PRE-UNIVERSITY STUDENTS' PERSONAL RELATIONSHIP WITH THE VISUALISATION OF SERIES OF REAL NUMBERS

Alejandro S. González-Martín

Université de Montréal, Canada

Our research focuses on the learning of series as a consequence of institutional choices for their teaching. Our analyses of textbooks and teaching practices led us to conjecture the existence of some implicit contract rules in the teaching of series: in particular, the teaching of series is made almost exclusively in the algebraic setting, with no importance given to visualisation or to the interpretation of visual images. The analysis of the students' responses to a questionnaire suggests that students learn series without developing any ability to visualise or to interpret images concerning series, which could have consequences on the learning of subsequent notions.

INTRODUCTION AND BACKGROUND

Infinite series of real numbers (*series* in what follows) are a key notion in mathematics: the idea of adding many terms was already present in ancient Greek mathematics and the use of infinite sums (either numerical or functional) allowed the development of Calculus. Series have many applications within mathematics (such as the calculation of areas by means of rectangles), and also outside of mathematics (as the modelling of situations such as the growth of interests in a bank account). These elements may explain why series are present in the introductory Calculus courses in many countries.

In Canada, each province has jurisdiction over the organisation of education and official curricula; education does not depend on the federal government. In the province of Québec, compulsory education finishes at the age of 16 and students who wish to pursue university studies need to follow two years of pre-university studies (called *collégial*) before they enter university. Students pursuing scientific or technical careers will have an introduction to Calculus during the *collégial* studies.

Research literature about the teaching and learning of series is scarce and it has mostly focused on their learning, but not on their teaching. Regarding their teaching, Robert (1982) already conjectured that teaching could have an impact in learning, and stated that the exercises used in teaching could be at the origin of the inadequate conceptions of convergence of sequences and series found in university students in France.

Regarding their learning, a summary of the main difficulties identified to learn series can be found in González-Martín, Nardi & Biza (2011). In particular, Alcock and Simpson (2004) suggest that students who regularly use visual images in their reasoning about real analysis, particularly using series and sequences, share some positive characteristics: "they all view mathematical constructs as objects, they all quickly draw conclusions about whole sets of objects, and they have confidence in their own assertions to the point of considering them obvious" (p. 29). They add that

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"those who use visual reasoning effectively do so because they build strong links between the visual and formal representations of real analysis concepts" (p. 30). Their results go in the same sense than much of the existing literature about visualisation, which underlines its importance in learning and doing mathematics, as well as in reasoning (Arcavi, 2003), and its crucial importance to experts and students alike, suggesting new results or potential approaches to proofs (Presmeg, 1986).

Our literature review led us to reflect upon whether or not the teaching of series takes into account the learning difficulties identified by research, and in particular whether the use of visualisation is encouraged by teaching practices. The first stage of our research involved the analysis of how series are presented in *collégial* textbooks, identifying some possible consequences of this presentation. We analysed a sample of 17 textbooks used in *collegial* studies in Québec from 1993 to 2008 (González-Martín *et al.*, 2011), paying special attention to the organisation of teaching. Our main results can be summarised as follows:

- *R1*: Series are usually introduced through organisations which do not lead to a questioning about their applications or their importance (*raison d'être*).
- *R2*: Organisations tend to introduce series as a tool in order to later introduce functional series, but the importance of series *per se* is usually absent.
- *R3*: These organisations tend to ignore some of the main difficulties in learning series identified by research.
- *R4*: The vast majority of tasks concerning series are related to the application of convergence criteria, or to the application of algorithmic procedures.

The second stage of the research consisted in analysing *collégial* teachers' practices and use of textbooks (González-Martín, 2010). Interviews with five teachers revealed that their practices tended to mostly reproduce what was presented in their textbooks.

As a consequence of the results of these two stages, we conjectured the existence of some implicit *contract rules* in the teaching of series in the *collégial* institutions in Québec, having a strong effect on students' learning. We have discussed some of these rules in previous papers: in González-Martín (2013a) we discussed two implicit rules implying that students do not need the definition of what a series is to solve the tasks given to them, and also that applications of series are not important; in González-Martín (2013b) we discussed the implicit rule implying that the notion of convergence is reduced to the application of convergence criteria. For the purposes of this paper, as we are interested in the use of visualisation, we only discuss the following rule

> *Rule 1*: "To solve the questions about series that are given, the use of visualisation (or any visual representation of series) is not necessary".

We conjectured the existence of this implicit rule guided by our analysis of textbooks and the interviews with teachers. We found that the number of visual images used by the textbooks to teach series was very low, especially in a conceptual way (we defined a *conceptual* image as that used to explain a concept, or to illustrate one step of a proof;

it might be part of the proving process and it is explicitly intended to help the student understand a notion or a mathematical argument) (González-Martín *et al*., pp. 572-574). In particular, the prototypical graphic representation of series used by textbooks was a variation of the image presented in Figure 1, used by textbooks in the proof of the Integral Test, which states under which conditions both $\int_a^\infty f(x)dx$ and $\sum_{n=1}^{\infty} f(n)$ are convergent or divergent. We noted that "these representations are not accompanied by an account that aims to link the representation with the algebraic and other symbolic representations of the concept used in the text. [and] the authors of the texts appear to take for granted that the students will instantly establish this connection and, for example, will interpret the rectangles appearing under a curve as representing the terms of the sum within a series" (p. 574).

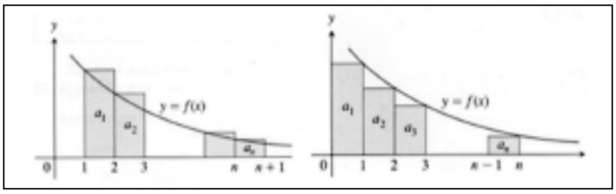


Figure 1: Prototypical representations of series found in textbooks.

We believe that *Rule 1* is a consequence of both *R3* and *R4*. The teaching of series is organised around the application of convergence criteria and algorithmic procedures (*R4*), hence activities promoting visualisation are scarce, and difficulties identified by research, as well as recommendations (as the use of visualisation), are not sufficiently taken into account (*R3*).

To verify whether *Rule 1* has an impact on *collégial* students' learning of series, we decided to create a sample of students and to apply a questionnaire. Let us define first the main elements of our theoretical framework, before clearly stating our objectives.

THEORETICAL FRAMEWORK

Chevallard's anthropological theory develops tools to better understand the choices made by an institution in order to organise the teaching of mathematical notions, as well as the possible consequences of these choices on what an individual learns. A fundamental notion in this theory is that of *institution*; an institution *I* is defined as a social organisation which allows, and imposes, on its *subjects* (every person *x* who occupies any of the possible positions *p* offered by *I*) the development of *ways of doing and of thinking proper to I* (Chevallard, 1988/89, p. 2). For instance, a classroom is an *institution* (with two main positions: teacher and student), as well as a school, or an educational system, are also *institutions*.

To analyse how an *institution* considers a notion, further definitions are required. An *object* is any entity, material or immaterial, which exists for at least one individual; in particular, any intentional product of human activity is an *object*. Every *subject x* has a *personal relationship* with any object *o*, denoted as *R*(*x*, *o*), as a product of all the interactions that *x* can have with the object *o* (using it, manipulating it, speaking of it…). This *personal relationship* is created, or modified, by entering in contact with *o*

as it is presented in different *institutions I*, where *x* occupies a given position *p*. From this *personal relationship*, a learner (if we consider an educational *institution*) will constitute what one could designate as being 'knowledge', 'know-how', 'conceptions', 'competencies', 'mastery', and 'mental images' (Chevallard, 1988/89). This notion of *relationship* is also transferred to *institutions*: given an object *o*, an *institution I*, and a position *p* in *I*, we define as the *institutional relationship* with *o* in position *p*, $R_I(p, o)$, the relationship with the object *o* which should ideally be that of the subjects in position *p* in *I*. By becoming a *subject* of *I* in position *p*, an individual *x* is subjected to the *institutional relationships* $R_I(p, o)$, which in turn will re-model his/her own *personal relationships*. This *institutional relationship* is mainly forged through the exercises (or tasks), and not only through the theoretical explanations. It is also forged through the use of elements (as symbols, images…) to refer to, or to manipulate, the mathematical notions to be constructed; these elements which allow to work concretely with abstract notions are called *ostensives* (Bosch & Chevallard, 1999).

The identification of the *institutional relationship* with a mathematical notion also allows to identify the existence of (sometimes implicit) *contract rules*, which are rules that the *institution* fosters through its practices around a mathematical notion and which contribute to determine the *institutional relationship* to a mathematical notion. This *institutional relationship* and its *contract rules* play an important role in the development of the learners' *personal relationship* with the mathematical notions s/he learns within the institution.

In our case, our objective is to have elements to characterise *collégial* students' *personal relationship* with the visualisation of series (and the use of *ostensives* in the graphic or geometric *settings*) and to see if this *personal relationship* seems to have a strong relation with the implicit *contract Rule 1* identified in the teaching processes.

METHODOLOGY

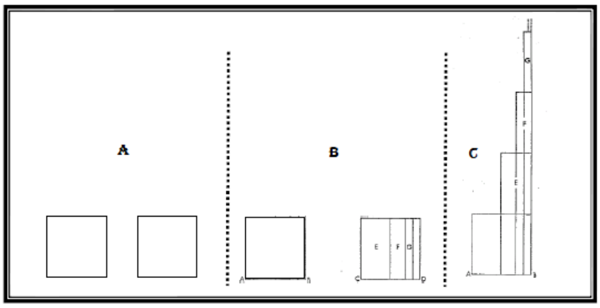
To verify the possible effects of *contract Rule 1*, among others, on *collegial* students' *personal relationship* with series, we created a sample of 32 students in their first year of *collégial* studies (where series are introduced) after the teaching of series had occurred. These 32 students come from three different mathematics teachers (named as A, B and C). Our sample consists of 4 students from teacher A (referred to as students A1 to A4), 14 students from teacher B (referred to as students B1 to B14), and 14 students from teacher C (referred to as students C1 to C14).

We constructed a questionnaire with 10 questions, aiming to assess the students' learning about series, as well as to verify our conjectures about the impact of different *contract rules* on their learning. The questionnaire was administrated in May 2011 during one of their courses (approximately 55 minutes in duration), and the students participated voluntarily.

In this paper, we discuss the students' responses to the two following questions:

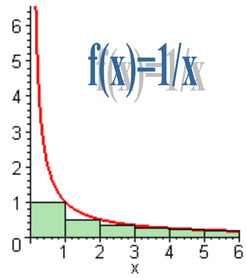
Question 8:

Which series is represented in the following image and which result does it allow visualising?
Describe the procedure represented in the image, and then write the series symbolically.



Question 10:

We know that $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ and that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. Taking into account these results, what can we say about the value of $\int_1^{\infty} \frac{1}{x} dx$ and of $\int_1^{\infty} \frac{1}{x^2} dx$?



Answer this question without making any calculation, only by using the following graph or by producing another graph if needed.

Figure 2: Questions 8 and 10.

In the next section, we present and comment on the results obtained from these questions.

DATA ANALYSIS

Question 8 (Q8)

The distribution of responses to this question is the following:

"A [square] is divided in half and one of the two halves is added to an initial identical [square]. Then, every remaining half is divided in two and added over the construction indefinitely" $\sum_{n=1}^{\infty} 1 \cdot (\frac{1}{2})^{n-1} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$; $\sum_{n=1}^{\infty} 1 \cdot (\frac{1}{2})^{n-1} = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = \frac{1}{(\frac{1}{2})} = 2$	C14
Describes correctly the image ("we divide a square by two, and then we re-divide it by two"), but unable to write it correctly symbolically	A4 B6, B9
Describes correctly the image, without attempting to write it symbolically	C8
"$\sum_{n=1}^{\infty} 1 + \frac{1}{2^n}$, but I cannot explain it" "$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$"	B1 C7

Other (wrong) attempts or (wrong) explanations	B2, B5, B7, B8 C1, C3, C5, C6, C12, C13
"I don't understand the image" "I don't understand the question"	B13 C11
No answer / ? / "Didn't have time"	A1, A2, A3 B3, B4, B10, B11, B12, B14 C2, C4, C9, C10

Table 1: Responses to Question 8.

These students have spent more than one week working with series and deciding the convergence or the divergence of quite complex series; however, confronted to a visual image of a simple series $\left(\sum_{n=0}^{\infty}\frac{1}{2^n}\right)$, only one student is able to describe it and to write it symbolically. Other four students (A4, B6, B9, C8) are able to describe it, without writing it symbolically, and two students (B1, C7) are able to write it symbolically, without describing it. Fifteen students (15/32) don't provide any answer, or acknowledge not understanding the question or the image. These results seem to go in the sense of *Rule 1*, and as the students do not need to interpret or to manipulate any visual representation of series to solve the algorithmic tasks they are usually given, they seem not to have developed any ability helping them to tackle or to interpret this type of *ostensive*. This seems to contradict the attitude of the textbooks, which seem to take for granted that students are able to interpret visual representations of series (González-Martín *et al.*, p. 574).

Question 10 (Q10)

The distribution of responses to this question is the following:

<table>
<tr><td colspan="2">Explicitly uses the graph to relate the behaviour of the series to that of the integrals</td><td>None</td></tr>
<tr><td rowspan="3">$\int_1^{\infty}\frac{1}{x}dx$ diverges and $\int_1^{\infty}\frac{1}{x^2}dx$ converges (explicitly or implicitly)</td><td>With no explanation</td><td>A1
B4
C2, C5, C8</td></tr>
<tr><td>Calculates the primitives (sometimes with errors)</td><td>B8, B13
C11</td></tr>
<tr><td>Other</td><td>A2, A3
B5, B9, B10, B12
C1, C6</td></tr>
<tr><td colspan="2">Imply (verbally or symbolically) that the value of the series and the corresponding integrals are the same</td><td>B1
C13</td></tr>
<tr><td colspan="2"></td><td>A4</td></tr>
</table>

Other interpretations	B6, B7, B11 C3, C7, C10, C12
No answer / "Didn't have time" / "I don't know"	B2, B3, B14 C4, C9, C14

Table 2: Responses to Question 10.

Again in this question, and in a more dramatic way, we see that the students are incapable of interpreting the given graph to relate the behaviour of series and integrals. However, the image used in this question should be familiar to the students, since it corresponds to the prototypical image used by textbooks to illustrate the integral test (see Figure 1). Nevertheless, even if students are supposed to be familiar with the image, and even if textbooks take for granted that students are able to interpret the image, our results seem to contradict these assumptions and the students of our sample seem to be totally incapable of interpreting and/or using the image. This image is present in the *institutional relationship* with series, but maybe because it is taken for granted, or not used in any specific task, students seem not to integrate it in their *personal relationship* with series.

FINAL REMARKS

Our analysis of textbooks and the teaching practices led us to conjecture the presence of *contract Rule 1*: abilities related to visualisation are not developed during the teaching of series. And as we conjectured, questions needing to manipulate or to interpret visual images implying series produce a very low level of correct responses in the students of our sample, seeming to confirm the presence of *contract Rule 1*. Even if the students spend a high amount of time deciding the convergence or the divergence of quite complex series, they seem incapable of interpreting visual images referring to very simple series, and their *personal relationship with series* seems to only consider the use of symbolic *ostensives*.

The lack of development of visual abilities concerning series could have serious consequences for students' learning, as the literature indicates: students might not develop a vision of series as objects and might not build strong links between the visual and formal representations of real analysis concepts (Alcock & Simpson, 2004), and they might also not develop adequately some reasoning abilities (Arcavi, 2003).

The results presented here, together with those shown in González-Martín (2013a, 2013b) seem to confirm the presence of *contract rules* influencing students' learning of series: they use series without being able to define them, or without knowing what they are useful for, reducing the notion of convergence to the application of criteria, and not developing abilities of visualisation and interpretation of series. These elements seem to be very clear in students' *personal relationship* with series, and this seems to be a consequence of the *institutional relationship* with series, which fosters the presence of the *contract rules*. The impact for the learning of subsequent notions seems too dramatic to be ignored, and research aiming to change this *institutional*

relationship with series and, as a consequence, students' *personal relationship* with series appears to be urgent.

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