# FROM SPECIFIC VALUE TO VARIABLE: DEVELOPING STUDENTS' ABILITIES TO REPRESENT UNKNOWNS

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*Third- through fifth-grade students participating in a classroom teaching experiment investigating the impact of an Early Algebra Learning Progression completed pre- and post-assessments documenting their abilities to represent or describe unknown quantities. We found that after a sustained early algebra intervention, students grew in their abilities to represent related unknown quantities using letters as variables.*

Algebra has historically served as a gateway to higher mathematics that—due to high failure rates—has been closed for many students. These failures have been due in large part to a narrow treatment of algebra as an exercise in symbol manipulation without much regard for the symbols' underlying meanings (Kaput, 1998). Mathematics education researchers (e.g., Davis, 1985; Kaput, 1998; Olive, Izsak, & Blanton, 2002) have argued that addressing this issue requires us to view algebra not as an isolated eighth- or ninth-grade course, but as a continuous strand spanning the entire K-12 curriculum. This is not to be interpreted as a call to shift traditional instruction in symbol manipulation to earlier grades, but rather as one to consider broadening our notions of what it means to think algebraically and introducing elementary school students to important ideas of algebra in the context of their study of arithmetic.

In response to this call, we are presently drawing from research findings and curricular resources in the area of early algebra to develop an Early Algebra Learning Progression (EALP) organized around five "big ideas": 1) Generalized Arithmetic, 2) Equations, Expressions, Equality, and Inequality, 3) Functional Thinking, 4) Proportional Reasoning, and 5) Variable.

We are conducting a one-year classroom-based study in grades 3-5 to gather initial efficacy data regarding the impact of EALP-based classroom experiences on elementary students' developing understandings of these big ideas. The focus of this paper will be our initial findings regarding students' developing understandings of particular aspects of variable. We will share pretest and mid-year assessment data documenting students' performance using variables to represent unknown quantities. We will additionally have end-of-the-year posttest data ready to share at the conference.

## Theoretical Framework

While traditional treatments of algebra often present the subject as one about "manipulating symbols that do not stand for anything" (Kaput, 1999, p. 134), we must acknowledge that much of algebra's power comes from the ability to represent unknown and varying quantities succinctly and manipulate these expressions without constant reference to their underlying

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meaning. The concept of variable must thus play a critical role in early algebra (Schoenfeld & Arcavi, 1988). Schoenfeld and Arcavi argue that rather than asking students to practice symbol manipulation and solving for unknowns, teachers should encourage students to view variables as shorthand tools for expressing already-understood ideas about varying quantities.

Despite its importance, variable is a concept with which many students struggle. Documented difficulties include believing that variables stand for names, labels, or attributes (e.g., *h* stands for height, *w* stands for weight) (Booth, 1988; Clement, 1982; Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005; MacGregor & Stacey, 1997; Weinberg et al., 2004) and being unable to operate with or even consider unknown quantities rather than specific values (Blanton, 2008; Carraher, Schliemann, & Schwartz, 2008).

While variables can take on several meanings in different mathematical contexts (Kieran, 1991; Küchemann, 1978; Usiskin, 1988), in this paper we focus on students' abilities to use variables to represent unknown quantities. We draw from Carraher et al.'s (2008) work, in which third-grade students were asked to represent the number of candies in two sealed boxes—one of which had three additional candies resting on top. The majority of the students initially assigned particular values to the amounts in each box (e.g., 3 and 6), suggesting they were unable to work with indeterminate amounts. Given a similar context, Blanton (2008) found that the presence of unknown quantities led students to consider such problems unsolvable. By drawing attention to multiple possible solutions to the *Candy Box* problem and encouraging students who refrained from assigning particular values to contribute to the discussion, Carraher et al. found that students' representations—which included drawings, tables, and verbal comments—were enriched and that eventually variable representations (e.g., $N$ and $N + 3$) were accepted.

Kinzel (1999) asserts that students' difficulties with algebraic notation originate from narrow conceptions of variable (e.g., viewing the symbol as a label) and that if algebra is to be viewed as a meaningful representational tool, further attention should be given to symbolizing and interpreting variables in classrooms. Carraher et al.'s (2008) work illustrates that young elementary students are capable of representing unknown quantities in sophisticated ways when these representations build on their informal representations and understandings.

## Method

### *Participants*

Participants include 301 students in grades 3-5 from two elementary schools in southeastern Massachusetts. The school district in which these schools reside is largely white (91%) and middle class, with 17% of students qualifying for free or reduced lunch. Six classrooms (two from each of grades 3-5 and all from one school) are serving as experimental classrooms and 10 classrooms (four grade 3, four grade 4, and two grade 5, all from both schools) are serving as control classrooms.

### *Classroom Intervention*

Students in the experimental condition are participating in an EALP-based classroom teaching experiment for approximately one hour each week for the majority of one school year. A member of our research team—a former third-grade teacher—is serving as the teacher during these interventions. Additionally, each lesson is observed by a member of the research team as a way to identify characteristics of students' thinking and issues of instructional design. The project team also meets twice weekly, once on-site to discuss initial observations about the teaching experiment, and once off-site to connect initial findings to the proposed EALP and revise the EALP. A typical one-hour lesson consists of a "jumpstart" at the beginning of class to review previously-discussed concepts, followed by group work centered on research-based tasks aligned with our EALP (see Figure 1 for an example). These tasks are designed to encourage students to reason algebraically in a variety of ways and justify their thinking to themselves and their classmates. All classroom tasks and assessment items presented in this paper are adapted from Carraher et al. (2008).

> Jack and Ava both have a box of candy. Each box contains the same number of pieces of candy, but we don't know how many in each. Ava is given four more pieces of candy.
>
> a) How could you represent the number of pieces of candy Jack has?
> b) How could you represent the number of pieces of candy Ava has?

**Figure 1. The Candy Boxes task**

Students in the control condition are participating in their usual classroom activity with their regular classroom teachers. District-wide, all classroom teachers are using "Growing with Mathematics" (Iron, 2003) curriculum materials. This curriculum does not include a particular focus on early algebra or tasks similar to the ones included in our intervention.

*Data Collection*

A pretest and (identical) posttest along with a shorter "mid-year" review were designed to measure students' understandings of algebraic topics identified across the five "big ideas" in the EALP. The constructs measured on our assessments are closely linked to the EALP. Item construction was research-based where multiple internal reviews of all assessment items were conducted by the authors in order to ensure consistency, coherence, and fidelity to that of the constructs mentioned in the EALP.

From these assessments, we will focus in this paper on two tasks—the *Piggy Bank* task from the pretest/posttest (see Figure 2) and the *Silly Bands* task from the mid-year review (see Figure 3)—that investigated students' abilities to represent unknown quantities in relation to other unknown quantities.

> Tim and Angela each have a piggy bank. They know that their piggy banks each contain the same number of pennies, but they don't know how many. Angela also has 8 pennies in her hand.
>
> a) How would you describe the number of pennies Tim has?
> b) How would you describe the total number of pennies Angela has?

**Figure 2. The Piggy Bank task**

> Carter and Jackson each have the same number of silly bands, but we don't know how many they have. Carter earns 3 more silly bands for cleaning her room.
>
> a) How would you represent the number of silly bands Jackson has?
> b) How would you represent the total number of silly bands Carter has?

**Figure 3. The Silly Bands task**

The pretest was administered to all participants prior to the start of the teaching intervention in early September 2010. The posttest will be administered at the intervention's conclusion, in May 2011. In this paper, we will be able to share results of the pretest and the mid-year review administered in late November 2010, after eight intervention lessons. Because the purpose of this mid-year review was to offer the research team formative feedback on the impact of the intervention, it was only administered in the experimental classrooms. As such, our focus in this paper will be exclusively on the experimental students' growth representing unknown quantities as evidenced in their performance on the aforementioned assessment items. A more complete experimental-control comparison will be included in the presentation as we will have posttest results to share by that time.

*Data Analysis*

Each item on the pre-test and mid-year review was first scored dichotomously (i.e., correct or incorrect). A response was scored as correct if a variable expression was written that correctly conveyed the given relationships (e.g. Jackson has $J$ silly bands; Carter has $J + 3$ silly bands), if a correct statement was made about the relationship between the items in question (e.g., "Carter has 3 more silly bands than Jackson" or even "Carter has more silly bands than Jackson"), or if a student stated a need to use a variable but did not specify a particular one (e.g, "I would represent Tim's number of pennies with a letter"). In addition, there were a few instances in which students wrote equations rather than just expressions that were scored as correct (e.g., "Jackson has $a + 3 = b$"). Equations such as $a + 3 = a$ were scored as incorrect because the same variable was used to represent two different quantities.

Next, each item on the pre-test and mid-year review was given an appropriate strategy code. For part a on both assessments, responses were assigned the code *Letter* if a student used a variable to represent the unknown quantity (e.g., "Tim has $x$ candies"), *Value* if a student assigned a specific value to the unknown quantity (e.g., "4 candies"), and *Picture* if a student drew a picture to represent the unknown quantity. Student responses in which the quantities in the task were compared using words and specific quantities (e.g. "Tim has 8 fewer pennies than Angela") were coded as *Comparison with Quantity.* Responses in which comparisons were made using words but no specific quantities (e.g., "Tim has fewer pennies than Angela") received a *Comparison without Quantity* code.

Responses to part b were coded in the same way as part a, with additional attention paid to whether the responses were related to part a or independent of part a. For example, if a student responded that "Tim has $x$ pennies" in part a and "Angela has $x + 8$ pennies" in part b, this second response would receive the code *Related Letter*. If, instead, a student responded that "Angela has $y$ pennies," this second response would receive the code *New Letter*. "I don't know" or blank responses received a *No Response* code, and all other responses were coded as *Other*. A chi-square analysis was conducted to check for a significant association between the correctness of responses on the pre-test as compared to the related mid-year review item.

To assess reliability of the coding procedure, a second member of the research team coded a randomly selected 20% sample of the pre-test data. The mid-year review data was fully scored by two coders. Pre-test scoring agreement between coders was 92% for both correctness and strategy on part a and 90% for correctness and 92% for strategy on part b. Mid-year scoring agreement between coders was 87% for correctness and 82% for strategy on part a and 81% for correctness and 84% for strategy on part b. All differences in scoring were discussed by the coders and resolved.

**Results and Discussion**

In this section, we report experimental student results from the *Piggy Bank* task (see Figure 3) and the *Silly Bands* task (see Figure 4). There was a significant association between the correctness of responses for all items on the pre-test as compared to the mid-year review. For part a, $\chi^2(1) = 85.85$, $p < .001$ where it was 15.08 times more likely for a participant to get *Silly Bands* part a correct on the mid-year review than to get *Piggy Bank* part a correct on the pre-test. For part b, $\chi^2(1) = 52.33$, $p < .001$ where it was 7.46 times more likely for a participant to get *Silly Bands* part b correct on the mid-year review than to get *Piggy Bank* part b correct on the pre-test. Both results were statistically significant.

*Percent of students who gave correct response and percent of students who used a variable correctly on part a on Piggy Bank and Silly Bands tasks*

| | Grade 3 | Grade 4 | Grade 5 |
|---|---|---|---|
| Piggy Bank (Pre-test) | (n = 39) | (n = 42) | (n = 42) |
| Gave correct response | 5.1% | 23.8% | 33.3% |
| Used variable correctly | 0% | 0% | 0% |
| Silly Bands (Mid-year) | (n = 39) | (n = 41) | (n = 41) |
| Gave correct response | 79.5% | 80.5% | 80.5% |
| Used variable correctly | 61.5% | 63.4% | 68.3% |

**Table 1. Part a results**

*Percent of students who gave correct response and percent of students who used a variable correctly on part b on Piggy Bank and Silly Bands tasks*

| | Grade 3 | Grade 4 | Grade 5 |
|---|---|---|---|
| Piggy Bank (Pre-test) | (n = 39) | (n = 42) | (n = 42) |
| Gave correct response | 5.1% | 23.8% | 35.7% |
| Used variable correctly | 0% | 0% | 0% |
| Silly Bands (Mid-year) | (n = 39) | (n = 41) | (n = 41) |
| Gave correct response | 66.6% | 70.7% | 68.3% |
| Used variable correctly | 51.3% | 48.9% | 58.5% |

**Table 2. Part b results**

From Tables 1 and 2, one can see that students initially struggled to produce correct representations or descriptions of unknown quantities, even with very liberal criteria. None of those students who did produce a correct representation or description used variables to represent the unknown. However, a majority of participants at all grade levels were able to use a variable correctly on part a and close to over half were able to use a variable correctly for part b by the time of the mid-year review. Furthermore, participants tended to use the same variable in part b of the *Silly Bands* task that they used in part a, showing an ability to consider the relationship between two unknown quantities. Tables 3 and 4 show the percent of students using the previously-discussed strategies in response to the pre-test and mid-year review tasks.

*Percent of students using each strategy in response to part a on Piggy Bank and Silly Bands tasks*

| Strategy | Grade 3 | | Grade 4 | | Grade 5 | |
|---|---|---|---|---|---|---|
| | *Piggy Bank* (Pre-test) | *Silly Bands* (Mid-year) | *Piggy Bank* (Pre-test) | *Silly Bands* (Mid-year) | *Piggy Bank* (Pre-test) | *Silly Bands* (Mid-year) |
| Letter (as variable) | 2.6% | 89.7% | 0% | 75.6% | 0% | 85.4% |
| Value | 33.3% | 5.1% | 45.2% | 12.2% | 42.9% | 4.9% |
| Picture | 0% | 0% | 0% | 2.4% | 0% | 4.9% |
| Comparison with Quantity | 5.1% | 5.1% | 11.9% | 9.8% | 23.8% | 4.9% |
| Comparison without Quantity | 0% | 0% | 11.9% | 0% | 7.1% | 0% |
| Other | 5.1% | 0% | 14.3% | 0% | 16.7% | 0% |
| No Response | 53.9% | 0% | 16.7% | 0% | 9.5% | 0% |

**Table 3. Part a strategies**

First, in response to part a, note that participants demonstrated no prior knowledge of representing an unknown quantity using variables, preferring instead to assign specific values. However, by the time of the mid-year assessment, at least 75% of students at all grade levels attempted to represent the unknown quantities using a variable, with the majority of them doing so correctly. We had predicted that students would use a pictorial representation (e.g., a piggy bank) or an empty box to represent an unknown amount, especially prior to instruction. However, only a small number of these pictorial and box representations surfaced. The decline in *No Response* codes was dramatic, especially in grade 3.

*Percent of students using each strategy in response to part b on Piggy Bank and Silly Bands tasks*

| Strategy | Grade 3 | | Grade 4 | | Grade 5 | |
|---|---|---|---|---|---|---|
| | *Piggy Bank* (Pre-test) | *Silly Bands* (Mid-year) | *Piggy Bank* (Pre-test) | *Silly Bands* (Mid-year) | *Piggy Bank* (Pre-test) | *Silly Bands* (Mid-year) |
| Related Letter | 2.6% | 59.0% | 0% | 53.7% | 0% | 61.0% |
| New Letter | 0% | 28.2% | 0% | 22.0% | 0% | 26.8% |
| Related Value | 25.6% | 2.6% | 35.7% | 9.8% | 35.7% | 4.9% |
| New Value | 18.0% | 5.1% | 11.9% | 4.9% | 2.4% | 0% |
| Related Picture | 0% | 0% | 0% | 2.4% | 0% | 2.4% |
| New Picture | 0% | 0% | 0% | 0% | 0% | 0% |
| Comparison with Quantity | 5.1% | 0% | 16.7% | 7.3% | 21.4% | 4.9% |
| Comparison without Quantity | 0% | 5.1% | 7.1% | 0% | 9.5% | 0% |
| Other | 7.7% | 0% | 14.3% | 0% | 16.7% | 0% |
| No Response | 41.0% | 0% | 14.3% | 0% | 14.3% | 0% |

**Table 4. Part b strategies**

In response to part b, we noted a shift in responses from assigning a specific value on the pre-test (related or non-related to the response in part a) to using a variable on the mid-year review. Well over half of participants across the grades wrote an expression using the same variable used in part a, with the majority doing so correctly. No students left this task blank or responded with "I don't know" on the mid-year review.

One particular category of response caught our attention. Some participants used variables but represented the unknown quantities with equations rather than expressions. For example, in response to the *Piggy Bank* task, one student represented the number of Angela's pennies as $n + 8 = n$. Such a response recalls Booth's (1988) assertion and Carraher et al.'s (2006) similar finding that students may struggle with "lack of closure" and feel the need to "simplify" or come to a "single term" answer. Such students appear to have difficulty holding operational and structural views of expressions simultaneously. As Linchevski and Herscovics (1996) suggest, expressions such as $n + 8$ are difficult for students because the addition operation cannot be performed. The expression must be viewed as the result of the operation and not merely a process. This tendency among some students to write incorrect equations was present even after several months of our instructional intervention, with 10.3%, 4.9%, and 2.4%, in grades 3, 4, and 5, respectively, answering in this manner on the mid-year review.

## Conclusion

As Blanton (2008) asserts, "It is as children work with symbols that they acquire meaning for them. They will experience a natural progression in their thinking that begins with a limited understanding of symbols and symbolizing" (p. 66). While students initially exhibited great difficulty representing unknown quantities in a general way (i.e., without assigning a specific value), over half of our participants at each grade level exhibited the ability to use a variable to represent an unknown quantity by the time of the mid-year review. While we have indications that several students had difficulty viewing variable expressions as objects not in need of further simplification, overall the majority of students seem to be learning to use variables in meaningful ways that were unknown to them at the start of the school year.

## Endnotes

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