# PRE-SERVICE ELEMENTARY TEACHERS' MISCONCEPTIONS OF PROOF AND COUNTEREXAMPLES AND THEIR POSSIBLE INFLUENCES ON THEIR INSTRUCTIONAL DECISIONS

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*How might pre-service elementary teachers' misconceptions of proof and counterexamples influence their teaching of proof? To investigate this question, two types of interviews—task-based and scenario-based—were designed to elicit pre-service elementary teachers' (PSTs) conceptions of proof and counterexamples and how those conceptions might impact their instructional decisions. A qualitative analysis of the data revealed that these PSTs had difficulties following or constructing formally presented deductive arguments and understanding how deductive arguments differ from inductive arguments. The data also revealed that the misconceptions that pre-service teachers held played an important role in their instructional decisions.*

## INTRODUCTION

Proof is considered an essential aspect of mathematics and mathematical reasoning and proof have gained an increasing level of attention in recent attempts to reform mathematics teaching (CCSSM, 2010; NCTM, 2000). More notably, there is a call for an enhanced notion of proof that elevates proof beyond a topic of study in advanced mathematics courses to a tool for studying and learning mathematics at all levels (Stylianides & Ball, 2008). Thus, student understanding of proof should be extended through consistent opportunities to reason about why something is true, make and test conjectures, and build mathematical arguments. Engaging in reasoning and proof enables students to make sense of new ideas and to develop habits that will be of lifelong importance (Hanna, 2000; Martin & Harel, 1989). In order to create such an environment for students, teachers must themselves have a deep understanding of proof. The purpose of this study is twofold: To investigate elementary pre-service teachers' misconceptions of proof and counterexamples, and to examine whether these misconceptions impact their instructional decisions. This study investigates the following two questions: 1) What are pre-service elementary teachers' misconceptions of proof and counterexamples in mathematics classrooms? 2) Do pre-service elementary teachers' misconceptions of proof and counterexamples influence their teaching practices? If so, how?

## FRAMEWORKS

### Proof Scheme

A fruitful approach to understanding students' difficulties with proof has been to classify these approaches along several dimensions (Balacheff, 1988; Harel & Sowder,

2014. In Nicol, C., Oesterle, S., Liljedahl, P., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 5, pp. 433-440. Vancouver, Canada: PME.

1998). Researchers have hypothesized that the development of students' understanding of mathematical justification is likely to proceed from inductive to deductive or from particular cases toward greater generality (Harel & Sowder, 1998; Simon & Blume, 1996) and various proof schemes have been proposed. We reviewed the literature in order to develop a taxonomy for teachers' conception of proof. While many studies have focused primarily on distinctions between inductive and deductive justifications (Chazan, 1993; Martin & Harel, 1989), some researchers have divided inductive and deductive justifications into further subcategories (Balacheff, 1988; Harel & Sowder, 2007; Simon & Blume, 1996). We followed that approach.

The taxonomy of proof schemes, external, empirical, and analytical, proposed by Harel and Sowder (1998), is a fundamental framework for research on students' conceptions of proof. It encapsulates the major categories included in other taxonomies and proposes further sub-categories. However, it is evidenced in the literature that some students may not even need to provide a justification, they may fail to produce a deductive argument even if they start with some deductions, or they may use a particular example—generic example—to express their deductive reasoning (Balacheff, 1988; Simon & Blume, 1996). Since these students do not hold external, empirical, nor fully developed analytical proof schemes, it may be hard to classify these students' proof schemes using Harel and Sowder's taxonomy. We propose Level 0, Level 2, and Level 4, described in Table 1, to be added to Harel and Sowder's taxonomy in order to account for a broader spectrum of proof schemes.

### Counterexamples

Zazkis and Chernoff (2008) argue that the existence of a counterexample should fit within an individual's proof scheme, therefore; what is convincing for one may not be convincing for others. They introduce the notions of pivotal and bridging examples to highlight the convincing power of counterexamples within an individual's example space. A pivotal example creates a turning point in the learner's cognitive perception, may introduce a conflict or may resolve it. A bridging example serves as a bridge from the learner's initial conceptions towards more appropriate mathematical conceptions. We use the notions of pivotal and bridging examples in our study of PSTs' conceptions of counterexamples.

## METHOD

### Participants

To select participants representing a broad spectrum in terms of knowledge and beliefs about proof, a proof questionnaire with open-ended questions was developed and administered to all students in one section of a geometry and measurement course and one section of a mathematics methods course at the beginning of the semester. After administering the questionnaire to all students in both courses, twelve PSTs, including five from the geometry course and seven from the methods course, were selected based on their responses so that there were participants displaying each of the following proof schemes: external, empirical, or deductive.

| Categories | Characteristics of Categories | |
|---|---|---|
| | Subcategories | Characteristics of Subcategories |
| Level 0 | Responses that do not address justification | |
| Level 1: External Proof Scheme | Responses appeal to external authority | |
| | (1) Authoritarian proof | Depends on an authority |
| | (2) Ritual proof | Depends on the appearance of the argument |
| | (3) Non-referential symbolic proof | Depends on some symbolic manipulation |
| Level 2: Naïve Reasoning | Responses usually with incorrect conclusions. Although, provers use some deduction, the arguments start with an analogy or with something that provers remember hearing, often incorrectly. | |
| Level 3: Empirical Proof Scheme | Responses appeal to empirical demonstrations, or rudimentary transformational frame | |
| | (1) Naïve Empiricism | An assertion is valid from a small number of cases |
| | (2) Crucial Empiricism | An assertion is valid from strategically chosen cases of examples |
| | (3) Perceptual Proof | An assertion is valid from inferences based on rudimentary mental images |
| Level 4: Generic Example | Responses expressed in terms of a particular instance (examples might be used to generalize the rules, but unlike an empirical proof scheme, the general rules are predicted based on deductive reasoning) | |
| Level 5: Analytic Proof Scheme | Responses appeal to rigorous and logical reasoning | |
| | (1) Transformational proof scheme | Involves goal-oriented operations on objects |
| | (2) Axiomatic proof scheme | Involves statements that do not require justification |

Table 1: Taxonomy of proof scheme

## Data Collection

The data was gathered in two distinct stages (Part I and Part II) with a different focus, and the primary sources of data were participants' semi-structured interviews. Part I and Part II interviews took place at the beginning and near the end of the semester, in order to detect any possible changes in PSTs' professed way of teaching proof that took place during the course. All twelve participants were interviewed individually, and interviews lasted approximately 60 minutes and were audio-recorded

Part I interviews focused on PSTs' (mis)conceptions of proof, including their way of producing proofs/counterexamples, as well as validations of different types of arguments ranging from empirical to formal. Thus, on PSTs as "knowers" of mathematics. During the interviews, participants were handed four tasks (five tasks during the post interviews), one at a time, and were asked to think out loud when determining the correctness of the tasks. For each task, participants were asked (1) to rate the level of their content understanding of the task using a four-point scale, (2) to determine whether the task is a correct statement or not, and (3) to rate the level of their confidence in terms of the validity of their evaluation using a four-point scale. Then, they were asked to produce a justification in cases where they believed the statements to be always true or to refute the statements where they believed the statements to be never true. After they provided an argument to justify or refute the statement and state their level of confidence in terms of the validity of their arguments, they were presented four brief arguments (five for the post interviews), varying in terms of level of justification; from empirical to deductive, one after the other, and asked to think out loud as they read each one, judge the correctness, and say to what extent each argument is convincing. Incorrect formally written arguments were added for each task for the post interviews. Finally, they were provided "Always," "Sometimes," "Never" cards and asked to assign the appropriate card to each argument presented as well as their own justification.

Part II interviews focused on the participants' usage of their conceptions that emerged from the analysis of Part I data. Part II interviews focused on pre-service elementary teachers as individuals who are going to be teachers of school mathematics. Knuth (2002) criticizes that research on teachers' conceptions of proof has tended to focus exclusively on teachers as individuals who are knowledgeable about mathematics rather than as teachers of school mathematics. Thus, in our study this stage focused primarily on PSTs' conceptions in the context of school mathematics. Participants' responses to questions about classroom scenarios and hypothetical students' questions were used to illuminate the process through which they would (1) validate proofs and counterexamples, (2) verify a statement's veracity, and (3) produce proofs and counterexamples as well as evaluate the validity of students' work. We also examined broader ideas and beliefs about how they plan to teach proofs in mathematics classrooms, including what types of arguments to incorporate in elementary classrooms.

## RESULTS

### Task-based interview results

The findings of this study outline a mixed picture of what constitutes proof and counterexample in the eyes of those twelve pre-service elementary teachers. The arguments that the participants constructed to justify the statements as well as the arguments presented to the participants after each task were coded according to the frameworks explained above. We now present our findings.

When asked to define proof, it was clear that pre-service teachers had some experience with proof and were using this to inform their judgments about what constituted a good proof. They had experience of seeing a proof being performed and were quoting these as examples of what was required. However, despite their experience seeing proofs in their classrooms, the majority of the participants failed to produce and/or recognize a proof. For instance, when given Task A—A kite is a quadrilateral with two distinct pairs of adjacent sides that are equal. Given this definition, justify whether or not the following statement is true. "*In a kite, one pair of opposite angles is the same*."—Only three out of seven students from the methods course were able to reproduce the proof that they learned in their previous geometry course correctly. Four students attempted to use triangle congruency to prove the statement as they learned in their geometry course. However, they either started with incorrect assumptions, such as trying to prove the wrong pair of angles as congruent, or they used incorrect reasoning to reach a correct conclusion. Only one out of 5 students from the geometry course was able to construct an argument that was coded as a deductive argument. The other four students came up with empirical arguments to justify the statement.

Not surprisingly, empirical approaches were by far the most common strategy employed by participants. Seven out of twelve students who participated in the study found empirical arguments as sufficient proof. Overall, pre-service teachers who were using an empirical approach to justify the statements recognized that they needed to test multiple examples. However, we should also note here that the participants tended to test fewer examples when they were familiar with the statement or the statement was initially believed to be true.

The fact that a generalization is found to be true in some cases does not guarantee − and thus does not prove − that it is true for all possible cases is a fundamental distinction between empirical and deductive arguments. However, we found that this distinction was not clear to the participants who constructed empirical arguments or found empirical arguments sufficient to prove. This is a fundamental difference between an empirical argument and the notion of proof in mathematics (Stylianides, 2007) and we believe it is necessary to learn it in order to move from an empirical proof scheme to a deductive proof scheme. We also found that some pre-service teachers failed to recognize that a proof always holds true.

Moreover, if participants could not make the distinction between empirical and deductive arguments, they tended not to recognize incorrect reasoning presented in formally written arguments and claimed that the argument would suffice as a proof. Similarly, some of the participants claimed that a counterexample could be found even after a proof was presented. In other words, some of the participants seemed to believe that a proof and a counterexample could exist for the same situation.

The participants also demonstrated various misconceptions refuting wrong mathematical statements, for example the belief that providing more counterexamples would make an argument more convincing. We also found that a counterexample, when presented to or created by the learner, may not create a cognitive conflict or result in refuting the statement. Instead, it may be simply dismissed or treated as an exception and as a result the need of seeing more counterexample may occur.

### Scenario-based interview results

In the scenario-based interviews, it was evident that the misconceptions described above played an important role when the pre-service teachers evaluated the classroom scenarios. We found that PSTs' decisions of whether an argument was a proof were influenced by the context, and PSTs'conceptions of proof differed when they switched from discussing proof from their own perspective to examining proof in the context of evaluating student work. We believe that this speaks to deep theoretical and practical concerns. The participants demonstrated the tendency of accepting empirical arguments as sufficient proofs in the context of elementary school, even if they did not display an empirical level of thinking about proofs.

Watson and Mason (2005) argued that examples could be seen as instances of a more general class or objects. In this study, PSTs treated examples as representation of a bigger class. In other words, the majority of the participants stated the importance of providing examples of different types to justify a statement in order to ensure the generality of the justification, thus, highlighting the importance of example space.

If students view proof as sufficient evidence to support a conjecture, one would expect the students' reasoning to end after generating a valid proof. While this was the case for the majority of the PSTs, some tested examples after generating/seeing a proof. It should also be noted that the majority of the PSTs stated that providing additional empirical checks could be helpful for students to better understand the proof and/or statement. Thus, almost all participants claimed that additional empirical checks were necessary. We interpret this finding in two possible ways: as a result of the conversation between the interviewer and the participant or it can be considered to be evidence that the students were not convinced by the generality of proofs.

## CONCLUSION AND DISCUSSION

Despite the growing emphasis on justifying and proving in school mathematics, a large body of research shows that students of all levels of experience use empirical arguments to prove statements in mathematics and/or they accept empirical arguments as valid proofs and that many students fail to understand the nature of what counts as evidence and justification. We found confirmation for these results as the majority of the participants in our study failed to recognize that testing examples is not sufficient for proof.

Several researches have focused on why many students possess these invalid proof techniques. Recio and Godino (2001) note that many such invalid proof techniques would be appropriate in non-mathematical domains. Reid and Knipping (2010) observe that reasoning about a concept using a prototypical example is common in our everyday experience. In this study, it was evident that some participants were over generalizing what they learned in other courses to mathematics. We believe that unless pre-service teachers realize the limitations of empirical arguments as methods for validating generalizations, they are unlikely to appreciate the importance of proof in mathematics (Stylianides & Stylianides, 2009). In order to achieve this learning objective, however, teachers must have good knowledge in the area of proof, for the quality of learning opportunities that students receive in classrooms depends on the quality of their teachers' knowledge (Ball, Thames, & Phelps, 2008).

Elementary teaching practices that promote or tolerate a conception of proof as an empirical argument may instill mental habits in students that significantly deviate from conventional mathematical understanding in the field. Martin and Harel (1989) state that if elementary teachers lead their students to believe that a few well-chosen examples constitute a proof, it is natural to expect that the idea of proof in high school geometry and other courses will be difficult for the students (pp. 41-42). It was clear in this study that those PSTs tend to believe that empirical arguments could be tolerated as proofs in elementary levels while they cannot be accepted as proofs in higher grade levels. Additionally, we found that the distinction between empirical arguments and deductive arguments was not clear for many of the participants. Thus we argue that unless teachers at all levels of schooling develop a good understanding of this distinction, it is unlikely that large numbers of students will overcome their misconception that empirical arguments are proofs.

There has been relatively little attention paid to the way PSTs conceptions of proof may depend on the particular context in which proof is being utilized. The results in this study indicate that this is an area worthy of further investigation as teachers' conceptions of proof in the context of teaching may be, and perhaps should be, different from the way they engage with proof in other settings.

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