HOW A PROFESSOR USES DIAGRAMS IN A MATHEMATICS LECTURE AND HOW STUDENTS UNDERSTAND THEM

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Mathematics education literature suggests that diagrams should be included in mathematics lectures, however few studies have empirically studied the use of diagrams in the undergraduate classroom. We present a case study investigating the use of diagrams in a university lecture and how students in the class understood them. Three archetypes of student understanding of diagrams are described and illustrated.

INTRODUCTION

Although one of the main objectives of advanced undergraduate mathematics courses is to help students learn to construct and understand proofs, mathematics majors have difficulty constructing proofs (e.g., Weber, 2001) and determining if a proof is correct (e.g., Selden & Selden, 2003). One possible way of investigating the sources of these difficulties is to consider how students are taught proof in these courses. In particular, given discussions on the importance of diagrams and informal arguments in the learning of mathematics and the construction of proof (e.g. Alcock, 2010; Thurston, 1994), some have called for the use of diagrams in lectures for undergraduate students (e.g. Zimmerman & Cunningham, 1991; Alcock, 2010).

In their review of the literature, Speer, Smith, and Horvath (2010) highlighted the dearth of research on college-level classroom teaching practices in mathematics. While some studies on undergraduate mathematics classrooms exist (Weber, 2004; Mills, 2012; Fukawa-Connelly & Newton; in press), there is a lack of research on how mathematics professors use diagrams in their lectures and the extent to which diagrams enhance students' understanding. The present study addresses these issues.

Theoretical Perspective

The literature outlines various theoretical benefits of using diagrams when presenting both definitions and proofs in the classroom. Using diagrams in the presentation of new definitions may enable students to develop an intuitive understanding of the definition (Vinner, 1991), perceive the connections between the formal symbolism of a definition and conceptual understanding of the definition (Zimmerman & Cunningham 1991), develop intuition of whether or not related conjectures are true (Vinner, 1991), and prove related conjectures (Vinner, 1991). Using diagrams in the presentation of proofs may enable students to gain an intuitive sense of why a statement is true (Barwise & Etchemendy, 1991), understand steps within the proof (Barwise & Etchemendy, 1991), and prove similar theorems using similar diagrams (Tall, 1991). While we make no claims that this list is exhaustive, we used these potential benefits to frame our investigation into how students understood diagrams.

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Research Questions

We consider the diagrams used in a lecture introducing the Riemann integral in an undergraduate real analysis course with the following research questions: 1) How did the professor use diagrams in this lecture and for what purpose? 2) What did the professor intend to convey by presenting these diagrams? 3) How did students interpret the diagrams and pictures that were presented in this lecture?

DR. A

The context for this case study is a real analysis course at a large public research university in the U.S. The course was taught by Dr. A (a pseudonym), a professor of mathematics with over three decades of teaching experience at the university level and a history of receiving high student evaluations. Dr. A had a reputation within the department of being a thoughtful and careful lecturer who frequently used diagrams in his lectures.

We videotaped a lecture in which Dr. A presented six diagrams. In this paper, we focus on the two diagrams presented in Table 1 (the diagram used when presenting the definition of upper and lower sums given a partition and the diagram used when presenting a proof of the claim that $\int_0^1 x\,dx = \frac{1}{2}$).

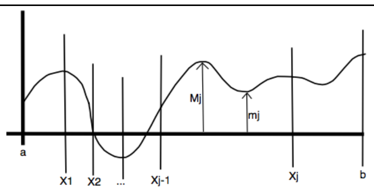
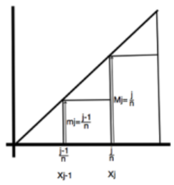
Diagram presented with the definition of upper and lower sums	Diagram presented with the proof of the claim that $\int_0^1 x\,dx = \frac{1}{2}$
	

Table 1: Diagrams presented by Dr. A

Dr. A was interviewed on his use of the diagrams in Table 1 and on his opinion on the use of the diagrams in mathematics in general. Dr. A was first asked why he chose to include the definition/proof and its associated diagram, and what he hoped to convey through their use. Dr. A was then was asked if he had hoped to convey each of the benefits discussed in our theoretical perspective, both through his general use of diagrams and, in particular, through his use of each of the two diagrams in Table 1.

Dr. A's Interview

In his interview, Dr. A reported having used the diagram illustrating the concept of upper and lower sums in order to help his students "associate concepts' symbols with geometrical pictures." He noted:

> The upper sum is approximation of the area by rectangles, which are larger than the area under the graph and the approximate by lower sums, again an approximation by rectangles, which have less area than the region under the graph of the function.

His goal of presenting this diagram was to convey the fact that upper and lower sums are approximations of area, since this concept will be essential when defining the

Riemann integral. When probed about the potential benefits of using diagrams, Dr. A agreed that he hoped the diagram would help his students develop a sense of intuition of the definition and prove related conjectures.

Next, Dr. A explained that he presented the proof of the proposition $\int_0^1 x\,dx = \frac{1}{2}$ as an example of using the approximation procedure that he outlined in the lecture. Dr. A reported that his goal in presenting this proof was to provide:

> A function where the areas a pretty clear, in the approximating rectangles can be easily seen to give the inequalities. … To give [the students] a concrete function to look at.

When probed about the benefits of using diagrams with proofs (listed in our Theoretical Perspective), Dr. A agreed that he hoped to convey each of these to his students through his use of diagrams including helping students write proofs about this concept. He suggested asking students to prove $\int_0^1 x^2\,dx = \frac{1}{3}$ would be an appropriate task to test students' understanding of his lecture.

STUDENT PARTICIPANTS

Five student participants for this study were recruited from Dr. A's class. Each of the students was pursuing either a major or minor in mathematics—the ages of the students varied from first to fourth years at the university. The goal of these interviews was to see how students understood the diagrams presented in the class and if the diagrams conveyed the mathematical insight that Dr. A intended. Each student was interviewed individually.

In the first task, we wanted to see how the participants understood the definition diagram use in lecture and whether this diagram conferred the benefits described in our theoretical perspective. Participants were first given a prompt with Figure 1 and were asked to draw the upper and lower sums on the partition, provide the definitions of upper and lower sums, and explain how the diagram was related to the definitions. Finally, to determine if the participants could use the diagram to infer properties about these concepts, each participant was asked what would happen to the sums if more points were added to the partition. If participants struggled with the first task, they were given the option to watch the video of Dr. A's presentation of the definition of upper and lower sums.

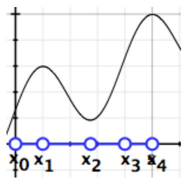


Figure 1: Diagram for the upper and lower sum task

In the second set of questions, we investigated how well students understood the proof that $\int_0^1 x\,dx = \frac{1}{2}$, particularly in relation to the diagram that Dr. A introduced in his lecture. After watching a video of Dr. A's proof presentation, participants discussed what they thought the professor was trying to convey in his presentation of the proof, what they remembered about the diagram, how the diagram affected their understanding of the proof, and what they thought the professor was trying to convey with the diagram. Each participant also was asked to explain Dr. A's diagram, including which parts of the diagram connected to which parts of the proof. Finally, each student was given the task of proving $\int_0^1 x^2\,dx = \frac{1}{3}$ which Dr. A thought students should be able to do if they understood the lecture.

Three Ways that Students May Understand Diagrams

We analyzed the student data to investigate how the students understood the diagrams from the lectures and to see the extent to which the students gained the insights the professor wished to convey. For the analysis of these data we followed the quasi-judicial procedure developed by Bromley (1986) for case study research, focusing on common patterns of student behavior to ultimately categorize them as cases of a certain type. The findings suggest three archetypes of student understanding of the diagrams presented in their course lectures: incoherent understanding, instrumental understanding, and integrated understanding.

Incoherent understanding

A student with an incoherent *understanding* of a diagram does not have a coherent understanding of how the components of the diagram relate to the formal mathematical theory. As a result, the student's responses to questions are geared toward imitating the behavior of the professor that he or she had previously witnessed. Three students evinced this type of understanding, which we illustrate with D3.

During the first task, D3 was able to correctly draw the upper and lower sums. However, when asked what information the graph provided, D3 responded:

> Well if I have both [the upper and lower sums], I could see that it will trace the function because if you put them on top of each other… It's basically this [upper sum] area minus this [lower sum] area and I feel like, I think it would give you this line [the function].

When asked to relate the graphs and definitions, D3 attempted to recall reasoning previously seen, "well, all I remember—all I keep thinking about is the function they give you, which is the upper sum minus the lower sum." Comments such as these reveal D3's belief that the difference of the upper and lower sums yields the function itself, illustrating D3's inability to connect the diagram to formal theory. Clearly D3 did not view the areas as approximations of the integral, as Dr. A intended.

Later, the student explained why the task was so difficult: "because I mean, during class we've never done any exercises like this. So I was really intimidated by like, I don't know, am I doing it correctly or not?" Feeling unfamiliar with the task, D3 had difficulty deciding how to respond to the task. D3's attempt to recall the reasoning presented in lecture and her inability to judge whether her responses made sense suggest D3 was relying on imitative reasoning.

Instrumental understanding

A student with an *instrumental understanding* of a diagram views the diagram as a tool to accomplish specific types of tasks, but does not understand the justification for why using this diagram yields the desired solution. Thus, following Skemp (1978), we say this student has an instrumental understanding of the diagram—the student knows what to do with the diagram to complete some tasks, but does not know why the solution is correct. In this archetype, the student does not have a strong understanding of how the diagram relates to the deductive mathematical theory. Hence, although the student may be able to flexibly use the diagram to accomplish some tasks, the student would not be able to draw novel inferences from the diagram, use the diagram to decide whether a statement is true or false, or connect the diagram to the logic of a proof that he or she observed. We illustrate this archetype with D2.

When asked how the upper and lower sums would be affected by a refinement of the partition, D2 reported that the upper sum would increase and the lower sum would decrease. When asked why this would occur, D2 explained, "as we increase… these areas [indicating areas between the lower sum and the curve of the function] will also increase, and also the denominator will also increase". This clearly illustrates a misunderstanding of how a refinement adjusts the upper and lower sums.

Next, when asked what the professor was trying to convey with the presentation of the proof that $\int_0^1 x\,dx = \frac{1}{2}$, the student responded "I think he's trying to show us how to prove that the… difference of the lower integral and the upper integral can be made small enough to show the area." When probed further:

Interviewer: Okay. Umm, is there anything else, or is that it?

D2: So that, that's it. Just the technique of how to show it.

We see that D2 believes the sole purpose of proof presentation is for the professor to communicate particular proving techniques to students.

Despite having a flawed understanding of how a refinement affects the upper and lower sums, D2 correctly produced a proof showing that $\int_0^1 x^2\,dx = \frac{1}{3}$. D2's description of how the diagram helped the proof construction highlights both the student's ability to relate the diagram to the proof and imitate reasoning:

> So for this I was just concentrating on the, how the curve would look like and what would be the relation of the upper and the lower, of the maximum and the min compared to the normal function, say like x. So like, since we could compare this function to x, I just had that in mind so we could use that partition.

The student further clarified that the professor's example had been in mind during D2's proof construction. D2 compared his diagram to Dr. A's proof diagram and appropriately adjusted the argument to construct a complete proof. D2 was successfully able to relate the diagram to the high-level ideas of the proof, utilizing Dr. A's reasoning to construct a similar proof. We note while D3 and D2 both illustrate imitative reasoning, they do so in different manners. D2 used Dr. A's reasoning and diagrams and adjusted the arguments to fit the new task, constructing a complete proof. This differs from D3's actions, which relied on mimicking exact actions and reasoning observed, regardless of the logical consequences.

Integrated understanding

A student with an *integrated understanding* of the diagram can use the diagram both to instantiate mathematical objects and mathematical logic; this student can form strong links between inferences drawn from the diagrams and deductive inferences drawn from the formal theory. One would expect that a student with an integrated understanding would be able to specify the components of a diagram, make inferences connecting the diagram and formal mathematical theory, and instantiate and apply the reasoning to proofs they observed and they wrote. So, not only is the student able to describe the mathematical objects being discussed at a basic level, but he or she is also able to build on the concepts. We illustrate this archetype with D1.

When asked how the student's diagram of the upper sum would be affected by a refinement, D1 was able to both relate the objects of the diagram to the formal theory and make inferences from the diagram. D1's responses throughout the first task demonstrated a clear understanding of upper and lower sums. However, despite D1's integrated understanding of the diagrams, D1 was unable to construct a complete proof of the claim that $\int_0^1 x^2\,dx = \frac{1}{3}$. D1's proof attempt began with choosing the partition of n^2 sub-intervals of length $\frac{1}{n^2}$. While correctly splitting the interval from 0 to 1 into sub-intervals with equal widths, this caused confusion when D1 did not correctly incorporate this when plugging in the maximums and minimums in the equations of the upper and lower sums, preventing D1 from constructing the proof. Nevertheless, when the interviewer asked what D1 was thinking while attempting to construct the proof, D1 explained why the integral should exist:

> Since your function is monotone increasing, every time you define a partition… [each rectangle is] going to the be upper for the one previous to it and the lower for the one after it… So as long as the partition is equidistantly spaced … You only have the last upper partition to consider. [Which] is just going to be the function value at that point, which is 1 times this infinitely thin slice, which is going to be $\frac{1}{n}$ as *n* goes to infinity so it should be nothing… it's monotone increasing so I'm always going to have this property.

D1 explained the monotonicity of the function leads the difference between the upper and lower sums to telescope to $\frac{1}{n}$, which goes to zero as n goes to infinity. Not only did this explanation demonstrate D1's ability to infer from the context of the diagram to the formal theory and to describe the relationship between them, but also justified the student's unusual choice of partition. Moreover, this monotonicity argument was not presented in lecture, illustrating D1's ability to make inferences and build further on concepts presented by the professor.

In Table 2, we present behaviors one may expect a student to exhibit as a result of their understanding archetype.

	Articulating the diagram's components	Inferring from the diagram	Linking the diagram to reasoning/proofs
Incoherent Understanding	One has an unstable or inconsistent interpretation of the diagram depending on the task evoked.	One may draw incorrect inferences, due to inconsistent understanding of diagram's components.	One cannot link the diagram to proofs in meaningful ways, since one may view proof as tightly tied to context.
Instrumental Understanding	One could specify the components of a diagram that relate to the mathematical objects discussed.	One may draw incorrect inferences, since understanding may not be integrated to the mathematical theory.	One may be able to relate the diagram to the high level ideas of the proof, but not specific logic of the proof.
Integrated Understanding	One could specify the components of a diagram that relate to the mathematical objects discussed.	One can draw inferences from the diagram, that are consistent with the formal theory.	One can use the diagram to instantiate reasoning and as a tool to construct proofs.

Table 2: Expected outcomes from the understanding archetypes

DISCUSSION

In this report, we presented a case study in which we studied how and why diagrams were used in a real analysis lecture by a highly regarded instructor, as well as how students understood the diagrams presented. In particular, we outlined three archetypes of how students may understand diagrams. There are two important observations that we have made. First, three of the five participants evinced an incoherent understanding of the associated diagrams. In particular, they were unable to see the partition diagram as representing an approximation of the area under the curve. Recall that Dr. A had a reputation as an excellent lecturer who valued diagrams. That three of five students had such a flawed understanding of the diagrams Dr. A used in his lecture illustrates the difficulties of incorporating diagrams into lectures and suggests that for many students, the presence of diagrams in lectures might not improve comprehension (cf., Alcock, 2010).

Second, we note that students might be able to use a diagram instrumentally to accomplish proving tasks without any deep understanding. We described D2, who flexibly used his diagram to prove a statement that he had not seen before. As proof is the primary means to assess performance in advanced mathematics, we imagine a professor would take D2's proof as evidence a deep understanding of the material. However, as we observed, he thought a refinement would increase the gap between upper and lower sums, implying that he could not possibly see how his proof established the existence of a Riemann integral. We contrast this with D1, who could not construct a proof despite seeming to have an integrated understanding of the diagrams. This reminds us that proof writing requires technical and algebraic expertise to complement the conceptual insights one might gain from a diagram.

Due to the small scale of the study, we make no claims of the exhaustive nature of the list of archetypes. We believe further research is necessary to investigate other archetypes for understanding diagrams and the proportion of students who fit each archetype. Such research would inform our understanding of the extent that diagrams can be used to improve understanding in lecture and how lectures might be improved.

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