# A FINE-GRAINED ANALYSIS OF PROOF SUMMARIES: A CASE STUDY OF ABSTRACT ALGEBRA STUDENTS

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*In this paper I explore eleven undergraduate students' comprehension of a proof taken from an undergraduate abstract algebra course. My interpretation of what it means to understand a proof is based on a proof comprehension model developed by Mejia-Ramos, et al. (2012). This study in particular examines the extent to which undergraduate students are able to summarize a proof using the proof's higher-level ideas. Additionally, eleven doctoral students in mathematics were asked to provide a summary of the same proof that the undergraduate students received. Undergraduates' holistic comprehension of the proof was then analyzed in light of summaries that the doctoral students provided. The main finding of the study is that undergraduates' comprehension of the proof was overall inadequate—notably, they demonstrated limited skills in summarizing a proof via the proof's key ideas. Moreover, undergraduates failed to recognize the scope of the method used in the proof.*

*Key words: Proof, Proof Comprehension, Abstract Algebra.*

In advanced undergraduate mathematics courses, students are expected to spend a significant portion of their time reading proofs. Although proof comprehension is a fundamental aspect of undergraduate mathematics education, studies by Conradie and Frith (2000), Rowland (2002) and Weber (2012) suggest that mathematicians rarely measure their students' comprehension of proofs. For example, Weber (2012) reports that mathematicians assess their students' comprehension of proofs by asking students either to reproduce the proof or prove a similar or a trivial consequence of proofs they presented in class. However, Conradie and Frith (2000) suggests that asking students to reproduce a proof may not be a pedagogically useful way of assessing students' understanding of a proof because students can and do correctly reproduce a proof by simply memorizing it word for word, with virtually no understanding at all.

Despite its importance in undergraduate mathematics education, research on undergraduates' comprehension of proofs is limited. In fact, much of the proof literature focuses on students' aptitude to construct or validate proofs and less on their ability to comprehend proofs (Mejia-Ramos et al., 2012; Mejia-Ramos & Inglis, 2009). Mejia-Ramos and his colleagues (2009) systematically investigated a sample of 131 studies on proofs and they found that only three studies focused on proof comprehension. They hypothesize that the scarcity of the literature on proof comprehension is perhaps due to the lack of a model on what it means for an undergraduate student to understand a proof. In this study, I adopt an assessment model for proof comprehension that was developed by Mejia-Ramos, et al. (2012) to explore undergraduates' comprehension of proofs. In particular, this study seeks to address the following research questions.

To what extent do undergraduates comprehend a proof? More specifically, to what extent do undergraduates:

- summarize a proof using its high-level ideas,
- recognize and appreciate the scope of a method used in a proof?

## Theory: Assessment Model for Proof Comprehension

Mejia-Ramos, et al. (2012) proposed that one can assess undergraduates' comprehension of a proof along seven facets. These seven facets are organized into two overarching categories: local and holistic. A local understanding of a proof is an understanding that a student can gain "either by studying a specific statement in the proof or how that statement relates to a small number of other statements within the proof" (p.5). Alternatively, undergraduates can develop a *holistic* comprehension of a proof by attending to the main ideas of the proof. Below, I will elaborate on what it means to understand a proof holistically.

## Assessing the Holistic Comprehension of a Proof

According to the proof comprehension model, the holistic understanding of a proof consists of being able to: (1) summarize the proof using the proof's main ideas, (2) identify the modular structure of the proof, (3) recognize and extend the method used in the proof, and (4) illustrate the method of the proof using a specific example or diagram. They developed these four facets of holistic comprehension of proofs based on: (a) mathematicians' perspectives on how and why they read and present proofs and what it means for them to understand a proof; (b) the proof literature on the role of proof; and (c) the recommendations by mathematicians and mathematics educators on proof presentations that would presumably improve students' proof comprehension. Below, I elaborate on (1) and (3).

**Summarizing a proof via its high-level ideas**. Mejia-Ramos, et al. (2012) state that "one way that a proof can be understood is in terms of the overarching approach that is used within a proof" (p.11)." Being able to summarize a proof via its high-level ideas entails understanding the proof's "top-level overview" or "big idea". A good summary of a proof may include what Raman (2003) describes as a proof's *key ideas.* Raman (2003) defines *key ideas* as "heuristic ideas which one can map to a formal proof with appropriate sense of rigor" (p. 323). *Key ideas* provide "a sense of understanding and conviction why a particular claim is true" (Raman, 2003,

323). Mathematicians can evaluate their students' understanding of this aspect of a proof in at least two ways. They can, for instance, directly ask students to provide a brief summary of the proof that includes the proof's higher-level ideas. Alternatively, they can provide students with a few summaries of the proof and ask them to choose which summary best captures the main ideas of the proof (Mejia-Ramos et al., 2012). In this study, I asked undergraduates to provide a proof summary using the proof's main ideas.

**Transferring the general ideas or methods to another context**. Mejia-Ramos, et al. (2012) suggested that identifying the scope of a method or technique used in a proof is an important aspect of proof comprehension. Mathematicians can assess this aspect of proof comprehension by asking students to (a) identify methods or techniques without which the proof would have collapsed, or (b) prove new claims by applying methods similar to those used in the original proof. In this study, undergraduates were asked questions to elicit their understanding of the scope of a method used in a proof.

## Review of the Literature

As noted earlier, educational research on proof comprehension in undergraduate mathematics has received little emphasis. Osterholm (2006) was among the first to look into student's comprehension of mathematical texts. He conducted a quantitative study of reading comprehension of abstract algebra students (he compared texts with one including symbols and another one not). He concludes that "mathematics itself is not the most dominant aspect affecting the reading comprehension process, but the use of symbols in the text is a more relevant factor"

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(p.325). His contention is based on the fact that the group of students who had almost no symbols and notations in their reading assignment outperformed, in a reading comprehension test, those whose reading assignments involved mathematical symbols and notations. Although Osterholm (2006) does point to difficulties student encounter while reading mathematical texts, it should be noted that his study only asked students to read mathematical texts and not proofs specifically.

Research also suggests that undergraduates are not successful in gleaning understanding from the proof they see during lecture (Selden & Selden, 2012; Lew et al, 2015). For example, students interviewed in Lew et al.'s (2015) study fail to comprehend much of the content the instructor desired to convey, including the method used in the proof. Students interviewed in Selden and Selden's (2012) study also failed to understand a proof holistically since they were fixated on verifying each line and put little emphasis in attending to the overarching methods used in the proof. One purpose of this study is to build on the growing body of research on proof comprehension.

## Methods

### Research Settings

This study took place in a large public university in the northeastern United States. The content of the proof used in this study come from an introductory abstract algebra course. In the chosen research setting the standard textbook used is *Abstract Algebra: An introduction* by Hungerford (2012). The goal of the course (as stated in the syllabus) is to introduce students to the theory of algebraic structures such as rings, fields, and groups in that order.

### Participants

**Undergraduate student participants.** Since the main purpose of this study is to explore undergraduates' comprehension of proofs—in particular, proofs that appear in an introductory abstract algebra course—I personally approached undergraduates who had taken or were enrolled in an introductory abstract algebra course. Eleven undergraduates agreed to participate in this study and were assigned pseudonyms S1-S11. At the time of the study, six of the eleven undergraduate participants (S3, S5, S6, S7, S8, and S9) were enrolled in an introductory abstract algebra course. Seven participants—S1, S2, S3, S5, S6, S7, S8, and S9—were pursuing a major in secondary mathematics education and said they intended to be high school mathematics teacher. The remaining four students were mathematics majors. Furthermore, each participant had taken a minimum of three proof-based courses and all but three (S1, S3, and S6) said they received an A or B in their introduction to proof course. Participants' responses on a background survey suggest that each participant spent at least two hours per week reading proofs outside of class.

**Doctoral students.** Eleven doctoral students at the aforementioned research site agreed to participate in this study. I used doctoral students to analyze undergraduates' summaries of a proof. At various times, I asked the doctoral students to provide, in writing, a summary of a proof using the proof's key or main ideas. To avoid confusion, in the remainder of this paper I will refer to these doctoral student participants as experts.

## Materials and Research Procedures

In this study undergraduates were asked to read a proof that shows that any finite integral domain is a field. This proof is given in appendix 1. I chose this proof because (a) it nicely draws connection between two important topics covered in abstract algebra: integral domains and fields, and (b) it uses a proof technique that I speculated most participants probably have not seen.

The principles I employed to write this pedagogical proof—a proof that is geared toward undergraduates for the purpose of pedagogy—is based on the Lai, Weber, & Mejia-Ramos (2012) study. Lai et al. (2012) report that mathematicians valued pedagogical proofs that (1) made assumptions and conclusions of the proof explicit, (2) centered on important equations to emphasize the main ideas, and (3) did not contain "true but irrelevant statements." (p.94). Also, when writing the proof, I consulted mathematics professors and made appropriate modifications.

Participants were asked to read the proof until they felt they understood it and were encouraged to write and/or highlight on the proof paper as well as to think out loud while reading. Once a participant finished reading the proof, she/he was asked to:

- to provide a good summary of the proof including the proof's main or higher-level ideas
- to indicate assertion(s) in the proof that would fail if $\boldsymbol{R}$ was infinite.

**Analysis**

Recall that eleven doctoral students in mathematics were asked to write a summary of the proof using what they think are the main ideas of the proof. Doctoral students were also asked to describe the key ideas of each proof. First, I carefully studied doctoral students' summaries of the proof. I then developed a synthesized summary for the proof. This synthesized summary, which will hereafter be referred to as the expert's summary, also incorporated all the key ideas that doctoral students described for each proof. That process resulted in the following summary of the proof.

> The proof shows that a finite integral domain $R$ is a field by showing that any non-zero element of $R$ has a multiplicative inverse. Let $a$ be any non-zero element of $R$. The absence of zero divisor in R taken together with the fact that R is finite gives us that left multiplication by a defines a bijective map from the integral domain to itself ($f_a: R \to R$ given by $f_a(x) = ax$ is a bijective). Surjectivity of this map guarantees that $a$ has a multiplicative inverse.

The key ideas identified in the expert's summary of the proof are:

- Overarching method: given an arbitrary non-zero element $a \in R$, show that there exists $b \in R$ such that $ab = 1_R$
- Approach: define a left multiplication by a non-zero element $a \in R, f_a: R \to R, x \mapsto ax$. Using kernel of $f_a$ one can show that it is injective.
- The finiteness of $R$ and the injectivity of $f_a$ to show that $f_a$ is a surjective map from $R$ to $R$.

The above expert's summary of the proof was eventually verified by two experienced researchers in mathematics education as to whether or not it indeed incorporated all the *key ideas* of the proof that doctoral students discussed; modifications were then made to the expert's summary, as needed.

Finally, using a rubric, two researchers, both with a master's degree in mathematics, independently conducted a comparative analysis of undergraduates' summaries of the proof against expert's summaries. When disagreement emerged, w engaged in discussion until a consensus was reached.

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**Results and Discussion Results on Undergraduates' Summaries of the proof**

Recall that undergraduates' summaries of the proof were analyzed in comparison to the expert's using a rubric that is omitted here for a shortage of space. Nearly all participants, nine out of eleven, provided a summary of the proof that suggested a limited proof comprehension. In particular, their responses indicated that they either *poorly* or *very poorly* understood the proof. The results of students' summaries of the proof is given in Table 1. Note, in table 1, that a majority, six undergraduates, provided a *very poor* summary of the proof, which implies that their summary failed to highlight the main ideas of the proof that was described in the expert's summary.

**Table 1: Undergraduate students' summaries of the proof**

| Evaluation | Undergraduate students |
|---|---|
| Very poor | S3, S5, S6, S8, S9, S11 |
| Poor | S1, S2 |
| Satisfactory | S4, S7, S10 |
| Good | None |

In order to give you a sense of a *very poor summary*, I will provide a detailed description of summaries given by S5, S8, and S9. When asked to provide a good summary of the proof including the proof's main ideas, S5 offered the following summary:

> $R$ is an integral domain and field. It is both surjective and injective to the kernel of the function that defines it. There is also $1_R \in R$ that allows it to have multiplicative inverse, thus units. I know a field's non-zero elements all make units, a field.

S5's summary consists of incomplete sentences and phrases that are either mathematically incorrect or appear to have been copied from the proof word for word. For instance, S5 begins by supposing the thing that needs to be shown—that $R$ is a field. Also, there is evidence of a misunderstanding about injectivity and surjectivity of a map. This is evident when he asserts that $R$, as opposed to $f_0$, is both surjective and injective. Finally, S5's summary doesn't mention how crucial assumptions in the proof such as the finiteness of $R$ are used in the proof. Thus, S5's summary is deemed to be *very poor*, which means it was considered to be very different from the expert's as it did not highlight main ideas of the proof. S8, likewise, provides a *very poor summary*:

> The proof basically gave a less textbook traditional explanation of a way to prove that all finite integral domains are fields. It used the understandings of kernals [sic], bijections, injection, surjections in order to prove facts about rings, where usually you learn the facts about rings before being introduced to functions

Her summary above makes no mention of the proof's high-level or *key ideas* and thus does not suggest a satisfactory comprehension of the proof. While she enumerates topics that are used in the proof, her summary does not illustrate how they were employed in the proof. S8's summary, for example, indicates that the concept of bijectivity is employed in the proof, but she does not explain how it is used. Along the same lines as S5, and S8, S9 also supplied a summary of the proof that did not suggest a satisfactory comprehension of the proof. For example, S9 provided the following summary:

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> Given a finite integral domain, you can prove that it is a field by showing it has a multiplicative inverse, no zero divisors, injective and surjective, kernels, and if there is a multiplicative inverse such that $ax = 1_R$ and $a \neq 1_R$ then $R$ is a field.

S9's summary above says very little beyond restating the claim. S9 essentially repeats phrases that appeared in the proof verbatim. He does not draw any connection between *key ideas* described in the expert's summary of the proof. Also, S9's summary states that *R* is first shown not to have zero divisors, but this is neither necessary nor true; *R* is assumed to be an integral domain and therefore it does not have zero divisors. Overall, S9's summary fails to mention the proof's key ideas and consequently shows limited comprehension of the proof.

While the majority, six out of eleven students, provided what is considered to be a *very poor summary* of the proof, S1 and S2 supplied a *poor summary* of the proof. S2, for example, provided the following summary of the proof:

> The aim of the proof is to show that if $R$ is a finite integral domain, then $R$ is a field. It then wants to show that $R$ has a multiplicative inverse, then that the kernel of $f_a: x \to ax$ to be trivial. The prof then shows that $x = 0$ since if $ax = 0$ then $a$ or $x$ is 0 but $a$ is not, thus $\ker f_a = \{0\}$ so $f_a$ is injective. Thus, $|R| = |f_a(R)|$, which shows it is surjective. It then proves $a$ has a multiplicative inverse, so $R$ is a field.

S2's summary is incoherent and appears to duplicate some parts of the proof word for word. Moreover, her summary includes way too much unnecessary information; for example, she repeats the argument for the triviality of $f_0$. While S2's summary does mention some *key ideas* that are noted in the expert's summary, it doesn't make the logical connection between those *ideas*. In fact, S2 appears to have the logic of the proof backward, as she seems to think the existence of multiplicative inverse is what guarantees the triviality of the kernel of $f_0$.

While no one provided a *good summary* of the proof, three students—S4, S7, and S10—provided a *satisfactory summary* of the proof. S7, for instance, summarized the proof as follows:

> First it is important to show each nonzero element of $R$ has a multiplicative inverse. Then we consider a nonzero element $a \in R$ and the map of $f_a$. We use the kernel of $f_a$ to prove $f_a$ is injective. Then from the fact that $f_a$ is injective and therefore $|R| = |f_a(R)|$, $f_a$ is also surjective. Finally, we show $f_a(x) = 1_R$ hence $a$ has a multiplicative inverse and therefore $R$ is a field.

Evidently, S7's summary above has significant resemblance to the expert's summary of the proof. In particular, S7 does mention some *key ideas* of the proof. However, she did not indicate the fact that the surjectivity of $f_0$ depends on the finiteness of *R*, which was a crucial idea that was noted in the expert's proof. Furthermore, the last line of her summary is incorrect in the sense that $f_a$ is not identically equal to the identity $1_R$. Also, based on what is stated at the very end of her summary, S7 does not seem to have understood how $a$ (the fixed nonzero element) has a multiplicative inverse. However, S7's summary overall does bear some resemblance to the expert's summary and suggests a satisfactory understanding of the proof. On the whole, while nearly all students provided a *poor* or a *very poor* summary of proof, no one provided a *good summary.* Indeed, only S4, S7, and S10 managed to provide a satisfactory summary of the proof.

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**Results on Undergraduates' Ability to Recognize or Appreciate the Scope of a Proof's Method**

No undergraduate demonstrated why $\boldsymbol{R}$ must be finite for the proof to be valid. When asked why the method of the proof would fail if *R* was assumed to be infinite, six out of eleven undergraduates offered no response or said "I don't know…" The other five students pointed incorrectly to an assertion that would fail if *R* is infinite. Table 2 below illustrates the various responses they provided:

**Table 2 Reasons undergraduates provided for why R must be finite**

| Reason why *R* must be finite | Participants |
|---|---|
| *R* would not be commutative (line two) | S1 |
| There wouldn't be exactly be the same number of | S3 elements in each |
| $f_a$ would not be injective (line seven) | S2, S8, S5 |
| No response/I don't know/Not sure | S4, S6, S7, S9, S10, S11 |

Table 2 shows that participants failed to pinpoint a specific assertion in the proof that would fail—the argument in line eight—if *R* was infinite. That is to say, if *R* is infinite, $f_a(R) \subseteq R$ and $|R| = |f_a(R)|$ taken together would not guarantee that $f_a(R) = R$, which would not make $f_a$ surjective.

To summarize, Mejia-Ramos and colleagues (2012) maintain that being able to provide a good summary of a proof is a key indicator of comprehension. However, undergraduates in this study showed a limited comprehension of the proof. In particular, in their proof summary undergraduates failed to highlight the proof's main idea. For a large number of participants, their proof summary was essentially a replica of a few sentences that appeared in the proof. Further research is needed to identify why undergraduates fail to summarize a proof using the proof's main ideas.

## Appendix 1

**Claim**: Let *R* be a finite integral domain. Then *R* is a field.

*Proof.*

1. Let *R* be a finite integral domain whose multiplicative identity is $1_R$ and whose additive identity is $0_R$.
2. Since *R* is a commutative ring, it suffices to show that every nonzero element in *R* has a multiplicative inverse.
3. Let *a* be a fixed nonzero element of $R$ $(a \neq 0_R)$. Consider the map $f_a: R \rightarrow R$ defined by $f_a: x \rightarrow ax$. We first show that the kernel of $f_a$ is trivial.
4. Note that kernel of $f_a = \{x \in R: f_a(x) = 0_R\} = \{x \in R: ax = 0_R\}$.
5. Since R has no proper zero divisors, $ax = 0_R \Longrightarrow a = 0_R$ or $x = 0_R$. But, $a \neq 0_R$ thus $x = 0_R$.
6. Therefore kernel of $f_a = \{0_R\}$ and so $f_a$ is injective.

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7. Next, note that $|R| \geq |f_a(R)|$. Since $f_a$ is injective, it follows that $|R| = |f_a(R)|$.
8. Because $f_a(R) \subseteq R$ and $|R| = |f_a(R)|$, we have that $f_a$ is surjective.

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