STUDENT UNDERSTANDING OF THE GENERAL BINARY OPERATION CONCEPT

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In this paper, we address the variety of ways in which students conceive of binary operations and the metaphors they might leverage when working with binary operations in group theory. We use open-ended surveys paired with interviews to qualitatively explore student's conceptions of binary operation. Through this analysis, we identified three central metaphors (function, arithmetic, and structure), as well as a number of attributes related to binary operation that students perceive as critical. This work has implications for our treatment of binary operation in advanced mathematics, and the messages we send about operation in the K-12 setting.

Keywords: Advanced Mathematical Thinking, Algebra and Algebraic Thinking, Post-Secondary Education

Binary operations are threaded throughout the mathematics curriculum starting with a focus on arithmetic, and then expanding to operations such as matrix multiplication and function composition. Although binary operations are prolific throughout mathematics in K-12, it is not until advanced courses in undergraduate mathematics that binary operation is formally defined. In courses such as group theory, students are put in a position to generalize the binary operation concept to better understand the structures that underlie our mathematical systems. Despite this occurrence, and the prevalence of binary operation mathematics, little research attention has been given to the general binary operation. This is disconcerting for two reasons: (1) according a survey of expert instructors, binary operation is considered to be one of the most important topics in group theory (Melhuish & Fasteen, 2016); (2) binary operations are one of the core concepts that can be connected back to the K-12 curriculum (Melhuish & Fagan, in press).

However, the survey of expert instructors also revealed a possible cause for the lack of treatment. They identified binary operation as a topic with low difficulty level (Melhuish & Fasteen, 2016). Yet, we found evidence to suggest that students' conceptions of binary operations can significantly interfere with performance on group theory tasks (Melhuish & Fasteen, 2016). This evidence was the impetus to study student understanding of binary operation more directly. In this paper, we address the variety of ways in which students conceive of binary operations and the metaphors they might leverage when working with binary operations across a number of task types. We present a qualitative study, guided by variation theory, in which we elaborate on the critical attributes of binary operations that students attend to while engaging with related tasks.

Literature Review

Binary operation has uneven treatment throughout the literature. Before we unpack this literature, we define what we mean by binary operation. Informally, binary operations can be thought of as rules that combine two elements within a set and produce a single element of the same set. For instance, addition is a binary operation on the set of integers because addition can be thought of as a rule for combining two integers to produce a single integer. Formally, binary operations are often defined as follows:

A *binary operation* $*$ on a set S is a function mapping $S \times S$ into S.

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This formal definition situates operation as a type of function. We therefore outline literature across three categories: (1) arithmetic and other specific operation literature, (2) function literature, and (3) literature pertaining to the general binary operation.

As most literature on student mathematical thinking connects to operation in some way, we focus on a few key theoretical contributions (that go beyond the specifics of a particular arithmetic operation.) One such example is the *operation sense* framework developed by Slavit (1998), which unpack a series of stages built around familiarity with standard arithmetic operations. In this framework, operations represent a standard process, such as the process of combining groups underlying the operation of addition. Slavit's work documented a path of increasing sophistication that could support students' in developing operation sense that moved beyond particular arithmetic process metaphors to an understanding of operation without a need for concrete referents. Such understanding can support students' transition from arithmetic to algebra. Gray and Tall (1994) provided a different view on this abstractness focusing on operation as a *procept*. That is, an operation can be conceived as a process and a concept. An expression such as "3+2" is both the object (the sum) and the process (adding the two numbers.) In light of this literature base, we identify (1) students may struggle to move beyond arithmetic operations on concrete referents and (2) students need to conceive of operations as processes and objects (which is non-trivial.)

We next turn to the literature on function to identify ways in which function understanding may be relevant to binary operation. First, there is a large amount of existing literature documenting the complexities of function (Oehrtman, Carlson, & Thompson, 2008). Students have been shown to possess numerous conceptions throughout their K-16 education. Due to space limitations, we note two that provide insight beyond arithmetic literature. First, students may interpret functions as necessarily having a written rule (Vinner & Dreyfus, 1989). Such an interpretation limits students' ability to leverage or make sense of non-symbolic representations. Additionally, function conceptions are tied to a wide range of metaphors (Zandieh, Ellis, and Rasmussen, 2017). One such metaphor is the ubiquitous input/output machine metaphor (Tall, McGowen, & DeMarois, 2000). In light of binary operations being a special case of function, we conjectured that similar representation and metaphor preferences might exist.

Finally, we attend to the literature on the general, abstract, binary operation concept. While majority of research related to abstract algebra and linear algebra implicitly treat binary operations (e.g., Larsen's (2009) work having students reinvent groups), few studies have explicitly focused on binary operation. These few studies are primarily theoretical breakdowns (with empirical instantiations) that are part of a larger attempt to map student conceptions. Brown, DeVries, Dubinsky, and Thomas (1997) and more recently Wasserman (2017) presented genetic decompositions of binary operations in which individuals may have an action (requiring concrete actions with individual inputs), process (seeing a binary operation as a general process on a domain), or object (seeing binary operation as something that can be acted on) conception of binary operations. Alternately, Novotná, Stehlíková, and Hoch (2006) use lens of *structure sense* to produce a framework of binary operation capturing shifts of attention where students can first recognize binary operation in familiar settings, then make sense of properties like closure, engage with unfamiliar operations, and compare differing representations. Other notable results include students have limited representational flexibility (Ehmke, Pesonen, & Haapasalo, 2010), and have limited example spaces (Zaslavsky & Peled, 1996). While these contributions provide some insight into student conceptions around binary operation, many questions remain.

As a whole, the body of research points to substantial reasons to identify binary operation as a potentially troubling topic with its nature as a function and operation. In the next section, we outline a theoretical lens that has aided us in further parsing students' activity and conceptions around binary operaiton: variation theory (Marton & Booth, 1997).

Theoretical Framework

One way to conceptualize learning is the perception of new attributes of a phenomenon or experiencing a phenomenon in a new way. Such an approach leverages a variation theory (Marton & Booth, 1997) lens. For a given individual, their understanding of a concept reflects which attributes of the concept are foregrounded for them. Students become aware of attributes through experiences of concepts (and non-examples of concepts) where attributes may be foregrounded via contrast. If a student never experiences a non-example of a binary operation with one or three inputs, they may not perceive two inputs as a critical attribute of binary operation. Alternately, if all examples students encounter have certain commonalities such as associativity, students may perceive associativity as a critical attribute of binary operation. In this way, variation theory ties learning tightly with the role of experiences and particularly contrasting experiences where students may discern attributes of a particular object.

From a variation theory stance, we then leverage several key ideas including: *object of learning*, *critical attributes*, and *metaphors*. Variation theory provides a complimentary tool to traditional cognitive or social investigations of learning via shifting from cognition to perception (Dahlin, 2001). This shift leads to questions about: What are the critical attributes of a given object of learning? What opportunities do students have to perceive (through variation) the critical attributes of an object? And ultimately, what critical attributes are in the *lived* object of learning for a student?

Vikström (2008) has expanded the use of variation theory to also attend to metaphors as the means for which students structure and communicate their lived object of learning. As students become aware of critical attributes of a phenomenon, metaphors are the mechanism to integrate and connect with the abstract notion of a particular concept. When students engage in tasks, they then rely on metaphors to connect a given situation to their greater understanding of a concept.

In accordance with this view on learning, the research questions guiding our study are: (1) What are the metaphors leveraged by students when communicating about binary operation?; (2) Which attributes are students attending to when engaged in binary operation related tasks?

Methods

Participants and Data Collection

The participants in this study consisted of students enrolled in three introductory, undergraduate-level abstract algebra classes (n =9, n = 12, and n = 12 respectively). Additionally, we interviewed six students, four of which come from the surveyed class, two from another class (that received a shortened survey that is not reported here.) Classes span two institutions and three unique instructors. Surveys were created to cover a variety of binary operations and non-binary examples. In particular, the questions were crafted in accordance with the following activity domains:

1. *Is or is not*. Determining if a given instantiation is an example of a concept (e.g. Ehmke , Pesonen, and Happasalo, 2011)
2. *Same or different*. Determining if two instantiations are mathematically the same (e.g. Novotna, Stehlikova, and Hoch, 2006)

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3. *Properties*. Determining what properties an example may or may not have. (e.g. Dubinsky Dautermann, Leron, & Zazkis, 1994)
4. *Generating*. Creating an example meeting some criteria (e.g. Zaslavsky & Peled, 1996).

Table 1: Example Survey Prompts

Category	Is or Is not	Same or Different	Property	Generating
Prompt (shortened)	$(a) = a^2$, on $\boldsymbol{R}$	$\boldsymbol{Z}_3$ and $\boldsymbol{Z}_6$ (presented tabularly)	Is ½(a+b) associative?	Define a binary operation on {1,2,4}
Purpose	Critical attribute: closure; Missing: 2 inputs	Calls on attribute: element-wise defined	Missing varying attribute: *E-O-E* formatting	Calls on varying attribute: non-symbolic representation

Each of the questions was open-ended and provided a prompt for the students to explain their reasoning. Across the classes, the surveys contained ten common questions, along with several variations designed to test the robustness of response types. For the scope of this report, we focus on the common questions.

In addition to the surveys, six semi-structured interviews were conducted to gain deeper insight into the students' mathematical thinking as they worked through the survey. The purpose of the interviews was twofold: (1) to validate our interpretations of written responses and (2) allow for a closer inspection of metaphors that are leveraged in verbal communication (and not always immediately apparent in written responses.)

Analysis

The analysis of the survey data was driven by methods of phenomenographic content analysis (Trigwall, 2006). Student responses were disconnected from the individual. The goal was to identify patterns in the different ways that students experience a given phenomenon. Each task was then analyzed for patterns in responses beginning with broad open codes. These were condensed to categories reflecting critical attributes of binary operation as experienced by students. We then independently coded all survey responses resolving all disagreements via discussion. After this analysis, we determined the degree to which a given student's attention to attributes was consistent across tasks. In the results section, we share the dominant critical attributes from student responses, as well as reports on consistency.

For the interview data, we analyzed transcripts to identify the metaphors they leveraged to communicate about binary operation. For each transcript, we identified each instance that a student communicated directly about binary operation noting varying patterns in language. We then returned to the literature to categorize metaphor clusters: arithmetic, function, and structure. We used these metaphors to code the transcripts to determine which of the three metaphors occurred in a given individual's communication.

Results

We begin by discussing the three metaphors that underlined students' communication.

Metaphors

Function Metaphor Cluster. We define the function metaphor cluster as being enacted whenever a student communicates about binary operation as if it were a function, that is, a mapping with inputs and outputs. Indicators include: reference to a domain and range, input/output language, mapping language, and referencing function specific properties such as

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one-to-one. An example of a student using this metaphor is the following description of a binary operation, "[the binary operation] takes two inputs and returns an output based on the operation." Notice this language evokes traditional function metaphor language where there is something (akin to a function machine) that takes inputs and produces outputs. Other examples of using the function metaphor include: "[The binary operation] does take two of the numbers and it will return a third," and "The division binary operation is going to take me from the Reals to the Reals." Such metaphors occurred across all of the interviews (see Table 2).

Table 2: Interviewed Students' Metaphors for Binary Operation

Metaphor	Function	Arithmetic	Structure
Student 1 (Class B)	x	x	x
Student 2 (Class B)	x	x	x
Student 3 (Class C)	x	x	x
Student 4 (Class C)	x	x	x
Student 5 (Class D)	x	x	x
Student 6 (Class D)	x	x	

Arithmetic Metaphor Cluster. We define the arithmetic metaphor as being enacted whenever a student communicates about binary operation as if it was an arithmetic operation. In arithmetic contexts, we tend to leverage a combining metaphor. For example, when adding 2 and 3, one metaphor would be combining collections of 2 items and 3 items to produce 5 times. This action is internal to the inputs rather than taking inputs and mapping to something new (as in a function metaphor.) Indicators of this metaphor include language around "answer" and "combining." For example, a student described the action: "Two numbers, take them, combine them". Another described comparing two operations on particular elements with: "They are not the same answer." As in the case of function, these metaphors occurred in all interviews (see Table 2).

Structure Metaphor Cluster. We define the structure metaphor cluster as being enacted whenever a student communicates about a binary operation as a mechanism for structuring a set. The structure may be noticed based on prior exposure to mathematics (such as a binary operation on a set forming a group) or the student may spontaneously create the structure (such as noticing a pattern or explicitly searching for a pattern within a set of outputs). Indicators include language about overall behavior, patterns, or structural properties. The structure metaphor tended to emerge in tasks in which students were asked to compare two binary operations to determine whether they were the same. For example, one student asked, "What pattern do we have going on there?" and another commented, "These two groups behave rather similarly." This metaphor can be quite productive when considering structural properties of a given set. Structure metaphors were frequent, occurring in five of six interviews.

These results reflect that the three metaphors are not a mutually exclusive categorization of a student's thinking about binary operations. A student can enact any of three metaphors depending on the task they are engaged with, and in fact it was common of the students to enact all three at some point. Furthermore, the same metaphors can be supportive or unsupportive depending on the critical attributes the metaphor is structuring.

Critical Attributes

In this section, we unpack some of these critical attributes. Our focus is on the survey results, but we note that are interview subjects provided similar profiles in terms of critical attributes. We cover two definitional properties (closure, two input elements), one additional critical attribute

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(binary operations are defined at the element level), and two non-critical representational attributes (symbolic rule and element-operator-element (E-O-E) formatting).

Table 3: Survey Responses when Determining if an Instantiation is a Binary Operation

	Attended to Closure			Attended to Two Input Elements		
	Always	Inconsistently	Never	Always	Inconsistently	Never
Class A	5 (62.5%)	2 (25.0%)	1 (12.5%)	0 (0%)	0 (0%)	8 (100%)
Class B	5 (41.7%)	5 (41.7%)	2 (16.7%)	1 (8.3%)	1 (8.3%)	10 (83.3%)
Class C	4 (33.3%)	2 (16.7%)	6 (50.0%)	6 (50.0%)	3 (25.0%)	3 (25.0%)
Total	14 (43.8%)	9 (28.1%)	9 (28.1%)	7 (21.9%)	4 (12.5%)	21 (65.6%)

Closure. The first of the critical attributes is closure. It was common for students to attend to closure of a binary operation on a set when asked to determine if a given operation was binary or not (see table 3). For example, when asked if $*(a) = a^2$ defined on $\mathbb{R}$ is a binary operation, one student responded, "yes, if a is real, the square of a is also real."

Two input elements. The second critical attribute is that a binary operation must be defined on two elements (or ordered pairs.) Some students perceived this critical attribute (as seen in Table 3). For example, when asked to determine if $\sqrt{a}$ on $\mathbb{Z}$ is a binary operation, one student responded, "No, only a is used, not both inputs." However, it was far more common for students to ignore this critical attribute and only attend to closure (as in the prior example).

Table 4: Survey Responses when Determining if two Given Operations were the same

	Attended to Element-Wise Differences		
	Across Prompts	Element-Defined Prompt Only	Never
Class A	4 (50%)	3 (37.5%)	1 (12.5%)
Class B	1 (8.3%)	8 (66.7%)	3 (25%)
Class C	6 (50%)	4 (33.3%)	2 (16.7%)
Total	11 (34.4%)	15 (46.9%)	6 (18.8%)

Defined element-wise. A particular binary operation is determined by where any two elements are mapped. However, students inconsistently treated this as a critical attribute of binary operation. This became clear on tasks comparing two operations. For example, when provided Cayley tables representing addition modulo 3 and 6 and asked if they were the same binary operation, one student responded they were not because whenever they "plug in 1 and 2 into both these operations [they] will get different results" reflecting attention to individual elements. In contrast, another student claimed that the binary operations were the same because the elements "behave the same." Over half of the students either never attended to element-definedness or only when provided a binary operation already defined that way (see Table 4).

Table 5: Survey Responses when Asked to Create a Binary Operation on {1,2,4}

	Non-symbolic Representation	Symbolic Representation	No Answer
Class A	0 (0.0%)	6 (66.7%)	3 (33.3%)
Class B	1 (8.3%)	4 (33.3%)	7 (58.3%)
Class C	9 (75.0%)	3 (25.0%)	0 (0%)
Total	10 (39.4%)	10 (39.4%)	13 (30.3%)

Symbolic Format. Students across the dataset reflected a preference for symbolic representations. In fact, the most commonly skipped tasks across all prompts where those in

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alternate representations. This preference was most notable in the final prompt asking students to generate a binary operation for set {1,2,4}. As seen in table 5, well over half of the students did not create a binary operation. Instead they often attempted to build a symbolic rule without any success. This desire for a symbolic rule is well documented in terms of functions, and it seems that students perceive a similar attribute as critical to binary operation.

Table 6: Survey Responses when Treating Associativity of ½(a+b) binary operations

	Focused on EOE Portion(s) of Operation (addition/multiplication)	Focused on Entire Operation
Class A	4 (44.4%)	5 (55.6%)
Class B	5 (41.2%)	7 (58.3%)
Class C	9 (75.0%)	3 (25.0%)
Total	18 (54.5%)	15 (45.5%)

E-O-E Format. The desire for a symbolic rule manifested further with students' perceiving a particular format as critical. This preference was seen throughout the survey responses but was most prominent when addressing the prompt to determine if ½(a+b) was associative. As seen in table 6, when placed in a situation without this formatting, students often defaulted to focusing on addition and multiplication as separate binary operations rather than holistically. Through the follow-up interviews, students who responded this way indicated that they could not discern the operation "rule" or that the operation was the "plus" portion reflecting the perception that E-O-E symbolism was critical.

Discussion and Conclusion

In this paper, we have addressed a variety of ways in which students conceive of binary operations in terms of critical attributes and identified metaphors they might leverage when working on tasks with binary operations. In particular, we found three metaphors that were commonly enacted by students: the function metaphor, the arithmetic metaphor, and the structure metaphor. Such metaphors can serve to be productive when they are robust and developed. However, that requires students to also have perceived the critical attributes of binary operation that can be structured and communicated via the metaphors. When leveraging a function metaphor, students may naturally attend to function attributes (such as the need for inputs), and not perceive a binary operation specific attribute (needing exactly two inputs). Similarly, students with an arithmetic metaphor may generalize from arithmetic examples and perceive attributes like E-O-E formatting as critical for general binary operations. As variation theorists have noted, variations are perceived not just synchronically, but also "by remembering earlier related experiences (diachronic simultaneity)" (Vikström, 2009, p. 212). As students structure their knowledge using metaphors, they are likely integrating attributes perceived from many prior experiences related to function and arithmetic in their earlier mathematics experiences.

While the links to function and arithmetic are immediate, the structure metaphor's link to prior experience is a less direct. Notably, the structure metaphor is a dominant metaphor for determining when mathematical structures are isomorphic. However, this is a new treatment at the undergraduate level. Yet many structural treatments exist throughout education such as finding patterns when adding even numbers, or connecting operations like multiplication and division. In fact, this played out robustly in our data. Many students identified multiplication and division as the same operation because of their similar behavior. A structural metaphor can support attending to important varying features (such as a structural property), but can also limit students' attention to the critical attribute of binary operations being element-defined.

These results have several implications. First, binary operations are not a trivial despite instructors classifying it as such. Instructors may wish to increase attention to binary operation in classes with this focus (such as abstract algebra and linear algebra.) Second, students may not perceive many critical attributes such as the need for two elements, and may perceive non-critical attributes as critical such as E-O-E formatting. From a variation standpoint, a potential solution is to expose students to examples, non-examples, and non-standard examples so that they have opportunities to contrast instantiations with varying critical and non-critical attributes. This type of activity can foreground these features so that students may perceive them. From a research standpoint, we may want to consider the degree to which these three metaphors, and attention to specific attributes may underlie student activity across the K-16 spectrum as they engage with common binary operations such as number arithmetic or function composition.

References

Brown, A., DeVries, D. J., Dubinsky, E., & Thomas, K. (1997). Learning binary operations, groups, and subgroups. *The Journal of Mathematical Behavior*, 16(3), 187-239.

Dahlin, B. (2001). The primacy of cognition–or of perception? A phenomenological critique of the theoretical bases of science education. *Science & Education*, 10(5), 453-475.

Dubinsky, E., Dautermann, J., Leron, U., & Zazkis, R. (1994). On learning fundamental concepts of group theory. *Educational Studies in Mathematics*, 27(3), 267-305.

Ehmke, T., Pesonen, M., & Haapasalo, L. (2011). Assessment of university students' understanding of abstract binary operations. *Nordic Studies in Mathematics Education (NOMAD)*, 15(4), 25-40.

Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity, and flexibility: A "proceptual" view of simple arithmetic. *Journal for Research in Mathematics Educatio*n, 25(2), 116-140.

Larsen, S. (2009). Reinventing the concepts of group and isomorphism: The case of Jessica and Sandra. *The Journal of Mathematical Behavior*, 28(2), 119-137.

Marton, F., & Booth, S. A. (1997). *Learning and awareness*. Psychology Press.

Melhuish, K. & Fagan, J. (Accepted / In Press). Connecting Group Theory Concept Assessment Results to Core Concepts at the Secondary Level. In N. Wasserman (Ed.), *Connecting abstract algebra to secondary mathematics, for secondary mathematics teachers (Research in Mathematics Education Series)*. Netherlands: Springer.

Melhuish, K, & Fasteen, J. (2016). Results from the Group Concept Inventory: Exploring the Role of Binary Operation in Introductory Group Theory Task Performance. In T. Fukawa-Connelly, N. Infante, M. Wawro, & S. Brown (Eds.), *19th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 1098–1103). Pittsburgh, PA.

Novotná, J., Stehlíková, N., & Hoch, M. (2006). Structure sense for university algebra. Paper presented at the Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education.

Oehrtman, M., Carlson, M., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. Making the Connection: Research and Teaching in Undergraduate Mathematics Education, 27, 42.

Slavit, D. (1998). The role of operation sense in transitions from arithmetic to algebraic thought. *Educational Studies in Mathematics,* 37(3), 251-274.

Tall, D., McGowen, M., & DeMarois, P. (2000). The function machine as a cognitive root for the function concept.

Trigwell, K. (2006). Phenomenography: An approach to research into geography education. *Journal of geography in higher education*, 30(2), 367-372.

Wasserman, N. H. (2017). Making Sense of Abstract Algebra: Exploring Secondary Teachers' Understandings of Inverse Functions in Relation to Its Group Structure. *Mathematical Thinking and Learning*, 19(3), 181-201.

Vikström, A. (2008). What is intended, what is realized, and what is learned? Teaching and learning biology in the primary school classroom. *Journal of Science Teacher Education*, 19(3), 211-233.

Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education*, 356-366.

Zandieh, M., Ellis, J., & Rasmussen, C. (2017). A characterization of a unified notion of mathematical function: the case of high school function and linear transformation. *Educational Studies in Mathematics*, 95(1), 21-38.

Zaslavsky, O., & Peled, I. (1996). Inhibiting factors in generating examples by mathematics teachers and student teachers: The case of binary operation. Journal for Research in Mathematics Education, 67-78.

Hodges, T.E., Roy, G. J., & Tyminski, A. M. (Eds.). (2018). *Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Greenville, SC: University of South Carolina & Clemson University.