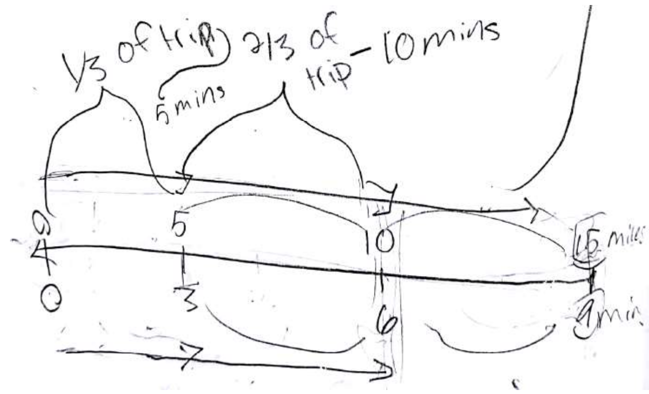

Figure 5: Joanna's Picture

Discussion and Conclusions

Now we point out some similarities in the students' ways of thinking, as well as differences that relate to students' multiplicative concepts. Notably, both Emily and Lisa were not sure how to create same speeds, and both took up groupmates' suggestions to double. However, Lisa's picture indicates that she conceived of doubling the quantities in a way that showed two smaller trips fitting into the larger trip. These relationships were not evident for Emily without interaction and support from a teacher to try to show relative size in her picture.

Students' multiplicative concepts are explanatory here. That is, both Emily and Lisa could repeat a distance and time to create a trip with double the distance and time. But Lisa appeared to have imagery of that larger trip as consisting of two smaller, equal trips—the smaller trip was both a part of the larger trip and also separate from it. This imagery is consistent with having constructed a disembedding operation where a unit (in this case, a segment representing a distance-time pair) can be both a part of a composite unit and also separate from it—a hallmark of the second multiplicative concept (Steffe, 2010). In contrast, Emily did not show obvious understanding of these embedded relationships. This phenomenon is consistent with MC1 students who conceive of a length as consisting of parts (smaller lengths), but once the original length has been separated into parts, they do not reunite the parts to create the original length or see the parts as being embedded in the original length (Hackenberg, 2013; Steffe & Olive, 2010).

In contrast to Emily, Lisa, and Sara, Joanna partitioned her distance and time quantities and seemed to view it as a logical necessity to partition each quantity proportionally (Thompson & Thompson, 1994). Her insight was that any numbers that were in a ratio of 5 to 3 would produce the same speed, so she saw more generally that 15 mi in 9 min was just one journey that was made from a multiple of 5 mi in 3 min. Her multiplicative concept can help account for her insight. That is, Joanna could view numbers and quantities as three-levels-of-units structures prior to working with them in a problem solving situation. So, she saw both 15 mi and 9 min as units of 3 composite units: 15 mi was a unit of 3 units of 5 mi, and 9 min was a unit of 3 units of 3 min. Being able to see both quantities in this way facilitated her thinking about how, since each 5 mi-3 min segment would have to be the same speed, then three of them strung together would be the same speed. Ultimately, she saw that any trip made from a multiple of this smallest whole number pair would produce the same speed.

Now we comment on the different numbers the students worked with. All distance-time pairs required taking thirds to get to the smallest whole number pair that could create the same speed (Table 1). Yet MC1 and MC2 students did not take thirds of their quantities: They doubled, tripled, and halved. We anticipated that many students initially would double and halve, and that is precisely why we gave pairs that could be "thirded" to produce the smallest whole number pair that could create the same speed. Thus, we learned that these number pairs were good choices in the sense of not being completely transparent to students. In addition, they supported MC3 students like Joanna to reveal the structural way she viewed and operated on both quantities.

Other researchers have used students' ways of thinking about speed as an avenue for supporting the construction of ratios and rates (Ellis, 2007; Lobato & Siebert, 2002; Thompson, 1994), which are extremely important mathematical ways of thinking in secondary school. However, to our knowledge researchers have not investigated how students with different multiplicative concepts construct these ways of thinking, as well as how to differentiate instruction for these thinkers in the same classroom. In this seventh grade classroom tiering instruction was successful in supporting the learning of each of these three thinkers, although what each thinker learned was different. Thus, we view this study as a step in supporting equitable approaches to an important aspect of classroom diversity, students' diverse ways of thinking. This kind of DI is an important component of inclusive, antiracist classrooms in which "equity is a priority" (Michael, 2015, p. 82) because all students are seen as mathematical thinkers and get what they need to learn.

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