DISMANTLING VISUAL OBSTACLES TO COMPREHENSION OF 2-D SKETCHES DEPICTING 3-D OBJECTS

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This study focuses on potentially misleading information (PMI) and potentially helpful information (PHI) embedded in the 2-D sketch of a 3-D geometric object. Our quest was to discover whether and how PHI and PMI capture visual obstacles, for high-school students, to desirable comprehension of the sketches used in teaching spatial geometry. We compared the anticipated difficulty of 24 sketches of cubes with different auxiliary constructions, according to their orientation and to the ratio #PHI/#PMI, with the actual difficulty reflected in the scores received by 174 high-school students for comprehending these sketches. The findings suggest: (a) deviations from a normative sketch of a cube affect spatial comprehension; (b) the ratio #PHI/#PMI accounts for a significant part of the students' visual difficulty.

THEORETICAL BACKGROUND AND RESEARCH QUESTIONS

Success in learning spatial geometry in high school is frequently attributed to a student's ability to visualize 3-D geometric configurations from 2-D sketches (Gutiérrez, 1996). This attribution is based on the presupposition that human vision and cognition have a high capacity of pattern recognition and synthesis (Gutiérrez, 1996; Christou, Pittalis, Mousoulides, & Jones, 2005). However, as shown by many researchers, this capacity by itself is not enough to enable an easy completion of missing information in the sketches (Parzysz, 1988; Kali & Orion, 1996; Gutiérrez, 1996; Bakó, 2003; Christou et al., 2005). In their study about perception of geological structures from 2-D drawings, Kali and Orion (1996) found that many students rely solely on external visual information, and fail to "penetrate" the 2-D sketches and construct desirable 3-D mental representations. Concurrently, in spatial geometry, Bakó (2003) found that learners mostly consider figural aspects, omitting conceptive didactical inference.

Gutiérrez (1996) suggests that far less information is visible from a 2-D static drawing than from rotating the 3-D object in reality, and learners are not always able to complete the missing information in their minds. Moreover, the learner may be under the illusion that the sketch precisely represents the real object, totally unaware of the loss of information in transit between the real object and the sketch (Parzysz, 1988). As a result, many learners face visual obstacles, often being unaware of their existence.

Further, a quick look on spatial geometry textbooks reveals a trend to orient spatial figures in a particular "normative" way. In planar geometry, existing conflicts between figural and conceptual aspects of geometrical objects may sometimes result in learners' incapacity to recognize a geometric figure when it does not coincide with a prototypical representation or is not placed in a normative position (Maracci, 2001; Larios, 2003). On the other hand, prototypes are not necessarily linked with visual barriers; sometimes the use of familiar prototypes may be advantageous, and permit meaningful learning (Solso & Raynis, 1979). Therefore, the existence of prototypical images has to be taken into consideration when examining spatial perception.

We argue that a better understanding of the visual obstacles' constituents, and the interaction between them, might be the key to improve spatial geometry instruction. Visual perception is undoubtedly influenced by many factors, some intrinsic to the learner, associated with individual knowledge, abilities and experience, while others extrinsic to the learner, related to the geometric problem itself and to the way it is presented (Parzysz, 1988; Arcavi, 2003; Christou et al., 2005).

In our research, we focused on two extrinsic visual aspects, embedded in the 2-D sketch of the 3-D object: (1) Potentially helpful information embedded in the 2-D drawing of a given 3-D geometrical configuration (will be referred to as PHI); (2) Potentially misleading information perceived from the 2-D drawing due to the chosen perspective angle (will be referred to as PMI). As a phenomenon influenced by many factors, visual obstacle investigation requires small and prudent steps. Focusing on simple geometric forms, familiar to high-school students, may increase chances of better understanding visual obstacles. Consequently, our research concentrates on cubes, and on basic shapes such as triangles and quadrilaterals contained in them (hereafter, auxiliary constructions). Perception and visualization undoubtedly consist of complex interactions between many aspects that may not be dismantled into isolated components. However, sometimes a simplistic approach has the power to facilitate and enable comprehension. Therefore, keeping in mind that the whole may be greater than the sum of its parts, we aimed to find answers to the following two questions:

1. Do deviations from the normative images of a cube affect spatial comprehension of auxiliary constructions?
2. Whether and how do the interaction between PHI and PMI capture visual obstacles, for high-school students, to desirable comprehension of 2-D drawings depicting 3-D objects, used in teaching spatial geometry?

DIFFERENT SOURCES OF VISUAL OBSTACLES

Visual obstacles are closely related to the information embedded in a 2-D geometrical sketch of a 3-D object, and to the way this information is perceived. Perception is the process of obtaining awareness, organizing and deriving meaning of sensory visual data, while visualization refers to the cognitive faculty of processing this information and forming an adequate mental image (Kirby, 2008). Perception and visualization are filtered by former experience, prior knowledge and personal expectations (Arcavi, 2003; Kirby, 2008), as well as by individual thinking skills and particular spatial abilities (Parzysz, 1988; Kali & Orion, 1996; Gutiérrez, 1996; Christou et al., 2005).

In spatial geometry, the angle from which a 3-D object is observed has the power to hinder or facilitate visualization. In particular, some projection angles may deform the object displayed almost beyond recognition, and present learners with a substantial visual obstacle. For instance, how can we identify the object in Figure 1? Is it an umbrella from bird's eye view, a right hexagonal-based pyramid seen from above, or a cube? All these interpretations are possible.

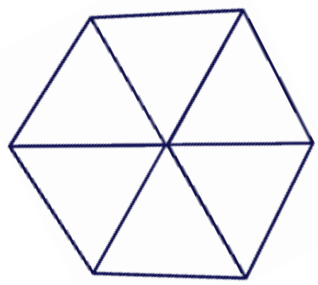


Figure 1: A 2-D sketch that can represent different 3-D objects

Potentially Misleading Information (PMI)

PMI comprises of two categories of geometrical regularities: (1) Hidden correct information (such as hidden from view vertices, edges, surfaces and intersection of edges), and (2) altered or added incorrect information (such as non-existing added intersections of edges, non-existing confluences of edges with a straight line, altered longitudinal ratios, altered angles and edges crossing above the surface and therefore hiding it). For instance, let us contemplate Figure 1 as a drawing of a cube. We may notice one vertex appears to be missing, while some edges appear lying on a beam of straight lines through another vertex. Counting PMI for different sketches of a cube led us to the realization that a cube's PMI values are minimal for the normative sketches frequently used for cubes in school textbooks (see the two examples in Table 1).

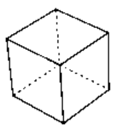
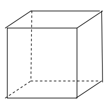
cube's sketch	hidden correct information				altered/added incorrect information			total PMI
	number of hidden vertices	number of hidden edges	number of hidden sides	number of hidden intersections of edges	number of non-existing added intersections of edges	number of non-existing confluences of edges with a straight line	number of geometrically altered sides	
	1	0	0	0	9	3	6	19
	0	0	0	0	2	0	6	8

Table 1: Counting PMI for two different cube sketches

However, typical high-school spatial geometry problems are not as simple, and contain auxiliary constructions. These constructions may add PMI. For example, cube ABCDA'B'C'D' in Figure 2a contains additional PMI: triangle DBB' may look isosceles in the 2-D sketch (DB=DB'), though in the desired comprehension it is not. Moreover, triangle DBB' does not look right-angled in the sketch, although in the desired comprehension it is.

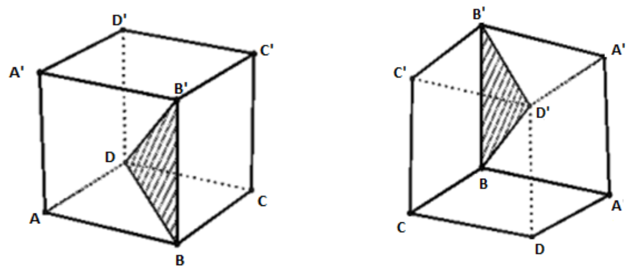


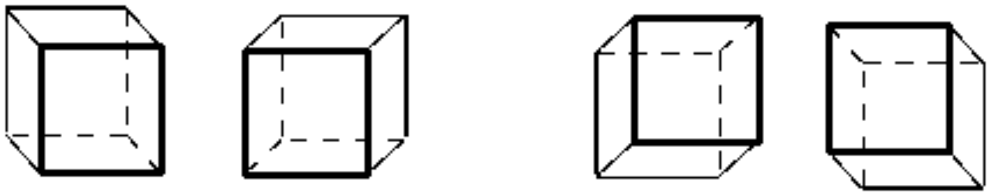
Figure 2: ABCDA'B'C'D' being a cube, what properties does triangle DBB' possess?

Potentially Helpful Information (PHI)

Two-dimensional sketches contain potentially helpful visual information (PHI) as well. Such information may supplement verbal data of the given spatial geometry problem, elicit visualization, and, support deductive reasoning and formal proof (Hadas, Hershkowitz & Schwarz, 2000). PHI comprises of geometrical regularities embedded in the drawing, such as vertices, edges, and diagonals, that auxiliary constructions share with the cube. Consider for example, triangle DBB' in Figure 2a: one side of the triangle coincides with the edge of the cube ABCDA'B'C'D', and the vertices of the triangle are simultaneously the vertices of the cube. Along with prior knowledge about the features of a cube, this visual information might help the learner reason that triangle DBB' cannot represent an isosceles triangle (though its sides on the drawing are equal) and has to be perceived as right-angled ($\angle ADD'$=90º).

Normative and Un-Normative Drawings

PHI and PMI may not be the only objective features of a 2-D drawing influencing one's vision; the orientation of the drawn object may affect perception as well (Larios, 2003). Sketches in spatial geometry textbooks tend to present students with cubes oriented in two out of four possible positions (Figure 3). Consistently with previous findings in planar geometry (Larios, 2003), we expect frequent use of normatively positioned cubes in spatial geometry instruction to form a prototypical image of cubes. Students accustomed to the normative orientation of cubes might find un-normative drawings, obtained by simply turning normative sketches upside down, less familiar and less coherent. Therefore, un-normative sketches may turn out more challenging than normative sketches, even though PHI and PMI remain invariant when changing a cube's orientation (see Figure 2a vs. Figure 2b).



Normative positions
Metaphorically "glued to the floor"

Un-normative positions
Metaphorically "attached to the wall"

Figure 3: Does a certain position of the cube seem more familiar to us?

METHODOLOGY

We started by finding out which geometric elements of 2-D sketches of cube-related configurations function as PMI and PHI. We developed and validated a counting method of PHI and PMI, based on the geometrical elements pointed out by three experienced mathematics high-school teachers. After defining and enumerating the various components of PHI and PMI, an unequivocal counting method was attained and peer-validated by a group of 20 experienced high-school teachers: enumeration was nearly identical (93%), with a kappa coefficient indicating adequate inter-rater reliability (k = .758).

Problem Difficulty Rating Test (PDRT) was constructed to accommodate our goals. To start with, we requested high-school students to draw a cube; since human drawings correspond to their memory representations of frequently encountered patterns (Solso & Raynis, 1979; Larios, 2003), we expected the cubes in the drawings to be normatively positioned, revealing an existing prototype.

In order to find appropriate items to be included in PDRT, auxiliary constructions comprised in 102 initial drawings of normative cubes were sorted out and divided according to #PHI/#PMI into six separate groups of 17 items each. Calculating the ratio #PHI/#PMI (note that #PMI > 0 under parallel projection because parallel projection does not preserve the ratio of lengths of non-parallel sides), seemed a reasonable suggestion for a formula apt to generate a comparative criterion for visual difficulty embedded in various sketches. Average #PHI/#PMI was calculated for each group, and two items having an approximately average #PHI/#PMI were chosen to be included in the test. Attempting to avoid Necker's illusionary dual perception, all drawings used intermittent lines for marking the cubes' hidden edges (Kornmeier & Bach, 2005). These twelve initial items were duplicated by turning the drawings upside-down (see Figure 2). The resulting 24 items were blended throughout the PDRT test. A significant correlation was found between the PDRT scores for corresponding normative and un-normative sketches: $r = .931$, $p<0.0001$. Calculating the estimated internal-consistency reliability of the 24-item test rendered high value as well: Cronbach's alpha was a = .887. Table 2 shows an example of #PHI/#PMI calculation for the two corresponding PDRT items presented in Figure 2 (note that calculations in Table 2 are identical for Figure 2a and Figure 2b).

<table>
<tr><th colspan="4">hidden correct information</th><th colspan="5">altered/added incorrect information</th><th rowspan="2">PMI derived from cube's perspective</th></tr>
<tr><th>number of hidden vertices</th><th>number of hidden edges</th><th>number of hidden surfaces</th><th>number of hidden intersections of edges</th><th>number of non-existing added intersections of edges</th><th>number of non-existing confluences of edges with a straight line</th><th>number of altered longitudinal ratios</th><th>number of altered angles</th><th>edges above surfac e</th></tr>
<tr><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>3</td><td>1</td><td>0</td><td>8</td></tr>
<tr><td colspan="10">total PMI: 12</td></tr>
<tr><td colspan="2">common vertices</td><td colspan="2">common edges</td><td colspan="2">side diagonals</td><td colspan="2">known locations</td><td colspan="2">edges behind surface</td></tr>
<tr><td colspan="2">3</td><td colspan="2">1</td><td colspan="2">1</td><td colspan="2">0</td><td colspan="2">1</td></tr>
<tr><td colspan="10">total PHI: 6</td></tr>
<tr><td colspan="10">#PHI/#PMI = 0.5</td></tr>
</table>

Table 2: Calculating #PHI/#PMI for two corresponding PDRT items

We administered the PDRT test to 174 high-school students, studying mathematics at the highest stream level in the 12th grade, and therefore familiar with cubes and their auxiliary constructions. Three scores were calculated for each respondent: average normative score (12 items), average un-normative score (12 items), and average general score (all 24 items).

RESULTS

Regarding our first question, 97% of the participants (174 out of 180) drew the same image of a cube, which was similar to a normative representation of a cube in the textbooks. We excluded the remaining 3% from our statistical analysis. Moreover, Fisher Test results indicate a significant difference between calculated correlations (see below) of #PHI/#PMI, normative and un-normative 2-D sketches ($z = 2.45$, $p = .014 < .05$), thus sustaining our anticipation that un-normative 2-D drawings, which do not match the prevalent prototype, may alter perceptual difficulty.

As to our second question, the findings show a significant correlation between the ratio #PHI/#PMI and the PDRT scores: $r = .703$, $p < .0001$ for normative drawings, $r = .543$, $p < .0001$ for un-normative drawings, and $r = .612$, $p < .0001$ for all 24 drawings. Thus, our hypothesis that the interaction between PHI and PMI captures the perceptual visual difficulty in spatial geometry for high-school students is highly supported by these significant correlations, not only for learners facing normative 2-D sketches, but also for learners presented with un-normative 2-D drawings: in both cases, spatial perception decreased, as the 2-D sketch exposed less PHI and more PMI.

DISCUSSION AND FURTHER RESEARCH

According to our findings, the ratio #PHI/#PMI accounts for a significant part of the students' obstacles to comprehension of 2-D sketches depicting a cube. This finding is a novelty that suggests a direction for further research, focused on a possibly unconscious mental pattern dominated by extrinsic visual stimuli, placing visual challenges to both, normative and un-normative spatial perception, beyond personal characteristics. Although further investigation is needed, spatial geometry instruction may already take advantage of this cognitive revelation and use #PHI/#PMI as a predictor of the visual difficulty embedded in drawings; different sketches should be adjusted to different educational purposes: minimizing PMI while maximizing PHI in 2-D drafts may help learners comprehend the 3-D geometric situation, and therefore assist visualization, while maximizing PMI and minimizing PHI may serve other pedagogical goals such as training students to cope with high visual difficulty, or spatial ability testing.

However, #PHI/#PMI serves as a better predictor for normative than for un-normative drawing, thus implying the involvement of an additional factor, disrupting vision in un-normative sketches. Evidently, our findings confirm the existence of a prototype representing a cube: the vast majority of the participants drew the same normatively-positioned cube frequently used during spatial geometry instruction. On one hand, the prototypical use of normative drawings of cubes in spatial geometry instruction may form a mental image meant to assist visualization. On the other hand, the prototypical model may not allow enough flexibility, and therefore hinder identification and manipulation of a 3-D geometrical situation in un-normative sketches (Larios, 2003). Further study is needed in order to determine the circumstances under which deviations from standard drawings affect perception. It may also be interesting to further investigate how prototypes influence our perception and whether it is appropriate to enrich the set of prototypes used in spatial geometry instruction. Still, we should denote an immediate instructional, pedagogical implication in classroom: special thought should be assigned to drawings' orientation; two students observing a geometric sketch from opposite directions may encounter different visual difficulty, since one of them is viewing a familiar, normative drawing, while the other is faced with a strange, un-normative sketch.

Next, we intend to examine exploration strategies employed by high-school students when trying to overcome visual obstacles by means of dynamic geometry software. We suggest that when rotating or measuring a computerized 3-D model of a geometric situation, the drawing's orientations, as well as PHI and PMI are altered, and consequently, the change may occur in the problem's difficulty.

We believe that, even though our findings are limited to cubes, and further research is needed for additional generalization to other 3-D geometric objects, the implications may be of interest for both research and practice not just within the area of mathematical education and technology-enhanced learning, but beyond.

References

Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics*, *52*, 215-241.

Bakó, M. (2003). *Different projecting methods in teaching spatial geometry*. Paper presented at the 3rd Conference of the European Society for Research in Mathematics Education, Bellaria, Italy. Retrieved from http://www.erme.tu-dortmund.de/~erme/CERME3/Groups/TG7/TG7_Bako_cerme3.pdf

Christou, C., Pittalis, M., Mousoulides, N., & Jones, K. (2005). Developing 3D dynamic geometry software: Theoretical perspectives on design. In F. Olivero & R. Sutherland, (Eds.), *Visions of mathematics education: Embedding technology in learning* (pp. 69-77). Bristol, UK: University of Bristol.

Gutiérrez, A. (1996). Visualization in 3-dimensional geometry: In search of a framework. In L. Puig & A. Gutiérrez (Eds.), *Proc. 20th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 1, pp. 3-19). Valencia: PME.

Hadas, N., Hershkowitz, R., & Schwarz, B. (2000). The role of contradiction and uncertainty in promoting the need to prove in dynamic geometry environments. *Educational Studies in Mathematics*, *44*, 127-150.

Kali, Y., & Orion, N. (1996). Spatial abilities of high-school students in the perception of geologic structures. *Journal of Research in Science Teaching*, *33*, 369-391.

Kirby, R. M. (2008). The need for verifiable visualization. *IEEE Computer Graphics & Applications, 28(5*), 78-83.

Kornmeier J., & Bach M. (2005). The Necker cube - an ambiguous figure disambiguated in early visual processing. *Vision Research*, *45*, 955-960.

Larios O. V. (2003). *Geometrical rigidity: An obstacle in using dynamic geometry software in a geometry course*. Paper presented at the 3rd Conference of the European Society for Research in Mathematics Education, Bellaria, Italy. Retrieved from http://fractus.uson.mx/Papers/CERME/TG7_draft/TG7_LariosOsorio_draft.pdf

Maracci, M. (2001). The formulation of a conjecture: the role of drawings. In M. H. A. M. Heuvel-Panhuizen (Ed.), *Proc. 25th Conf. of the Int. Group for Psychology of Mathematics Education* (Vol. 3, pp. 335-342). Holland: PME.

Parzysz, B. (1988). 'Knowing' vs. 'seeing': Problems of the plane representation of space geometry figures. *Educational Studies in Mathematics*, *19*, 79-92.

Solso, R. L., & Raynis, S. A. (1979). Prototype formation of imaged, kinesthetically and visually presented geometric forms. *Journal of Experimental Psychology: Human Perception & Performance*, *5*, 701-712.