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Remediating Difficulty with Fractions for Students with Mathematics Learning Difficulties

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Competence with fractions is foundational to acquiring more advanced mathematical skills. However, achieving competency with fractions is challenging for many students, especially for those with mathematics learning difficulties who often lack foundational skill with whole numbers. Teaching fractions is also challenging for many teachers as they often experience gaps in their own fractions knowledge. In this article, the authors explain the sources of difficulty when learning and teaching fractions. Then, the authors describe effective instructional strategies for teaching fractions, derived from three randomized control trials. Implications for practice are discussed.

Keywords: *Fractions, instructional strategies, mathematics learning difficulties*

Competence with fractions is foundational to acquiring more advanced mathematical skills, such as algebra (Booth & Newton, 2012; Booth, Newton, & Twiss-Garrity, 2014; National Mathematics Advisory Panel [NMAP], 2008). However, achieving competency with fractions is challenging for many students, and the difficulties associated with learning fractions have been documented widely (e.g., NMAP, 2008; Nunes & Bryant, 2008; Stafylidou & Vosniadou, 2004). For example, in a national survey of algebra teachers, fractions was rated as the second most important deficit area explaining students' difficulty in learning algebra (Hoffer, Venkataraman, Hedberg, & Shagle, 2007). Accumulating data from the National Assessment of Educational Progress (NAEP) also provide evidence for students' difficulties with fractions. According to the 2017 NAEP, only 32% of fourth graders correctly identified which fractions were greater than, less than, or equal to a benchmark fraction, 1/2. In 2009 NAEP, only 25% of fourth graders correctly identified a fraction closest to 1/2. As demonstrated by the performance on the two related fraction NAEP items, difficulty with fractions is not new; it persists over time.

Learning fractions can be especially challenging for students with mathematics learning difficulties. Namkung, Fuchs, and Koziol (2018) found that students with severe mathematics learning difficulties, as indexed by their whole-number competence below the 10th percentile at fourth grade, were 32 times more likely than students with intact whole-number knowledge to experience difficulty with fractions. Students with less severe mathematics learning difficulties (between the 10th and 25th percentile) were five times more likely to experience difficulty with fractions than students with intact whole number knowledge. Likewise, Resnick et al. (2016) found that students with inaccurate whole-number line estimation performance were twice as likely to show low-growth in fraction magnitude understanding compared to those with accurate whole-number line estimation skills. Therefore, a critical need exists to improve fractions learning for students with mathematics learning difficulties. The first purpose of this paper was to explain why learning and teaching fractions present major challenges for students and teachers; the second purpose was to describe effective instructional strategies for teaching fractions, derived from our randomized control trials examining the efficacy of fractions intervention for students with mathematics learning difficulties.

Difficulty with Learning Fractions

Traditionally, difficulty with fractions has been attributed to fundamental differences between whole numbers and fractions. This can lead to whole-number bias, which refers to students' overgeneralization of whole-number knowledge to fractions (DeWolf & Vosniadou, 2015; Ni & Zhou, 2005). That is, students assimilate whole-number concepts into understanding fractions, which subsequently leads to misconceptions about fractions due to the inherent differences between whole numbers and fractions. For example, whole numbers are represented with one numeral whereas fractions are represented with two numerals and a fraction bar. One common misconception that arises from whole-number bias is that students often view numerators and denominators as independent whole numbers, instead of interpreting a fraction as one number. This often results in common errors, such as adding both numerators and denominators across two fractions (e.g., 2/3 + 4/6 = 6/9).

A second distinction between fractions and whole numbers is that whole numbers can be counted and placed in order; by contrast, there is infinite density of fractions in every segment of the number line. Thus, when comparing whole numbers, students can use counting strategies to identify a greater number as each number in the counting sequence always has a greater value than the previous number (e.g., 3 > 2). In fractions, the same counting strategies are not productive. Thus, common errors include students misapplying the whole-number properties to compare the value of fractions. For example, students often think that 1/12 is greater than 1/2 since 12 is greater than 2. Accordingly, Malone and Fuchs (2017) found that 65% of errors in ordering fractions at fourth grade were due to students misapplying whole-number ordering to fractions (e.g., 1/8 > 1/6 > 1/3). A third distinction is that fraction operation procedures differ from whole number operations. Adding and subtracting fractions require a common denominator whereas multiplying or dividing fractions do not require a common denominator. Further, quantities decrease with multiplying fractions and increase with dividing fractions whereas the opposite is true for whole numbers. Due to these distinctions between whole numbers and fractions, learning fractions has been originally conceptualized as distinct from the learning of whole numbers (e.g., Cramer, Post, & delMas, 2002; Cramer & Wyberg, 2009; Vosniadou, Vamvakoussi, & Skopeiliti, 2008).

Even so, the literature is mixed on whether prior whole-number knowledge interferes with or facilitates fractions learning. More recent studies reveal that strong whole-number knowledge supports fractions learning (e.g., Namkung et al., 2018; Resnick et al., 2016; Rinne, Ye, & Jordan, 2017). Students with a strong foundation in whole-number magnitude understanding had more accurate fraction magnitude understanding than those who did not (Resnick et al., 2016), and whole number magnitude understanding also predicted an accurate strategy use for comparing fractions. Further, Namkung et al. (2018) found that students with strong whole-number calculation skills were less likely to develop difficulties with fractions.

These findings support the integrated theory of numerical development proposed by Siegler and colleagues (Siegler, Thompson, & Schneider, 2011). In contrast to the whole-number bias framework, the integrated theory of numerical development posits that fractions understanding develops as part of gradual expansion and refinement of understanding of number systems that all numbers, including fractions, have magnitudes that can be assigned to specific locations on number lines. That is, although whole-number bias may cause some challenges with the initial learning of fractions, the development of fraction knowledge is not independent from that of whole numbers.

Instructional Practices in Teaching Fractions

With this shift toward conceptualizing fractions in terms of an integrated system of numbers, a shift has also occurred in how fractions are taught. Fractions instruction in the United States had predominately relied on teaching part-whole understanding (Fuchs, Sterba, Fuchs, & Malone, 2016c; Ni & Zhou, 2005; Thompson & Saldanha, 2003). Part-whole understanding refers to conceptualizing fractions as representing one or more equal parts of an object or set of objects. Thus, instruction on part-whole understanding often focuses on equal-sharing (e.g., one slice of a whole pizza when sharing equally among five friends to represent 1/5) and area models (e.g., one shaded part of a rectangle divided equally into five parts to represent 1/5). Both equal-sharing and area models teach fractions as part of one whole and have great advantages of being concrete and accessible for initial learning of fractions (Siegler et al., 2011).

Unfortunately, viewing fractions as only part of a whole limits students' understanding, especially for fractions greater than one (i.e., improper fractions, 9/5) and for fractions with large numerators and dominators (Siegler et al., 2011; Tzur, 1999). Further, part-whole interpretation encourages students to separate numerators from denominators, which reinforces the common misconception of treating a fraction as two independent whole numbers (Fuchs, Malone, Schumacher, Namkung, & Wang, 2017).

Combined with the emphasis on understanding fractions as numbers with numerical magnitudes, as reflected in the integrated numerical development theory, prior research shows that understanding fractions as measurements of quantity improves fractions learning (e.g., Keijzer & Terwel, 2003; Rittle-Johnson & Koedinger, 2009; Siegler & Ramani, 2009). Measurement understanding refers to conceptualizing fractions as numbers that reflect cardinal size, the core component of the integrated numerical development theory (Hecht & Vagi, 2010). Measurement understanding, which is often represented with number lines (e.g., 1/2 is half way between 0 and 1 on a number line), promotes deeper understanding of fractions. Number lines can teach proper and improper fractions in conceptually similar ways that teaching improper fractions easily make sense (e.g., 3/2 is 1 plus 1/2 way between 1 and 2 on a number line; Wu, 2009).

This form of fraction magnitude understanding has been found to predict not only fraction-related skills, such as fraction computations and conceptual understanding of fractions (e.g., Hecht, 1998; Hecht & Vagi, 2010; Siegler et al., 2011; Vamvakoussi & Vosniadou, 2010), but also other mathematics skills, such as algebra and overall mathematics performance (Bailey, Hoard, Nugent, & Geary, 2012; Booth & Newton, 2012; Booth et al., 2014; Siegler et al., 2011; Siegler & Pyke, 2013). The NMAP posited that improvement in the measurement understanding of fractions may be a key mechanism for achieving competency with fractions (NMAP, 2008).

Difficulty with Teaching Fractions

Although we have a better understanding of what should be the focus of our fractions instruction, teaching fractions can also pose significant challenges because many adults and even expert mathematicians sometimes have difficulty with fractions (e.g., DeWolf & Vosniadou, 2015; Lewis, Mathews, & Hubbard, 2015; Obersteiner, Van Dooren, Van Hoof, & Verschaffel, 2013; Schneider & Siegler, 2010). Therefore, it is not surprising that weak fractions knowledge has been documented with preservice and inservice teachers. Siegler and Lortie-Forgues (2015) examined conceptual understanding of fraction computations among preservice teachers, middle school students, and mathematics and science majors at a university. Although preservice teachers had good understanding of magnitudes for fractions less than 1, they showed minimal understanding of multiplication and division of fractions less than 1. In fact, preservice teachers and middle school students showed similar levels of fractions understanding.

Research also documents that many elementary school teachers lack fraction competence with ordering fractions, adding fractions, and explaining computations for fractions (Garet et al., 2010; Ma, 1999). This is unfortunate given that teachers' mathematics knowledge predicts student learning (e.g., Hill, Rowan, & Ball, 2005; Kersting, Givvin, Thompson, Santagata, & Stigler, 2012; Kunter, Klusmann, Baumert, Richter, Voss, & Hachfeld, 2013). Therefore, both the NCTM and the Institute of Education Sciences (IES; Siegler et al., 2010) emphasized the importance of improving teachers' understanding of fractions. However, increasing teachers' fraction knowledge via professional development has failed to improve teachers' or students' rational number knowledge (Garet et al., 2011). This calls for more effective, alternative ways, such as structured intervention programs, to guide teachers and improve both teachers' and students' knowledge about fractions. In support of this view, Malone and Fuchs (2019) found that a structured fractions intervention program not only improved struggling students' fractions knowledge, but also improved the tutors' own fractions knowledge.

Effective Fractions Intervention for Students with Mathematics Learning Difficulties

Taken together, prior research suggests that whole numbers and fractions develop in an integrated way despite fundamental distinctions between them. This has implications for students with mathematics learning difficulties who often lack strong whole-number foundations, with research indicating that students with weak whole-number competence are at a great disadvantage for learning fractions (Namkung et al., 2018; Resnick et al., 2016). Second, a critical component of numerical development is understanding number magnitudes. This includes whole numbers and fractions. Based on this, instruction on fractions has begun to emphasize understanding fractions as numbers with magnitudes. Such an emphasis has been also found to improve fractions learning of students with mathematics learning difficulties (e.g., Fazio, Kennedy, & Siegler, 2016; Fuchs et al., 2013, 2014, 2016c). Third, not only learning fractions, but teaching fractions also pose significant challenges because preservice and interservice teachers often have gaps in their fraction knowledge. A structured fractions intervention may be an alternative way to guide teachers to effectively teach fraction concepts to their students and improve both their own and their students' fractions knowledge.

We next summarize findings from three randomized controlled trials we have conducted (Fuchs et al., 2016a, 2016b; Wang, Fuchs, Fuchs, Gilbert, Krowka, & Abramson, 2019) that estimated the effects of fractions intervention for students with mathematics learning difficulties. The core fractions intervention across these studies emphasizes measurement understanding of fractions via fraction comparison, fraction ordering, fraction word-problem, and fraction calculation activities. We begin by discussing two randomized control trials (Fuchs et al., 2016a, 2016b), in which fourth graders with mathematics learning difficulties were randomly assigned to a control group (business as usual, schools' fractions instruction) or a fractions intervention group. In these two studies, trained research assistants implemented the core fractions intervention program, *Fraction Face-Off!* (Fuchs, Schumacher, Malone, & Fuchs, 2015; See www.frg.vkcsites.org for the materials and sample lessons).

Description of *Fraction Face-Off!*

This Tier-2 core fractions program is designed to promote fractions understanding of fourth graders with mathematics learning difficulties in multiple ways. First, the program mainly emphasizes the measurement interpretation of fractions with instruction on understanding the magnitude of fractions, comparing fraction sizes, ordering three fractions, placing fractions on number lines, finding equivalent fractions, adding and subtracting fractions, and converting fractions (mixed number to improper fractions and vice versa). Second, concrete manipulatives, such as fraction circles, fraction tiles, and number lines are used throughout the lessons. Third, the intervention relies on explicit instruction, such as scaffolding, providing immediate and corrective feedback, and optimizing student attention and motivation with self-regulated learning strategies. Each lesson starts with modeling, in which tutors introduce concepts, skills, and strategies with concrete and representational manipulatives, followed by guided practice, in which students take turns completing problems cooperatively with tutors' prompts. Then, students independently complete problems, on which tutors provide immediate, corrective feedback. Students also earn "Fraction Money" for working hard and completing activities correctly during the intervention sessions. This "money" can be spent on prizes at the end of each intervention week. Details of the scope and sequence of the core intervention are described below.

Each lesson is approximately 30-35 min and is implemented three times a week for 12 weeks (36 lessons). In Weeks 1-2, students are introduced to fraction vocabulary (e.g., denominator, numerator, unit), naming and reading fractions, and fractions equivalent to one whole. Students also learn how to compare fractions with the same denominators (e.g., 7/12 > 3/12) or the same numerators (1/4 > 1/8). In Weeks 3-5, students learn fractions equivalent to 1/2 (e.g., 2/4, 4/8, 5/10) and learn to use 1/2 as a benchmark for comparing fractions, building off prior lessons on comparing fractions with the same denominators (e.g., 2/6 < 1/2 because 2/6 < 3/6). Students learn how to order fractions (e.g., 2/8, 1/2, 3/4) and place fractions on a 0-1 number line marked with 1/2 (e.g., A fraction [4/12] less than 1/2 is placed in between 0 and 1/2 on the number line). In Week 6-8, students are introduced to improper fractions (e.g., 9/8) and mixed numbers (e.g., 1 3/4) on a 0-2 number line, and how to covert between improper fractions and mixed numbers. Improper fractions and mixed numbers are also integrated into comparing and ordering activities. In Week 9, students learn fraction calculations: adding and subtracting with like denominators, with unlike denominators, and with mixed numbers. In Week 10, students continue to work with number lines, but with benchmarks (1/2 on 0-1 number line and 1 on 0-2 number line) deleted. In Weeks 11 and 12, students practice previously learned skills via a cumulative review game.

Multiplicative Fractions Word-Problem Solving Strategy

In Fuchs et al. (2016b), two variants of the core fractions program were designed to examine the effects of two forms of fraction word-problem solving components: multiplicative word problems and additive word problems. However, the primary focus was on the multiplicative word problems because multiplicative thinking is more difficult to achieve than additive understanding and is central to understanding fraction equivalencies. These word-problem components are integrated into the core fractions program. The word-problem instruction relies on schema-based instruction, an evidence-based strategy for teaching word-problem solving (e.g., Fuchs et al., 2003, 2009; Jitendra et al., 2009, Jitendra & Star, 2011). As with schema-based instruction, students are taught to identify word problem types that share structural features and represent the underlying structure with a number sentence or visual display.

The multiplicative word-problem intervention focuses on two types of multiplicative word problems: "splitting" and "grouping." In splitting, a unit is divided, cut, or split into equal parts (e.g., Melissa had two lemons. She cut each lemon in half. How many pieces of lemon does Melissa have now?). By contrast, in grouping, fractional pieces are combined to form a unit (e.g., Keisha wants to make eight necklaces for her friends. For each necklace, she needs

1/2 of a yard of string. How many yards of string does Keisha need?). The additive word-problem intervention also focuses on two types: "increase" and "decrease." In increase, something happens to the initial amount to increase the starting amount (e.g., Marria bought 4/10 of a pound of candy. Later she bought another 3/10 of a pound of candy. How many pounds of candy does Maria have?). By contrast, in decrease, something happens to the initial amount to decrease the starting amount (e.g., Jessica had 5/6 of a cake. She gave 2/6 of a cake to her friend. How much cake does Jessica have now?).

Participants. A total of 213 fourth graders with mathematics learning difficulties (i.e., performing below the 35th percentile on a broad calculations test) were randomly assigned to one of three conditions: the core fractions program with the multiplicative word-problem solving component (n = 72), the core fractions program with the additive word-problem solving component (n = 71), or the business-as-usual control (n = 70). Table 1 provides detailed demographic information by each condition.

Outcome measures. Students were assessed on three outcome measures: Fraction Number Line 0-2 (Hamlett, Schumacher, & Fuchs, 2011, adapted from Siegler et al., 2011), fraction calculations, and fraction word problems. On Fraction Number Line, students are asked to place proper fractions (e.g., 1/4, 5/6), improper fractions (e.g., 3/2, 15/8), and mixed numbers (e.g., 1 11/12 , 1 5/6) on a 0-2 number line on a computer. On fraction calculations, students solve 12 fraction addition problems (five with like denominators and seven with unlike denominators), and 12 fraction subtraction problems (six with like denominators and six with unlike denominators). On fraction word problems, 12 problems focus on multiplicative word problems (6 splitting problems and 6 grouping problems) and 12 problems focus on additive word problems (6 increase problems and 6 decrease problems). Each question is read aloud to students.

Intervention conditions. Students in the intervention conditions received intervention for 35 min per session, three times a week, for 12 weeks. Trained research assistants implemented the intervention with pairs of students. Of each 35-minute session, 28 min are identical across both multiplicative and additive word-problem conditions. The first 7 minutes of the tutoring session differ by the word problem condition. Students in the multiplicative word-problem condition received instruction on multiplicative fraction word problems, and students in the additive word-problem condition received instruction on additive fraction word problem. During the 7 min of word-problem instruction, students identify the problem type (splitting vs. grouping in the multiplicative condition, increase vs. decrease in the additive condition), represent the underlying structure of the problem type with a number sentence (the additive condition) or visual display (the multiplicative condition), and solve for the unknown and write a numerical answer with a word label.

In both conditions, one type of problem (splitting in the multiplicative condition and increase in the additive condition) is first introduced without a missing part. Then, tutors explicitly teach the underlying structure of the problem type and teach students to recognize, identify, and explain the problem type. About halfway through the intervention, another problem type (grouping in the multiplicative condition; increase in the additive condition) is introduced and is taught in a similar way. In both conditions, distractor word problems that require students to identify the greater or less fraction (Compare problems; e.g., Ruby ate 1/3 of the pizza, and Bob ate 1/8 of the pizza. Who ate less pizza?) are introduced to prevent overgeneralization and to promote students' recognition of nonexamples. Every intervention session was audiotaped

Table 1
Demographics

	Fuchs et al., 2016b			Fuchs et al., 2016a			Wang et al., 2019		
Condition	M-WP (n = 72) %	A-WP (n = 71) %	Control (n = 70) %	M-WP (n = 69) %	EXP (n = 73) %	Control (n = 70) %	FCI (n = 23) %	SR (n = 23) %	Control (n = 23) %
Females	54	54	61	55	58	48	60	52	43
Race									
African American	57	58	59	57	49	43	52	61	44
Hispanic	18	10	19	23	27	30	9	9	26
White	21	24	17	19	19	21	30	30	22
Other	4	8	5	1	4	6	9	-	4
Free/reduced lunch	93	86	86	90	93	87	96	91	96
English-language learner	11	23	16	13	7	13	22	35	-
School-identified disability	17	17	10	13	7	13	13	-	9

Note. M-WP = Multiplicative word problem; A-WP = Additive word problem; EXP = Supported self-explaining; FCI = Fractions core intervention; SR = Self-regulation and randomly sampled to check fidelity of implementing the intervention. The mean percentage of implementation ranged from 97.54 to 98.74.

Control condition. The classroom fractions instruction relied on the district's fourth-grade mathematics curriculum, enVisonMATH (Pearson Education, 2011). The enVisionMATH program primarily focuses on part-whole understanding by using shaded regions and other area model manipulatives. Adding and subtracting fractions are taught procedurally. For fraction word problems, additive word problems and equal sharing word problems are predominantly represented with little emphasis on multiplicative word problems. The word-problem solving strategies in the control group focused on drawing pictures, making tables, and suing key words. In addition, the control group instruction addressed advanced skills not covered in the intervention, such as fraction estimation, and taught a broader range of fractions in general and for teaching equivalent fractions and reducing activities.

Results. Students in both intervention conditions outperformed the control group on the number-line task (effect sizes [ESs] = 0.81 for the additive condition and 1.10 for the multiplicative condition) and fraction calculations (ESs = 1.70 for the additive condition and 1.22 for the multiplicative condition), corroborating prior findings on the efficacy of the core fractions program (Fuchs et al., 2013, 2014). That is, the core fractions program that focused on the measurement interpretation of fractions significantly improved students' fraction magnitude understanding and fraction calculations compared to the control group, and effects were large. In terms of two contrasting word-problem conditions, although the multiplicative condition produced a larger ES on fraction magnitude understanding (1.10) than the additive condition (0.81), such difference was not statistically significant.

On the multiplicative word problem measures, students in the multiplicative word problem condition outperformed the control group (ES = 1.06) and the additive word problem condition (ES = 0.89). Similarly, students in the additive word problem condition outperformed the control group (ES = 1.40) and the multiplicative word problem condition (ES = 0.29). However, students in the multiplicative word problem condition outperformed the control group on additive word problems (ES = 1.10) whereas students in the additive word problem condition performed comparably on the multiplicative word problems (ES = 0.16). That is, the multiplicative word problem instruction improved both multiplicative and additive word-problem solving skills whereas the additive word problem instruction was limited to improving only additive word problem solving skills. This indicates that multiplicative word-problem intervention is a more effective instructional strategy for improving fraction word-problem solving skills.

Supported Self-Explaining Strategy

Expanding on the multicomponent core fractions intervention explained above, Fuchs et al. (2016a) investigated the effects of using self-supported explaining in comparing fraction magnitudes. Explaining mathematical ideas and reasoning are emphasized in the current college- and career-ready standards and NCTM standards (NCTM, 2000; Common Core State Standards, 2010). Students are often expected to use writing to answer and explain how to solve problems, and to construct mathematical arguments on high-stakes mathematics assessments. When students explain mathematics work, their understanding is thought to deepen (Kilpatrick, Swafford, & Findell, 2001; Whitenack & Yackel, 2002).

Three types of self-explaining exist: spontaneous self-explaining, elicited self-explaining, and supported self-explaining. Spontaneous self-explaining refers to students generating explanations for the to-be-learned materials without any prompts. With elicited self-explaining, students are prompted to invent explanations. However, prior research found that these types of self-explaining do not produce deeper levels of understanding. Rather, students are limited to describing procedures or stating whether the answer is correct (Rittle-Johnson, 2006). Besides, generating explanations is especially difficult for student with mathematics learning difficulties who not only have poor mathematics skills, but also often have weaknesses in cognitive processes, such as poor language and poor reasoning. These cognitive processes are involved in generating high-quality explanations.

Therefore, Fuchs et al. (2016a) focused on supported self-explaining, in which students are provided with high-quality explanations to work on. With supported self-explaining, high-quality explanations are modeled, and students practice analyzing and applying the explanations. In addition, students discuss important features of the explanation and are encouraged to elaborate on their explanation.

Participants. A total of 212 fourth grade students with mathematics learning difficulties (i.e., performing below the 35th percentile on a broad calculation assessment) participated in this study (See Table 1 for detailed demographics). The students were randomly assigned at the individual level to one of the three conditions: the core fractions program with supported self-explaining (n = 73), the core fractions program with multiplicative word-problem solving (n = 69), and the business-as-usual control (n = 70).

Outcome measures. Students were assessed on four outcome measures: Fraction Magnitude Comparisons/ Explanations (Schumacher, Namkung, Malone, & Fuchs, 2013), fraction word problems, fraction number line estimation, and fraction computation measures. On Fraction Magnitude Comparisons/Explanations, students compare nine pairs of fractions (three with the same numerator, three with the same denominator, and three with a different numerator and denominator) and places the greater than or less than symbol between fractions. Then, students explain why the fractions differ in magnitude by writing or drawing pictures. On word problems, students solve 12 multiplicative fraction word problems (six grouping and six splitting) and two compare problems. Each question is read aloud to students. On the Fraction Number Line 0-2 (Hamlett et al. 2011, adapted from Siegler et al., 2011) task, students place proper fractions (e.g., 1/4, 5/6), improper fractions (e.g., 3/2, 15/8), and mixed numbers (e.g., 1 11/12 , 1 5/6) on a 0-2 number line on a computer. On fraction calculations, students solve 12 fraction addition problems (five with like denominators and seven with unlike denominators), and 12 fraction subtraction problems (six with like denominators and six with unlike denominators).

Intervention conditions. Students in the intervention conditions received intervention for 35 min per session, three times a week, for 12 weeks. Trained research assistants implemented the intervention with pairs of students. Of each 35-minute session, the first 7 minutes of the tutoring session differ by the intervention condition. The other 28 min rely on the core fractions intervention and are identical across both supported-self explaining and fraction word-problem conditions. Students in the self-explaining condition received instruction on how to provide high-equality explanation for comparing fraction magnitudes. To control for intervention time, students in the contrasting intervention condition received multiplicative fraction word-problem instruction as described above. That is, in the multiplicative word-problem condition, students identify a problem type (splitting vs. grouping), represent the underlying structure of the problem type with a visual display, and solve for the unknown and write a numerical answer with a word label.

During the 7 min of supported self-explaining instruction, tutors model explanations in four steps using explicit instruction, scaffolding, and gradual release of responsibility. In the first step, students write "same D" for the same denominator comparison (e.g., 3/10 and 7/10), "same N" for the same numerator comparison (e.g., 3/4 and 3/12), or "both diff" for different numerator and denominator comparison (e.g., 1/2 and 5/6). In the second step, students explain and draw fraction models with two units of the same size, and correct number of parts divided and shaded. In the third step, students label drawings with a numerical value and describe how the parts of the fractions differ. For example, students explain both fractions have the "same size parts" for the same denominator comparison because the unit is divided into the same number of parts. For the same numerator comparison, students explain a fraction with fewer parts has "bigger parts" because the unit is divided into different number of parts. For the different numerator and denominator comparison, students rewrite the fractions with the same denominators and apply the same denominator reasoning. In the last step, students write a short sentence about why one fraction is greater than the other (e.g., more same size parts means greater fraction). Tutors provide corrective feedback. Tutors also work on discriminating between viable explanations for the same denominator versus same numerator comparison problems by solving them side by side and discussing distinctions between the two. Every intervention session was audiotaped and randomly sampled to check fidelity of implementing the intervention. The mean percentage of implementation ranged from 98.43 to 98.57.

Control condition. The majority of fractions instruction (76%) in the control group relied on part-whole understanding using fraction tiles/circles, pictures of shaded regions, and blocks with little emphasis on number lines (21%). Fraction magnitude understanding instruction focused heavily on finding common denominators and using cross multiplication for comparing fractions. In addition, teachers reported that students in the control group explained their fraction work approximately four times per week. The control group's fraction word-problem solving instruction also mainly focused on using key words, drawing pictures, writing equations, using words to explain thinking, and making tables.

Results. Students in both intervention conditions outperformed students the control group on fraction number line (ESs = 0.63 for the supported self-explaining condition, 0.71 for the word problem condition), and fraction calculations (ESs = 1.98 for the supported self-explaining condition, 2.08 for the word problem condition). That is, the core fractions program significantly improved students' general fractions knowledge, fraction magnitude understanding, and fraction calculations compared to the control group. Again, effects were large. On fraction multiplicative word problems, students in the word problem condition outperformed students in the control group (ES = 1.20) and students in the supported self-explaining condition (ES = 1.48), providing additional evidence that the multiplicative word-problem intervention improves fraction word-problem solving skills.

On Fraction Magnitude Comparisons/Explanations, students in both intervention conditions outperformed students in the control condition on both accuracy with which students identify greater or less fractions (ESs = 1.37 for the supported self-explaining condition, 0.89 for the word problem condition) and the quality of explanations about why fraction magnitudes differ (ESs = 1.37 for the supported self-explaining, 0.60 for the word problem condition). However, students in the supported self-explaining condition outperformed students in the word-problem condition on both accuracy (ES = 0.43) and the quality of explanations (ES = 0.93). Given that students in both intervention groups received the same instruction on efficient strategies for comparing fraction magnitudes and other important fraction skills as described above, supported self-explaining was significantly more effective for comparing fraction magnitudes and also producing high-equality written explanations about fraction magnitudes for the struggling fourth graders.

Next, we discuss another randomized control trial (Wang et al., 2019), in which third graders with mathematics learning difficulties were randomly assigned to a control group or a fractions intervention group. With the adoption of the career- and college-ready standards, a strong emphasis on fractions has been placed in third grade. Therefore, the core fractions program, *Fraction Face-Off!* (Fuchs et al., 2015), described above was modified for third graders with mathematics learning difficulties. The third-grade fractions intervention program, *Super Solvers* (Fuchs, Malone, Wang, Fuchs, Abramson, & Krowka, 2015) has the same focus on magnitude understanding of fractions and using schema-based instruction to teach fraction word problems as with *Fraction Face-Off!*, but the program's scope was simplified to address the third-grade curriculum.

Description of *Super Solvers*

In *Super Solvers* (Fuchs et al., 2015), each lesson is approximately 35 min and is implemented three times a week for 13 weeks (39 lessons) with pairs of students. Some of the components are similar to those addressed in *Fraction Face-Off!* (Fuchs et al., 2015), such as the emphasis on the measurement interpretation of fractions, explicit instruction principles, and concrete manipulatives. In terms of the fraction contents addressed in the intervention program, we highlight similarities and differences between *Super Solvers* and *Fraction Face-Off!*. As with *Fraction Face-Off!*, students are introduced to fraction vocabulary, naming and reading fractions, and fractions equivalent to one whole and 1/2. Students also learn how to compare fractions with the same denominators or the same numerators. Then, students build on these strategies to compare fractions with different denominators and different numerators. They first learn to identify fractions less than, equal to, or greater than 1 and 1/2 (i.e., benchmark fractions). Students next learn to place fractions on 0-1 number line using 1/2 as a benchmark fraction. However, improper fractions and mixed number (converting between improper fractions and mixed numbers; placing them on number lines) are not taught in *Super Solvers*.

In addition to these differences, *Super Solvers* (Fuchs et al., 2015) focuses on two additional skills, multiplication and fraction word problems. Weeks 1-3 focus on teaching multiplications, in which students learn strategies, such as skip counting and decomposition, to build fluency with whole-number multiplication facts. The additive fraction word-problem instruction described above is incorporated in Weeks 4-13. In Weeks 4-7, students are taught compare and change (i.e., increase and decrease) problems types using schema-based instruction. Students identify a word problem type and its underlying structure to solve the word problem. Lastly, a fraction curriculum-based measurement (CBM) progress monitoring system is incorporated in *Super Solvers*. Starting in Week 3, students complete the CBM every two weeks, in which they solve 20 fraction problems and receive immediate, corrective feedback.

Embedding Self-Regulation Strategy

Self-regulation refers to the cyclic process of proactively initiating thoughts, planning, evaluating, and adjusting the use of skills and strategies to attain personal goals (Zimmerman, 2000), and self-regulation is positively correlated mathematics performance (e.g., Cleary & Chen, 2009; Rosario, Nunez, Valle, Gonzalez-Pienda, Lourenco, 2013). In the academic context, self-regulation is often interpreted as having a growth mindset reflecting the belief that intellectual and academic abilities can be developed, goal-setting, self-monitoring, using strategies to engage motivationally, metacognitively, and behaviorally (Lezak, Howieson, Bigler, & Tranel, 2012; Lin-Siegler, Dweck, & Cohen, 2016; Yeager & Dweck, 2012; Zelazo, Blair, & Willoughby, 2016).

However, students with learning difficulties often have deficits in memory, attention, and self-regulation that they have a limited repertoire of strategies, immature metacognitive abilities, low motivation, and fail to monitor their performance (Montague, 2007). Therefore, based on prior research documenting that students with learning difficulties lack self-regulation and can benefit from skill-based instruction that incorporates self-regulation strategies (e.g., Fuchs et al., 2003; Graham & Harris, 2003; Montague, 2007; Wong, Harris, Graham, & Butler, 2003), Wang et al. (2019) investigated whether embedding self-regulation strategies to *Super Solvers* (Fuchs et al., 2015) had added value in promoting fractions understanding.

Participants. A total of 79 third graders with mathematics learning difficulties (i.e., performing below the 22th percentile on a broad calculations test or performing below the 31st percentile on the broad calculations test and scoring less than three on a basic addition and subtraction fluency test; See Table 1 for detailed demographics) participated in the study. These students were randomly assigned at the individual level to one of three conditions: the core fractions program, which focuses on magnitude understanding and word-problem solving (n = 23), the same core fractions program with the embedded self-regulation component (n = 23), or the business-as-usual control (n = 23).

Outcome measures. Students were assessed on four outcome measures: whole-number multiplication, Fraction Number Line 0-1 (Hamlett et al., 2011, adapted from Siegler et al., 2011), fraction ordering, and fraction word problems. On whole-number multiplication, students solve 30 single-digit multiplication problems in 5 min. On fraction number line, students are asked to place proper fractions (e.g., 1/4, 5/6) on a 0-1 number line on a computer. On fraction ordering, students order 12 sets of three fractions from least to greatest (two items with fraction the same numerator; one has fractions with the same denominator; the remaining nine include 1/2 as one of the three fractions [e.g., 9/12, 1/2, and 3/8]). On fraction word problems, 12 problems focus on additive word problems (six increase problems and six decrease problems), and six problems focus on compare word problems (e.g., In art class, Maria used 5/12 of a bottle of blue paint and 3/4 of a bottle of red paint. What paint color did she use more of?). Each question is read aloud to students.

Intervention conditions. Students in the intervention conditions received intervention for 35 min per session, three times a week, for 12 weeks. Trained research assistants implemented the intervention with pairs of students. Students in the core fractions program received intervention that emphasizes measurement understanding and fraction word-problem solving. Students in the self-regulation condition received the same core intervention, but with embedded self-regulation components. To control for intervention time, students in the core intervention solved extra word problems while students in the self-regulation condition participated in the self-regulation instruction.

In the self-regulation condition, a comic series, which features school-aged students who struggle with learning fractions and face other school and life difficulties was incorporated. Tutors present an episode from the comic series at each lesson and discuss key self-regulation components, such as goal setting, asking for and providing help, using help cards and other tools only when necessary, planning own learning activities, and tracking own progress. Students also take a fractions CBM every two weeks, set goals to beat their highest score, develop and discuss strategies to meet goals, and graph their progress. In addition, tutors teach students to check their work for mistakes, evaluate the sources of errors, check for misunderstandings of fraction concepts and strategies, and encourage students to use the mistakes to plan and select practice items to reach their goals. Although students in the core program also take a fractions CBM, tutors only score assessments, but do not provide feedback or guide reflection on student progress. Every intervention session was audiotaped and randomly sampled to check fidelity of implementing the intervention. The mean percentage of implementation ranged from 92.05 to 96.58.

Control condition. The fractions instruction in the control group primarily relied on state standards in addition to using the district's mathematics curriculum (enVisionMath). Although the amount of time spent on mathematics instruction in the control group was similar to that of the experimental group, the control group instruction focused on part-whole understanding of fractions. For fraction word problems, the control group instruction focused on operational procedures and using key words.

Results. Students in both intervention conditions outperformed the students in the control condition on whole-number single-digit multiplication (ESs = 0.97 for the core condition and 1.08 for the self-regulation condition), fraction word problems (ESs = 0.92 for the core condition and 0.88 for the self-regulation condition), fraction ordering (ESs = 1.23 for the core condition and 1.21 for the self-regulation condition), and fraction number line (ESs = 1.00 for the core condition and 1.07 for the self-regulation condition). Although both intervention conditions produced superior outcomes over control with large ESs, there were no significant differences between the intervention conditions. That is, the intervention with embedded self-regulation was equally effective as the contrasting core intervention.

Conclusions

In this paper, we discussed the importance of achieving competence with fractions, the challenges of learning and teaching fractions, and effective strategies for teaching fractions to students with mathematics learning difficulties based on our prior intervention work. Initially, students rely on their whole-number knowledge to understand fractions, which often leads to misconceptions and errors about fractions. Instruction that focuses on the measurement interpretation of fractions via number lines can help students integrate whole numbers and fractions, and expand their conceptualization of numbers as a continuum. Because teachers often have limited understanding of fractions, using a structured intervention program may be an alternative, more effective way to guide teachers to improve their own and students' understanding of fractions.

Based on the review of our randomized control trials examining the effects of fractions intervention, explicit Tier-2 fractions intervention programs that focus on teaching measurement understanding of fractions with concrete and representational manipulatives improved fraction competence for students with mathematics difficulties with large effect sizes (Fuchs et al., 2016a, 2016b; Wang et al., 2019). In addition, providing supported self-explaining improved students' accuracy in comparing fraction magnitudes and providing high-quality explanations for why one fraction is greater/less. Teaching multiplicative word-problem solving using the schema-based instruction also improved both multiplicative and additive word-problem solving skills. Lastly, although our study did not find added value of embedding a self-regulation strategy, operationalized as building students' growth mindset, goal-setting, planning, and self-monitoring, self-regulation strategy may help some students overcome their weaknesses in motivation, self-regulation, and executive functions, and produce better fraction outcomes for those students.

Limitations

Several limitations should be noted. Students who participated in the three intervention studies all came from the same school district and shared similar demographic characteristics. Although our samples across three studies were diverse in terms of race, ELL status, and disability status, they were predominately from low SES backgrounds. The findings from our studies may not be generalizable to other students with different demographic characteristics. In a related way, there may be other factors that could have affected student learning, such as quality of teacher instruction and support from parents that were beyond the scope of our studies. Future studies should explore identifying additional factors that contribute to fractions learning and response to fractions intervention.

Another limitation pertains to control group instruction. Our description of fractions instruction in three control groups across the studies was based on the review of the district's curriculum and teacher reports. Teachers' self-reports may have been biased and may not accurately reflect the actual fractions instruction in classrooms. Future studies should document the control group instruction via direct observations.

Implications for Practice

Despite some limitations, our findings across the studies provide implications for educators. First, although not sufficient, achieving mastery with foundational whole-number skills should be emphasized for successful learning of fractions. Second, as our studies demonstrate, third and fourth graders with mathematics learning difficulties who struggle with fractions can benefit from small-group fractions intervention that relies on explicit instructional principles and magnitude understanding of fractions.

Third, word-problem solving strategies for fractions should focus on using schema-based instruction and teach students how to identify a problem type and its underlying structure. Although both additive and multiplicative problems types are equally effective in improving fraction magnitude understanding, the multiplicative word problem instruction is more effective for improving fraction word-problem solving. Nonetheless, many teachers reported using key words to teach fraction word problems. We strongly discourage educators from using the ineffective key word strategy. The key word strategy focuses on identifying a single keyword from a word problem and tying the key word to a single operation (e.g., in all means to add; Mevarech, 1999), rather than engaging students in understanding the entire text.

We encourage educators to explicitly teach how to provide high-equality explanations. Such explicit instruction involves modeling high-equality explanations, discussing important features of the explanation, and practicing analyzing and applying the explanations with struggling students. Supported self-explaining not only improves students' written explanations about fraction magnitudes, but also fraction comparisons. Lastly, although the intervention with embedded self-regulation was equally effective as the contrasting core intervention, we encourage educators to consider promoting self-regulation with a focus on positive mindset, goal setting, evaluation, and planning given that students with mathematics learning difficulties often lack self-regulation and motivation.

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