WHY LECTURES IN ADVANCED MATHEMATICS OFTEN FAIL

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This case study investigates the effectiveness of a lecture in advanced mathematics. We video recorded a lecture delivered by an experienced professor. Using video recall, we then interviewed the professor to determine the content he intended to convey and we analyzed his lecture to see if and how this content was conveyed. We also interviewed six students to see what they understood from this lecture. The students did not comprehend much of the content that the professor intended to cover in his lecture. We propose three reasons for why students failed to grasp much of the content that the professor intended to convey.

INTRODUCTION

This paper investigates the seeming paradox that an excellent mathematics teacher delivering a high-quality lecture may not result in student learning gains. The specific context we study is the proof-based real analysis course. There is a widely held belief amongst mathematics educators that most lectures in advanced mathematics are ineffective for developing students' understanding of mathematics (e.g., Davis & Hersh, 1981; Dreyfus, 1991; Rosenthal, 1995). Perhaps the most common complaint is that the predominance of definitions, theorems, and proofs in lectures leads the lecturer to pay scant attention to other important types of mathematical thinking (e.g., Davis & Hersh, 1981; Dreyfus, 1991). Consequently issues such as informal ways of understanding mathematical concepts (e.g., graphical or diagrammatic interpretations of concepts), why theorems appeared plausible to mathematicians, and how these proofs could have been constructed are (purportedly) largely ignored in advanced mathematics lectures. Yet, extant case studies (e.g., Fukawa-Connelly & Newton, in press; Weber, 2004) show lecturers use informal representations of concepts such as examples and diagrams to help students understand the content. Similarly, the interview data of Yopp (2011) and Weber (2012) found that mathematics professors claimed to focus on things such as providing explanation and illustrating proof methods, rather than a formal proof.

Research has generally not explored how mathematics majors comprehend or gain understanding from the proofs that they read (Mejia-Ramos & Inglis, 2009). This case study examines the presentation of proof in a real analysis lecture and what students might learn from it via the following research questions:

1. What content did the professor intend to convey in his lecture?
2. How was this intended content presented in his lecture (if at all)?
3. What did the students in this class perceive to be the important content in the proof and did it align with the professor's goals?

2014. In Oesterle, S., Liljedahl, P., Nicol, C., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 3, pp. 129-136. Vancouver, Canada: PME.

4. In cases where students' interpretations of the lecture differed from the professor's intent, what factors could explain these discrepancies?

THEORETICAL PERSPECTIVE

According to de Villiers (1990), mathematicians engage in the activity of proving for five different purposes: (1) to *verify* that a theorem is true and that the conclusion of a theorem being proven is a necessary consequence of the premises of that theorem (although de Villiers emphasized that this was not the primary function of proof); (2) to go beyond verifying that a theorem and *explaining why* it is true (Hanna, 1990, and Hersh, 1993, argued that explanation should be the primary function of proof in the classroom); (3) to *discover* new ideas and methods that will help mathematicians solve problems that they are working on (Mejia-Ramos and Weber, in press, reported that mathematicians claim this is one of the main reasons they read proofs); (4) to *communicate* new mathematical ideas, tools, and proof techniques with other mathematicians; and (5) to *systematize* a body of mathematical knowledge by showing how new definitions or axiom systems can account for results that are known to be true (cf., Weber, 2002). In this report we focus on the second and third purposes: using a proof as an *explanation* for a particular mathematical idea and as a way to *discover* new methods students could use to solve other problems.

We also follow the New Literacy Studies movement (Gee, 1990) and treat the totality of a lecture, including the words spoken by the professor, his chalk inscriptions and kinesthetic movements, as a single coherent piece of text. Our interest is in the meanings that the professor attempted to imbue in the text, the meanings that students constructed from reading this text, and discrepancies that may arise between the two. Our theoretical analysis suggests three reasons that students might fail to understand a proof in lecture: (i) the professor may not believe conceptual explanations and methods are important and not include them; (ii) the professor might fail to encode the content into the text, (iii) the students might lack the tools to interpret the text.

METHODS

The lecture

This research took place at a large American state university, in a real analysis course, which is, as is typical for the U.S., a junior-level course required for mathematics majors. We studied a section of the course taught by Dr. A (a pseudonym), a highly-experienced and well-respected instructor, videotaping one of his lectures. This study focuses on the proof from that lecture that we felt was the most conceptually interesting. To avoid ambiguity, we refer to the *blackboard proof* as the text that Dr. A inscribed on the blackboard and the lecture proof as the totality of the 10-minute segment. Our analysis of the *lecture proof* suggested that there is substantial content that can be learned from it. That content could focus on explanation, methods (i.e., discovering how to find new theorem), or conviction/validity. For the sake of brevity, we focus on the use of Cauchy sequences (methods) and that Cauchy sequences are those that bunch up (explanation).

After the initial analysis of the text, the 2nd author met with Dr. A for an audio-recorded interview. The interview focused on the main ideas he wished to convey via the proof presentation and used video recall to prompt him to reflect on how he attempted to convey those ideas. When we analyzed Dr. A's comments, if they were consistent with what we observed, we would fold them into the categories that we formed in our analysis of the lecture. If he introduced new ideas or described the content that we observed in a different way, we would form a new category.

Student data

We collected notes and interviewed six students. The interviews were with pairs of students and video recorded. Pair 1 consists of S(tudent)1 and S2, Pair 2 is S3 and S4, Pair 3 is S5 and S6. From Dr. A's perspective, these students displayed a wide range of performance, but were collectively above average in their class. We asked the students to consider the lecture proof in three passes. First, we asked them to describe what they learned from the lecture based upon their notes to see what they could reconstruct. Second, we showed them the entire proof on video in order to explore their interpretations of what Dr. A considered the main ideas of the proof. In the third pass, we showed the students short clips of the lecture and after each clip, asked what they understood to measure whether the participants had the means to interpret what Dr. A considered to be the main content of the proof. In each pass, we compared their claims to the conceptual meaning that Dr. A ascribed to the proof presentation.

THE LECTURE

First, we note that Dr. A's lecture proof was more detailed than his blackboard proof. The latter was a polished proof that might appear in a textbook. However, in the lecture proof, he supplemented the blackboard proof with many oral comments about the proof writing process and his thinking about concepts. That is, all statements about the methods and content he intended to convey were stated orally, not written on the blackboard.

In the theorem about sequences that Dr. A proved in class, a specific sequence is not given. Rather, the theorem states that the sequence has the property that the distance between any two consecutive elements x_n and x_{n-1}.is less than r^n, where r is a constant with $0 < r < 1$. One cannot prove that such a sequence converges simply by applying the definition of convergence (given that we cannot know what the limit will be), so another approach is needed. In the proof presented by Dr. A, the sequence is shown to be convergent by demonstrating that it is a special type of sequence called a Cauchy sequence (in a previous class, students had seen a proof that all Cauchy sequences are convergent sequences). A key point stressed at several points in Dr. A's lecture proof is that this theorem was useful to apply when one wanted to prove a sequence was convergent, but could not determine what the limit of the sequence was.

ANALYZING THE TEXT VIA THE PERSPECTIVE OF ITS AUTHOR

When asked why he chose to present this proof, Dr. A gave an 11 minute response, situating Cauchy sequences along students' mathematical progression starting with calculus and concluding with the study of measurable functions in graduate school. He tied this to the importance of repetition of ideas, suggesting that students do not gain intuition and understanding the first time they view a proof. Rather, he believed students came to grasp ideas through repeated exposure. Describing the main things he intended to convey to students with this proof, Dr. A emphasized thinking of Cauchy sequences in terms of pictures, using the word "picture" 32 times. He began:

> What has to be emphasized over and over again is that these definitions, which you might write down in symbols, are not going to make sense to you unless you have a picture associated with it.

However, when viewing the proof, Dr. A was surprised that he actually did not include any pictures, saying "this is a poor example. There are no pictures here!" When asked what content he was trying to convey while presenting this proof, he cited that he wanted students to view these sequences as "bunching up," which from our perspective implied that the terms of the tail of the sequence would become arbitrarily close together and "bunch up" around a particular point. Dr. A explained:

> If you go far enough out in the sequence, the difference between any two terms whose index, the m, the n, are large enough. Will always be less than epsilon. What that says is that they bunch up [Dr. A places hands vertically and parallel to one another and slowly moves his two hands towards each other]. So the Cauchy property for a sequence is, the property says they bunch up [Dr. A repeats the gesture described above] in some place.

STUDENTS' PERCEPTIONS OF THE LECTURE

Pass 1: Students' recall of the content of the lecture from their notes

First, we note that five of the students recoded only what was written on the board in their notes. The sixth student (S1) was an exception: she recorded nearly everything Dr. A said aloud, as well as what he wrote. The students did not mention the content that Dr. A aimed to convey in this proof, although their summaries did, generally, have mathematical value. No student mentioned the critical point, emphasized thrice in the Dr. A's presentation, that using Cauchy sequences to establish convergence was specifically useful if one did not know what value to which the sequence converged. Perhaps they did not recall this content from their notes because it was part of Dr. A's oral but not written presentation and they only recorded the written proof.

Pass 2: Students' perceptions of the content after viewing the proof

Students' comments in this pass through the data (i.e. after showing them a video-recording of Dr. A's presentation of the proof) were more detailed than in the first pass. Although all pairs of students highlighted important content in the proof, none of the students mentioned that showing a sequence is Cauchy is an important method for proving the sequence is convergent *particularly* when one does not know what the limit of the sequence is. Two pairs of students mentioned that showing the sequence was Cauchy was a way of establishing convergence and S4 observed the repetition of this proof structure:

> Other than showing that a contractive sequence is a Cauchy sequence, I think it's more. He's showing more of the structure of the proof […] A lot of the proofs that we did over the last nine or so weeks basically have the same structure […]

But no student mentioned the conditions under which this was likely to be useful even though Dr. A emphasized these conditions at three separate points in his lecture.

Pass 3: Students' interpretations of specific video clips

In Clip 1, Dr. A claimed to be trying to give students some geometric intuition for what was being asserted in the theorem and why the theorem was true (the sequence, like Cauchy sequences, bunches up), our analysis of his presentation suggested that such content was available from his lecture proof, but not from the blackboard proof. By this pass, S2, S5 and S6 indeed believed this clip was trying to establish geometric intuition for why the sequence converged, that is, that the sequence was 'bunching up'. S2 said, "I mean, this is fairly intuitive. You look at it and the *r* to the *n*'s are going to keep going up and so this interval is going to keep shrinking, so of course it would be natural to suggest Cauchy sequence." The idea of the interval shrinking is what Dr. A meant by 'bunching.' Both S3 and S4 said that Dr. A was trying to convey that one can show a sequence is Cauchy without knowing its limit, but only describe Cauchy sequences as 'bunching up' after the interviewer directly asked them if the clip suggested that.

In Clip 2, Dr. A introduced the idea of Cauchy sequences as a way to show that a sequence is convergent. Dr. A asks the students what types of sequences converge even if the limit cannot be determined, saying:

> There's no mention of what the definition is of the sequence, so there's no way we're going to be able to verify the definition limit of a convergent sequence, where we have to produce the limit. So what do we do? […] What kind of sequences do we know converge even if we don't know what their limits are? It begins with a 'c'.

Both Pair 1 and Pair 2 believed Dr. A was trying to convey that one can show a sequence is convergent by showing it is Cauchy, which is useful if you do not know the limit of the sequence. For instance, S1 said, "we should recognize it, like to figure out it's a Cauchy, we should know that it's converging, but its limit is not necessarily given." However, Pair 3 did not mention this.

In Clip 3, Dr. A explicitly highlights that one can show a sequence is Cauchy without knowing what the limit is:

> We will show that this sequence converges by showing that it is a Cauchy sequence [writes this sentence on the board as he says it aloud, then turns around to face class]. A Cauchy sequence is defined without any mention of limit.

Pair 1 and Pair 2 repeated that the intent here was to remind students that one can show a sequence was Cauchy without knowing its limit. Pair 3 again made no comment of this type.

In Clip 4, Dr. A again reiterates that one cannot find a limit for the sequence in the theorem and showing the sequence is convergent involves showing that it is Cauchy.

> And now we'll state what it is we have to show. ... See there is no mention of how the terms of the sequence are defined. There is no way in which we would be able to propose a limit L. So we have no way of proceeding except for showing that it is a Cauchy sequence or a contractive sequence. So let's look and see how we proceed.

Only Pair 2 remarked that Dr. A was trying to convey that one needed to use Cauchy sequences to establish convergence because one could not propose a limit of the given sequence. Both students in Pair 1 were unsure of the intention of the clip.

For Pair 3, S6 mentioned that the limit of the sequence could not be determined. He said, "he wanted to emphasize that there is no mention of limit whatsoever, so we won't like confuse it with the concept of limit". Our interpretation of this excerpt is that they perceived Dr. A as noting that their previous approaches to showing convergence, which relied on knowing the limit of the convergent sequence, would be inadequate for this problem. Pair 3 did not, however, mention that showing a sequence was Cauchy would be useful since the definition of Cauchy sequences did not involve the definition of limit even when specifically asked whether the clip suggested that showing the sequence was Cauchy would be useful.

Summary of students' perceptions of the lecture

When shown specific clips that Dr. A highlighted, the students were collectively much better at identifying what Dr. A aimed to convey, than when recalling the content of the lecture just from their notes (Pass 1) or after showing them a video-recording of Dr. A's presentation (Pass 2). As this content concerned conceptual explanation and method, the findings above indicate that these students could decode some of this content that Dr. A expressed orally, if asked to do so immediately after viewing his comments regarding that specific content.

First, all three pairs of students observed that the proof illustrated a new way to show a sequence was convergent—namely by showing that it was Cauchy—and Pair 1 and Pair 2 remarked on this in the first pass through the proof. However, the conditions under which this was useful, when a limit for the sequence could not be proposed, were less prevalent in students' responses. Pair 1 and Pair 2 did not mention this until the third pass through the data and Pair 3 did not discuss this content in any pass through the data. Similarly, Pair 1 and Pair 3 that Cauchy sequences 'bunch up' but only when shown the specific short clips in which this was mentioned while Pair 2 only stated this as true when the interviewer specifically asked them if it would be seen in the data. Finally, there were two instances where students described content but at a more shallow level than Dr. A intended.

DISCUSSION

Consistent with claims from the literature (Yopp, 2011; Weber, 2012), Dr. A emphasized conceptual explanations and method content when discussing this proof. Dr. A valued conceptual explanatory content in the form of pictures, but did not include any in his proof which aligns with findings suggesting that while mathematicians value such informal ways of thinking, their actual decisions about teaching might de-emphasize them (Alcock, 2010; Lai & Weber, 2013). Also, Dr. A would state his method content orally, but not include it in his blackboard proof, which is consistent with other findings from the literature (Fukawa-Connelly, 2013; Weber, 2004). Further investigation is necessary to see how common this practice is.

Students were able to say more about some of the content of the proof when presented with a short clip in which Dr. A encoded this content than in the first two passes. Thus, we claim students possessed the means to interpret the lecture proof, but did not use them when watching the lecture proof in its entirety. There are several possible reasons why this may have occurred, ranging from students essentially ignoring this content, not having time or cognitive resources to attend to it, or, simply not prioritizing it in their discussions of the proof. Five of the students only transcribed written content into their notes. Combined with the fact that most of Dr. A's conceptual explanations were stated orally but not written, this suggests a reason for why comprehension was not occurring; students did not see this content as valuable to attend to. Finally, there was some content that both the research team and Dr. A felt was important that students seemed to lack the means to interpret.

There are two significant limitations of this study. The first is that this was a case study studying a single lecture proof. More research is needed to determine how common these themes are with other professors or with the presentation of other proofs. Second, identifying why comprehension fails to occur does not necessarily imply how lectures in advanced mathematics can be improved. Although, some of our projects have demonstrated that students have inadequate beliefs and strategies for reading mathematical proofs (e.g., Weber & Mejia-Ramos, in press) and focus on developing interventions that will help students understand proofs better.

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