

## What can be learned from teachers assessing mathematical reasoning: A case study

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Recently, mathematical reasoning has grown in prominence in curriculum documents and professional learning programs. However, the assessment of reasoning actions continues to be an elusive task for many teachers. Research has shown that many primary teachers focus only on explaining. This case study examines the salient behaviours of two Year 6 primary teachers employing the *Assessing Mathematical Reasoning Rubric*. Results indicated the teachers gained deeper insights into the diverse nature of reasoning through the employment of rubric. Therefore, it provides teachers with a vehicle for a more nuanced examination of reasoning beyond explaining and is a launching pad for lesson planning.

The emphasis on reasoning in curriculum documents is reflected in the reasoning focus of professional development for pre- and in-service teachers through: demonstration lessons (Herbert, Vale, Bragg, Loong, & Widjaja, 2015); and workshops (Hilton, Hilton, Dole, & Goos, 2016); teachers' use of reasoning language (Clarke, Clarke, & Sullivan, 2012); peer-learning-teams (Herbert & Bragg, 2017); and, mathematics teacher educators modelling of reasoning focused lessons in primary classrooms (Livy & Downton, 2018).

Previously, we noted teachers focused on reasoning as *explaining* (Herbert et al., 2015). This focus on *explaining* disregards the multifaceted nature of reasoning. Therefore, we developed the *Assessing Mathematical Reasoning Rubric* (See Figure 1) to foster a more nuanced examination of children's reasoning. This paper specifically explores the research question, "What can be learned from teachers' employment of the Assessing Mathematical Reasoning Rubric?"

### Literature review

Teachers often grapple with the nuances of mathematical content and do not have strategies for helping their students to recognise or utilise it to solve problems (Hilton et al., 2016). Extending this thinking beyond content to proficiencies, if teachers struggle with understanding reasoning, then it may be difficult for them to teach it effectively. This section outlines the background literature which informed this current study, including a discussion of reasoning, and its assessment.

#### *Mathematical Reasoning*

In mathematics, reasoning is viewed as "the glue that holds everything together, the lodestar that guides learning" (Kilpatrick, Swafford, & Findell, 2001, p. 129). It "involves making, investigating and evaluating conjectures, and developing mathematical arguments to convince oneself and others that the conjecture is true" (Goos, Vale, & Stillman, 2017, p. 37), thus allowing students to go beyond routine procedures towards an appreciation of the interconnected, logical and meaningful aspects of mathematics (Mata-Pereira & da Ponte, 2017). These views of mathematical reasoning are consistent with the Australian Curriculum: Mathematics (AC:M) which states: 2018. In Hunter, J., Perger, P., & Darragh, L. (Eds.). *Making waves, opening spaces (Proceedings of the 41<sup>st</sup> annual conference of the Mathematics Education Research Group of Australasia)* pp. 178-185. Auckland: MERGA.

Students develop an increasingly sophisticated capacity for logical thought and actions, such as analysing, proving, evaluating, explaining, inferring, justifying and generalising. (ACARA, 2017, Key ideas, para. 5).

Analysing is described in the AC:M to occur when “students explain their thinking, ... they adapt the known to the unknown, ... transfer learning from one context to another, ... and when they compare and contrast related ideas and explain their choices” (ACARA, 2017). Students generate specific cases or examples to satisfy the conditions of a problem drawing on prior knowledge, such as recalled facts, to construct examples or cases (Vale et al. 2017). Examples are compared and contrasted to form conjectures explaining similarities and differences between examples (Jeannotte & Kieran, 2017). Mason (2003) recommended that teachers use prompts “such as ‘What is the same and what different about...?’” (p.24) to support learners to connections between cases or examples.

Forming conjectures and generalising are essential components of the teaching and learning of reasoning (Lannin, Ellis, Elliot, & Zbiek, 2011). Furthering the reasoning action of analysing, generalising identifies commonalities across cases, extending beyond the original case (Kaput & Blanton, 1999). Lannin, et al. (2011) merged conjecturing and generalising to proffer four key understandings of generalising: (1) developing statements [forming conjectures], (2) identifying commonality and extending beyond original cases, (3) recognising a domain for which the generalisation holds, and (4) “clarifying the meaning of terms, symbols and representations” (p. 12).

Justifying is more than explaining “what”, including “why” (Vale, et. al., 2017) to verify a claim (Sowder & Harel, 1998). A mathematical justification is a logical argument based on accepted procedures, properties, concepts, and mathematical ideas (Mata-Pereira & da Ponte, 2017). As students’ complexity of reasoning grows, they are able to offer a mathematically and sound logical argument to support a claim (Jeannotte & Kieran, 2017).

Despite the complexity of reasoning, teachers mainly focus on *explaining*. Clarke, Clarke, and Sullivan (2012) found nearly all 104 teachers surveyed regularly used explaining, with less use of other reasoning words. Therefore, there is a need to extend teachers’ awareness of a broader range of reasoning actions. This paper reports on the efficacy of the *Assessing Mathematical Reasoning Rubric* in assisting teachers to gain a deeper view of reasoning, than merely explaining, by interrogating the data collected in two post-lesson discussions where two teachers utilised the rubric to assess the reasoning capacity of their students.

### *Assessing Reasoning*

Assessment in mathematics is the process of examining evidence about student learning to reveal student knowledge and skills (Heritage, Kim, Vendlinski, & Herman, 2009) and to plan for subsequent action with a goal to improve their student’ conceptual understanding (Binkley, et al., 2012). In our larger study exploring teachers’ knowledge of reasoning and enriching their understanding of reasoning through a professional learning program, it was noted that primary teachers struggle to define, recognise, and implement reasoning (Loong, Vale, Herbert, Bragg, & Widjaja, 2017). Consequently, without an understanding of the complexity of reasoning it is challenging to notice and thereby assess when reasoning takes place.

Figure 1. Assessing Mathematical Reasoning Rubric. Version 1 (Herbert & Bragg, 2017).

	Analysing	Forming Conjectures and Generalising	Justifying and Logical argument
Not evident	<ul style="list-style-type: none"> <li>• <b>Does not notice</b> numerical or spatial structure of examples or cases.</li> <li>• Attends to <b>non-mathematical</b> aspects of the examples or cases.</li> </ul>	<ul style="list-style-type: none"> <li>• <b>Does not communicate</b> a common property or rule for pattern.</li> <li>• <b>Non-systematic</b> recording of cases or pattern.</li> <li>• <b>Random</b> facts about cases, relationships or patterns.</li> </ul>	<ul style="list-style-type: none"> <li>• <b>Does not justify.</b></li> <li>• <b>Appeals</b> to teacher or others.</li> </ul>
Beginning	<ul style="list-style-type: none"> <li>• <b>Notices</b> similarities across examples</li> <li>• <b>Recalls</b> random known facts related to the examples.</li> <li>• Recalls and <b>repeats patterns</b> displayed visually or through use of materials.</li> <li>• Attempts to <b>sort</b> cases based on a common property.</li> </ul>	<ul style="list-style-type: none"> <li>• Uses body language, drawing, counting and oral language to <b>draw attention to and communicate</b>: <ul style="list-style-type: none"> <li>○ a single common property</li> <li>○ repeated components in patterns.</li> </ul> </li> <li>• <b>Adds</b> to patterns displayed verbally and/or visually using diagrams or through use of materials.</li> </ul>	<ul style="list-style-type: none"> <li>• <b>Describes</b> what they did and why it may or may not be correct.</li> <li>• <b>Recognises</b> what is correct or incorrect using materials, objects, or words.</li> <li>• Makes <b>judgements</b> based on simple criteria such as known facts.</li> <li>• The <b>argument may not be coherent</b> or include all steps in the reasoning process.</li> </ul>
Developing	<ul style="list-style-type: none"> <li>• <b>Notices</b> a common numerical or spatial property.</li> <li>• <b>Recalls, repeats and extends</b> patterns using numerical structure or spatial structure.</li> <li>• <b>Sorts and classifies</b> cases according to a common property.</li> <li>• <b>Orders</b> cases to show what is the same or stays the same and what is different or changes.</li> <li>• <b>Describes</b> the case or pattern by labelling the category or sequence.</li> </ul>	<ul style="list-style-type: none"> <li>• <b>Communicates</b> a rule about a: <ul style="list-style-type: none"> <li>○ <i>property</i> using words, diagrams or number sentences.</li> <li>○ <i>pattern</i> using words, diagrams to show recursion or number sentences to communicate the pattern as repeated addition.</li> </ul> </li> <li>• <b>Explains</b> the meaning of the rule using one example.</li> </ul>	<ul style="list-style-type: none"> <li>• <b>Verifies</b> truth of statements by using a common property, rule or known facts that confirms each case. May also use materials and informal methods.</li> <li>• <b>Refutes</b> a claim by using a counter example.</li> <li>• Starting statements in a <b>logical argument</b> are correct and accepted by the classroom.</li> <li>• <b>Detecting and correcting errors</b> and inconsistencies using materials, diagrams and informal written methods.</li> </ul>
Consolidating	<ul style="list-style-type: none"> <li>• <b>Notices more than one</b> common property by systematically generating further cases and/or listing and considering a range of known facts or properties.</li> <li>• <b>Repeats and extends patterns</b> using both the numerical and spatial structure.</li> <li>• Makes a <b>prediction</b> about other cases: <ul style="list-style-type: none"> <li>○ with the same property</li> <li>○ included in the pattern.</li> </ul> </li> </ul>	<ul style="list-style-type: none"> <li>• <b>Identifies</b> the boundary or limits for the rule (generalisation) about a common property.</li> <li>• <b>Explains</b> the rule for finding one term in the pattern using a number sentence</li> <li>• <b>Extends</b> the number of cases or pattern using another example to explain how the rule works.</li> </ul>	<ul style="list-style-type: none"> <li>• Uses a correct <b>logical argument</b> that has a complete chain of reasoning to it and uses words such as ‘because’, ‘if...then...’, ‘therefore’, ‘and so’, ‘that leads to’ ...</li> <li>• <b>Extends the generalisation</b> using logical argument.</li> </ul>
Extending	<ul style="list-style-type: none"> <li>• <b>Notices and explores</b> relationships between: <ul style="list-style-type: none"> <li>○ common properties</li> <li>○ numerical structures of patterns.</li> </ul> </li> <li>• <b>Generates examples</b>: <ul style="list-style-type: none"> <li>○ using tools, technology and modelling</li> <li>○ to form a conjecture.</li> </ul> </li> </ul>	<ul style="list-style-type: none"> <li>• <b>Communicates</b> the rule for any case using words or symbols, including algebraic symbols.</li> <li>• <b>Applies</b> the rule to find further examples or cases.</li> <li>• <b>Generalises</b> properties by forming a statement about the relationship between common properties.</li> <li>• <b>Compares</b> different symbolic expressions used to define the same pattern.</li> </ul>	<ul style="list-style-type: none"> <li>• Uses a <b>watertight logical argument</b> that is mathematically sound and leaves nothing unexplained.</li> <li>• <b>Verifies</b> that the statement is true or the generalisation holds for <i>all</i> cases using logical argument.</li> </ul>

Although it is known that student outcomes are improved when rubrics are used (Panadero & Jonsson, 2013), little is known about how the use of existing rubrics for assessment may build teachers’ knowledge. So, to support teachers with the complex task of assessing reasoning the Assessing Mathematical Reasoning Rubric (hereafter also referred to as the “rubric”), with five levels (Not Evident; Beginning; Developing; Consolidating; and, Extending) for the three reasoning actions: Analysing; Generalising; and, Justifying (see Figure 1), was developed. Dot points in each cell are intended to assist teachers to identify a student’s level for each of the reasoning actions.

This rubric was developed by a team of academics at Deakin University as one aspect of the resources created for the reSolve: Mathematics by inquiry (Australian Government Department of Education and Training, 2017). It was trialled and refined by teachers at four Victorian primary schools, using it to assess the reasoning demonstrated by their students during specifically designed reasoning lessons.

## Methodology

A case study provides “an intensive, holistic description and analysis of a single instance, phenomenon or unit” (Merriam, 1988, p. 21). It is “an empirical inquiry that investigates a contemporary phenomenon within its real-life context; when the boundaries between phenomenon and context are not clearly evident; and in which multiple sources of evidence are used” (Yin, 1984, p. 23). A case study is used to “explore those situations in which the intervention being evaluated has no clear, single set of outcomes” (Baxter & Jack, 2008, p. 548). In this paper, case study is being employed as it was intended, by utilising this approach a deeper understanding may be gained of specific issues associated with teachers’ awareness of the diversity of actions encompassed in the broader term ‘mathematical reasoning’ through the assessment of students’ work samples in the post-lesson discussion facilitated by the rubric.

### *Description of the case*

The participants in this case study are two Year 6 teachers and their children (approximately 50 children) engaged in the classroom enactment of the painted cube task at a suburban primary school on the outskirts of Melbourne. They were chosen to assist the researchers understand about the challenges in assessing mathematical reasoning because of their involvement in trialling tasks and resources for the reSolve project: Assessing Mathematical Reasoning.

#### Painted Cube Task

The painted cube task (Driscoll, 1999) (Figure 2) was adapted to provide children with the opportunity to share and debate their algebraic thinking as they searched for patterns and generalisations. The multiple layers of the problem offered incremental developments in the sophistication of the children’s reasoning (Koellner, Pittman, & Frykholm, 2008/2009).

Imagine a cube made up of 27 smaller cubes ( $3 \times 3 \times 3$ ). Imagine that you dip the cube in paint. If you now separate it into 27 small cubes, you will notice that some of the small cubes are painted. Which small cubes have been painted on 3 sides, on 2 sides, on 1 side, and not painted at all – and how many are there?

Fill in the grid for a  $3 \times 3 \times 3$  cube.

Consider and complete the grid for different size cubes,  $2 \times 2 \times 2$ ,  $4 \times 4 \times 4$ ,  $10 \times 10 \times 10$ , etc.

Create a rule for predicting the answers for larger cubes without counting all the small cubes  $n \times n \times n$  ?

Describe the patterns that you see. What changes, and what stays the same?

Figure 2. The painted cube task.

### *Analysis*

The post-lesson discussion was audio-recorded. The work samples and their assessment via the rubric were collected. Transcripts were jointly read to establish common coding consistent with the *Assessing Mathematical Reasoning Rubric*. The findings arising from the data analysis are presented in the following section in narrative form.

## Results

This section presents the results of the analysis of the transcript of the post-lesson discussion. It is structured according to the reasoning actions: Explaining, Analysing, Generalising, and Justifying. A total of 47 statements were recognised as related to reasoning actions in the transcript of the post-lesson discussion. The statements were further coded into Explaining (7); Analysing (11), Generalising (23) and Justifying (6). The examples below are illustrative of how the teachers utilised the rubric to assess students' work samples, thus demonstrating their attention to a wider range of reasoning actions than explaining.

### *Explaining*

As is typical of what other researchers have noted, the teachers did talk about "explaining" as a reasoning action. For example:

Lee: He was trying to explain his formula at the end yeah. He wasn't very clear though.

Rosie: So, she was explaining that in here

While teachers did refer to explaining there were also many instances of their focus on other reasoning actions as they attempted to use the rubric to assess their students' work samples.

### *Analysing*

Both teachers noticed the students' analysing and were able to articulate their interpretation of their students' actions. In the quote below, Rosie has noticed the attention paid by students to the pattern related to the number of unpainted cubes. This evidence indicates that she can see the students are analysing this problem, i.e., noticing a common property, describing the pattern and exploring relationships between the examples they are generating.

Rosie: They started to have a look at the pattern of the cubes not painted at all and looking at the connection between this column and then the total number of small cubes.

The next two quotes demonstrate Lee's iterative contemplation of the levels in the Analysing column of the rubric. In this way, the rubric's wording assisted him to notice analysing in his students' work. Firstly, he reviews the 'Consolidating' description 'Makes a prediction about other cases with the same property'.

Lee: I'm just looking at "Consolidating" now. I can see for sure that they can predict.

He recognises that this student has met that level's indicator but may also meet the indicators of a higher level. Lee reviews the next level in the Analysing column 'Extending' where he reads 'Numerical structures of patterns'.

Lee: So, I'm just going to move down to "Extending" and just see if they fit that. "Numerical structures of patterns," yes, I think that's evidence by the actual formulas they've written out.

In the next quote, Rosie is grappling with the idea of what constitutes analysing. Her paraphrasing of the words in the rubric demonstrates she is building her language related to reasoning. This is different from Lee's use of the words in the previous quote where Lee is reading the rubric dot points verbatim, whereas, Rosie is embedding the wording from the dot point in her articulation of analysing.

Rosie: I've got her as "Consolidating" in Analysing. But I wasn't sure whether to put her in "Developing" or "Consolidating" because I suppose she made a prediction that it would work with

any other numbers, but she didn't really elaborate on that. She didn't use different examples, she just used what was already here.

Later in the post-lesson discussion we notice Lee beginning to appropriate the words into his understanding and expression of analysing.

Lee: So, she [student name] has begun to find the pattern but she's doing them one at a time I've noticed instead of going down. So that indicates to me that she's not maybe going any further with the pattern. She's doing one at a time still whereas with [a different student] you can see that he's actually gone [filled the column] all the way done.

The quote indicates Lee's deepening understanding of the nature of analysing as he compares these two students, noticing that generating further examples in a pattern is considered a higher level of Analysing.

### *Generalising*

As with Analysing, teachers used the words embedded in the rubric to assist in assessing their students' level of Generalising. Below Lee and Rosie expressed their evaluation of their students' work in terms of the rubric's language.

Rosie: He can "explain the meaning of the rule using one example", and he can add to the pattern, and he can "communicate a single property and repeated components".

The teachers demonstrated their growing awareness of the nature of reasoning in their noticing of their students' generalising, for example:

Lee: So, I think for this one he is actually using algebraic symbols here. That's evidenced by the actual formulas they've written out. He's actually explained the formulas for the first 2 columns.

Rosie: So, they noticed that you've got 8 cubes with a 2 by 2 by 2 and then in a 4 by 4 by 4. That's how many cubes that aren't painted. So, they started to notice that connection. They've just begun to make that connection and come up with a formula

The rubric assisted Rosie to evaluate the complexity of the students' generalising capabilities through a comparison of two students' work samples.

Rosie: I've put her [one student] for "Developing" in Generalising because she was talking about the rules and the patterns. ... Definitely not in "Extending" because she [another student] didn't talk about other examples.

### *Justifying*

The results revealed that justifying, whilst not the focus of this task, was identifiable in the students' reasoning actions as noticed by these teachers. This is evident in Rosie's articulation below:

Rosie: Using a "logical argument that has a complete chain of reasoning" and yeah, she used the words, "just, because, if, then, therefore". That's why I would sort of put her in the middle of those two [levels]. Definitely not in "Extending" because she didn't talk about other examples. ... When I asked them about it they were trying to work out the rule.

The words Rosie used, are embedded in the Consolidating level for Justifying. These words indicate that she is using the language of the rubric to identify justifying and to categorise this student's reasoning. Of note, Rosie is moving beyond describing this student's reasoning as "explaining".

## Discussion and Conclusion

Clarke, et al. (2012) uncovered primary teachers' focus on explaining as the key action of reasoning observed and promoted in their classrooms. With this concern, regarding the limitation in viewing the role of reasoning, in mind, our professional learning goal was to develop and utilise a rubric to shift teachers' understanding, enactment, and assessing of reasoning actions from explaining to encompass Analysing, Generalising, and Justifying. In learning from the teachers' employment of the rubric, we noted that whilst explaining, rightly, was articulated during the teachers' feedback on their students' actions, evidence of teachers' recognising the complex nature of reasoning was apparent. The use of the rubric pushed the teachers beyond explaining and allowed them to notice the students' developing arguments, conjecturing, generalising, and convincing others (Goos, et al., 2017). While we acknowledge that the main focus of the task was to generalise, pleasingly, the teachers were able to capture examples of their students' exhibiting the actions of analysing and justifying, thus, we witnessed in the teachers' appraisal of the students' actions the interconnectedness of reasoning (Mata-Pereira & da Ponte, 2017).

Not surprisingly, we would anticipate that using a rubric specifically designed to examine multiple actions of reasoning would result in the teachers' noticing the selected actions in the rubric. However, explaining is a feature of the rubric, and yet encouragingly was the number of accurate examples of the other reasoning actions within the rubric which the teachers were able to articulate, without returning to the holdall of "explaining". Prior experiences with similar Victorian teachers noticing reasoning had resulted in them describing complex reasoning actions predominantly as explaining (Herbert et al., 2015). Therefore, the rubric offered the teachers a nuanced vocabulary to describe the reasoning actions they were witnessing, thus leading to a deeper understanding of these reasoning actions.

In the course of creating the *Assessing Mathematical Reasoning Rubric*, we were concerned that the heavily detailed rubric would deter teachers from employing the rubric. Thus, we have developed a briefer, less-detailed rubric. However, as a result of our investigation in this study of the usefulness of the detailed rubric, one implication for further research is to reconsider how to balance the effectiveness of the shorter rubric as a tool for quickly assessing students' reasoning versus the detail-rich *Assessing Mathematical Reasoning Rubric* which offers more breadth in supporting teachers' noticing of the complexity of reasoning.

The Australian Curriculum: Mathematics (ACARA, 2017) and others (Goos, et. al, 2017; Kilpatrick, et al. (2002) in describing reasoning as multifaceted, encourages teachers to facilitate learning with tasks that reflect the complex nature of reasoning. The *Assessing Mathematical Reasoning Rubric*, in this case study, appears to be successful in providing teachers with a tool to notice and assess the complex nature of reasoning exhibited by their students.

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