VISUAL REPRESENTATIONS IN MATHEMATICS PROBLEM-SOLVING: EFFECTS OF DIAGRAMS AND ILLUSTRATIONS

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Many types of visual representations are used in math textbooks but not all of them contain mathematically relevant information. Little research directly addresses the effects of different types of representations on mathematics performance. Theories offer differing perspectives about how visual representations such as illustrations influence student learning. Here, we investigated the effects of diagrams and contextual illustrations on trigonometry problem solving. Diagrams helped all students, but the effect of contextual illustrations depended on students' backgrounds. Additionally, not all subgroups of students accurately assessed the effect of illustrations on their performance. We emphasize the need to consider how different types of visual representations interact with student characteristics and the problem-solving task.

Keywords: Problem Solving

Mathematics textbooks use a wide variety of visual representations, including diagrams, tables, graphs, decorative images, and photographs. Given students' frequent use of textbooks and the large number of visual representations in these textbooks, understanding how different types of visual representations affect problem solving and learning is critical.

Most research on the effects of visual representation has been conducted using scientific texts. It has focused primarily on diagrams and illustrations accompanying expository texts about causal phenomena. In such contexts, graphically integrating visual and verbal information is found to be beneficial, as is removing irrelevant information (e.g., Mayer, 2009). However, mathematical and scientific problem solving differ in many important ways, including different emphases on causality, spatial relationships, procedural and conceptual knowledge, and analytic methods. Thus, findings from research on science learning may not apply straightforwardly to math.

Existing research about the effects of visual representations in mathematics is based primarily on studies with elementary-age students, and it presents a complex and mixed picture. Some studies suggest that contextual illustrations hurt performance for particular subgroups of students (e.g., Berends & van Lieshout, 2009). Other studies suggest that decorative illustrations do not affect performance (e.g., Berends & van Lieshout), or that certain types of illustrations can benefit performance (e.g., Hegarty & Kozhevnikov, 1999; McNeil, Uttal, Jarvin, & Sternberg, 2009). Many studies also suggest that the usefulness of visual representations depends on students' ability levels (e.g., Booth & Koedinger, 2011; Berends & van Lieshout).

Theoretical Frameworks

In making sense of research on visual representations, two theoretical frameworks are particularly relevant: the Cognitive Theory of Multimedia Learning (e.g., Mayer, 2005, 2009) and Cognitive Load Theory (e.g., Sweller, 2004, 2005). Both theories address the processing and learning of information presented in different formats.

The Cognitive Theory of Multimedia Learning (e.g., Mayer, 2009) is based on three assumptions: (a) a limited capacity for processing information, (b) separate visual and verbal pathways through which information enters the cognitive system, and (c) meaningful learning arising from active processing. Cognitive Load Theory (e.g., Sweller, 2005) focuses on the cognitive load—the mental effort from the task itself, the processing required to integrate new and old material, and the processing required to work with a task's format. Overall, one idea is that the structure of the cognitive system imposes limits on how learners select, organize, and integrate information. These approaches have been used to guide instructional design.

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Two principles derived from these theories are particularly relevant to the research reported here. The *multimedia effect* holds that words and pictures are better than just words (e.g., Butcher, 2006; Mayer & Anderson, 1992) based on the assumption of separate visual and verbal channels which can then be integrated for deeper learning. The *coherence effect* captures the performance benefits that occur when extraneous or seductive features of the material are eliminated (e.g., Harp & Mayer, 1997, 1998). Adding interesting but irrelevant material can overload the visual or verbal pathways or create too much extraneous load, thereby disrupting learning (Sweller, 2005). The coherence effect applies to both text and visual material.

Contextualization Perspective

Another theoretical perspective applicable to the current study focuses on how contextualizing or "grounding" math problems in real-world scenarios can help learners (Goldstone & Son, 2005; Koedinger & Nathan, 2004). Contextualization is thought to help students build a model of the situation underlying a problem. In addition, realistic content or greater familiarity with the content may promote generalization or facilitate reasoning because it fosters integrating the current problem with prior knowledge. Some studies have suggested that contextualization is more beneficial for simpler problems (Koedinger, Alibali, & Nathan, 2008), whereas other studies have suggested that it is more beneficial for difficult problems or lower ability students (Walkington, 2012). This body of research has typically involved contextualizing problems by adding verbal information to text, but contextualization can also be accomplished through accompanying visual representations.

Current Study

It remains an open question as to how the multimedia principle, the coherence principle, and the notion of contextualization apply to visual representations used in mathematics. The current research involves trigonometry problems accompanied by 4 types of visual representations: combining diagram presence (or not) with the presence of contextual illustrations (or not). The contextual illustration could add extraneous details through the graphics, but it also could ground the problem situation. We use the term *contextual illustration* since the illustrative features correspond to the spatial layout necessary to solve the problem. The perspectives discussed above vary in their predictions about which visual representations will be most helpful (see Table 1). We consider these effects in terms of student performance and evaluations of the problems.

Table 1: Predictions from Applicable Theoretical Frameworks

	Theoretical Prediction		
Multimedia Principle	Visual representations will help	→	Text by itself will be hardest
Coherence Principle	Extraneous information hurts performance	→	Problems with illustrations will be harder
Multimedia + Coherence Principles	Provide visuals but avoid extraneous information	→	Diagram by itself will be easiest.
Contextualization Perspective	Illustrations further ground problem solving	→	Problems with illustrations will be easier.

Method

Participants

Participants were 93 undergraduates, who received credit in introductory psychology for their

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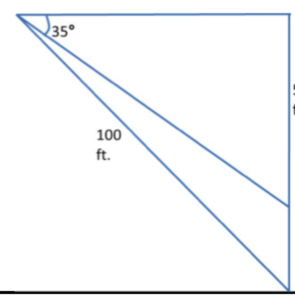
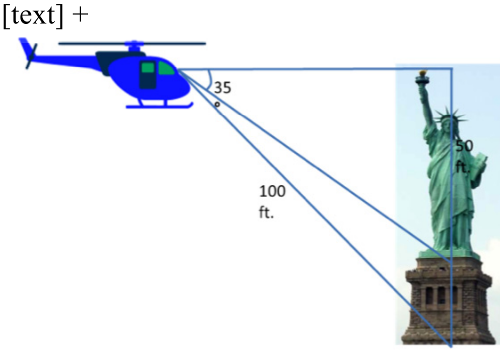
participation. The majority (63%) had completed middle school math in the United States. Of those who had not, most (82%) had their earlier math education in an Asian country. Over two-thirds (69%) intended to major in a math or science field.

Participants were divided into subgroups based on their intended major (math/science field or not) and where their previous math education occurred (U.S. or non-U.S.). Students who were math/science majors were fairly evenly split into those who were previously educated in the U.S. (*n* = 34) and those who were previously educated outside the U.S. (*n* = 30). The vast majority of participants who were not math/science majors were educated in the U.S. (*n* = 25). Only 4 participants previously educated outside the U.S. were not math/science majors; this small group was excluded from the analyses reported here.

Design and Materials

Each participant received 4 problems based on a 2 (Diagram Presence) × 2 (Illustration Presence) within-subjects design, yielding 4 conditions: text alone, diagram alone, illustration alone, and illustration with diagram overlay (see Table 2). Condition order was counterbalanced across participants.

Table 2: First Background Story, Shown for Each Visual Condition

	No Diagram	**Diagram**
No Illustration	The parks department is putting a statue on a base. The statue is some distance away, and you are in a helicopter, eye level with its top. The angle of depression to the bottom of the statue (i.e., the top of the base) is 35 degrees. The height of the statue is 50 feet. If someone were to stretch a string from the bottom of the base directly to you, it would be 100 feet long. How tall is the base?	[text] + 
Illustration	[text] + 	[text] + 

Each of the 4 problems each participant received involved a different cover story. All required applying trigonometric relations to overlapping right triangles to solve for an unknown dimension. The different stories had varied combinations of sides and angles, such that the solution processes were not identical for any two problems. These types of problems were selected as they lend themselves well to concrete situations and are at an appropriate difficulty level for undergraduate participants. The order of the cover stories (and thus of the mathematical solutions) was held constant across participants. Each problem was on its own page, with the text and visual representation (if present) at the top of the page.

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The illustration corresponded to the problem situation. As shown in Table 2, although it did have decorative features, it was also mathematically relevant because it indicated the spatial layout of the components of the story problem.

Procedure

Participants received a reference handout (with text and equations but no diagrams) of information about triangles and trigonometric formulas. The information was available throughout the study, and participants were told that not all of it would be needed. Participants worked through each of the four problems at their own paces. After completing the problems, they rated how difficult each problem was, how clear it was, and how willing they would be to do more problems like it. They assessed these characteristics on a 5-point Likert scale. While making these ratings, participants were permitted to look back over the problems but not to change any of their answers. Finally, participants completed a questionnaire about their attitudes towards mathematics, their math abilities, and their math background.

Results

A sizeable proportion of participants answered all or none of the problems correctly, and these rates depended on participant subgroup. For instance, 40% of the students who were not math/science majors answered *no* problems correctly and 30% of the math/science majors who received their previous education outside of the U.S. answered *all* the problems correctly. The results below include all participants; however, the patterns also hold for the subset of participants who did not perform at floor or ceiling (i.e., correctly answered 1–3 of the 4 problems).

Did Visual Condition Affect Accuracy?

We analyzed the dichotomous measure of accuracy on each problem using mixed models logistic regression (Bates & Maechler, 2009) in R. The best fitting model included the fixed factors of diagram presence, illustration presence, educational background / major (henceforth *participant subgroup*), and the interaction between illustration and participant subgroup. We also included participant and cover story as random factors; cover story significantly improved the model's fit ($p < .0001$). Coefficients and odds for the model are reported in Table 3.

Table 3: Coefficients from Regression Model for Accuracy Ratings

Fixed effects:	Estimate (logit)	SE	Odds	z value	Sig
Intercept	-0.53	0.68	0.59	-0.78	0.44
Diagram – no	Reference				
Diagram – yes	1.49	0.28	4.42	5.23	<.0001
Illustration – no	Reference				
Illustration – yes	0.996	0.47	2.71	2.1	0.04
Subgroup – outside US & math/science major	Reference				
Subgroup – US & math/science major	-0.39	0.64	0.68	-0.61	0.54
Subgroup – US & not math/science major	-1.15	0.70	0.32	-1.65	0.10
Illustration – yes x Subgroup – US & math/science major	-0.62	0.64	0.54	-0.96	0.34
Illustration – yes x Subgroup – US & not math/science major	-1.56	0.73	0.21	-2.13	0.03
Model: Accuracy ~ DiagramPresence + IllustrationPresence * Subgroup + (1 | CoverStory) + (1 | ID)					
Random effects (Intercepts): Variance of participant = 3.26 Variance of cover story = 0.91					

Participants performed significantly better on problems with diagrams than without, $p < .0001$ (see Figure 1; *SE* are corrected via procedure in Morey, 2008, to reflect within-subjects design). This effect existed for all three subgroups and did not interact with illustration presence.

However, the effect of illustration varied across participant subgroups. Participants who received their math education outside the U.S. (and were also math/science majors) performed significantly better with illustrations ($p < .04$) than without. This improvement differed significantly ($p = .03$) from the slightly negative effect of illustration on the US subgroup not majoring in math/science. The accuracy level of the subgroup who were not math/science majors was significantly lower than the accuracies the other two subgroups when there was an illustration (*p*s < .02), but this pattern did not reach significance when there was no illustration.

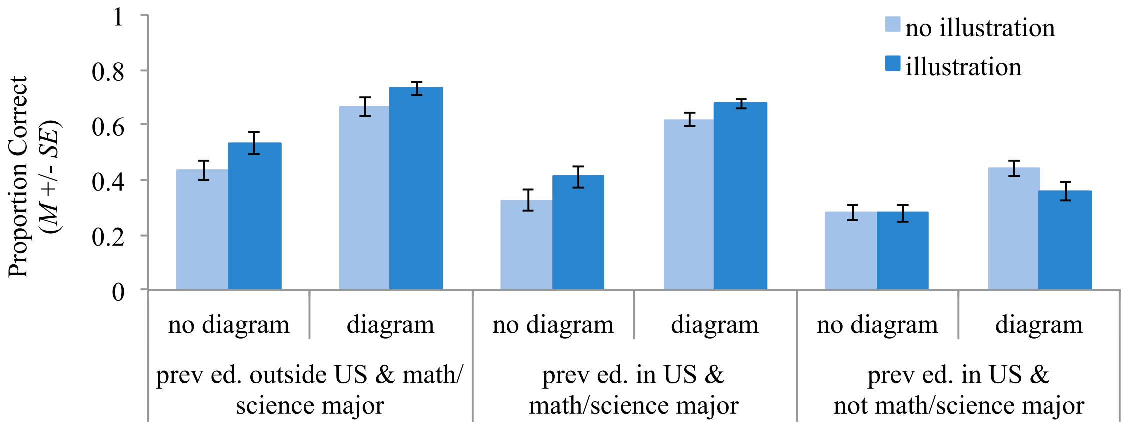

Figure 1: Average accuracy (+/– *SE*)

Did Visual Condition Affect Participants' Ratings of the Problems?

We combined participants' ratings of each problem's clarity and difficulty (reverse-coded) as well as their ratings of how willing they would be to do more problems similar to those completed. This composite measure offered an assessment of a participant's overall favorability towards a problem type. Correlations among the three measures ranged from .34 to .56, *p*s < .0001. The best fitting mixed effects model for participants' ratings included the fixed factors of diagram presence, illustration presence, and participant subgroup. We also included participant and cover story as random factors; cover story significantly improved the model's fit ($p < .001$).

As shown in Figure 2, participants viewed problems with diagrams significantly more favorably than those without, and they viewed problems with illustrations significantly more favorably than those without; respectively, each of these factors improved the fit of the model, $\chi^2(1) = 7.79$, $p = .005$ and $\chi^2(1) = 10.5$, $p = .001$. However, the magnitude of these effects was relatively small. Comparisons of the subgroups indicated that participants who were not math/science majors rated the problems significantly lower than participants with math/science majors (*t*s > 2.71), whose subgroups did not differ from one another ($t = 1.70$).

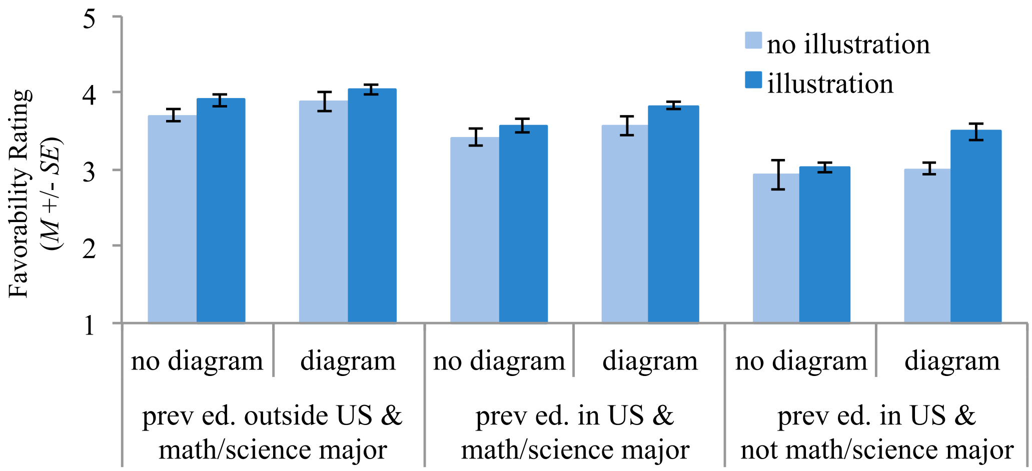

Figure 2: Average favorability (+/– *SE*)

Discussion

In this study, participants performed more accurately on trigonometry problems with diagrams. The effect of illustrations was mixed. Illustrations yielded a slight improvement in performance for students who intended a math/science major, but illustrations slightly hurt performance among students who were not intending to major in a math- or science-related field. These findings highlight that ability differences affect the use of visual representations.

The *multimedia principle* predicts that problems with visual representations would be solved more successfully than problems presented as text only. This was clearly the case for diagrams. The more mixed influence of contextual illustrations can be considered with respect to the *coherence principle* and the *contextualization perspective,* which make opposite predictions. Indeed, each prediction fit a subset of the participants. As predicted by the contextualization perspective, having an illustration benefited performance for participants who were math/science majors. In contrast, as was predicted by the coherence principle, illustrations hurt performance for those who were not math/science majors. Overall, though, the effects of illustration presence were relatively small.

These findings indicate that the coherence principle, which has been supported in multiple studies using science material (see Mayer, 2009), may not apply so straightforwardly in math. However, the coherence principle stresses the removal of extraneous details. Not all details are extraneous, and added visual details do not necessarily harm everyone's performance. This research used contextual illustrations that could have assisted students in mapping the problem content to the visual representation and thus does not necessarily contradict the coherence principle. It is also worth noting that the contextualized illustrations we used were more relevant to mathematics than the majority of illustrations that are found in American mathematics textbooks (Cooper et al., 2012; Mayer et al., 1995). Addressing the impact of purely decorative illustrations will be an important extension of this research.

Focusing on the cognitive load required by these problems offers a possible way to combine the two perspectives on the effect of illustrations and understand the dependence of the effect on subgroup. The cost of encoding and integrating the extraneous information (such as the design of the base of the statue) conveyed in illustrations may outweigh any possible benefits from contextualization if cognitive load surpasses the available cognitive resources. Illustrations might be more helpful for individuals with more math experience because such individuals can construct a contextualized mental representation of the problem scenario without exceeding their available cognitive resources. However, other research on contextualization has found grounding problems to offer greater benefits for students of lower math ability (see Walkington, 2012).

Diagram presence increased the favorability with which participants viewed the problems, as did illustration presence. Comparing this with performance data indicates that all participants' metacognitive beliefs about problems with diagrams matched their actual performance. However, only students intending a math/science major accurately perceived the effect of illustration presence. Participants who were not math/science majors performed the same or worse when an illustration was present, despite their more favorable view of these problems. This pattern of findings is particularly important to consider in light of the motivation-based argument that textbook visuals will help engage learners, particularly those with low math interest (see Durik & Harackiewicz, 2007, for related findings). However, it aligns with the research arising from the Cognitive Theory of Multimedia Learning and from Cognitive Load Theory, which hold that these extraneous but interesting details can be problematic for learning. As noted above, this may hold true especially when an individual's resources are taxed, which is more likely to occur for individuals with lower background knowledge.

The overall differences we observed in accuracy between students of different backgrounds are not surprising in light of the well-documented finding that students from many foreign countries outperform American students in math (Fleischman et al., 2010). What is more interesting is that students of different backgrounds were differentially affected by visual representations. The underlying constructs tapped by our measures of students' backgrounds need to be characterized with greater precision. We collected data on the intended majors and the country in which they received their middle school education. These measures may simply reduce to experience and interest in math; however, further research on students' backgrounds and how they affect performance is needed.

It is also worth noting that overall levels of performance in this study were not high, even in the highest performing subgroup. The problems we used were quite complex, and many components needed to be performed correctly in order to reach an accurate final answer. Students needed to know how to map information from the problem content to the visual representation and from the visual representation to their mental representation of the problem. Students also needed to identify what quantity to solve for, figure out the steps needed to reach the solution, and correctly apply the trigonometric formulas to reach a final answer. Understanding the differential effects of the type of visual representation on these different components of problem solving is an important arena for future research (see Butcher, 2006; McNeil et al., 2009).

In sum, this work highlights the need for a continued focus on the ways in which visual representations support learners' strategic problem solving and learning. Rather than asking simply which types of illustrations serve learners better, it is important to identify how learners with different backgrounds and skill levels utilize visual representations when solving problems.

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