Understanding Strategy Change:

Contextual, Individual, and Metacognitive Factors

Martha W. Alibali

University of Wisconsin – Madison

Sarah A. Brown

Boston University

David Menendez

University of Wisconsin – Madison

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Abstract

Learning, development, and instruction often involve changes in the strategies that learners use to solve problems. In this article, our focus is on mathematical problem solving in both children and adults. We offer a selective review of research on three classes of factors that may influence processes of strategy change in mathematical problem solving: contextual factors, individual factors and metacognitive factors. *Contextual factors* involve information that learners encounter in the learning context, such as feedback about prior strategies and examples of alternative strategies. *Individual factors* involve the abilities, dispositions, and knowledge that learners bring to the learning context. *Metacognitive factors* involve knowledge *about* strategies and factors that affect the application of strategies—including perceptions of problem difficulty, confidence in the strategies one already knows, and judgments about qualities of alternative strategies. These factors operate both independently and in combination to influence learners' behavior. Therefore, we argue that scientific progress in understanding strategy change will require comprehensive conceptual models that specify how different factors come together to explain behavior. We discuss several such models, including vulnerability-trigger models, cumulative risk models, and dynamic systems models. We argue that each of these types of models has insights to offer, and that research guided by such models will contribute to greater progress in understanding processes of strategy use and strategy change.

Keywords: strategy use, strategy change, feedback, individual differences, metacognition, mathematics

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I. Cognitive development as change in strategies

Why do learners change their approaches to solving problems? And how does strategy change occur? These questions are important ones for understanding learning, development, conceptual change, the acquisition of expertise, and students' responses to instruction (see, e.g., Lemaire, 2017; Siegler, 1996). However, these questions are also vexing ones, for several reasons. First, there are many dimensions of strategy change to be explained, including the acquisition of new strategies, the adaptiveness with which people choose among alternative strategies, and the breadth and efficiency with which people apply particular strategies (Lemaire & Siegler, 1995; Siegler, 2000). Second, patterns of strategy use and strategy change are often highly variable across individuals. Third, there are many factors that are potentially involved in processes of strategy use and strategy change.

With growth, learning, experience, and the development of expertise, learners change their strategies in a wide range of problem domains, including locomotion (e.g., Adolph, Vereijken, & Denny, 1998), memory (e.g., Schneider, Kron-Sperl, & Hunnerkopf, 2009), scientific reasoning (e.g., Boncoddo, Dixon, & Kelley, 2010), and mathematics (e.g., Fazio, DeWolf, & Siegler, 2016; van der Ven, Boom, Kroesbergen, & Leseman, 2012). In this article, our focus is on mathematical problem solving in both children and adults. We consider several classes of factors that are implicated in processes of strategy change in mathematical problem solving, and we consider several possible conceptual models for how such factors can be integrated to account for strategy change.

Factors that are implicated in strategy change can be grouped into classes of factors that may operate in similar ways. In this article, we focus on three broad classes of factors: *Contextual factors* involve information that people encounter in the learning context—including feedback about prior strategies, examples of alternative strategies, and so forth. *Individual factors* involve the abilities, dispositions, and knowledge that people bring to the learning context, which may influence their likelihood of taking in new information or their responsiveness to other factors that promote or prevent change. *Metacognitive factors* involve knowledge *about* strategies and factors that affect the application of strategies—including perceptions of problem difficulty, confidence in the strategies one already knows, and judgments about qualities of strategies, such as their appropriateness and efficiency. Of course, these three classes of factors are not the only factors that matter for strategy use and strategy change; other factors have also been considered in the literature, such as sequential effects in the learner's history of problem solving (Lemaire, 2016) and features of the social context (Bjorklund, Hubertz, & Reubens, 2016).

Many studies of strategy change focus on only a single factor or a small number of factors that may influence change, for example, the provision of feedback about current strategies. Such studies typically take an experimental approach, varying that one factor to determine whether different levels of the factor lead to different patterns of change. Such studies provide valuable information about factors that *can* cause strategy change in particular contexts; however, patterns of strategy change in authentic, real-world settings are undoubtedly more complex, with many factors operating simultaneously.

In this article, we offer a selective review of research on three classes of factors that may influence processes of strategy change in the domain of mathematics—contextual factors, individual factors and metacognitive factors. We then discuss several different conceptual models for integrating multiple factors in more comprehensive models of strategy change.

II. Contextual Factors

Many aspects of context may affect learners' strategy use and strategy change, including aspects of the physical setting, the social setting, and the specific information available to the learner. Here, we limit our consideration of contextual factors to task-relevant information that people encounter in the learning context, though in doing so, we do not wish to imply that other contextual factors are unimportant. Here, we consider four types of contextual factors that may influence strategy change: feedback about current performance, exposure to alternative strategies, information that highlights problem structure, and variations in problem presentation. After discussing each factor, we briefly consider other factors that may moderate its effects.

A. Feedback about performance

Feedback can be defined as information provided by an external agent about one's task performance (see, e.g., Kluger & DeNisi, 1998). Feedback can be provided in a range of ways. It can involve information about the correctness of solutions or about the appropriateness of strategies. It can be provided on a problem-by-problem basis or in summary form after a set of problems. It can be provided by another person, such as a teacher or a peer, or by a computer-based agent, such as a software program. Feedback is typically direct (e.g., a teacher marking solutions as correct or incorrect), but in some cases, others' actions can serve as indirect forms of feedback. For example, noting that another person solves a problem using the same strategy you have just used can be a form of indirect feedback that your strategy is likely to be correct (Brown, Menendez, & Alibali, in preparation).

At first blush, it seems that feedback should naturally lead to improvements in performance—such as more frequent correct solutions and more frequent use of optimal strategies. However, the literature yields a much more complicated picture, which reveals that the effects of feedback on strategy change and learning are not uniformly positive (e.g., Hattie & Timperley, 2007; Kluger & DeNisi, 1996). For example, in their meta-analytic review of feedback interventions, Kluger and DeNisi (1996) reported an overall positive effect of 0.4 *SD*, but they also reported a large amount of variability, with approximately one-third of all effect sizes being negative.

Factors that may moderate the effects of feedback. One key factor that moderates the effects of feedback is learners' prior knowledge. Several studies suggest that feedback is often beneficial for learners with low prior knowledge, but harmful for learners with high prior knowledge (Fyfe & Brown, 2017). In one representative study, Fyfe and Rittle-Johnson (2016a) examined the roles of feedback and prior knowledge in children learning to solve mathematical equivalence problems (e.g., 3 + 4 + 6 = 3 + __). They manipulated prior knowledge by teaching some children a strategy for solving the problems at the outset of the study, and they provided some children with feedback about whether their solutions were correct. There were significant interactions of these factors on several outcome measures, including use of correct strategies at posttest. Interestingly, feedback hindered use of correct strategies for students who had prior knowledge of a correct strategy, but it promoted use of correct strategies for students who did not have prior knowledge of a correct strategy.

Other research suggests that the effects of feedback may differ based on the source of that feedback. In studies of children learning to solve mathematical equivalence problems, feedback provided by a human tutor and feedback provided by a computer can have different repercussions for later performance, with computerized feedback being more effective, especially for children with low prior knowledge (Fyfe & Rittle-Johnson, 2016b; 2016a). One potential explanation is that feedback may be more effective when it focuses attention on the task at hand, rather than on the learner (Kluger & DeNisi, 1996).

Metacognitive factors may also moderate the effects of feedback. For example, in a study of children learning about mathematical equivalence, Brown and Alibali (2018) found that the effects of feedback depended on children's confidence in their existing approaches. For children who reported being uncertain about their existing strategies, feedback led to a significant increase in the likelihood of strategy change. In contrast, for children who reported being highly confident in their existing strategies, feedback led to a non-significant decrease in strategy change.

B. Exposure to alternatives

One factor that can provoke strategy change is learning about alternatives to one's existing approach. Exposure to new strategies can occur in a variety of ways. Learners can be exposed to new ideas through direct instruction, through observing others' problem solving, or through discussions with others. Direct instruction often involves introducing new strategies, sometimes with explicit instructions to use the new strategies in particular situations. Not surprisingly, this sort of exposure to new approaches often leads to strategy change (Alibali, 1999; Star & Rittle-Johnson, 2008). Similarly, instruction with worked examples typically involves showing learners a new method via demonstration or by providing a written example that shows problem solving steps. This type of exposure also often leads to strategy change; learners who study worked examples often adopt the strategy used in the examples, and they tend to perform better on posttest assessments than learners who spend comparable amounts of time simply solving additional problems (Calin-Jageman & Ratner, 2005; Matthews & Rittle-Johnson, 2009; Siegler, 2002).

Even simple exposure to alternative strategies, without any information about whether those strategies are correct or incorrect, can also provoke strategy change (Brown & Alibali, 2018; Brown, et al., in prep). Thus, exposure to new strategies can lead to change, even outside of the context of direct instruction, and even when there are minimal or no cues to the validity of the new strategies.

Factors that may moderate the effects of exposure to alternative strategies. Although exposure to alternative strategies is a powerful driver of strategy change, learners will not readily adopt just any strategy. Once they have started to use correct strategies, learners are unlikely to adopt novel incorrect strategies to which they are exposed (Brown & Alibali, 2018). Even when considering only correct strategies, learners are more likely to adopt some strategies than others (Brown et al., in prep), for reasons that are not yet fully understood. Children's metacognitive judgments about the qualities of strategies (such as their complexity and their efficiency) may be part of the reason.

The source of the novel strategies may also matter. Children are less likely to adopt a novel strategy when it is presented as linked to a specific person (e.g., "Molly's strategy") than when it is presented impersonally (e.g., "the continuous strategy") (Riggs, Alibali, & Kalish, 2015; 2017). It seems likely that children may be more likely to adopt a novel strategy demonstrated by a teacher than one offered by a peer; however, empirical data on this point are lacking.

C. Information that highlights problem structure

Many studies have documented close ties between how learners encode problems and the strategies they use to solve problems (e.g., McNeil & Alibali, 2004; Rittle-Johnson, Siegler, & Alibali, 2001; Siegler, 1976). In light of these relations, it stands to reason that contextual factors that highlight the structure of problems will support accurate problem encoding and thereby affect processes of strategy use and strategy change. Many factors can influence attention to problem structure, including activating relevant prior knowledge (Crooks & Alibali, 2013) and providing environmental supports that highlight problem structure, for example, via color coding (e.g., Joh & Spivey, 2012) or via gestures (e.g., Yeo, Cook, Nathan, Popescu & Alibali, 2018).

Several studies have shown that support for encoding the perceptual structure of problems can promote generation of new problem-solving strategies. In mathematical equivalence problems (e.g., 3 + 4 + 5 = 3 + __), children often fail to accurately encode the position of the equal sign (McNeil & Alibali, 2004). One study supported children's encoding of the equal sign by using a contrasting color to highlight the equal sign during a feedback intervention (Alibali, Crooks, & McNeil, 2017). Some students solved problems in which the equal sign was printed in red ink, whereas other students solved problems in which the equal sign was presented in ordinary black ink; all children also received feedback that their prior strategies were incorrect. Students in the red equal sign condition were more likely to generate correct strategies for solving equivalence problems in an immediate posttest and at a follow-up test approximately four weeks later.

In another study of children learning to solve mathematical equivalence problems, one intervention involved drawing an analogy between the equations and a teeter-totter (or see-saw), both verbally and with a schematic drawing of a teeter-totter under the problem, with the fulcrum at the equal sign (Alibali, 1999). The teeter-totter analogy supported children's encoding of two problem features that children often have difficulty encoding—the position of the equal sign and the sides of the problem (see Rittle-Johnson & Alibali, 1999). This intervention, in combination with feedback, led to greater use of correct strategies on a posttest and transfer test than an intervention that involved feedback alone.

Other techniques that highlight problem structure may also scaffold performance by supporting accurate problem encoding. Indeed, the beneficial effects of schematic diagrams on reasoning and problem solving, which have been well documented (Butcher, 2006; Cooper, Sidney, & Alibali, 2018), may be due, at least in part, to diagrams supporting accurate encoding of the perceptual structure of problems. Likewise, the beneficial effects of labels on information presented in visual representations may be due to the labels supporting accurate encoding—an interpretation supported by data on adults' eye movements to tables of values presented with and without labels (Clinton, Morsanyi, Alibali, & Nathan, 2016).

Factors that may moderate the effects of highlighting problem structure. Although scaffolds that highlight the problem structure are helpful for many learners, the extent of their beneficial effects may not be uniform across learners. Indeed, in one study of undergraduates who solved trigonometry problems with different types of visual representations, Cooper and colleagues (Cooper, Sidney & Alibali, 2018) found that schematic diagrams were beneficial overall, but the strength of their beneficial effect varied depending on how much participants valued mathematics. Participants who had more positive attitudes about the value of math reaped greater benefits from the presence of schematic diagrams than did participants who had less positive attitudes.

Other individual differences may also moderate the effects of diagrams on problem solving performance. For example, in an algebraic symbolization task, Bartel and Alibali (2018) found that schematic diagrams were more beneficial for individuals with low mathematics ability and for those with poor visuospatial skills. Along similar lines, Booth and Koedinger (2012) found that diagrams supported algebra problem solving among seventh- and eighth-grade students, but not among sixth-grade students, suggesting that students may need to develop skills for interpreting and using diagrams in order for them to be beneficial. More generally, the effects of manipulations that highlight problem structure seem to vary depending on students' abilities to discern that structure on their own.

D. Problem presentation

Environmental supports that highlight problem structure (such as the red equal sign in equivalence problems or schematic diagrams for algebraic expressions) are a form of *scaffolding* that helps students to generate and use appropriate strategies. Supporting learners' attention to problem structure is one form of scaffolding; other techniques may scaffold learners' use of correct strategies via other mechanisms, such as by helping learners manage memory demands or by protecting learners from particular sorts of errors.

Different ways of presenting problem information make different sorts of demands on long-term and working memory. For example, Koedinger, Alibali, and Nathan (2008) investigated students' solving of algebraic problems presented either in equation format or in story format. For simpler problems, they found that students performed better on story problems than on corresponding equations—which they termed a "verbal advantage" (see also Koedinger & Nathan, 2004). Based on analysis of students' strategies and errors, they argued that the more grounded, story format reduced demands on long-term memory for recalling the meanings of the algebraic symbols, because the words and scenarios used in the stories were familiar and evoked everyday experiences, such as sharing the cost of an item among several people. Koedinger et al. (2008) also argued that the story format protected students from making errors by providing redundant cues to meanings. As evidence, they showed that students made fewer errors on story problems than on corresponding equations, even when they generated those very equations to solve the story problems. Thus, for simpler problems, presentation formats that made fewer long-term memory demands and that provided redundancy in processing afforded better performance.

Studies of other sorts of problems have demonstrated similar advantages for problems presented in grounded formats, relative to more abstract formats. One classic study involved the Wason selection task, in which participants are given a rule of the form "if *p*, then *q*" and a set of exemplars (*p*, *not p*, *q*, and *not q*) and asked to determine which exemplars need to be checked in order to determine whether the rule is true. For example, participants might be given the rule "If there is an *A* on one side of the card, then there is a *3* on the other side," along with exemplars *A*, *B*, *2*, and *3*, or they might be given a grounded version of the same problem: the rule "If a person is drinking beer, then the person must be over 19 years of age" along with exemplars *drinking a beer*, *drinking a Coke*, *16 years of age* and *22 years of age*. Griggs and Cox (1982) demonstrated that people used more accurate strategies to solve the task when it was presented in the familiar, everyday context. Cox and Griggs (1982) argued that this enhanced performance was due to people recalling relevant past experiences with the problem content, the relation expressed, and potential counterexamples (e.g., someone under age 19 drinking beer)—an interpretation that aligns with the idea that familiar content makes fewer memory demands.

Another study involved undergraduate students' interpretations of contingency tables, such as that presented in Figure 1 (Osterhaus, Magee, Saffran & Alibali, in press). Students were more likely to use the accurate and appropriate *conditional probabilities* strategy for problems presented in a grounded context (i.e., whether people were more likely to show improvement in their blood pressure after taking or not taking a particular medicine) than for problems presented in an abstract context (i.e., whether output Y was more likely, given input X or not given input X). Osterhaus and colleagues suggested that, by activating knowledge of familiar situations (e.g., situations in which someone took medicine but did not get well), the grounded context may have supported participants in allocating attention to cells in the table in which the effect does not occur, enabling them to apply a strategy that required integrating information from all four cells.

Insert Figure 1 about here

Factors that may moderate the effects of problem presentation. Despite the many studies demonstrating advantages for grounded representations over abstract representations, grounded representations are not always better. Some evidence suggests that the beneficial effects of grounded representations hold true only for simpler problems. In their study of algebra problem solving, Koedinger, and colleagues (2008) found that for more complex problems (ones that involved multiple references to the unknown), the data pattern was reversed: students performed better on equations than on corresponding story problems—which they termed a "symbolic advantage". They argued that for complex problems, the working memory demands of manipulating quantities were less when manipulating abstract symbols (such as $0.15x$) than when manipulating grounded, meaning-rich concepts (the number of students in a class, the price of a pair of jeans). For such complex problems, presentation formats that made fewer memory demands afforded better performance.

Thus, different ways of presenting problems may help solvers to select effective strategies or to implement strategies accurately, and the specific mechanisms at play may depend on the complexity of the problems. Problem complexity, in turn, depends on the skill level of the learners. What is difficult for one group of learners may be easy for another.

Indeed, the effects of variations in problem presentation may also depend on characteristics of learners, such as their mathematical or visual-spatial ability or their relevant prior knowledge. Koedinger and colleagues (Koedinger & Nathan, 2004; Koedinger et al., 2008) found that the specific benefits that accrued to learners as a function of more grounded representations varied for students with different levels of prior knowledge of algebra. For students who had little prior knowledge, the verbal advantage was due largely to students solving story problems successfully using informal strategies and failing to respond at all on equations (Koedinger & Nathan, 2004). For students with intermediate knowledge of algebra, the verbal advantage was largely due to students' using correct strategies to solve story problems successfully but making conceptual errors in applying formal algebraic strategies to solve equations (Koedinger et al., 2008, Experiment 1). For students with strong prior knowledge of algebra, most students used formal strategies for both story problems and equations, and the verbal advantage was due to students' making fewer arithmetic errors on story problems than on equations (Koedinger et al., 2008, Experiment 2). Thus, the specific effects of different kinds of problem presentation depended on learners' level of prior knowledge.

III. Individual factors

As reviewed in the previous section, many different contextual factors are influential in processes of strategy use and strategy change. However, also as noted in the previous section, in many cases, the operation of these contextual factors depends on what learners bring to the learning situation. Individual difference factors may exert independent effects, or they may interact with contextual factors to influence strategy use and strategy change . Some individual difference factors are stable characteristics of learners, such as their working memory capacity, executive function skills, mathematics ability, or spatial ability. Other individual difference factors are more transitory characteristics, such as prior knowledge or attitudes toward the content domain in question.

A. Stable individual differences

Stable individual differences in cognitive abilities are associated with variations in performance in many domains. Some strategies require certain abilities for their implementation or generation. For example, implementing a complex, multi-step strategy may require a certain minimum level of working memory capacity, and individuals who do not have the requisite capacity may choose not to adopt or apply that strategy, even if they receive direct instruction about it. Likewise, implementing a visuo-spatial strategy for solving an algebraic problem—or understanding that strategy when it is demonstrated—may require a particular level of spatial ability. People's abilities may constrain their strategy generation or strategy adoption—or may potentiate those same processes.

A large literature documents the importance of working memory in mathematical tasks (for a review, see Raghubar, Barnes, & Hecht, 2010), including some work on the role of working memory in strategy implementation (Imbo & Vandierendonck, 2007, 2008). Other work documents the roles of other basic cognitive abilities in strategy use and strategy change. For example, in the study of adults' strategies for interpreting 2x2 contingency tables described above (Osterhaus, et al., 2018), the odds of participants' using the most sophisticated strategy (the conditional probabilities strategy) were greater for participants with higher scores on a standardized test of mathematics abilities.

Other stable individual difference factors may influence strategy change in a more general way, by affecting people's overarching tendency to try to new things. *Need for cognition* is the tendency to engage in effortful or challenging cognitive activity, and to derive enjoyment from doing so (Cacioppo & Petty, 2005; Sadowski & Gulgoz, 1992). Learners who are high in need for cognition may be more likely than their counterparts who are low in need for cognition to generate new approaches to solve problems, or to try out new approaches when they see them demonstrated. Indeed, some recent work suggests that individuals high in need for cognition are particularly likely to adopt new strategies to which they are exposed, particularly strategies that are not highly intuitive (Menendez, Brown & Alibali, 2018).

Stable individual differences may also influence learners' tendencies to rely on retrieving answers from memory or on applying back-up strategies in tasks such as arithmetic or spelling. Siegler (1988) identified three subgroups of learners, whom he referred to as *good* students, *not-so-good* students, and *perfectionists*. These subgroups differed, both in the quality of their knowledge (i.e., in the strengths of the associations they had between problems and specific answers) and in terms of the level of confidence in a retrieved answer that they required in order to advance that answer, rather than apply a back-up strategy—what Siegler termed their "confidence criterion". Although we are not aware of data that address this point directly, it seems likely that individual differences in people's confidence criteria could stem from stable individual differences, perhaps related to conscientiousness, one of the "Big Five" personality dimensions.

B. Transitory individual differences

Other relevant individual differences relevant to strategy use and strategy change involve more temporary characteristics of individuals, such as whether they have relevant prior knowledge about the task at hand. Like stable factors, transitory factors may exert independent effects, or they may interact with contextual factors to influence strategy use and strategy change.

Some of the work reviewed thus far has highlighted the critical role of prior knowledge as a transitory factor that affects patterns of strategy change. Unsurprisingly, individuals who have prior knowledge of a particular strategy are more likely to apply that particular strategy (Fyfe & Rittle-Johnson, 2016a). More surprising, however, is that for individuals who have prior knowledge of a correct strategy, receiving feedback is often detrimental, compared to not receiving feedback (Fyfe & Rittle-Johnson, 2016a; Fyfe, Rittle-Johnson, & DeCaro, 2012).

Transitory factors other than prior knowledge may also influence people's strategy use. For example, one recent study of adults solving probability problems showed that participants' expectations about task difficulty influenced the effectiveness of feedback about correctness (Fyfe & Brown, 2018). For learners who expected the problems to be easy, feedback was beneficial for transfer; however, for learners who expected the problems to be difficult, feedback was detrimental.

IV. Metacognitive factors

More than two decades ago, Crowley, Shrager, and Siegler (1997) argued that strategy change could be viewed as a process of "competitive negotiation" between two types of processes: associative processes, which track the effectiveness and efficiency of strategies, and metacognitive processes, which involve consideration of the demands of particular tasks, the qualities of particular strategies, and learners' own resources and competencies. From this perspective, Crowley et al. argued that metacognitive factors play a critical role in processes of strategy discovery and generalization. With experience, learners accumulate relevant knowledge about tasks, strategies, and their own competencies, and they use this knowledge to guide their selection and application of existing strategies and to invent new strategies, if they deem it necessary to do so.

The view that metacognitive factors play a role in strategy use and strategy change has received support from studies in a range of content domains. For example, in a series of studies on children's acquisition of memory strategies, Waters and Kunnman (2010) demonstrated that conditions that allow for metacognitive reflection—in particular, conditions in which children's resources are *not* taxed by strategy implementation, leaving some resources available for reflection—are most likely to lead to discovery of new strategies and generalization of newly-learned strategies.

Research on strategy choice and change in mathematical problem solving has focused on three primary types of metacognitive knowledge: learners' perceptions of the difficulty of problems, learners' degree of confidence in their existing strategies, and learners' evaluations of qualities of particular strategies, such as their efficiency and their intuitiveness.

A. Perceived difficulty of problems

Many studies have demonstrated that people choose strategies *adaptively* (e.g., Lemaire & Siegler, 1995). This means that they strategically select strategies based on the perceived difficulty of the task at hand in order to maximize their reward or reduce their effort. For example, Walsh and Anderson (2009) found that, when choosing strategies for solving multiplication problems, adults chose mental multiplication strategies more often for easier problems, and they chose calculator-based strategies more often for more difficult problems. This adaptive use of strategies suggests that they made judgements about the difficulty of problems and based their strategy choices on these judgements.

But is adaptive strategy use based on metacognitive reflection about problem difficulty, or is it based on more simple processes, such as the need to use backup strategies when retrieval fails? One source of evidence on this issue comes from research comparing conditions in which participants are allowed to choose their own strategies, and conditions in which participants are directed to use a particular strategy (sometimes called a choice/ no choice design; Siegler & Lemaire, 1997; see also Luwel, Onghena, Torbeyns, Schillemans, & Verschaffel, 2009). In the Walsh and Anderson (2009) study, participants were faster at mentally solving problems when they had a choice of which strategy to use, compared to when they were directed to use the mental multiplication strategy. Based on this finding, Walsh and Anderson argued that participants were engaging in metacognitive reflection when they had a choice of which strategy to use—they chose to solve easy problems mentally, because they knew that they could solve such problems faster using this method than by using a calculator. This finding implies that participants made metacognitive judgements about how difficult the problems were and used these judgements to choose which strategy to use.

B. Judgements about current or possible strategies.

People may reflect, not only about the difficulty of problems, but also about the strategies they do or could use. For example, learners can provide judgements about how confident they are that their existing strategies are correct As noted above, learners' confidence in their existing strategies can moderate the effects of contextual factors, such as feedback (Brown and Alibali, 2018).

Learners can also readily provide judgements about the legitimacy or quality of other potential strategies. For example, Siegler and Crowley (1994) asked 5-year-old children to judge different strategies for solving addition problems as *smart*, *kind of smart*, or *not-so-smart*. They found that five-year-old children judged novel correct strategies as smarter than novel incorrect strategies. Children also judged a novel correct strategy as equally smart as their existing correct strategy, even though the novel strategy was more efficient. This pattern was replicated with 9-year-olds' judgements about strategies for winning at tic-tac-toe—with the exception that the 9-year-olds judged the novel, correct strategy as smarter than their existing, correct strategy.

Based on these findings, Siegler and Crowley (1994) argued that children possessed a *goal sketch*—a basic understanding of the goals and causal structure of the task at hand, and of the objectives that appropriate strategies for the task must meet. Thus, the goal sketch embodies metacognitive knowledge about the task domain that has implications for evaluating potential strategies, including one's own strategy and other novel strategies. According to Siegler and Crowley (1994), children's goal sketches both enabled them both to recognize "good" strategies when they saw them and provided them a basis for judging incorrect strategies as poor. Siegler and Jenkins (1989) argued that goal sketches help children to avoid generating or using illegitimate strategies, though it may not always enable them to appropriately judge strategies more advanced than those that they use (Baroody, Tiilikainen, & Tai, 2006).

Indeed, relatively little is known about factors that inform learners' judgements of alternative strategies. Contextual factors are surely relevant; for example, Brown and Alibali (2018) found that negative feedback decreased learners' confidence in their existing strategies for mathematical equivalence problems. Likewise, in a study of children's evaluations of possible strategies for solving mathematical equivalence problems such as $3 + 4 + 5 = 3 + \_\_$, children who received perceptual support for encoding the position of the equal sign or the right side of the problem (via highlighting problem features in red ink) showed more appropriate strategy evaluations following the manipulation (Alibali & Prather, 2007). Specifically, children who received perceptual support evaluated the incorrect *add-all* strategy (i.e., add all the numbers) more negatively, and evaluated the correct *equalize* (i.e., make both sides equal) and *add-subtract* (i.e., add the numbers on the left side and subtract the number on the right) strategies more positively (see Figure 2). Children who received perceptual support showed significant improvements in overall strategy judgement scores, relative to children in a control condition, who did not receive perceptual support—indicating that contextual support for encoding problem structure can lead to improved metacognitive judgements of alternative strategies.

Insert Figure 2 about here

People can evaluate strategies, not only in terms of correctness or overall “smartness”, as these previous studies have done, but on a range of dimensions. Brown, Menendez and Alibali (2018) asked adults to judge three correct strategies for solving algebraic story problems about constant change. They asked participants to judge the strategies on six relevant dimensions: goodness, length, easiness to remember, complexity, commonness, and whether the strategy “made sense”. A factor analysis revealed that participants’ judgements cohered into two correlated factors, one focusing on the *intuitiveness* and the other on the *efficiency* of the strategies. Do individuals differ in their desire for strategies that are intuitive vs. efficient? And how do such judgements affect participants’ willingness to adopt new strategies that vary along these dimensions? These questions are important arenas for future work.

V. Integrating factors in conceptual models of strategy change

This selective review has highlighted three classes of factors—contextual factors, individual difference factors, and metacognitive factors—as important to understanding processes of strategy use and strategy change in mathematics. As noted at the outset of this article, these classes of factors are not the only ones that matter, but they are ones that any integrative theory of strategy change must consider.

The rich complexity of strategy change processes and the sheer number of different factors that may be relevant in any learning situation can be daunting for scholars who wish to understand—and potentially, to intervene in—processes of strategy change. How can one make sense of this complexity?

A. Identifying causal factors in experimental and quasi-experimental studies

One common way to conceptualize the integration of multiple factors in explaining behavior, including strategy generation and strategy change, relies on the standard analysis-of-variance framework that is common in psychology. From this perspective, change can be conceptualized in terms of the main effects and interactions of various causal factors. Such models are typically tested using generalized linear models in an analysis-of-variance or regression framework.

This framework has guided much of the existing research on strategy change. It is particularly useful in establishing whether or not specific factors, alone or in combination, can cause strategy change. Thus, it is most useful in *identifying* potential causal factors. However, as the number of factors under consideration increases, the number of potential interactions increases multiplicatively, and the sample sizes needed for adequately-powered tests of main effects and interactions can quickly become prohibitive.

Nevertheless, many studies rely on this framework to identify causal factors, particularly contextual factors that can be readily manipulated, and in some cases, transitory individual difference factors, which can also be manipulated. As one example, Fyfe and Rittle-Johnson (2016) manipulated children's prior knowledge of a correct strategy by teaching children that strategy at the outset of the study, and they then investigated whether prior knowledge moderated children's responses to feedback. As a second example, Fyfe and Brown manipulated participants' expectations of task difficulty, and examined whether these expectations moderated their responses to feedback. These studies show that it is possible to manipulate some individual differences, such as prior knowledge and task expectations. However, many individual differences, even ones that are transitory, are challenging to manipulate in the laboratory. And, more stable individual difference factors, such as need for cognition or mathematics anxiety, are not generally amenable to experimental manipulation, although they can be tested in quasi-experimental designs.

Meta-cognitive factors may also be amenable to experimental or quasi-experimental study using factorial designs. For example, Brown and Alibali (2018) examined the moderating role of confidence in one's existing strategy on children's responses to feedback, by measuring confidence at the outset of their study and including it in their statistical model. It might also be possible to manipulate how participants evaluate strategies; for example, do participants privilege efficiency or intuitiveness when determining whether to adopt a new strategy? Some research has suggested that providing students with different sorts of goals for learning—e.g., goals for getting problems right or acquiring deep understanding of a domain—may influence what children learn from lessons (McNeil & Alibali, 2000). Thus, manipulations of students' goals—e.g., to solve problems as quickly as possible or with as few errors as possible—might shift their criteria for strategy selection and use.

B. "Diathesis-stress" or "vulnerability-trigger" models

Within the analysis-of-variance framework, one kind of conceptual model for integrating different classes of factors is "diathesis-stress" models, which have been widely used in work on the development of psychopathology (e.g., Walker & Diforio, 1997, Zuckerman, 1999; see also Belsky & Pluess, 2009). Broadly speaking, such models hold that psychopathology emerges only in cases where there is both an underlying genetic vulnerability (the diathesis) *and* an environmental trigger (the stress). The basic idea is that individual factors may constitute a *vulnerability* to change, and external factors may serve as a *trigger* that potentiates change.

Children's generation or adoption of novel strategies could be conceptualized in a similar way, as suggested by Alibali, Crooks and McNeil (2017). In their view, there may be "two key ingredients necessary for strategy generation to occur: a vulnerability to change …. and a 'trigger' that actually provokes change" (p. 164). They conceptualized vulnerability to change in terms of perceptual readiness, which they defined as encoding key structural features of the target problems (but not using them appropriately in existing strategies), and they conceptualized feedback about the correctness of existing strategies as a "trigger" for strategy generation. From this perspective, strategy generation would be likely, only if children *both* encoded key structural features of the problems (i.e., they were vulnerable to change), *and* received feedback that their prior strategies were incorrect (i.e., they encountered a trigger for change).

How might such a model be used to integrate contextual, individual, and metacognitive factors to account for strategy change? Individual factors could be viewed as conferring vulnerability to change, while contextual factors might serve as environmental triggers that provoke change. This perspective can therefore naturally account for why the same contextual factors might yield change for some individuals but not for others—those individuals might differ in their initial vulnerability to change.

From this perspective, both stable and transient individual factors may confer vulnerability change. For example, a high level of need for cognition might be conceptualized as a stable individual difference that creates a high baseline level of vulnerability to change. Indeed, as noted above, individuals high in need for cognition may be more likely to adopt new strategies to which they are exposed than individuals low in need for cognition (Menendez, et al., 2018). Along similar lines, relevant prior knowledge might be conceptualized as a more transitory individual difference factor that may temporarily create a high level of vulnerability to change. Indeed, as noted above, some work has documented that individuals with differing levels of relevant prior knowledge are differentially receptive to feedback about whether their strategies are correct (Fyfe et al., 2012; Fyfe & Rittle-Johnson, 2016a).

Where might metacognitive factors fit into such a conceptual model? One possibility is that metacognitive factors might contribute to vulnerability; for example, lack of confidence in their current strategies could make learners more vulnerable to change. Alternatively, metacognitive factors might influence the particular strategies that learners adopt, once they become vulnerable to change. For example, learners who value efficiency might choose to adopt different strategies than do learners who value intuitiveness.

C. *Cumulative risk models*

Conceptual models of strategy change that attempt to isolate main effects and interactions of distinct factors have been useful in identifying critical factors and in specifying how multiple factors might work together to produce patterns of strategy use and strategy change. However, as the number of factors under consideration increases, it becomes more difficult to establish the causal relevance of various factors or combinations of factors. Moreover, it is possible that some factors may have small effects that would be difficult to discern in traditional experimental designs—but that nevertheless, accumulations of many such factors might lead to strategy generation or strategy change. Thus, another conceptual model for the integration of different classes of factors conceptualizes strategy change as due to the accumulation of multiple factors that may “push” an individual toward change or that may “pull” an individual away from change. Such models, termed *cumulative risk models*, have been used in large scale, naturalistic studies of health and behavioral outcomes, such as psychiatric diagnoses, physical illness, or adolescent sexual debut (e.g., Raviv, Taussig, Culhane, & Garrido, 2010; Price & Hyde, 2009). In such models, a single “risk” factor is not expected to provoke change on its own, but the accumulation of many factors—many small pushes—has the potential to provoke change.

From a cumulative risk perspective, contextual factors, individual difference factors, and metacognitive factors may each constitute potential “risk factors” for strategy change—and the cumulation of such risks increases the likelihood of change. For example, in conceptualizing the “risk” of generating a new strategy for solving algebraic story problems, one might consider a broad set of possible risks for strategy change, including relevant prior knowledge, a positive attitude about the value of mathematics, high need for cognition, and contextual exposure to a correct strategy. The cumulation of such risks might predict strategy change, such that a learner with none of these risk factors might be very unlikely to adopt a new strategy for solving the problems, whereas a learner with all of these risk factors might be almost certain to do so.

This perspective can also explain why similar contextual conditions yield change in some individuals, but not others—because such factors represent just one source of potential “risk” for strategy change. In the absence of other accumulated “risk factors”, change may be unlikely, but in the presence of other risks, change is highly likely to occur.

In thinking about cumulative risk for strategy change, it is important to consider both “pushes” and “pulls”—that is, both risk factors, which promote change, and protective factors, which may prevent change. A range of different factors may contribute to change resistance, and the specific factors may depend on the particulars of the task. For example, in children learning to solve mathematical equivalence problems, strongly activated arithmetic knowledge or entrenched misconceptions about the meaning of the equal sign may prevent children from generating or adopting correct strategies. (McNeil, 2014; McNeil & Alibali, 2005). Once children have started using correct strategies, metacognitive knowledge may prevent children from shifting to incorrect strategies (Brown & Alibali, 2018). Thus, cumulative risk models may need to integrate both the presence of risks that promote change, and the absence of protective factors that prevent change.

Of course, there are many limitations to cumulative risk models. Most notably, such models treat risk as additive, thus excluding the possibility of interactive risks, and they treat each risk as independent, excluding the possibility of collinearity. In addition, because cumulative risk models treat risks additively, they do not allow for differentiating the strength of various risks, and information about the intensity of risks is not captured (Evans, Li, & Whipple, 2013).

Nevertheless, cumulative risk models can be an effective model for conceptualizing the integration of multiple risk and protective factors in accounting for the emergence of discrete behaviors, such as strategy generation or change. Thus, cumulative risk models may be a valuable conceptual tool for understanding the integration of contextual, individual, and metacognitive factors in naturalistic settings in which strategy change may occur.

As one example of the power of a cumulative risk approach, we reanalyzed data from a study of children who worked in pairs to solve mathematical equivalence problems, and who then solved posttest and transfer problems independently (see Gutiérrez, Brown, & Alibali, 2018, for a description of this study and a qualitative analysis of a subset of the data). We focused on children with low knowledge at pretest, and we considered three risk factors for use of a correct strategy on the posttest and transfer items: presence of some minimal prior knowledge at pretest (as opposed to no such knowledge, an individual difference factor), high confidence in one's pretest strategy (a metacognitive factor), and being paired with a partner who had strong knowledge about equivalence (a contextual factor). As seen in Figure 3, as the number of risk factors increased, the likelihood that children solved the posttest and transfer problems correctly increased systematically, $B = 1.30$, $z = 2.92$, $p = .004$, odds ratio = 3.66. Thus, the accumulation of risks predicted whether children would shift to using correct strategies after working with a partner, and the accumulated risks included contextual, individual and metacognitive factors.

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D. Dynamic systems models

Dynamic systems theories conceptualize development as change within a complex system that involves interactions of multiple factors at different levels and on different timescales (e.g., Smith & Thelen, 2003; Spencer, Austin, & Schutte, 2012). Thus, dynamic systems theories are well suited to conceptualize the interactions of multiple factors in processes of strategy change. Within dynamic systems theories, one key idea is *self-organization*, which is the idea that patterned behavior emerges out of the interactions of multiple elements of the system. Thus, patterns of behavior are not specified in advance, but instead, they are "softly assembled" in the moment, depending on the specifics of the task, the context and setting, and the immediate and developmental history of the individual. Further, within any set of possible behavior patterns, some patterns are more likely to emerge than others. These common patterns are conceived as "attractor states", to which the behavior of the system is drawn.

Empirical studies motivated by dynamic systems theories focus on how different factors interact to drive performance (Spencer, Perone & Buss, 2011). Within dynamic systems theories, these factors are sometimes termed *control parameters*, because they "control' which behavior pattern emerges—that is, they control which of the possible behavioral forms emerges. Control parameters may include factors at different levels and on different timescales. For example, in research on infants' reaching, Clearfield and colleagues (Clearfield, Dineva, Smith, Diedrich, & Thelen, 2009) considered factors that operate on the time scale of the individual reach (such as the salience of a cue to reach to a particular location), as well as factors that operate over the set of reaching trials within a session (such as motor memory for previous reaches).

A similar approach could be applied in conceptualizing how different sorts of factors interact to drive strategy use and strategy change. Relevant factors operate on the time scale of the individual problem (e.g., is there contextual support for encoding problem structure? Was feedback provided on the previous trial?), and factors that operate over longer time scales (e.g., does the learner possess relevant knowledge for implementing a particular strategy? Is the learner highly confident that a particular strategy is correct?). Common strategies can be viewed as “attractor states”. Certain levels of particular factors may “push” the learner toward particular attractor states (i.e., towards using particular strategies) or “pull” the learner away from particular attractor states (i.e., away from using particular strategies).

As a representative example, consider mathematical equivalence problems with addends on both sides of the equal sign (e.g., 3 + 4 + 6 = 3 + __). The common incorrect and correct strategies that learners use can be conceptualized as attractor states; for example, the *add-all-the-numbers* strategy is one attractor state, and the *make-both-sides-equal* strategy is another. Certain levels of specific control parameters may push learners toward the *make-both-sides-equal* attractor state; for example, contextual support for encoding the position of the equal sign encourages learners to generate the *make-both-sides-equal* strategy (Alibali et al., 2017). When solving a problem, if a learner lacks confidence in the strategy used—a metacognitive judgement which might reflect that the attractor state that gave rise to the strategy was relatively weak—this may make generation of the novel strategy in the context of perceptual support even more (or less) likely. Likewise, if a learner does not have the requisite mathematical skills to implement a particular strategy (an individual difference factor), this may prevent that behavioral form from emerging.

From a dynamic systems perspective, individual difference factors, contextual factors, and metacognitive factors represent control parameters that may push the system toward or away from particular attractor states. These factors converge for any given individual at any particular moment—sometimes in non-additive, nonlinear ways—to yield behavior that (usually) aligns with a patterned form of behavior that is recognizable as a strategy. Thus, changes in the levels of different control parameters may—or may not—yield changes in the behavior that emerges. Factors of different types and at different timescales interact to give rise to behavioral forms that reflect particular strategy choices, and that may represent consistent strategy use or strategy generation for a given individual at a given point in time.

Some scholars have sought to formalize dynamic systems accounts in computational models (e.g., Samuelson, Spencer & Jenkins, 2013; Simmering, 2016). Such models can be used make predictions about performance and change that can then be put to empirical test. If applied to research questions about strategy change, such an approach holds promise for shedding light on how multiple factors interact to explain processes of strategy use and strategy change.

V. Conclusion

Processes of strategy change are influenced by many factors. In this article, we have considered research on three classes of factors—contextual factors, individual factors and metacognitive factors—that are implicated in strategy change in mathematics learning. These factors operate both independently and in combination to influence learners' behavior—yet scientific progress in understanding how these factors are integrated has been slow.

In our view, progress will require comprehensive conceptual models that specify how different factors come together to explain behavior. In this article, we have discussed several such models, including diathesis-stress or vulnerability-trigger models, cumulative risk models, and dynamic systems models. We believe that each of these types of models has insights to offer, and that research guided by such models—including research that utilizes computational models to test predictions of different models—will contribute to greater progress in understanding strategy change.

It seems likely that general processes of change will be similar for different types of problems, but that the specifics may differ in important ways. Thus, one path forward that seems likely be to fruitful is to work toward more specific and comprehensive models of change within a narrow domain—perhaps one about which much is already known, such as learning of mathematical equivalence or algebra problem solving. Lessons learned within a specific domain can then be tested more broadly, with other types of problem content and with learners of different ages.

In sum, we encourage researchers not to shy away from vexing questions about the complexity of strategy change, but instead to address those questions head-on. One way to do so is to develop conceptual frameworks and analytic tools that acknowledge, honor and organize that complexity. The value in answering those vexing questions seems clear, because a deeper understanding of strategy change is critical for understanding—and potentially enhancing—learning, development, and education.

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Figure Captions

Figure 1. Sample item that requires interpreting a contingency table.

Figure 2. Mean pretest to posttest change in “smartness” ratings for the add-all, equalize and add-subtract strategies, as a function of which elements of the structure of the equations were highlighted in red ink during an intervention.

Figure 3. Proportion of posttest and transfer items solved with a correct strategy, as a function of the number of “risks” for strategy change (out of 3 possible).