PMENA 2013

CHICAGO • NOVEMBER 14-17

Broadening Perspectives on Mathematics Thinking and Learning

Proceedings

Thirty-Fifth Annual Meeting

of the North American Chapter of the International Group
for the Psychology of Mathematics Education

November 14–17, 2013

Chicago, Illinois

UIC
UNIVERSITY
OF ILLINOIS
AT CHICAGO

Eds:

Mara V. Martinez,
Alison Castro Superfine

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Martinez, M. & Castro Superfine, A (Eds.). (2013). *Proceedings of the 35th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Chicago, IL: University of Illinois at Chicago.

Citation

Martinez, M. & Castro Superfine, A (Eds.). (2013). *Proceedings of the 35th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Chicago, IL: University of Illinois at Chicago.*

ISBN

978-0-615-86464-8

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Cover design by An Lai, Dawn Design Studios

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The major goals of the International Group and the North American Chapter are:

- To promote international contacts and the exchange of scientific information in the psychology of
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- To promote and stimulate interdisciplinary research in the aforesaid area, with the cooperation of
- psychologists, mathematicians, and mathematics teachers;
- To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

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Preface

These Proceedings are a written record of the research presented at the 35th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (PME-NA 2013) held in Chicago, Illinois, November 14-17, 2013.

The theme of the conference is Broadening Perspectives on Mathematics Thinking and Learning. Dr. Nicolas Balacheff focuses on a model, cK¢, to think about learners´ mathematical conceptions. Dr. Na'ilah Suad Nasir focuses on race and culture in mathematics education. Dr. Tobin White and Dr. Melissa Gresalfi address technology in mathematics education in a panel format.

The Proceedings include papers from 1 plenary talk, 2 panel presentations, 68 research reports, 121 brief research reports, 208 posters, and 14 working groups. The plenary and working group papers are the first and last chapters, respectively. Papers from the research reports, brief research reports, and posters are organized into chapters by topics. Each paper is indexed by authors and keywords. Underlined author indicates presenting author.

We would like to thank Janet Liao Kornas for her dedication to the technical details of putting together a high quality document. We are pleased to present these Proceedings as an important resource for the mathematics education community.

Alison Castro Superfine
&
Mara V. Martinez
Co-Chairs

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Plenary Papers

cK¢, A MODEL TO REASON ON LEARNERS' CONCEPTIONS

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Understanding learners' understanding is a key requirement for an efficient design of teaching situations and learning environments, be they digital or not. This keynote outlines the modeling framework cK¢ (conception, knowing, concept) created with the objective to respond to this requirement, with the additional ambition to build a bridge between research in mathematics education and research in educational technology. After an introduction of the rationale of cK¢, some illustrations are presented. Then follow comments on cK¢ and learning. The conclusion evokes key research issues raised by the use of this modeling framework.

Keywords: cK¢, Conception, Misconception, Knowing, Milieu, Didactical Situations, Conceptual Fields, Learner Modeling, Design Experiment, Technology Enhanced Learning

A Short Story

The model I will introduce has a long history. Its construction started at the end of the 80s with the project of bridging artificial intelligence and didactics of mathematics, and the objective of enhancing the design of computer-based learning environments. The design of these environments had a ternary structure including a model of the learner, a model of the content to be learned and an instructional model. These models are nowadays either effectively implemented, or only involved in the design phase. For the last two, research has constantly been very active with some promising progress. On the contrary, modeling the learner proved to be a real challenge, and actually it is still the case despite expectations raised by recent research on educational data mining and learning analytics.

The theoretical framework within which I was working, the Theory of Didactical Situations (TDS), is based on ideas that some may consider as precursors of "learning games" now celebrated by researchers in educational technology:

> "*Modeling a teaching situation consists of producing a game specific to the target knowledge among different subsystems: the educational system, the student system, the milieu, etc*."
> (Brousseau 1986/1997 p.47).

Within this approach, the teacher is "*a player faced with a system, itself built up from a pair of systems: the student and, let us say for the moment, a 'milieu' that lacks any didactical intentions with regards to the student*" (ibid. p.40). A systemic approach to learner modeling makes possible expressing learning as adaptation and adaptation as construction. However, while research went quite far in modeling didactical situations, the progress on modeling the learner "*subsystem*" was rather limited. Indeed, there was a lot of research within different frameworks, with different concepts and a variety of terms, reporting observations of learners' behaviors. But the distance between the content of these reports and models that we could use to inform the design of learning environments, was quite significant.

As a researcher in mathematics educations, I had another motivation. I started my research on the learning of mathematical proof in the 70s, with an exploration of the structure of proofs produced by students, using graphs to represent them. This approach was intrinsically limited, so it was not too difficult to understand the critiques of Guy Brousseau and Gérard Vergnaud. The former drew my attention to the role of the situation; the latter pointed to the cognitive

complexity of producing proofs. Then, I engaged in a completely different direction which resulted at the end of 80s in a first picture of what could be the genesis of mathematical proof. While this picture could help designing didactical situations, it didn't shed light on the underlying mental processes. In other words, it was a didactical study, not a psychological study. Still, there was a weak point: it left open a gap between proving and knowing. To close this gap I had to find a way to model the learner's ways of knowing. The Vergnaud model of concept and conceptual field offered a possible solution:

> "[…] *from a developmental point of view, a concept is altogether: a set of situations, a set of operational invariants (contained in schemes), and a set of linguistic and symbolic representations.*" (Quoted from Vergnaud 2009 p.94, but this characterization goes back to the early 80's)

Vergnaud introduced the notation: C=(S, I, $\mathcal{S}$) in which the components refer respectively to each of the three sets mentioned above. He emphasized that these components cannot be separated; they have to be considered all at the same time when studying learners' development. This characterization has direct connections with the TDS description of the relation between a learner and a milieu based on different forms of knowledge (Brousseau ibid. p.61):

[1] The models for action governing decisions.
[2] The formulation of the descriptions and models.
[3] The forms of knowledge which allow the explicit "control" of the subject's interactions in relation to the validity of her statements.

Apart from the set of situations (S) which is implicitly shared, common elements are related to action (I) and formulation ($\mathcal{S}$). Then, one element is missing which corresponds to the terms "control" used in the TDS description. I mentioned this lack when discussing the Vergnaud concept of theorem-in-action. A theorem, and the same applies to a theorem-in-action, is both a tool and a statement: "if A then B" is a tool to obtain B if A is valid, it is also a statement which has a truth value. This duality of "*the operational form and the predicative form of knowledge*", as Vergnaud expresses it, facilitated keeping implicit the control dimension in the characterization he proposed. However, after Polya and a long tradition of research on metacognition, Schoenfeld (1985 pp. 97-143) has shown the crucial role of control in problem-solving. The suggestion I made consists in introducing explicitly this dimension of control in Vergnaud model. This is the origin of the quadruplet I describe below.

Before entering the main content of this keynote, I would like to address an issue which led me use the term "conception" and not the term "knowledge" as it is classical in educational technology and mathematics education as well. Most of our research is based implicitly or not on the hypothesis that learners act as rational subjects. But, one often is faced to rational thinking co-existing with knowledge which looks contradictory (from the observer's point of view). Bourdieu (1990) proposed a solution to this paradox: "*The calendar thus creates ex nihilo a whole host of relations [...] between reference-points at different levels, which never being brought face to face in practice, are practically compatible even if they are logically contradictory*" (ibid. p. 83). The key elements are time on one hand, and on the other hand the diversity of situations. Time organizes the subjects' decisions sequentially in such a way that even contradictory, they are equally operational because appearing at different periods of their history: contradictory decisions can ignore each other. The diversity of the situations introduces

an element of a different type. It is a possible explanation insofar as one recognizes that each decision is not of a general nature but that it is related to a specific sphere of practice (we would say, nowadays, that it is situated) within which it is acknowledged as efficient. Within a sphere of practice learners are coherent and successful; they are non-contradictory.

Contradictions (and failures) appear when learners are faced with situations foreign to their sphere of practice but in which they have nevertheless to produce a response (e.g. a question from an interviewer). They mobilize what they have available which worked elsewhere, but more often than not this ends in systematically making errors. The classical position in the 80s was to consider these errors as symptoms of misconceptions. This term used to come with expressions like "naive theory", "private concepts", "beliefs" or even "mathematics of the child". Such views missed the fact that "*a child may not be 'seeing' the same set of events as a teacher, researcher or expert. […] many times a child's response is labeled erroneous too quickly and […] if one were to imagine how the child was making sense of the situation, then one would find the errors to be reasoned and supportable* " (Confrey 1990 p.29). Agreeing with this position, I renounced using the term "misconception". Still, recognizing that learners may have different models-in-action to mobilize for (what we consider as) the same piece of knowledge, I needed a term, but one different from "knowledge" because of the issue raised by the observation of possible contradictions in learners behaviors. A possible term was "conception" largely used in science education to denote theory-in-action. Most often than not conception functioned as a tool in discourses but it was not taken as an object of study as such (Artigue 1991, p.266), although there was an acknowledged need (e.g. Vinner 1983, 1987) for a better grounded definition of conceptions, and for tools allowing analyzing their differences and commonalties. In the following section, I propose a definition of "conception", and then describe a model revisiting the Vergnaud's triplet.

Behavior, Conception and Knowing

The only indicators one has to get an insight into learners' understanding are their behaviors and products which are consequences of the conceptions they have engaged. Such evaluations are possible and their results are significant only in the case where one is able to establish a valid relationship between the observed behaviors and the invoked conception. This relation has been relatively "hidden" as such for a long while as a result of the fight against behaviorism, but it has always been present in educational research at least at the methodological level. Indeed, the key issue is that *the meaning of a piece of knowledge cannot be reduced to behaviors, whereas meaning cannot be characterized, diagnosed or taught without linking it to behaviors*.

Being a tangible manifestation of the relationships between a person and her environment, a "behavior" depends on the characteristics of this person as well as on the characteristics of her environment. A now well documented example is that of an instrument which at the same time facilitates action if the user holds the required competence, and on the other hand limits this action because of its own constraints (Rabardel 1995, Resnick & Collins 1994, p.7).

The words "person" and "environment", which I am using here, refer to complex realities whose aspects are not all relevant for our investigations. One may want to ignore the clothes the person wears and the shape of the room in which he or she stands (although we have always to be prepared to consider seriously features initially downplayed). What is of interest is the person from the point of view of his or her relationship to a piece of knowledge. For this reason I will refer from now on to the learner as a reduction, if I dare saying so, of the person to her cognitive dimension. In the same way, I do not consider the environment in all its complexity, but only those of its features that are relevant with respect to a given piece of knowledge. Actually, this

corresponds to the TDS concept of *milieu*, which is a kind of projection of the environment onto its epistemic dimension: the milieu is the learner's antagonist system in the learning process (Brousseau, 1997 p.57)

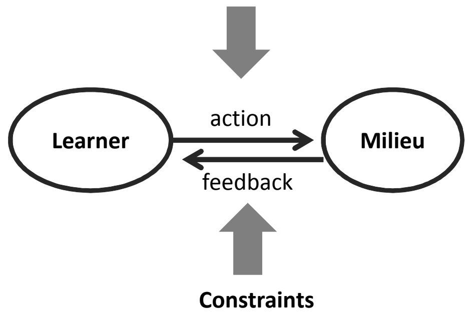


Figure 1. learner/milieu system

This situatedness nature of a conception suggests not considering it as a property which can be ascribed only to the learner but as a property of the interacting system formed by the learner and his or her antagonist milieu, to which I will refer as the "learner/milieu system". What is requested for this property to be valid is that the system satisfies the necessary conditions for its viability. I mean that the system has the capacity to recover equilibrium after a perturbation which otherwise would cause its collapse, or that it can transform itself or reorganize itself. This is another formulation of Vergnaud's postulate that problems (perturbed system) are the sources and the criteria of knowing (Vergnaud 1981 p.220). It is important to realize that nothing is said about the process leading to the recovery of the equilibrium under the said constraints. They are proscriptive (Stewart, 1994 pp. 25-26), which means that they express necessary conditions to ensure the system viability, but not prescriptive, which means that they do not say in what way an equilibrium must be recovered.
Hence, a definition of conception:
A conception is the state of dynamical equilibrium of an action/feedback loop between a learner and a milieu under proscriptive constraints of viability.
The study and characterization of a conception will be based on observable behaviors of the system (action, feedback) and outcomes of its functioning. It requires evidence of the assessment of the equilibrium, which depends on the possibility to elicit the learner's control of the interaction and of the milieu's reification of failures and success by adequate feedback.

Geometry provides many good examples: constructing a diagram on a sheet of paper with a pencil is permissive to empirical adjustments, while dynamic geometry software allowing messing up a diagram by dragging points can reify the failure of conforming to geometrical properties (Healy et al. 1994)—but still, "students may modify the figure 'to make it look right' rather than debug the construction process" (Jones1999 p.254).

The situated nature of a conception means that for different situations considered conceptually the same or for problems claimed isomorphic, one may associate different conceptions with the same learner. There is a large documentation of this phenomenon in the literature, for example under the theme of transfer, or from research in ethnomathematics. Anyhow, in the researcher's referential system, these different states of the observed systems [learner in a situation] should be labeled in the same way. For this reason, I define a *learner's knowing* as the set of conceptions which can be activated by different situations the observer considers conceptually the same—a qualification that indeed one will have to clarify. I realize that using "knowing" as a noun is rare, but it helps keeping distance with the word "knowledge" which has in education a strong authoritative connotation.

Having this definition of conception and of its relation to knowing, in the next section, I propose a model inspired by Vergnaud's formalization which I develop in the next section. Some examples will illustrate the model and facilitate clarifying the intention of my research program.

Outlines of a Model

What is a model could be discussed at length within the PME multidisciplinary research community. I take here a pragmatic position, looking for something likely to facilitate our collaboration with research in educational technology, but also as a means to make more efficient our own research. The objective is to contribute to a better understanding of learners' understanding and to have eventually a practical value for teachers and designers.

It should be emphasized that the terms "conception", "knowing", "concept" and several others appearing in the description of the model are abstract terms, whose meaning is that of their functions and relations within the model. Indeed, we must then discuss how far the proposed formalization makes sense when confronted with "reality", and if it is an adequate tool for our research. The examples I will present and some additional comments will hopefully partly respond to this preoccupation.

I call "conception" a quadruplet (P, R, L, Σ) in which:

- P is a set of problems.
- R is a set of operators.
- L is a representation system.
- Σ is a control structure.

The first three elements are almost directly borrowed from the Vergnaud triplet. The vocabulary is different to avoid confusion with the vocabulary of psychology. In particular, operators correspond to actions one can observe in the functioning of the learner/milieu system; they are not schemes in psychological terms. The representation system is formed of all the semiotic tools which allow representing problems, supporting interaction and representing operators if formulations were required. The characterization of the set of problems P is more complex than expected. Two opposite solutions have been proposed: (i) to include all problems for which the conception provides efficient tools (Vergnaud 1991 p.145), but for basic concepts this option is too general to be effective; (ii) to consider a finite set of problems from which other problems will derive (Brousseau 1997 p.30), but this option opens the question of establishing that such a generative set of problems exists for any conception. A solution familiar to most researchers consists of deriving the description of P from both the observation of students in situations and from the analysis of historical and contemporary practices of mathematics. Actually, what one does when working on specific conceptions is to open a window on P by making explicit a few good representatives of its potential elements. These representatives work as kind of prototypical problems; this is a pragmatic implementation of Brousseau's proposal.

The forth element of the quadruplet, the control structure, includes behaviors such as making choices, assessing feedback, making decisions, judging the advancement of a problem solving process. These metacognitive behaviors are more often than not silent and invisible, hence rarely accessible to observation. It is why, to overcome this difficulty, one uses specific experimental settings, for example inviting learners to work in pairs, with the expectation that this will be enough to elicit these behaviors.

It is worth noticing that the quadruplet is not more related to the learner than to the milieu with which he or she interacts: the representation system allows the formulation and the use of the operators by the active sender (the learner) as well as the reactive receiver (the milieu); the control structure allows expressing the learner's means to assess an action, as well as the criteria of the milieu for selecting a feedback. It is in this sense that the quadruplet characterizing a conception is congruent to the conceptual definition of a conception as a property of the learner/milieu system.

This formalization not only allows characterizing conceptions and hence providing a framework to discuss their diagnosis, it has also the potential of helping to establish links among conceptions more precisely.

Shaping Relations Between Conceptions

Arithmetic, From Fingers to Keystrokes

Addition has been widely studied, so there is enough resources to document what learners' conceptions could be like. This first example shows how eliciting the four dimensions of the quadruplet provide a synthetic and precise picture of the conceptions chosen for the purpose of the illustration (there are several others).

C1: Verbal counting IIIII & IIII
P -- *prototype*: "You have 5 pebbles, I give you 4 more, how many have you now?" (Objects are present or represented in an analogical way, both numbers are small).
R – match fingers and objects, match fingers and number names, pointing to objects,
L -- body language (finger counting, pointing), number naming, verbal counting
Σ – not counting twice an object, counting all the objects, order of the number names
C 2: Counting on 16 & 4
P -- *prototype*: "You have 16 pebbles, I give you 4 more, how many do have you now?" (The numbers are given, but the collections are not present, one of the numbers must be small enough)
R -- choose the greater number, count on to determine the result.
L -- body language (finger counting), number naming, verbal counting.
Σ -- order of the number names , match of fingers to number names
C3: written addition 16+23
P -- adding two integers
R -- algorithm of column addition
L -- decimal representation of numbers
Σ -- check the implementation of the algorithm, check the layout of number addition
C4: Pocket calculator [1][6][+][2][3][=]
P -- adding two integers, the result is bound by the size of the screen
R – keystroke to represent a number, to process number addition
L -- body language (keystrokes), decimal representation of numbers on the screen
Σ – keystrokes verification, order of magnitude.

Table 1: Four Examples of Conceptions of Addition

First, the quadruplet puts to the fore the domain of validity of these conceptions, none of which can be claimed wrong but might be badly adapted outside their spheres of practice. For example, C1 will not be reliable in the domain of C3 and really difficult to implement in that of C2. For very large numbers, C4 will not work unless extended with additional strategies to deal with the limits imposed by the technology (e.g. screen display). Beyond the remark that the two first concern quantities and the two others concern numbers, one can compare the conceptions on each dimension of the quadruplet and analyze in an accurate way their commonalities and differences. One can also express relations often considered among conceptions; for this it is necessary to introduce a function allowing passing from one system of representation to another. Let us take the case of generality:

[*C is more general than C' if there exists a function of representation*
f: $L' \to L$ so that $\forall p \in P'$, $f(p) \in P$]

So, one can show that C3 is more general than C2 and C4, but for different reasons although in both cases the size of the numbers is at stake. Indeed, we obtain in this way something that intuitively and informally would have been seen without such a sophisticated formulation. Isn't there here a flavor of pedantry? I hope not. It is just good to check that the formalism can express

familiar facts but, indeed, it must go far beyond that and be capable of revealing less obvious relations.

The Challenge of Translation

The exercise of formalizing generality puts on the fore the importance of *translation*, a process we engage anytime we analyze learners activities. To emphasize the importance of the manipulation of representation systems, I take another example from the history of mathematics: ancient Egyptian arithmetic. The mathematical papyruses have been translated in the contemporary mathematical language, including the processes used to solve some problems. Figure 2 displays the sequence of steps to compute “10 times 1/5” (Couchoud 1993 pp.21-22). I will not here explain how the scribes obtained the result, 2, out of this sequence and how they moved from one line to the other. Looking closer to these processes suggests that the translation of  by 1/5 is misleading. What is denoted by the Egyptian sign is “five parts of the whole”, hence an integer but integers which could not be added as integers are. These representations were not computed, instead scribes used tables to establish the correspondence between two numbers to be multiplied and the result. The control structures associated to this ancient conception of “fractions” and the modern one are completely different. Moreover, if one wants to consider passing back from the modern conception to the Egyptian one, that is expressing any fraction as a sum of unitary fractions, he or she will enter a new mathematical chapter. Several algorithms are available to compute an Egyptian decomposition for any fraction. For example, for 4055/4093 one will get the shortest and unique additive decomposition: [1/2 + 1/3 + 1/7 + 1/69 + 1/30650 + 1/10098761225]. Unfortunately, Egyptians could not write the last term. Analyzing these conceptions along the dimensions of the quadruplet in a systematic way makes easier figuring out what separates them.

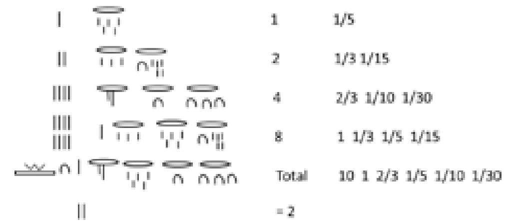


Figure 2: 10 times 1/5

Discussing learners’ conceptions in the context of the mathematical curriculum is more difficult because operators and representation systems are often very similar. In this case, the control structure may be the discriminating element. As a matter of fact, this touches the foundation of conceptions because of the legitimacy controls provide by validating them.

Let us take a case in school algebra. In his research questioning the “production of meaning for Algebra”, Romulo Lins (2001 p.47) observes the activity of students to whom he proposed the following task:

To calculate how many oranges will fill into each box, we divide the total number of oranges by the number of boxes, i.e.:

orange per box = $\dfrac{\textit{number of oranges}}{\textit{number of boxes}}$

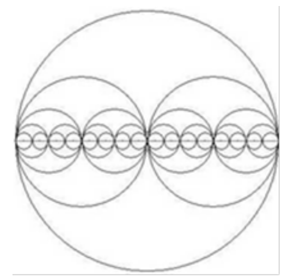
If I tell you the total number of oranges, and the number of oranges in each box, how would you calculate the number of boxes used?

Justifying the task, Lins writes: “*The reason for presenting the ‘algebraic’ formula was to ascertain whether the pupils would constitute it into an object, dealing with it in the process of solving the problem; neither of them made any reference whatsoever to this formula*” (ibid.) In a very pragmatic way students manipulated oranges and boxes: “*They always used a number of something*” (ibid.). They dealt with quantities and not numbers. The control on their reasoning

comes from the concrete reference the context makes possible. Actually, if algebra had been called up, literals would have been used to speak about actions on objects of a referent world made of boxes and oranges. This phenomenon is familiar, as Boero (2001 p.108) reports following a research he carried out in a different context: "*some students seem to transform the problem situation by thinking about the number of sheet and the weight of the envelope as physical variables*" whereas others "*put into a numerical equation the problem situation and transform the equation*".

Invited to write a postscript to the book "*Perspective on School Algebra*" (Sutherland et al. 2001), where I analyzed the reports of Lins, Boero and several others, I introduced the expression "*symbolic arithmetic*" to distinguish from algebra those conceptions in which symbols are manipulated and rules used with the supervision of a control structure grounded in the referent context.

In the following case, the role of the control structure clearly makes a difference between two conceptions which on the other hand seem to share the representation systems and operators.



Questioning Controls to Understand Representations

This case, which I have often presented, is as an excellent prototypical example of the complexity of identifying conceptions. The excerpt is borrowed from the work of Bettina Pedemonte (2002) on argumentation, cognitive continuity and proof. She chosen the problem described below and proposed it to pairs of students—the idea of having pairs was driven by the expectation to get spontaneous comments about actions, choices and decisions taken during the problem solving process.

"*Construct a circle with AB as a diameter. Split AB in two equal parts, AC and CB. Then construct the two circles of diameter AC and CB… and so on.*

How does the perimeter vary at each stage?

How does the area vary?"

Using the formulas they know well both students, Vincent and Ludovic, express the perimeter and the area for the first steps in the series of drawings. They agree to conjecture that the perimeter will be constant and that the area will decrease to zero. But soon Vincent notices that "*the area is always divided by 2…so, at the limit? The limit is a line, the segment from which we started…*" This observation raises a conflict about the value of the perimeter which, in the opinion of Vincent, should be the length of the segment:

41. *Vincent*: It falls in the segment… the circles are so small.
42. *Ludovic*: Hmm… but it is always $2\pi r$.
43. *Vincent*: Yes, but when the area tends to 0 it will be almost equal…
44. *Ludovic*: No, I don't think so.
45. *Vincent*: If the area tends to 0, then the perimeter also… I don't know…
46. *Ludovic*: I will finish writing the proof.

Although Vincent and Ludovic collaborate well and seem to share the mathematics involved, the types of control they have on their problem-solving activity differ. Ludovic is working in the algebraic setting (Douady 1985) where control is based on a constant checking of the correctness of the symbolic manipulations conforming to the syntax of elementary algebra. Vincent is working in a symbolic-arithmetic setting where the control comes from a constant confrontation between what the formula "tells" and what is displayed by the drawings. So, both understand the initial situation in the "same" way, both manipulate the symbolic representations (i.e., the

formulas of the perimeter and of the area) following the right syntax, but their controls are radically different. The symbolic representation supports the cooperation of the problem-solvers but it does not impose a shared understanding: as a *boundary object* it is flexible enough to adapt to the different meanings but robust enough to work as a tool for both students. To identify the differences beyond the apparent commonalities of representations one has to question learners' decisions and choices, which means identifying the control grounding their activity.

Conception, Knowing and Concept

Understanding learners' conceptions requires their interpretation from the perspective of our own conception which we claim related to the same content of reference; one may say: the same concept. This can be expressed within the terms of the model, putting on the fore the role of translation which is more often than not implicit in our research practice. Let's take the case of "falsity" which is defined in the model in the following way (with a natural coding of the respective quadruplets):

> [*C is false from the point of view of C' if there exists a function of representation* $f: L \to L'$, *and there exists* [$p \in P$, $r \in R$, $\sigma \in \Sigma$, $\sigma' \in \Sigma'$] *so that* $\sigma(r(p))$=*true and* $\sigma'(f(r(p))$=*false*]

In other words, there exists a problem from the sphere of practice of C which has an accepted solution but which is assessed "false" from the point of view of C'.

"Generality" and "falsity" are not properties of conceptions but relations between two conceptions whose validity depends on the translation from one system of representation to the other. This is a general situation often hidden by the fact that we tend to read the production and the processes learners carry out directly in mathematical terms. Not being aware of this may make understanding learners difficult, as illustrated by Linns remarks on students' inability to escape the concrete reference of a situation he sees himself as mathematical. More generally, we have a tendency, often implicit, to consider ourselves as privileged knowers entitled to judge and evaluate other people's knowings. Such ambition requires at least that we can claim that the conception one assesses and the conception one holds are—so to say—ontologically compatible; they are concerned by the same object. This is difficult in mathematics where the only tangible things one manipulates are representations, and representations of representations. This can be solved within the model, taking Vergnaud's postulate as a grounding principle: problems are sources and criteria of knowings (1981 p.220):

> *Let C and C' be two conceptions and* C_a *be a conception more general than C and C'. This means, with a natural coding of the respective quadruplets, that there exist functions of representation* $f: L \to L_a$ *and* $f': L' \to L_a$ *which relate C, C' and* C_a. *Then:*
>
> [*C and C' have the same object with respect to* C_a *if for all p from P there exists p' from P' such that* $f(p)=f'(p')$, *and reciprocally*]

The fact that two conceptions have the same object does not mean that they have another type of relationship (one being false with respect to the other, or more general, or partial, or else), it may be the case that problems of P' (resp. P) cannot be expressed with L (resp. L'); and if they are, the translated problems may not be part of the sphere of practice of the other conception (e.g. the case of the conceptions of addition, Table 1). Eventually, conceptions have the same object if their defining problems (or their spheres of practice) can be matched from the point of view of a more general conception which in our case is the conception of the researcher. Research on mathematics learning must start with an introspection of researchers' own conception of the content at stake; questioning this conception is the first methodological tool to assess the validity

of what can be said about learners' conception. This corresponds to the a priori analysis in the methodology associated to the TDS.

"To have the same object with respect to a conception C_a" sets an equivalence relation among conceptions. Let's now claim the existence of a conception Cμ more general than any other conception to which it can be compared; this seems to be an abstract declaration, but pragmatically it corresponds to a piece of a mathematical theory.

I call "concept" the set of all conceptions having the same object with respect to Cμ.
This definition is aligned with the idea that a mathematical concept is not reduced to the text of its formal definition, but is the product of its history and of the set of practices in different communities. Indeed, there is no agent holding the concept and no way to ensure that we can enumerate a complete list of these conceptions. So, a last definition will allow reducing the distance between this abstract definition and the needs we have to have a practical model:

> *I call "knowing" any subset of a concept which can be ascribed to a cognitive subject or a community.*

In practical terms, this definition of conception and knowing provides a framework which preserves learners' epistemic integrity despite contradictions and variability across situations. In a way which might seem more usual, I could summarize the ideas presented here in the following way: a *conception* is the instantiation of a *knowing* by a situation (it characterizes the subject/milieu system in a situation), or a *conception* is the instantiation of a *concept* by a pair (subject/situation).

The name cK¢ comes from the names of the three pillars of the model: *conception*, *knowing*, *concept*. I keep the word "knowledge" to name a conception which is identified and formalized by an institution (which is a body of an educational system in our case).

Problems, Conceptions and Learning

Indeed, most problems are not solved by activating just one conception. So we need to be able to express the relation between a problem and conceptions which contribute to its solution, but without having to give details about this solution because of the cost of a too thin granularity. For this reason, I propose the more general idea of a *set of conceptions solving a problem*:

> *Let p be a problem, and {C1,…, Cn} a set of conceptions.*
> *{C1,…, Cn} solves p iff there exists a sequence of operators* $(r_{i1}, \ldots, r_{im})$ *whose terms are taken in one of the* R_i *so that the sequence* $(p_1=r_{i1}(p), \ldots, p_{im}=r_{im}(p_{im-1}))$ *verifies that there exists* σ *from* Σ_{im} *so that* $\sigma(p_{im})$*=solved.*

From this general characterization, one can derive more precise properties, for example expressing that a conception is specific to a problem (any set of conceptions solving it contains this conception), or that conceptions are equivalent from a problem solving perspective (one can replace the other without changing its property vis-à-vis the problem). Exploring this possibility evidences that problems and conceptions are of a dual nature: on the one hand conceptions need problems as constituents of their characterization, and on the other hand problems get their meaning from the conceptions contributing to their solutions. This duality suggests a natural connection between conceptions by the mediation of problems: this is exactly the idea of Vergnaud's conceptual field.

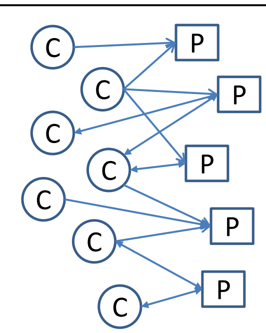


Figure 3: Duality of problems and conceptions

Learning is a process whose outcome is an evolution of conceptions being reinforced, questioned or transformed. The motor of this process is problems, which are (in our terms) destabilizations of the learner/milieu system. This destabilization can be obtained by modifications of constraints on the interaction between the learner and the milieu or modifications of the characteristics of the milieu (cf. Figure 1)—indeed, the learner is a "subsystem" on which no direct action is possible. The most difficult task is to find problems questioning the control structure and/or the representation system because the former is mostly implicit in the activity of learners and the role of the latter is invisible to their eyes once they are familiar with it. To overcome these difficulties is the *raison d'être* of the TDS situations of formulation and validation, which are based on a social organization of the class and a play on the characteristics of the milieu. I will not elaborate on this relation between cK¢ and the TDS, a part from noticing briefly here that within a didactical *problématique*, learning can be modeled as a transition function on a bipartite graph of conceptions and problems. Problems are the means to activate and (i) diagnose a conception, (ii) destabilize a conception to obtain an evolution, (iii) reinforce a conception.

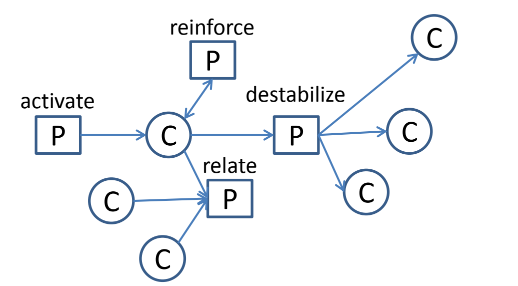


Figure 4: sketch of a bipartite graph conceptions/problems

Let C be the current diagnosed conception and C_t a targeted conception (the expected learning outcome). The most critical evolution is the one to be obtained when C is false with respect to C_t. To engage the learning process, it is necessary to find a problem for which a representation is possible within both C and C_t, and which could be a means to reveal a conflict: a solution is conceivable from the perspective of C, but is not accessible in a way that the milieu witnesses and the subject recognizes. It may appear that such a problem does not exist and that intermediary problems, and possibly intermediary conceptions, are necessary to "reach" C_t. Learning is often not a single step but a path in the graph. To identify and create the conditions to bring this path to reality within a learning situation, in particular within a classroom, is one of the core objectives of the TDS.

Let's take the classical and well documented case of the sum of the angles of a triangle. The most common initial conception is that the bigger the object the bigger its measurable characteristics (e.g. area, perimeter), hence the sum of the angles. The operators are those involved in the manipulation of geometrical instruments (rule, compass, protractor, etc.) and symbolic arithmetic, the control structure includes visual control of actions and checking of computations. To activate this conception, one can ask students to draw triangles, measure angles and add up the obtained results. The variety of the results in the class will not raise questions since triangles are different; students will be reinforced in the confidence that they have the capacity to achieve the task. But measuring angles is not sufficient to destabilize the conception and give room to the targeted conception which is rooted not in the manipulation of the geometrical "object" but of its property. A possible way out is of asking learners to repeat the task, all with the same triangle (e.g. reproduced on a worksheet); the problem of deciding of the results for *this* single triangle will emerge and challenge the operators of the initial conception. More is needed to question its core theorem-in-action, this will come from the orchestration of the confrontation of the outcomes of collective workshops on triangles with contrasting shapes (small, large, sharp, flat). The destabilization of the initial conception can be overcome only by

engaging in geometrical arguments of a theoretical nature. At this point, operators and controls are questioned, but the ambivalent nature of the triangle being both an object (of the spatio-graphic space) and a (geometrical) representation will (probably) remain unsolved.

Concluding Forewords

cK¢ proposes a framework for "learners modeling" taking up the challenge of providing a model of an epistemic relevance to bridge research in mathematics education and research on educational technology. It responds to a need for representations both understandable by researchers in mathematics education and computationally tractable. The formalism it dares should enhance the way one informs the design of technology enhanced learning environments, complementing descriptions generally available in natural language with no standardized narrative structure.

A European multidisciplinary assessment project (Baghera 2003) has been an occasion to check the efficiency of cK¢ in supporting a productive conversation between researchers in education and in computer-science. But probably more interesting for us is the powerfulness of this framework to think and develop our own research. Research in mathematics education develops jointly theories and experimentations, in this context models serve as mediators between theories of which they require an articulate and precise understanding, and experiments of which they frame the design and drive the collection of data. However, both theories and experiments raise difficult issues. On the side of theories, one has to deal with a complex discourse which rarely makes explicit all details and hence gives room to non-univocal interpretations. On the side of experiments, the practical implementation is always richer and more complex than what the design of models anticipates. Moreover, in the case of conceptions, one is confronted with issues (that Toulmin already noticed when proposing a model of argumentation): distinguishing operators from controls is not absolute (e.g. theorems can be activated as tools or predicates), controls are more often than not implicit. Such difficulties require further theoretical as well as methodological investigations.

Initially based on the Theory of Didactical Situation and the Theory of Conceptual Field, the cK¢ modeling framework is not restricted to them. For the purpose of its development and in order to enhance its efficiency it is necessary to integrate other theories to strengthen its components (e.g. representation, control system). But cK¢ holds other promises; it facilitates building a bridge between knowing and proving, constructing a link between control and proof, hence facilitating understanding the relation between argumentation and proof. But this is another topic which connects the research agendas I have had along my career, first on the teaching and learning of mathematical proof, then modeling learners' conceptions for the design of learning environments.

Acknowledgements

I would like to express my thanks to Professor Patricio Herbst, Professor Anna Sierpinska and Dr. Bettina Pedemonte for their comments on an earlier draft of this paper, and Professor R. Knott for his "Calculator to convert a fraction into an Egyptian fraction" [http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fractions/egyptian.html] retrieved 06-06-2013.

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TECHNOLOGY IN MATHEMATICS EDUCATION: A DISCUSSION OF AFFORDANCES

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This paper details a framework for evaluating the potential of particular forms of technology for supporting student engagement with mathematics. I propose that, focusing on the strength of affordances to engage in particular ways allows us to begin to build an analytic frame that serves two purposes: (1) to begin to document learning as an interaction with larger bodies of data; and (2) to make and evaluate conjectures about the potential of different technologies for supporting high quality engagement with mathematics. I describe design iterations from previous work that leveraged this framework as an example of how this framework could be used and why it has proven to be valuable.

Keywords: Technology, Data Analysis and Statistics, Learning Theory, Design Experiments

Introduction

We seem to be in an upswing of technology adoption, with the general belief that more technology is better. We can see this in one-laptop-per child programs, charter schools that advocate the use of technology such as "New Tech High," the enthusiasm for "blended learning," and even the popular BYOT programs at schools, which encourage children to bring in any form of technology that they have at their disposal. I find this enthusiasm puzzling, to say the least. Technology, as a general term, refers to such a broad range of tools that it is difficult to understand why anyone would immediately believe (or doubt) that it would improve education. COULD technology become a resource to support student learning? Of course. Could it also thwart student learning? Absolutely. It's all in which forms of technology, how they're being used, by whom, and when. The questions we need to pose involve not whether technologies are useful, but which technologies, and under what circumstances.

In this paper I discuss a framework for considering the potential of particular forms of technology for supporting student learning, drawing on work that I have published that has focused specifically on particular forms of student engagement. I then share some of my own work on designing and studying immersive videogames, and leverage this framework as a means of making sense of the potential of these technologies. In so doing, my goal is not to take a position about whether technology is useful for supporting mathematical learning or not, but instead, to suggest a way of thinking about technology that would better allow us to evaluate whether, when, and why leveraging technology would make a difference for mathematics learning.

A Theory of Context

In 2008, Jim Greeno and I published a chapter that focused on considering "opportunities to learn" as a set of *affordances* for learning. Drawing on Gibson's (1979) theory of perceptual affordances as a framework, this paper described human activity as a co-constructed set of possible actions defined by: 1) the affordances of the environment for a particular action; 2) the intention of the agent to take up those affordances; and 3) the effectivities of the agent to actually

realize those affordances (c.f. Barab, Cherkes-Julkowski, Swenson, Garrett, Shaw, & Young, 1999; Gresalfi, Barnes, & Cross, 2012). *Affordances* refer to the set of actions that are made possible by a particular object; for example, a chair affords sitting, but a door does not. Of course, the extent to which an affordance can be acted on has to do with one's *effectivities*, an individual's ability to realize those affordances. To return to the earlier example, because of their quite different effectivities, a chair affords sitting for a human, but not for an alligator. And finally, the extent to which an affordance is realized depends on the dynamic *intention* that emerges among the elements of the system; just because one could sit on a chair does not mean that one does so, depending on the situation. Thus, understanding what happens at a particular moment requires unpacking what was afforded, and how those affordances were recognized and realized. It is important to note that something can only be afforded if it can be recognized and acted on; what makes an affordance actionable is inherently a dynamic relation between the environment and person (Greeno & Gresalfi, 2008). This focus on interaction acknowledges that a particular task might make something possible, but doesn't make it obligatory.

This framework helps to suggest a way of documenting "situated learning," that is, the assumption that what someone is able to do and, eventually, learn, is inextricably tied to the particular context and circumstances in which one is acting (Greeno & MMAP, 1998; Lave & Wenger, 1991). Situative and sociocultural theories of learning receive a lot of attention in educational research in general, and in mathematics education in particular. However, it is one thing to claim that learning is an interactional accomplishment; it is quite another thing to demonstrate how that is so. Much of the research in this area has relied on micro-level analyses of discourse and gesture, painstakingly demonstrating that what one person does cannot be separated from the other things that were done to/for/around that person at the same time (Engle & Conant, 2002; Esmonde & Langer-Osuna, 2013; Esmonde, Takeuchi, & Radakovic, 2011).

Despite their quality, these kinds of analyses are not scalable; they cannot be applied to large numbers of students with the goal of understanding whether and why a particular intervention worked as it did. Instead, for those large-scale questions, we often rely on assessments that target one particular moment in time and offer that moment (or a difference between two such moments) as evidence of learning. Condensing an entire learning activity to a single moment is problematic, but even more problematic is that these assessments are themselves part of an interaction (that also includes the learner), and thus understanding what someone demonstrated that they "know" cannot be reasonably accomplished without a careful analysis of what or how the assessment itself contributed to that moment of knowing. This kind of analysis is rarely, if ever, done, and thus we are left in a situation wherein we can offer detailed, textured analyses of learning as an interaction among a small number of students, but can offer no such analysis with larger numbers of students. What is missing, it seems, is a theory of context that could be leveraged in a scalable way such that larger studies of learning could also functionally offer analyses of learning as an interaction.

The Strength Of Affordances

In my own work, I have considered context as having affordances that make it more or less likely that someone will act in a particular way. In documenting what people do in relation to these affordances, we come to understand someone more nuanced about moments of knowing. In Gresalfi (2009), I proposed a framework for characterizing the nature of affordances that focused on the relative strength an affordance offered to enact a particular activity (see also Gresalfi, Barnes, & Cross, 2012). Specifically, an affordance was characterized as "strong" when it

required students to comply or when not complying would have been a social or rule violation. In contrast, a weaker affordances provides an opportunity for something to happen, but does not make it imperative that the affordance is realized. A videogame can offer a nice example of the contrast. In some videogames, the player is encouraged to collect particular objects as they play, which can be used for some purpose in later game play. In some contexts, the players are merely encouraged to be on the lookout for these objects—picking them up, or not, is voluntary. In that case, the affordance that is offered is relatively weak; picking up objects is possible, but not mandatory. In contrast, much stronger affordances could be offered if the game prevents a player from moving on if objects have not been collected. In this case, the affordance for collecting objects is very strong, because not doing so prohibits continued engagement with the game.

In this work, characterizing the strength of affordances, that is, attempting to make progress on a theory of *context,* allowed a different understand *individual* performance. As an explanation of why, consider a paper written by Ilana Horn in 2008 about "turnaround" students. Horn followed seven students at two schools, all of who had initial experiences of mathematical success despite having previous histories of lower mathematical achievement. She labeled this year their "turnaround" year, and then followed these seven students through the rest of their high school career. Four students were in a school that offered tracked mathematics classes, while the other three students were in a school that did not offer tracked mathematics classes. Horn (2008) offers great detail about the differences between these schools in terms of the *figured worlds* (Holland et al, 1998) that they offered to students; a very brief summary is that the tracked school offered a narrative of success being possible for some, but not all, while the de-tracked school offered a narrative that anyone could be successful if they put forth the effort. Horn documented that, in their "turnaround" year, all seven students were in classes that supported a shift in their identities to consider themselves to be good at math. However, in subsequent years, in the absence of this class structure, students in the tracked school reverted to their earlier mathematical performances and self-perceptions, ultimately enrolling in remedial math or dropping out of math altogether. In contrast, students in the de-tracked school all enrolled in college preparatory math, and completed, at a minimum, the courses they needed for college acceptance. Horn's (2008) goal was to document how structure can influence identities, and the particular importance of curricular structure and teacher beliefs in shaping the kinds of figured worlds that allow identities to develop and persist. However, equally interesting is the question of what was similar about students' "turnaround" experiences, and how that was related to what happened in later courses.

One could characterize this finding as illuminating the difference between students when they are offered strong opportunities to engage with mathematics through the activity of meaning making, in 9th grade, and then later only weak opportunities. Specifically, students in the tracked school were working with an extremely skilled teacher who was awarded the Presidential Award for Excellence in Mathematics and Science Teaching. She has been the focus of significant analyses for her skill at maintaining rigor while broadening participation and collaboration (Staples, 2007). In short, she was a very skilled teacher who created strong opportunities for students to experience success in her classroom. However, when moved to classrooms with teachers who arguably offered weaker opportunities to engage, these students, for whatever reason, failed to take up those opportunities. In contrast, the students who continued to be successful worked both in their 9th grade year *and* throughout high school with a department of math teachers who shared an approach to teaching that intentionally welcomed diverse forms of participation, leveraged heterogeneous grouping practices, and believed that all students could

succeed (c.f. Boaler & Staples, 2008; Staples, 2008). Like the students in the tracked school, students in the detracked school were offered strong affordances for students to engage deeply with mathematics and experience success. However, the students in the detracked school *continued* to be offered strong opportunities to engage in these meaningful ways. Suddenly these students are not "bad at math," but rather need stronger opportunities to engage in order to be successful. But crucially, if offered those opportunities, they *are* indeed successful. This recasting offers a clear way of conceptualizing performance as a shared accomplishment between person and environment, and helps us to recast failure and success as something that lies not just on the shoulders of teachers, or students, but both.

To summarize, the potential of building a theory of context is that it helps us to understand what individual behavior is responding to. A theory of context in no way guarantees that we will know what people will do, but it gives us a working hypothesis about what could happen. It is for this reason that I find this framework compelling not only as an analytic tool, but also as a design tool.

Affordance to do what? One challenging thing about creating a theory of context that focuses on potential is that it is possible to do many different things in any given moment. One could think about an affordance to comply or not, to try or not, to work with others or not (the list is endless). For that reason, when attempting to design a theory of context, it is imperative to select a specific focus. In my work, I have focused on the nature of student engagement with content. This is a meso level focus; a macro focus might look at participation versus non participation or opposition (see Hand, 2010), while a micro focus might look at the nature of what, specifically, students understand about mathematical content (see Empson, 1999). A meso level looks at the ways that students are engaging with mathematics itself, characterizing that engagement in terms that could be applied across mathematical content. Specifically, I have considered four types of engagement: procedural, conceptual, consequential, and critical (Gresalfi & Barab, 2011; Gresalfi, Barab, Siyahhan, & Christensen, 2009). Procedural engagement involves using procedures accurately (Rittle-Johnson, Siegler, & Alibali, 2001); conceptual engagement additionally involves understanding why an equation works the way it does. Consequential engagement involves recognizing the usefulness and impact of disciplinary tools; being able to connect particular solutions to particular outcomes. For example, students are engaging consequentially when they not only solve a problem by calculating the mean, but also explain what the mean tells them about the situation. Finally, critical engagement concerns agency in problem solving, and involves explicitly choosing particular tools and interrogating their impact on a solution. This way of engaging builds on the others; students can engage conceptually or consequentially without making an explicit choice about how they are solving a problem; realizing that their approach to a problem impacts its solution is a significant shift in engagement.

To illustrate the difference between these levels, consider the problem below (Figure 1), taken from Henningsen & Stein (1997), which they characterized as having the potential of offering high cognitive demand. At the macro level, this problem (assuming it is offered in the context of a classroom activity, with a grade attached to its completion), offers a strong affordance to attempt to complete the task. Not attempting to complete the task, in that context, would generally represent a significant breach of classroom expectations. At the micro level, this problem offers strong affordances to make connections between fractions, percents, and decimals, simply because the question includes the same question in these three terms. What's more, the problem offers a strong affordance for students to consider what .725 of the rectangle

actually looks like, as they are asked to shade this area in the rectangle. This task is made more challenging because the rectangle contains 80 squares, rather than 100, so students will have to do some actual mathematical work to determine what .725 of 80 actually is, and what it looks like in the representation.

1. a. Shade .725 of the area of this rectangle.
   b. What fractional part of the area is shaded?
   c. What percentage of the area is shaded?

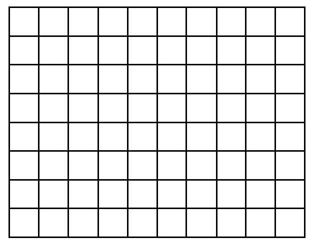

Figure 1: Sample Problem Offering High Cognitive Demand

At the meso level, we start to think about the nature of students' engagement with the problem—how they could go about solving it and what implications that might have for how they would understand it. A student could answer the question without considering the figure, as the fractional equivalent of .725 can be calculated procedurally, as 725/1000, and the percentage could also be calculated procedurally, by moving the decimal point two places to the right. These are standard procedures that are commonly taught to students to enable them to determine equivalency between fractions, decimals, and percentages. Thus, this problem offers strong affordances for engaging procedurally. However, the addition of the figure, and the request to shade the figure, creates opportunities for students to consider what .725 actually *looks like*. If the students were given a 10x10 square, this would be relatively easy. But because the square is 10x8, more calculation and thought is required. There are many ways that students could approach this task, but ultimately, successful completion requires both conceptual and, possibly, consequential engagement.

Conceptually, figuring out what .725 of 80 looks like requires moving beyond the rote execution of a procedure. The value of .725, or 72.5%, is not a quantity that is typically worked with. Students could solve the problem by figuring out what 75% of 80 is (60), and then estimating that it would be a bit less. Depending on the classroom context, this kind of solution might be acceptable and would surely demonstrate the students were taking up the affordance to think conceptually about what it means to take .725 of something. If a more exact solution was required, students might multiply 80 by .725. In either case, there is a strong affordance for students to move beyond simply using a procedure, either through estimating or calculating the area. As a brief note, the context of the classroom plays a significant role in the ways these affordances might be realized (Henningsen & Stein, 1998; Gresalfi, Barnes, & Cross, 2012).

Consequentially, the addition of the grid in this problem offers another strong affordance, which is to consider whether the solution that a student might come up with actually makes sense. Students often have trouble remembering the "rules" for decimal multiplication (for various predictable reasons, c.f. Martinie & Bay-Williams, 2003; Rathouz, 2011). Thus, the "simple" task of multiplication could yield a variety of solutions, few of which match onto the grid that the students are asked to shade. In this case, being asked to shade the area offers a

potential source of feedback for students, as it is impossible to shade 58000, 5800, or even 580 squares (the correct answer is 58). To make the strength of this affordance a bit clearer, contrast this task with one that asks to *determine* how much .725 of 80 is (without offering the grid). From a micro analysis (above), the two problems would offer identical affordances. From a meso analysis, however, the problems are quite different, as one contains a strong affordance to reason consequentially by testing out whether the calculational result actually makes sense.

Translating to Technology

Although the analysis of tasks is a well-worn area in mathematics education, the addition of technology is much newer. Technologies offer their own affordances, with the potential to support or thwart deeper forms of engagement. Potentially more problematically, most forms of technology offer not just different affordances for engagement, but also, more. Tools such as geometer's sketchpad or Tinkerplots create opportunities not just for productive engagement with mathematical content, but also offer a sandbox-like space that also allows for play and experimentation that can distract students from engaging with key ideas. More open technologies, such as video games, have the potential to offer even more affordances, both productive and not. In this context, with new tools that offer so many spaces for engagement, it is particularly important that designs be subjected to some kind of theory of context, with the goal of understanding what forms of engagement are truly likely to result.

Below, as an illustration, I offer an example of design changes that I made in the context of designing an educational videogame, which focused specifically on increasing the strength of an affordance to engage critically with content. These changes were made to the overarching dilemma of the game, which changed across three implementations to offer increasingly strong affordances to engage critically with content. As a reminder, critical engagement with content involves being reflective about the impact of a mathematical move on a designed context; in the case of statistics, one is engaging critically when they are able to discuss how the choice of a particular analytic tool (such as the mean) impacted what one could see in data, and how that might be different if a different analytic took (such as the median) was used.

Because the goal of this paper is to focus on design, I do not share detailed information about the implementations themselves. A much more extensive discussion of the evolution of the unit, details about design-based research, and specifics about the implementations can be found elsewhere (Gresalfi & Ingram-Goble, 2008; Gresalfi, in review).

Evaluating Technology in Terms of Affordances for Engagement

The curriculum discussed in this paper takes place in the context of an online, interactive and immersive videogame called Quest Atlantis (QA) (www.QuestAtlantis.org). QA is an educational videogame that uses a 3D multi-user environment to immerse children, ages 9-12, in meaningful learning trajectories across multiple disciplines. The QA virtual environment involves engaging a core dilemma that requires leveraging disciplinary content for its resolution. In some ways, these dilemmas provide a framework much like a well-developed problem or project might. Specifically, in designing these games, our goal is to *situate the learner* as having the intention of making decisions that impact the dilemma, *situate the content* as information that is legitimately required to resolve the dilemma, and *situate the context* as modifiable and responsive to learner choices (Barab, Gresalfi, & Ingram-Goble, 2010).

Interactive, immersive educational games create several opportunities for student engagement that are frequently lacking in traditional schooling. In particular, the immersive

context can embed students' inquiry in contexts that can have consequence, thus transforming targeted concepts from rules to be remembered to tools that can be applied. In addition, the fantasy of online videogames support pedagogically useful tropes that can serve as legitimate feedback for students' thinking; rather than having to look to a teacher for whether one's solution is accurate, the situational context can change in response to students' decisions, thus serving to push back on student thinking and solutions. In QA, students take on particular roles and engage various plot lines, which can evolve and change based on the choices that they make. In this way, students are central players in an unfolding storyline that requires participation in academically meaningful activities, either in the real or simulated world. Although students are "playing," their actions have more consequence than their "real world" interactions in schooling. In this way, well-designed videogames hold the potential to position students in relation to content in particular ways by affording different kinds of engagement, while simultaneously setting out a trajectory of participation that has implications for more enduring relationships with content and a new way of seeing the world (Gee, 2003; Lee & Hoadley, 2007; Shaffer, 2006).

Ander City

The statistics unit discussed here, called Ander City, was created to support students' engagement with the following content standards (NCTM, 2000):

- Select and use appropriate statistical methods to analyze data.
- Describe the shape and important features of a set of data and compare related data sets, with an emphasis on how the data are distributed;
- Use measures of center, focusing on the median, and understand what each does and does not indicate about the data set;
- Compare different representations of the same data and evaluate how well each representation shows important aspects of the data.
- Develop and evaluate inferences and predictions that are based on data.
- Propose and justify conclusions and predictions that are based on data and design studies to further investigate the conclusions or predictions.

The specific and somewhat unusual goal of the unit was to understand the role of statistical tools in allowing one to reveal—or hide—information. Students were afforded opportunities to consider how particular tools allowed some stories to be told, and not others. Although the exact storyline changed through cycles of design, as will be discussed below, the overarching issues remained consistent. Students were asked to make three crucial decisions for the city, all of which involved comparing two data sets that represented different companies or approaches. As an example, the first activity that students completed in all three revisions of the unit involved deciding which brand of bicycle the city should offer for rental in the park, based on which was the safest. All dilemmas were designed to support different conclusions depending on the statistical method leveraged for analysis (e.g. students might make a different recommendation if they calculate the mean of a data set than if they consider the mode or look at the distribution of the data set). As students help to redesign the city by making arguments for or against particular decisions, they engage with increasingly advanced statistical content, and are challenged to offer increasingly sophisticated explanations and justifications of their decisions.

To summarize briefly, looking across the iterations of the design (see Figure 2), we found that as we increased the strength of the affordance to engage critically with content, more students demonstrated critical engagement in their written recommendation.

Figure 2: Average total scores for Procedural, Conceptual, Consequential, and Critical Engagement across the Implementations.

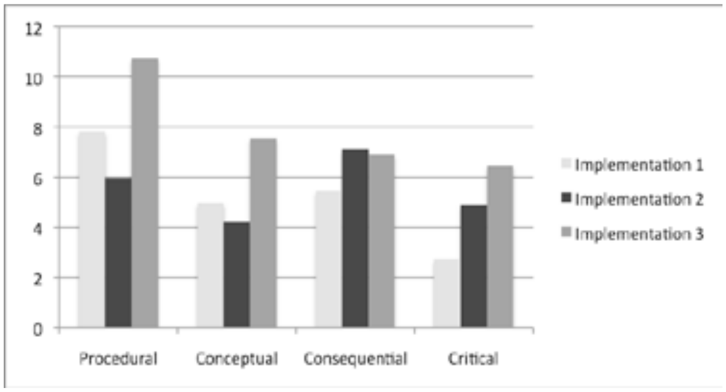

Table 1 summarizes the overall design decisions that were made. Below, I analyze the first and last of these decisions in terms of their affordances and share outcome data, particularly with respect to how those outcomes shaped our revisions for future designs.

Table 1: Overview of changes in narrative, intentionality, and outcome

Design	Narrative	Intention	Outcome
1	Mayor Enoch needs help making good decisions for the city.	Help Mayor Enoch	None
2	Mayor Enoch is being accused of deception, his opponent claims that he is making biased decisions that don't really benefit the entire city.	Defend Mayor Enoch against skeptics	Help Mayor Enoch keep his position
3	Mayor Enoch and Mr. Grant BOTH claim that they are making the right decision. They disagree with each other, although they both have data to support their positions. Is someone lying?	Determine which candidate is making better recommendations; or who might be lying.	Determine who will ultimately be elected mayor.

First Design

The storyline of the first version of the statistics unit centered on the fictional character of Mayor Enoch, mayor of Ander City, who claimed to need help. He was concerned that the citizens of the town were unhappy and were leaving the town. He felt that he had an obligation to make sure that he was making decisions that would make the city a better place to live, and would ensure that its citizens stayed. He thus needed assistance in making three crucial decisions: the first involved which brand of bikes they should offer for rental in the park, the second involved which brand of swings they should install, and the final issue involved the kinds of snacks they should make available in the snack cart that was adjacent to the playground in the park. These contexts became the decision points for students; they were invited to take a look at the data and make a decision to help Mayor Enoch based on their analysis of those data. The specific dialogues that occurred between Mayor Enoch and the player can be seen in Figure 3.

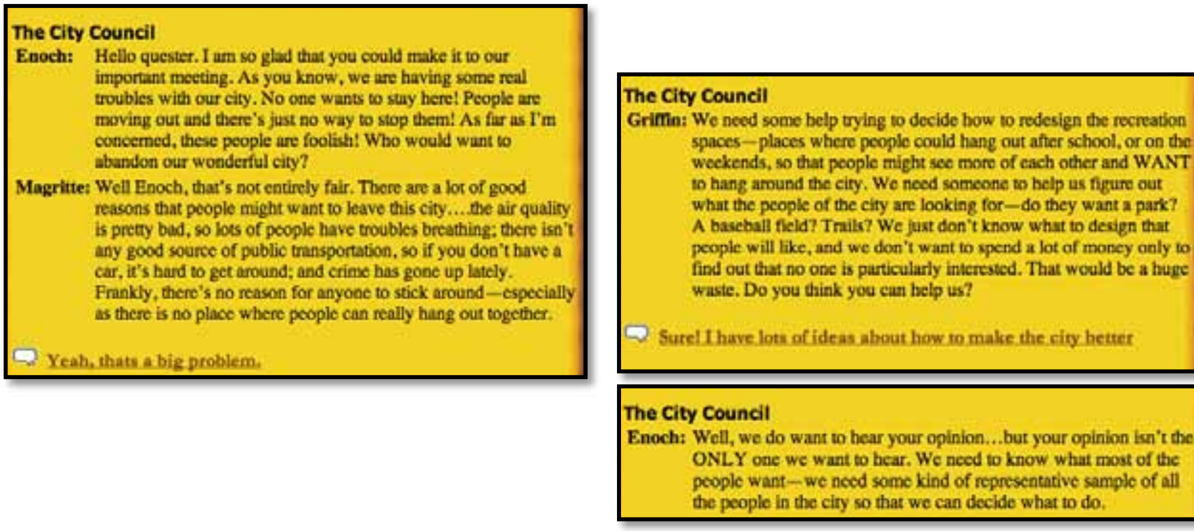

Figure 3: Narrative Framing for Design 1.

The narrative was designed with the goal of attuning students to the importance of making decisions that were *defensible*; decisions that could be justified with data. In creating a scenario wherein the Mayor of the city was in somewhat of a crisis, our initial conjecture was that the emotional salience of the situation would create a strong affordance not just to analyze data, but to be intentional about *how* those data were analyzed (critical engagement). We conjectured that this external audience (the citizens of Ander City) would raise the standards for students' recommendations, as they would take their activity to be about *convincing* rather than simply *complying* (Engle & Conant, 2002).

Uptake of Affordances. In this first implementation, we found that while students' critical engagement improved over the three decisions they made, there was still very little critical engagement overall. Specifically, students became accomplished at selecting one way to analyze the data they were given (the analytic method was always left open to the students), and then explaining what their analysis meant in terms of the situation (what we called consequential engagement). However, very few students seemed to realize that the way they analyzed the data impacted what they were ultimately able to say about those data. This suggested that what we had originally considered to be a strong affordance for engaging critically was in fact a fairly weak one; this was evidenced by the fact that so few students engaged in this way. This lack of uptake was in spite the fact that, in whole class discussions, they were able to compare and contrast solutions and therefore could clearly compare different solutions. In other words, we did not believe that there was a problem in students' effectivities for engaging these affordances.

Final Design. In our redesign, we attempted to raise the salience for the need for justification and considering both sides. We conjectured that the largely invisible audience of the citizens of Ander City did not provide a sufficiently strong affordance for engaging. Thus, we changed the narrative so that the mayor was running for re-election against a character who was maligning the mayor and his decisions. In the new scenario, students were asked to decide which candidate was making better choices for the city, and in particular, if one candidate was lying. We conjectured that shifting the role of the skeptic to the students themselves would serve to shift the agency for being critical directly to the students, as it would be impossible to consider who was

lying without evaluating both perspectives. In this way, the affordance for engaging critically was much stronger than in the initial design, as NOT engaging critically would constitute only a partial solution. The revised unit began with the following letter from another student in the town, seen in Figure 4.

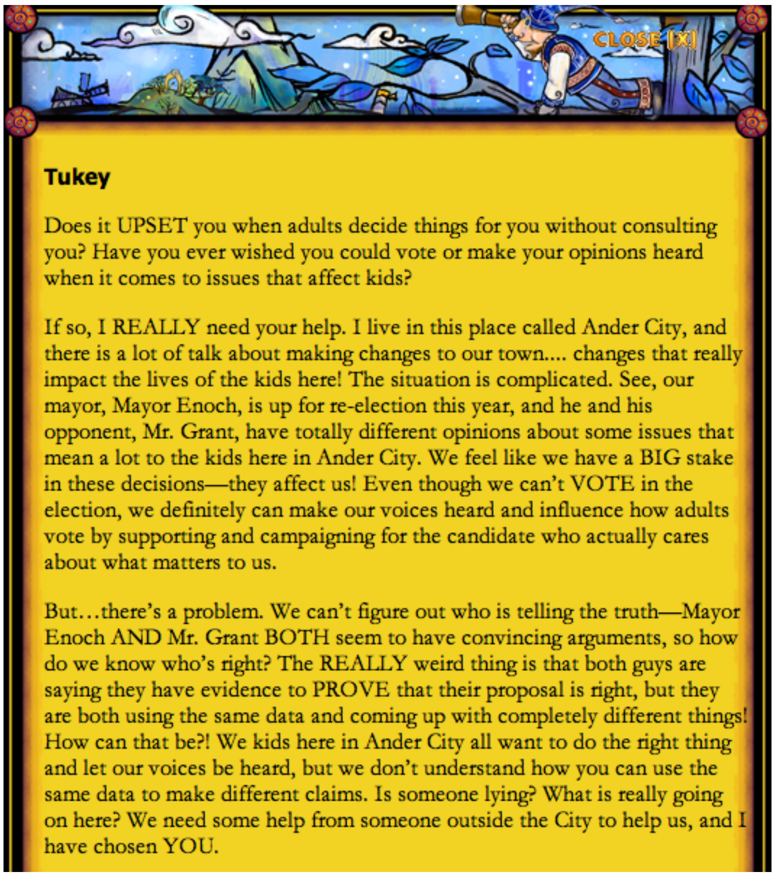

Figure 4: Narrative Framing for Final Design

Uptake of Affordances. In the final design, many more students engaged critically with the content, offering explanations of not only what they did, but also what they might have found if they had done something different. An example of an excerpt from a student's recommendation is below; the student explains that she used two different tools (total first, and then range), and justifies why she believes range is a better measure for the situation under consideration.

> "I analyzed the data by making a bar graph then added each 10 skids up and came out with speedy spokes for the answer as which is most reliable bike to stop soon enough before wrecking into something. But I just noticed the that the Speedy Spokes skid can be anywhere from 30 in. to 54 in. and the Rollin Steady skids were from 43 in. to 55 in. I would much rather be on a Rollin Steady bike because it is more consistent to what we are looking for here in Ander City so you don't run into the cars at stop sign. The reason why I chose range for the analysis is because I just thought that with a more consistent stopping bike that you wouldn't be worrisome about you stopping before the traffic or in the traffic this is why I chose the range for my analysis."

This student made it clear that she had considered the data in two different ways, each of which revealed something different about the data. Specifically, one way of looking at the data (total stopping distance) suggested that one brand of bike tended to stop sooner, but another way of looking at the data suggested that the other brand of bike was more predictable. These are both reasonable ways to think about safety, and thus it was left for the student to decide which analytic frame made more sense. Importantly, this student clearly felt that she had the agency to determine how to analyze the data, and was able to reason about what particular numbers (total stopping distance, range) actually meant in terms of the situation. Although not all students engaged critically even by the final implementation (the average score for critical engagement was 6.2/12, which is only over 50%), there was certainly a marked increase in the number of students who were able to approach data analysis in this more empowered, critical way.

Discussion

The examples above were intended to demonstrate how, in considering the strength of affordances, we can make sense of what students to in relation to what they have opportunities to do. In the case of the first design, very few students engaged critically in their written responses despite the fact that every single decision they were asked to make had two very plausible different solutions. One could imagine an analysis that suggests that students aren't able to consider two perspectives simultaneously, or, more insidiously, that students aren't thinking but are instead just following rules. Both explanations assume that there is a problem with students' interest or ability. Considering how engagement changed in light of the shift in the strength of the affordance to engage critically, a different interpretation is possible; specifically, that students *are* able to engage critically when they have a reason to do so—when the affordance is sufficiently strong that taking it up is required to successfully solve the problem. The shift in engagement that we saw across implementations clarifies that there wasn't a problem with students' effectivities, but rather, that a stronger affordance was required in order for students to recognize it and realize it.

An important question to be posed of this framework is whether or how one might actually measure the strength of an affordance. Indeed, the design trajectory shared above demonstrates compellingly that it is difficult to know whether students will take up an affordance, and thus initial determinations of something as being "strong" might ultimately be mischaracterizations. To be sure, there are parameters that we can use to offer initial conjectures about the potential strength of an affordance, but of course this can only be seen by examining a task or tool in interaction. This might seem a bit of a tautology: an affordance is strong if someone takes it up, and someone takes it up if it is strong. But this is not the case. In understanding what someone does in relation to what they have opportunities to do, we cannot simply examine a single individual separate from the classroom context, nor can we examine a single moment in time. If only one or two students act on an affordance, we can reasonably assume that the affordance was weak. Therefore, initial considerations of the strength of affordances must be grounded in small-scale analyses that document patterns in participation. Following this close analysis, this framework can begin to be used at larger scales, allowing for consideration of how the performance of one group, or many groups, differs from other groups, and to what we can attribute that difference. It is in this way that this framework can begin to bridge the large gap between case studies and "big data."

All learning theories, despite their vast differences, define learning as a change of some kind. Considering the opportunities to learn framework, learning can be considered as a shifts in

students' sensitivities to affordances such that, over time, strong affordances for engagement are not required. A student who had had multiple opportunities to work on statistical data analysis by engaging critically might become increasingly sensitive to opportunities, thus requiring only weak opportunities to engage in this way (Gresalfi, 2009). Reconsidering learning in this way offers a different interpretation of success and failure, and, equally importantly, offers immediately actionable recommendations for students who are failing to engage mathematical content as we would like them to. Specifically, as demonstrated above, by subjecting a task, technology, or even small design decision to a consideration of the strength of its affordance, we can begin to develop conjectures about what students will do. Following data collection and analysis of student participation, we can develop a clearer understanding of how the context is contributing to what we ultimately see students doing, knowing, and learning.

Acknowledgements

The work discussed in this paper was supported with funding from the MacArthur Foundation and the Gates Foundation. The development and research of the Ander City unit occurred in collaboration with Sasha Barab, Adam Ingram-Goble, Edward Gentry, and Janis Watson.

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EXTENDED NETWORKS: MATHEMATICS CLASSROOM COLLABORATION WITH MOBILE DEVICES

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This paper explores the potential of mobile devices to support novel forms of mathematics classroom activity. In particular, I consider two salient aspects of contemporary practice with mobiles—networking/communication, and picture taking—and explore technology designs that might exploit their potential for supporting related forms of mathematical practice. I present data from two cycles in a design-based research project focused on collaborative learning in classroom networks, one featuring graphing calculators, and the other using iPods.

Keywords: Algebra and Algebraic Thinking, Classroom Discourse, Modeling, Technology

Introduction

Modern mobile computing devices—smartphones, tablets, and the like—have rapidly emerged as central tools in the daily practices of many American youth. Moreover, they increasingly prevalent in the hands of k-12 students at school. According to a recent report, more than half of all high school students now bring a smartphone to school (Grunwald, 2013), and Los Angeles Unified, the nation's second-largest school district, plans to give an iPad to every one of its 650,000 students by the fall of 2014.

But what might this proliferation of new mobile digital tools mean for teaching and learning mathematics? Handheld computational devices, in the form of four-function, scientific and graphing calculators, have been commonplace in mathematics classrooms for years. So for math, the potential for instructional novelty more likely lies in the newer features unique to the latest generation of tools—internet connectivity, communication and information-sharing resources, photo and video capture capabilities, customized apps. Indeed, many forms of digital practices with mobile devices popular with today's teens have clear analogs with aspects of mathematical practice highlighted in current standards (White, Booker, Martin & Ching, 2012).

In this paper, I present results from an ongoing design-based research (Brown, 1992; Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Design-Based Research Collective, 2003) program focused on using classroom networks of handheld devices to support student interaction and collaboration in secondary mathematics classrooms. In particular, I focus on two of these distinctive capabilities of contemporary mobile devices—communication and information sharing via wireless networking, and camera-capturing photo and video images—as resources for designing tools and learning activities that might have the potential to integrate informal digital and formal mathematical activity.

Communicating and Collaborating in Classroom Networks

Perhaps the most essential purpose of mobile devices, and the fundamental reason for their near-ubiquity among today's youth, is as tools for communication: talk, text, email, video chat, and social media interface. As such, these devices have the potential to facilitate learners' participation in critical aspects of mathematical activity. Recent national mathematics education frameworks such as the NCTM and Common Core practice standards have stressed the

importance of providing students with opportunities to organize and express their own mathematical thinking as well as critically examining that of others. In this spirit, promoting student participation in mathematically rich classroom discourse has been a central theme in mathematics education research and practice over the last two decades (Ball, 1993; Lampert & Blunk, 1999; Yackel & Cobb, 1996). Often, instructional activity in this vein takes the form of teacher-facilitated whole-group conversations in which mathematical meanings, arguments and standards of evidence are established collectively (see, e.g., Forman, et al., 1998; Staples, 2007). In other instances, students work in pairs or small groups on collaborative problem-solving tasks, and thus have opportunities to discuss ideas and strategies, negotiate and coordinate interpretations, and provide peer tutoring (e.g., Barron, 2000; Boaler & Staples, 2008; Moschkovich, 1996; Leikin & Zaslavsky; 1997).

Recently, several innovative research and design projects have begun to map out ways that handheld devices connected to local computing networks might support and enrich these forms of communication and discursive interaction in mathematics classrooms. Classroom networks can support students' agency and participation in collective mathematical activity (Ares, Stroup & Schademan, 2009), attention to and identification with dynamic mathematical representations (Hegedus & Penuel, 2008), and opportunities to draw on diverse cultural and linguistic resources for participating in classroom discourse (Ares, 2008). Likewise, networked handheld devices in small group collaboration can facilitate greater communication, coordination and negotiation among peers (Zurita & Nussbaum, 2004), and expand and enrich avenues for active participation in joint problem-solving activity (White, 2006; White, 2009; White & Pea, 2011).

Contemporary mobile devices offer the potential to extend these networks beyond the classroom to connect students, teachers, and community members even as they move across contexts. Networked devices offer new possibilities for interaction; they present ways of rapidly distributing information, exchanging ideas and constructing shared artifacts. Just as in-classroom device networks have powerfully demonstrated the potential for merging peer communication with dynamically linked mathematical representations (Hegedus & Moreno-Armella, 2009; White & Pea, 2011), connections between mobile devices that extend the sharing of mathematical objects and ideas beyond the schoolyard represent unique opportunities for blending learners' informal digital activity with conventional forms of mathematics classroom discourse. In particular, photos, videos and other artifacts or data collected outside the classroom using mobile devices may form a particularly powerful resource for supporting learner's efforts to find conceptual coherence between real-world phenomena and school mathematics—especially if those captured artifacts or data allow learners to make mathematical meaning of personally relevant objects and experiences.

The *Graphing in Groups* Design

To explore these possibilities, this paper reports on two implementation cycles of a design for collaborative mathematics activities using a classroom network. The learning environment described in this paper is one among a family of designs greated according to the guiding principle that the *social* should be mapped to the *mathematical*—that collaborative relationships among students should be organized around the mathematical relationships that the learning activity seeks to help students understand (Stroup, Ares & Hurford, 2005). These designs have two primary pedagogical objectives: 1) to link each student participant in a small group with objects in the shared space of the classroom network to make important mathematical relationships salient, and 2) to encourage collaborative interactions among students by posing

tasks and challenges that require participants to coordinate their individual contributions in order to jointly manipulate shared objects. We develop these activities using the NetLogo modeling environment (Wilensky, 1999) and HubNet network tools (Wilensky & Stroup, 1999) in concert with classroom sets of student devices–Texas Instruments graphing calculators or Apple iPod/iPads. This classroom network situates each student's device within a server-defined small group, and a screen projection from the server at the front of the classroom displays mathematical objects linked to both individual student devices and to student small groups. Exemplars of this approach include each member of the small group examining different dynamically linked representations of the same mathematical function displayed on their respective devices (White, 2006; White & Pea, 2011), transforming alternate sides of a shared equation (Sutherland & White, 2011), manipulating different vertices of a jointly constructed quadrilateral (Lai & White, 2010), or moving respective points in a shared graphing space in order to jointly manipulate a curve (White & Brady, 2010; White, Wallace & Lai, 2012).

In the present design, called *Graphing in Groups*, students are assigned to work in pairs. Each student uses the directional arrow keys on her calculator or iPod to adjust the Cartesian coordinate location of a point, which is displayed both privately on the student's device and publicly in a graphing window projected from the teacher's computer. Two pairs of students are assigned to each in an array of such graphing windows in the public space, as shown in Figure 1.

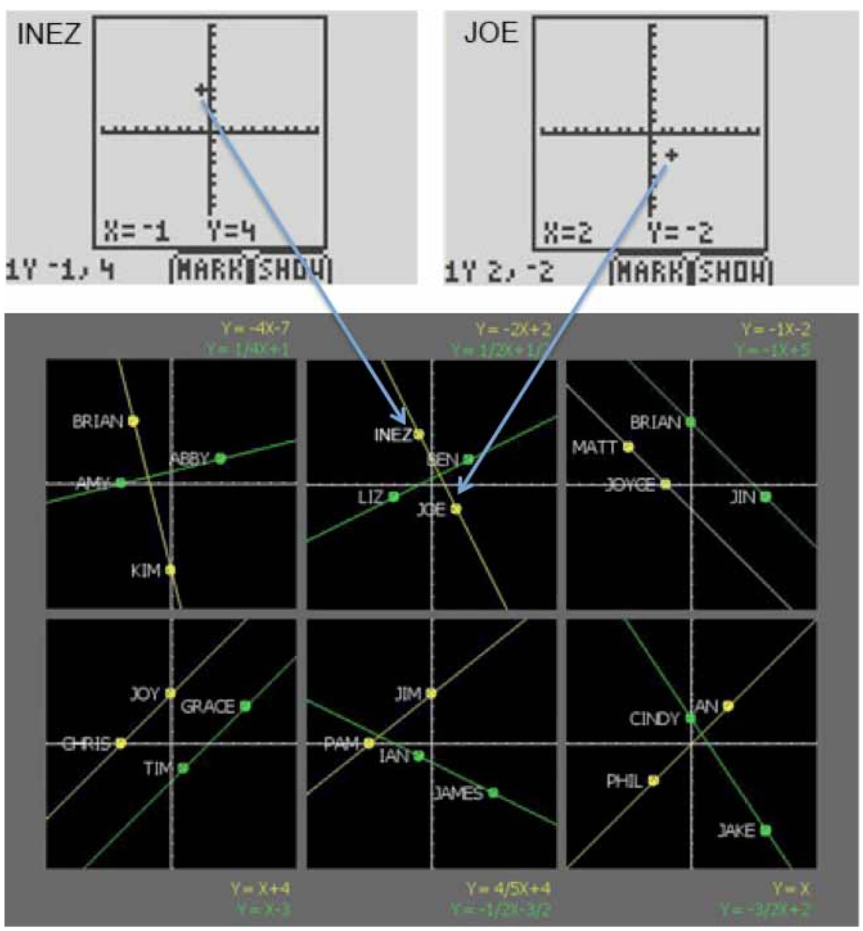


Figure 1: Two student Cartesian coordinate locations, on calculator screens at top, form a line drawn in a group window of a *Graphing in Groups* public display.

As each member of a pair marks a new coordinate location, the corresponding line between the students' respective points is dynamically redrawn and its slope-intercept-form equation updated in the public display. Classroom activities in this environment typically revolve around student pairs' successive efforts to construct lines with particular characteristics: a slope of three, an equation y=(4/3)x-2, x- and y-intercepts both equal to seven without either student placing her point on an axis, etc. In the iPod version of the Graphing in Groups design, students have the additional capability of sending photos taken using their devices up to the server, where they become the background for the group's graphing display (Figure 2).

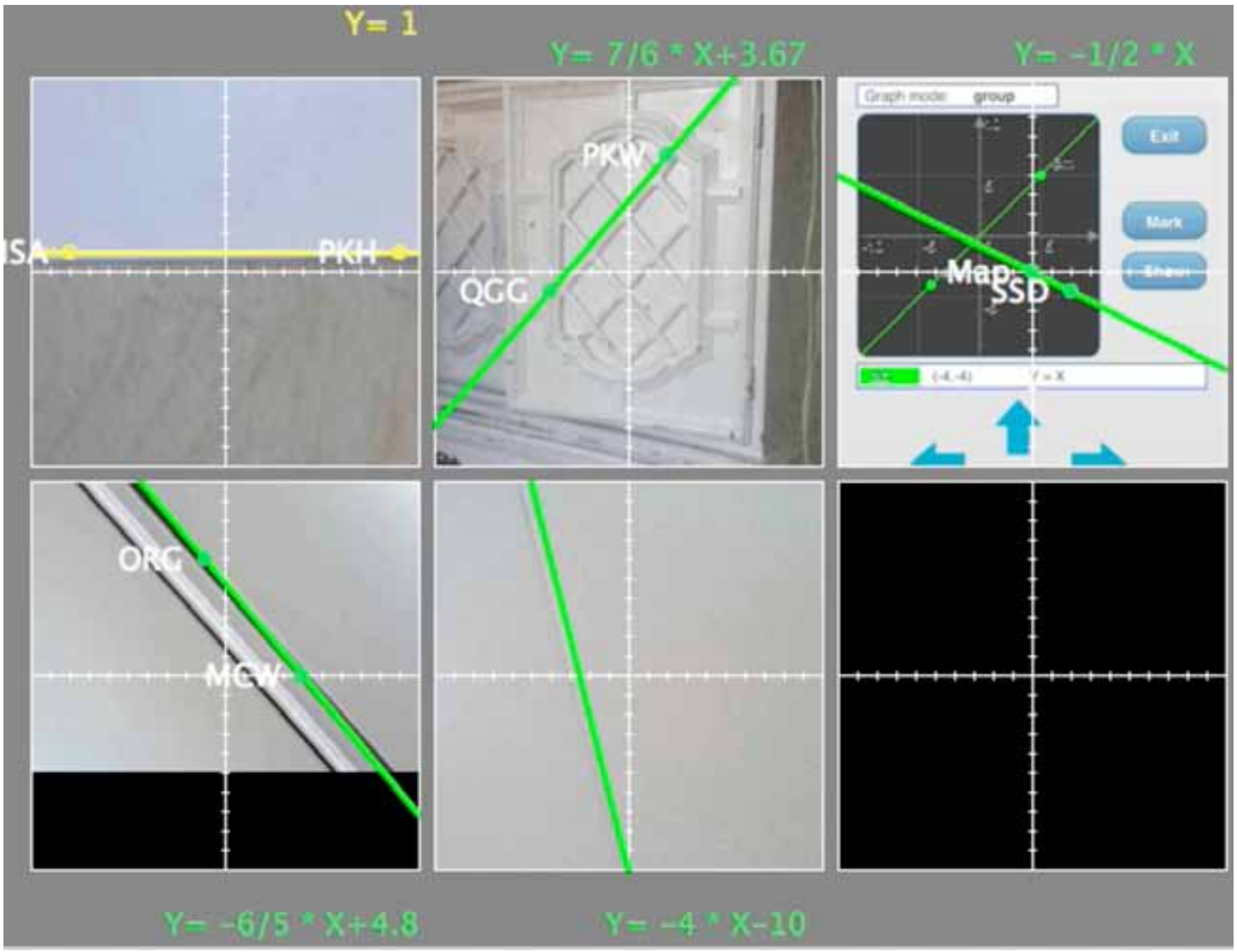


Figure 2: *Graphing in Groups* Display with Student Photo Backgrounds

Method

This paper presents data from two successive classroom-based design experiments using the *Graphing in Groups* environment at the same school site, first using the graphing calculator platform in the spring of 2010, and then again using the iPod Touch version in 2013. In the first cycle, six days of *Graphing in Groups* activities were part of a year-long project in which students participated in classroom network activities for a one-hour session each week as a supplement to their regular mathematics program. The author served as the teacher for all these

class sessions, and sixteen 9th grade Algebra I students participated regularly throughout the year. The second cycle was conducted as a shorter instructional unit featuring 4 sessions with a different group of sixteen students in a mixed-age cohort spanning grades 7 to 9. Two other researchers shared teaching duties with the author in this second study. Two to three student pairs in each class were selected as focus groups and videotaped during all activities. All screen states of the public computer display were recorded as a video file for each class session, and an additional camera with a wide zoom setting captured this projected display along with the whiteboard at the front of the room, as well as whole-class discussions and other teacher moves.

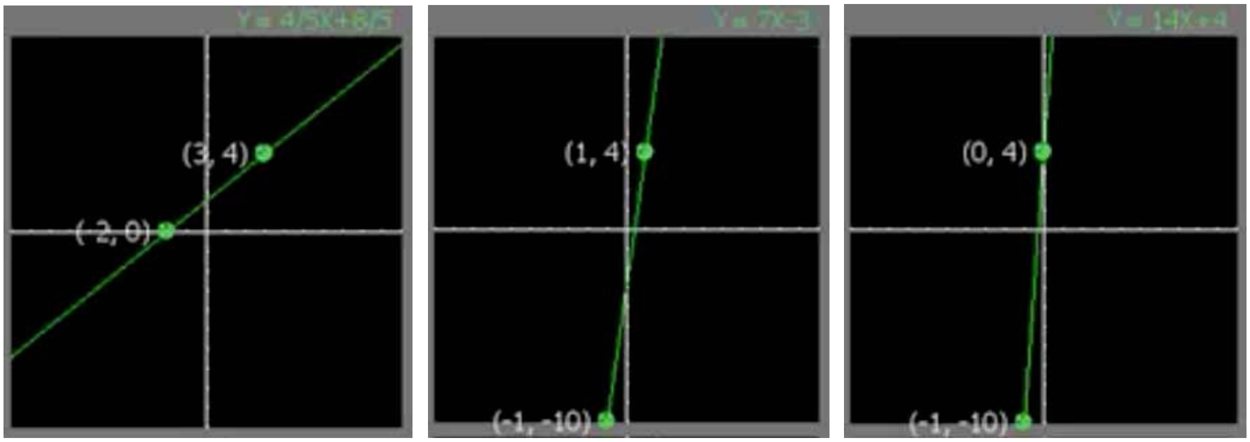
Results

To illustrate the kinds of mathematical activity supported by classroom networking tools, I will detail two brief episodes of classroom interaction as students worked together with the *Graphing in Groups* design. The first episode, taken from the study cycle featuring graphing calculators, highlights some of the complex interactions among students, teachers and mathematical objects typical of this environment. The second episode, from the iPod cycle, examines the potential for bridging these classroom mathematics activities with mathematical and digital experiences drawn from outside the classroom.

Episode 1: Classroom collaboration—supporting connections within and across groups

Below, I present a sequence of three consecutive segments of classroom dialogue during a line construction task in *Graphing in Groups*. Together, these segments span just over two minutes of classroom activity, as students worked in pairs to create lines with as large a slope as possible given the constraint of points confined to a graphing window with x- and y-min=-10 and x- and y-max=10. The first segment focuses on two students, Monica and Jamal, as they begin this task.

1. *Teacher*: Now, I want you to make the biggest slope that you can.
2. *Monica*: The biggest slope that we can?
3. *Jamal*: Oh, ok, I know how to do this. [From his initial location at (-2, 0) (Figure 3), begins moving his point to the y-axis and then down].
4. *Monica*: You know how to do this?
5. *Jamal*: Yeah. Alright, watch. [Continues to move down to (-1, -10)].
6. *Monica*: Where am I going? [Begins moving right from (3, 4)]
7. *Jamal*: Alright, hold on, go back, go back, go back, go back… [Monica moves her point back to (3, 4)]. Closer, closer… to the left. Go left… alright, go back. [Monica moves her point to (0, 4), then back to (1, 4)]. Yeah, right there. Stay right there!
8. *Monica*: Right there?
9. *Jamal*: Mark it. Yeah. [Both students mark points to form the line y=7x-3 (Figure 4)].
10. *Monica*: Seven? I think I can make it bigger than that. [Moves one unit left, to (0,4), and marks to form y=14x+4 (Figure 5)]. Fourteen!
11. *Teacher*: Group 3 has a slope of fourteen.
12. *Jamal*: Yup. [Sets his calculator down on the table in front of him.]
13. *Teacher*: That's pretty big.
14. *Jamal*: I think that's the biggest, cause…
15. *Teacher*: You think that's the biggest you can make? Let's see if anybody can make it bigger.



Figures 3, 4 and 5: Successive Lines Generated by Monica and Jamal

In this excerpt, Jamal opened with the assertion that he “knows how to do this” (line 3), and quickly began to move his point down to the bottom of the graphing window, presumably seeking to maximize the Δy between their points. When Monica began moving right, he directed her back to the left, likewise decreasing the Δx (lines 6-7). When both students marked their new points to form a line with slope m=7, Monica proposed that she could “make it bigger than that,” moving left one unit to form a line with slope m=14 (lines 9-10). As the teacher announced to the class that their group had formed the steepest line yet (line 11), the pair appeared satisfied with their efforts (lines 12-14) until challenged by the teacher to “make it bigger” (line 15).

Successfully completing tasks posed in the *Graphing in Group*s environment typically involves students’ establishing and maintaining a shared understanding of a solution strategy involving how to move their respective points to jointly construct the desired line. Several common characteristics of student actions and peer interactions in this environment (some of which are elaborated in greater detail in White, Wallace & Lai, 2011) are salient in this opening segment. First, ‘inhabiting’ this mathematical space (Noble, Nemirovsky, Wright and Tierney, 2001) takes embodied and multimodal form; students working on these tasks tend to rely on the coordination of spoken utterances with electronic actions in the shared graphical space (as when Jamal invites Monica to “watch” as he moves his point downward in line 5, and Monica likewise asks “where am I going?” even as she begins moving her point to the right in line 6).

Second, students often accomplish this coordination through careful negotiation and sequencing of their respective moves, rather than simultaneous action. Thus Jamal narrates his own problem-solving efforts in lines 3 and 5, then directs Monica’s movements “back,” “to the left,” and “back” again in line 7, then prompts them both to “mark” in line 9; Monica likewise announces her own successive strategic move in line 10.

Finally, the public nature of the group’s graphing display allowed the teacher to join them in assessing the product of their work, both by pointing out the value of their slope to the rest of the class (line 11) and prompting Jamal and Monica to consider whether the slope could be steeper. The next segment explores this public, whole-class dimension of the activity in greater depth. To do so, we widen our analytic lens, zooming out from one student pair (Monica and Jamal, Group 3) to include two other groups (Group 1, Juan and Miguel, and Group 4, Felix and Byron) working on the same task in parallel. In Table 1 below, the utterances of the teacher and the simultaneous dialogue within each pair are synchronized over a 32-second interval. Figure 6

shows the state of the whole class display projected at the front of the room as the next segment begins, immediately following the Teacher's invitation in line 15 above.

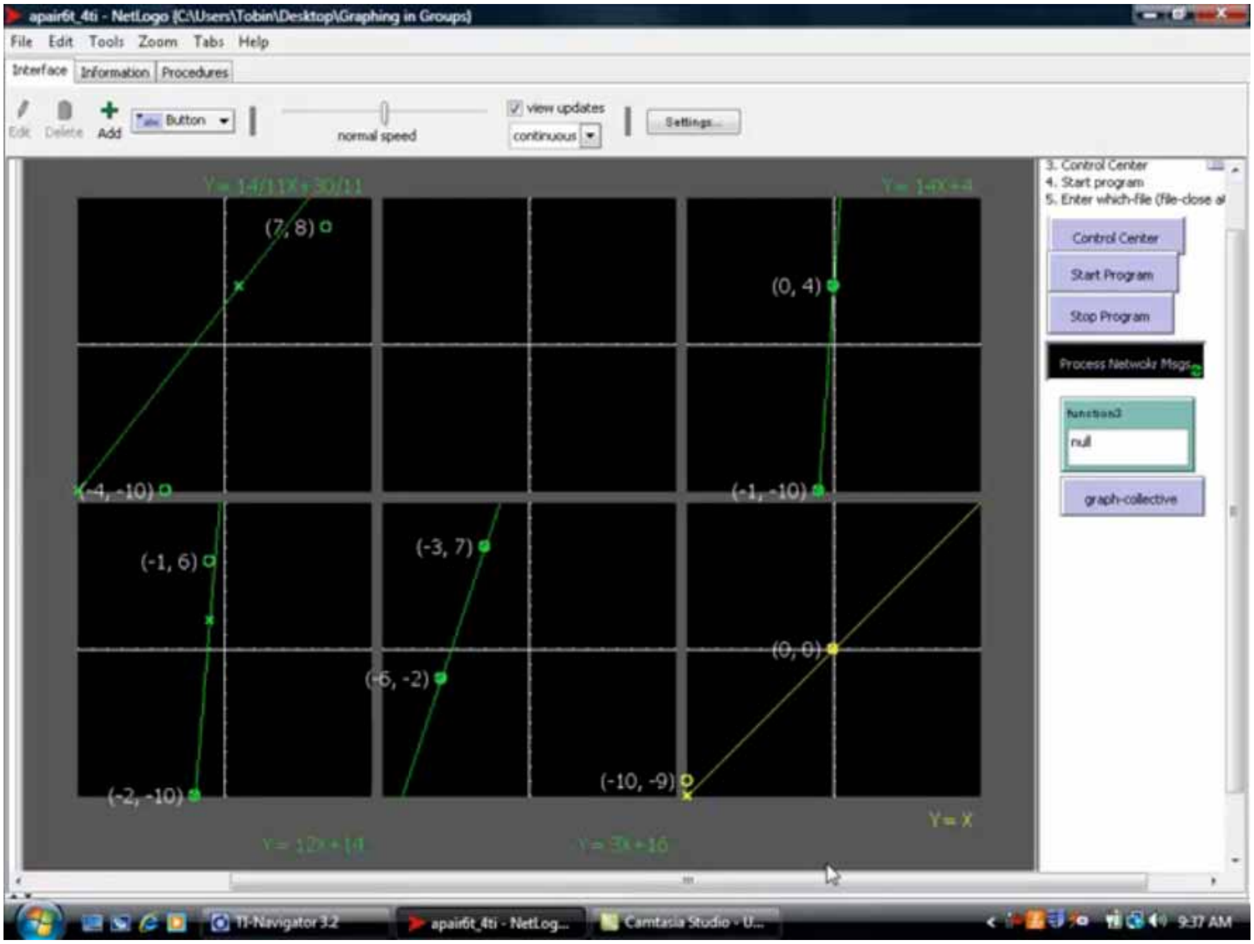


Figure 6: Graphing in Groups Whole-class display. Group 1 is in upper left corner, Group 3 in upper right, group 4 in lower left.

Table 1: Simultaneous Transcripts from Three Classroom Groups

Time (sec)	Teacher	Group 3	Group 1	Group 4
1	Group 4's got a slope of twelve.	*Monica*: [moving left to (-1, 4) as J moves right to (0, -10)]. Let me see, hold on, don't move.		[Byron marks at (-1, 10) to form y=20x+30]
2	Ooh, group 4's got a slope of twenty.	*Monica*: [as both students mark their new points] That ought to be a little bit better.		
3				[Byron punches the air in triumph]
4		*Monica*: Really then.	[J marks at (2, -10)] to form y=-14x+18]	
5		Go over to the, 10…	*Miguel*: Aww!	
7		[J begins moving to the left]		
8	Group 1, negative fourteen.		[J marks at (1, -10) to form y=N/A]	

9				
10	Group 3, negative fourteen.	[J marks at (-10, -10) to form y=(7/6)x+5/3]	[J marks at (0, -10) to form y=14x-10]	*Byron*: That's the lowest you can get.
11				
12			*Juan*: Go up. [M moves up one unit, to (7, 9)]	
14		[M marks at (2, 4) to form y=7x-10]	*Juan*: [Starts moving up] Go to the same one,	[F moves left and marks at (-3, -10)]
15			just go a little higher. [M begins moving left]	Byron: Just stop moving.
16			Same one you had.	
17		*Jamal*: Alright, hold on. Go all the way to the right corner.	*Miguel*: Oh, like right here [moves to (1, 9), directly over last mark	[F moves back to (-2, -10) and marks again]
18		*Monica*: [moving right to (7,4)] That's not going to be a very big slope, though.	at (1,4)] but higher? [moves up to (1, 10). J stops at (0, 2)]	
19		[M begins moving back to the left]	*Juan*: [M moves to (2, 10)] Yeah, right there.	
20		*Monica*: I have to be, like, closer to you.	[M moves back to (1,10)]	
21			*Miguel*: There.	[B begins drumming the desk with his thumb]
22		Jamal: [moving up] Oh, hold on, hold on. I've got an idea.	[J starts moving down]	
23			Juan: Wait, hold on, hold on.	
24			Yeah, right there.	
25		[J marks at (-8, -5) to form y=(9/10)x+11/5]	*Miguel*: Oh, do I mark it?	
26		[M marks at (-9, 4) to form y=-9x-77]	*Juan*: [still moving down] Yeah, mark it.	
27		*Monica*: That's only 9. [Begins moving right]	[M marks at (1, 10) to form y=20x-10]	
28		*Jamal*: [moves right] Alright, hold on. Go to the [waves finger to the right] right [points again] side.	*Juan*: [moving back down Yeah, you can't get higher than 20.	
29	Group 1's got 20.	[J marks at (-6, -5)]		
32		[M marks at (-3, 8) to form y=(13/3)x+21]		*Felix*: (to Group 1) Copycatters!

In the first two seconds of this segment, Monica and Jamal both tried further reducing Δx to increase the slope, producing a line with m=-14 even as the teacher announced Group 4's creation of a line with slope m=20. This news prompted different reactions from all three groups, as Byron thrust a hand forward in triumph (3s) while Monica mused "really then" (4s) and Miguel simply groaned (5s). Monica and Jamal then initiated a series of unsuccessful attempts at further increasing the slope of their own line, Jamal moving to (-10, -10) (10s) and directing Monica to the opposite corner (17s) even as Monica asserted that they should move closer together (20s). Meanwhile, Juan appeared to draw inspiration from Group 4's success, repositioning his own point at the bottom of the screen and then directing Miguel into a configuration that matched Byron and Felix's in order to likewise forma line with slope m=20.

This segment illustrates two additional characteristics common to *Graphing in Groups* classroom activity. The first is that the public display serves as a resource for the teacher not only to track the progress of all student groups as they work in their respective graphing windows, but also to comment on and provide feedback regarding their progress. The second is that the display also allows groups to attend to one another, sometimes modifying their own solution strategies based on their observations of others', as Juan appeared to do based on Group 4's work, and Jamal may well have likewise been attempting when he went to the lower left corner and directed Monica to the upper right after seeing a similar line constructed by Group 6 (lower right window in Figure 6).

In the moments that followed, Jamal and Monica continued to experiment with their own strategies until they converged on a solution:

16. *Monica*: I am just going to keep going until it makes it bigger. [Moves to the right]
17. *Jamal*: [Moves up and right, marks at (-5, -4)] Alright, hold on, let me go this way. [Moves to the left, marks at (-8, -4) as Monica marks at (-1, 8) and then (0, 8)] Holy crap. [Marks at (-9, -4), (-8, -4) again, and then at (-7, -4) as Monica marks at (-2, 10) to form y=(14/5x+78/5] Alright, fourteen…
18. *Monica*: [Moves left and marks, in quick succession, at (-3, 10), (-4, 10) and (-5, 10).] Don't move. [Marks at (-6, 10) to form y=14x+94, then moves to mark at (-7, 10) just as J marks at (-10, -6), forming y=(16/3)x+142/3] Why are you moving?
19. *Jamal*: Oh, sixteen…
20. *Monica*: [Marks at (-8, 10) to form y=8x+74] Eight.
21. *Jamal*: We had sixteen.
22. *Monica*: [Marks at (-9, 10) to form y=16x+154] Wait, now, sixteen.
23. *Teacher*: Sixteen, good, getting bigger.
24. *Monica*: [Marks at (-10, 10) just as J moves right and marks at (-9, -7) to make y=-17x-160] Stop it! Stop it!
25. *Jamal*: Seventeen, [marks at (-9, -8) to form y=18x-170] eighteen, [marks at (-9, -9) to form y=-19x-180] nineteen…
26. *Monica*: Twenty! (laughs) [J marks at (-9, -10) to form y=-20x-190] Twenty.
27. *Jamal*: Negative twenty, alright.
28. *Monica*: I think 20 is the biggest you can make it.
29. [Jamal, Miguel and the teacher have a brief exchange about the y-intercept]
30. *Jamal*: [marks at (-10, -10) to form y=N/A] Alright, hold on. Go right one
31. *Monica*: What did you do? Oh my God, Jamal. [moves right to (-9, 10), marks to form y=20x+190]
32. *Teacher*: Alright, there's another 20. Can anybody get bigger than 20?
33. *Monica*: I don't think it's possible.
34. *Jamal*: No, I don't think so.
35. *Teacher*: I don't think so either. Why don't you think so?
36. *Jamal*: Cause, that's the height [holds both hands in front of him, then waves right finger up and right, left finger down and left as if to mimic the respective positions of Monica's and his points (Figure 7)] like, the graph doesn't go any bigger.
37. *Monica*: The graph doesn't get any bigger.

Figure 7: Jamal's Gestural Depiction of the "Height"

Monica's opening comment sums up the approach she adopts in this last excerpt, moving two steps to the right and marking after each, and then reversing direction to systematically move and mark at every point from (-2, 10) to (-10, 10) (lines 17, 18, 20, 22, 24)—keeping on "going until it makes it bigger" (line 16). This kind of incremental variation represents another common strategy students use when trying to construct lines in *Graphing in Groups*, gradually tinkering with a line and feeling their way to a solution when they could not see a more direct analytic approach. Jamal, however, was not initially in synch with her efforts, instead moving and marking more haphazardly (lines 17, 18, 24) without consulting Monica. This failure to coordinate their efforts clearly frustrated Monica, who repeatedly asked Jamal to stop moving (lines 18, 24). By the time she reached the end of her leftward path, Jamal appeared to have recognized the progress she was making as he, Monica and the teacher all commented (lines 21-23) on their arrival at a slope m=16, their largest yet. Indeed, Jamal then took up the same iterative and incremental approach, moving to (-9, -7) and marking, then repeating the sequence for each step down to (-9, -10) as he and Monica called out the increasing (negative) slopes.

This segment thus highlights important additional aspects of action and interaction in *Graphing in Groups*; the embodied sense in which students act on these graphical objects through electronic point movements facilitates both individual exploration of the mathematical space, and nonverbal communication, as peers observe, react to and take up one another's patterns of motion. Moreover, the interdependence of the shared line on their respective

Cartesian points imposes constraints on individual action; students' common discovery that failure to coordinate their point movements undermines their individual problem-solving efforts and investigations, and thus builds a need for coordinated collaborative interaction into the learning environment.

Finally, Jamal's gestural activity during this episode further illustrates the embodied dimensions of learners' meaning-making in this virtual graphical setting. In the previous excerpt, Jamal's successive waving of his finger twice to the right in precise synchronization with the words "right" and "side" (28s) as he sought to orchestrate Monica's graphical movement highlighted the "semiotic bundling" of utterance, gesture and representation common to mathematical activity (Arzarello, Paola, Robutti & Sabena, 2009; Radford, 2003; 2009). Likewise in this segment, Jamal's carefully synchronized movements of each hand as he sought to explain why the line's slope cannot be greater than 20 while the points remain in the present graphing window (line 36) provide a visual rendering of the graphical situation, his hands abruptly stopping at the ends of their up- and downsweeps as if confined, just like their points, to the "height" of the available display (Figure 6). In this sense, these gestures with his hands at once appear to encapsulate the sequence of point movements necessary to construct the line, enact the relative positioning of his and Monica's points, and summarize the argument that the maximum slope has been achieved. That they are articulated via embodiment of two points, rather than a pitched hand or forearm to display the resulting line, suggests that the salient features of student action in this particular designed space provide specific resources for student reasoning about the corresponding mathematical space.

Episode 2: Extending Networks Beyond the Classroom

This section examines an excerpt from the work of another student pair in the *Graphing in Groups* environment, this time using iPods. This episode finds these students on their third day in the study unit. In the previous two days, they had been introduced to the iPods and the graphing tools; at the end of the previous session, the teacher gave a homework assignment in which he asked them to use their iPods to take photos of lines with different slopes to be shared and investigated when they returned to class. The excerpt below begins just after the teacher asked each pair to select one of their photos to upload to the network for display in their group's graphing window, and focuses on two students, Olivia and Jane, as they start this task:

1. *Olivia*: Okay, so these are the photos that I have [shows her iPod to Jane]. I have um…that…[opens Figure 8 on her iPod]
2. *Jane*: That would probably be good.
3. *Olivia*: That…[swipes through photo album to display Figure 9]
4. *Jane*: Is this stuff in your room?
5. *Olivia*: And that [swipes again to show Figure 10]. No, it's my sister's room. So which one do you wanna use, the one with the [holds hand up and makes a pitched line] stripes?
6. *Jane*: Yeah. Then we have different [trace a crisscross with her index finger] lines to go to. [A cropped version of the cabinet door from Figure 9 appears in Jane and Olivia's graphing window in the public display (Figure 11)].

Figures 8, 9, and 10: Candidate Photos on Olivia's iPod

In lines 1-6, we get some insights into the ways students made sense of this photo assignment. Olivia captured three images—one of a grid that might be a tile floor (Figure 8), one of a cabinet door (Figure 9), and one of a decorative string of globe lights hanging from wires at diagonals to one another (Figure 10). In each case, Olivia seems to have made a point of seeking out not just a single line to photograph, but multiple lines in parallel, perpendicular and/or skewed sets. In other words, she chose scenes that allowed her to fulfill the requirements of the assignment (find lines with different slopes) within a single image. She and Jane then further applied these same criteria to select the second image because it had the most pronounced slanting and "different lines to go to" (line 6).

As students' photos began appearing in the public display, Jane and Olivia proceeded to construct a line that matched one of those on the cabinet door:

7. *Jane*: [begins moving her point up and down along the y-axis, from the origin]
8. *Olivia*: What do you think the great…? [Jane pauses her point at (0, 3), right at the intersection of two lines in the photo image. Olivia begins moving left, approaching one of those two lines] Let's do the biggest one right [moves left again to (-4, -2), so that her point is now directly over the same line as Jane's] there.
9. *Jane*: This one?
10. *Olivia*: Like that?
11. *Jane*: Mark it?
12. *Olivia*: You can go higher if you want. [Olivia moves her own point up one unit] You see—you see where I am? Go on that line. You should go probably up two [Jane moves her point up two units to 0, 5) and then across, like…I don't know…one, two,…
13. *Jane*: [moves to the left] Like that?
14. *Olivia*: No, across the other way.
15. *Jane*: [laughing at the picture that just appeared in Group 3's window, moves right to (1,4)]
16. *Olivia*: One more. Go up.
17. *Jane*: [moves over and up to (2, 5), then back down] Ah!

18. *Olivia*: It's okay. [Takes the iPod from Jane and moves her point to (3, 7), hovering just above and beyond the tip of the line on the cabinet door in a position that matches Olivia's own relative to the other end of the same line]. There we go. [returns the iPod, then picks it up again to mark on it and then on her own to form y=(8/7)x+3.57 (Figure 11)]

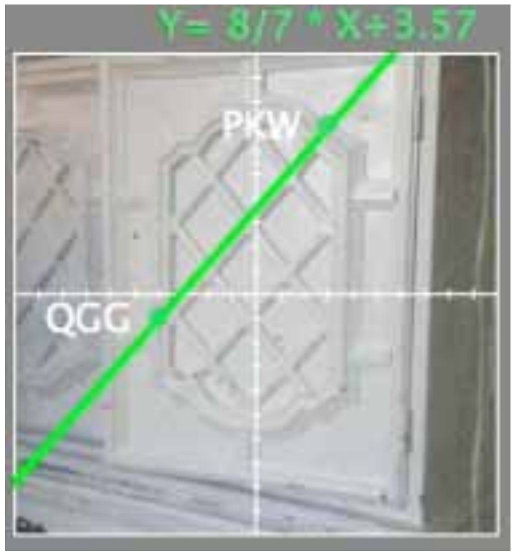


Figure 11: Olivia and Jane's Graphing Window with Cropped Photo Background

Beginning in line 7, we can observe how these students set about making mathematical sense of the images they collected. In particular, they were quick to identify one of the longest line segments in their selected image (line 8), and to use it as a frame of reference for forming their own *Graphing in Groups* line. Importantly, their efforts in this segment do not reveal substantive reflection on or interpretation of the slope of the line they constructed or its relationship to the photographed image. Moreover, and in contrast to the more open-ended and more collaborative process through which Jamal and Monica sought to make the "biggest slope" they could in the previous episode, this segment depicts a relatively closed mathematical task, and a cooperative process in which one student—Olivia—took most of the initiative and did most of the work, directing Jane where to move and mark (lines 12, 14, 16), and even taking her iPod to complete the final steps herself (line 18).

Discussion and Conclusion

This paper offers a detailed look at the nature of classroom interaction and collaboration in the context of a local network of mobile devices, as well as a more cursory glance at the possibilities for extending that classroom network to incorporate and support mathematical investigation of material such as photos. The first episode illustrates several key elements of student participation in the *Graphing in Groups* environment: that accomplishing line construction tasks in this environment involves coordinated action between participants, that achieving that coordination can require careful and complex orchestration of utterance, gesture, and electronic action, and that shared constructions dynamically displayed on a public screen affords additional layers of both teacher orchestration and peer interaction. The second episode reveals some of the possibilities and the challenges associated with integrating student-generated

images into this environment: they clearly yield personally relevant contexts to which learners bring their own aesthetic and mathematical lenses, and they clearly also require carefully constructed tasks that scaffold students' mathematical interpretations of those images and their collaborative interactions around them.

While the calculator version of this classroom networking design has been extensively studied and implemented in a variety of classrooms over several years, the iPod version and its photo and other extensions are still in a pilot phase. Our research group at UC Davis is actively developing these and other tools and accompanying classroom learning activities. The photo activity described here, for example, represents what we see as a bridging activity to begin connecting students' engagement with linear graphs with their experiences with linear phenomena in a variety of settings; the next and more substantive phase in this learning sequence involves students capturing video of phenomena that vary linearly over time, and then use tools on their mobile devices—and mathematics—to analyze that variation in greater depth.

Acknowledgements

This material is based upon work supported by the National Science Foundation under Grant No. DRL-0747536. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

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Chapter 2: Curriculum & Related Factors

Research Reports

Brief Research Reports

Poster Presentations

CONNECTING TEACHING AND LEARNING IN CURRICULUM ADAPTATIONS

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This study of six teachers focuses on the ways they organized the classroom discourse, attended to student thinking, and adapted complex tasks from a Standard-based middle school curriculum. The study explores Cohen's (2011) premise that the knowledge teachers develop is related to their attentiveness to student teaching. This study explores the relationship between the extent to which teachers were successfully able to elicit and organize instruction around student strategies and their ability to productively adapt tasks in terms of being responsive and maintaining cognitive demand. The results show that teachers with the most student-centered discourse practices were also able to provide the most detailed justifications for task adaptations and to productively adapt tasks from the Connected Mathematics Program (CMP) curriculum.

Keywords: Instructional activities and practices, Classroom Discourse

The ways teachers attend to student thinking impacts the kinds of knowledge they are likely to develop. This poses a significant change for teacher educators, as Cohen (2011) notes that most instruction "is marked by little close attention to learners' thinking and little effort to design instruction to advance it" (p. 27). Cohen attributes the rareness of attending to student thinking to the inherent uncertainty associated with listening and responding to another person's thinking, in contrast to the relative certainty and predictability of helping students to practice and remember predefined and narrow forms of knowledge. However, teachers' attention to how students make sense of mathematical concepts is, as Cohen notes, central to *attentive* teaching. Furthermore, Confrey et al. (2008) state that attentive teaching is an essential practice with respect to using the curriculum programs aligned with the National Council of Teachers of Mathematics (NCTM) Standards documents (NCTM, 1989, 1991) [hereafter referred to *Standards-based curriculum programs*]. Given the large U.S. investment in the Standards-based programs, it is imperative to research practices that lead to productive uses of the materials.

Cohen (2011)'s essential premise is that teaching needs to be connected with student learning to be effective. That is, teachers need to pay attention to how their teaching practices influence how students reason about academic content. He analyzes the relationship between teaching and learning by exploring connections between the knowledge developed by teachers, the organization of the classroom discourse, and the ways teachers attend to student thinking. He describes how teachers who engage in more expansive forms of discourse have opportunities to develop a greater sense of how students reason about content and can extend student learning to more complex forms of knowledge than would otherwise be the case, stating that "teachers who attend only a little or narrowly to students' knowledge constrain their opportunities to make intellectual connections that may advance learning" (p. 39).

The connections between teachers' discourse practices, their attention to student thinking, and their use of Standards-based curriculum materials are poorly understood. Most of the research on Standards-based materials focuses on early stages of implementation, which has limited the ability to show the longer term impact of teachers' instructional practices on their understanding of how the materials engage students. In short, Cohen's (2011) basic premise is

largely conjectural with respect to teachers' understanding and use of Standards-based programs. Consequently, this study explores Cohen's conjecture that there is a connection between teachers' discourse practices, their attention to student thinking, and the ways they design instruction to intentionally provoke forms of student learning. The study focuses on the ways that teachers' attention to student thinking is not only helpful in the moment but is associated with more productive uses of the materials over time. The study is set in the context in which an NSF-funded middle school curriculum was implemented, building from the idea, as Confrey et al. (2008) note, that Standards-based programs were designed to be sensitive to the ways teachers build instruction around student thinking.

Curriculum Adaptations

Adaptations to tasks as represented in written materials are inevitable as they are transformed in dynamic classroom contexts (Remillard, 2005). Spontaneous adaptations – those that happen during enactment – are influenced by the teacher's ability to purposefully improvise on the fly, which has been shown to be a high-capacity practice. In this study, the focus is on adaptations to curriculum materials that occur prior to enactment, as teachers use and adapt written materials to plan lessons (Remillard, 1999; Stein, Grover & Henningsen, 1996). In these planned adaptations, teachers often tinker with written tasks in ways that are intended to improve the efficiency with which students are able to complete tasks, but which neglect the conceptual development of the lesson (Kennedy, 2005). Other adaptations are in response to constraints in the local context, often in ways geared toward easing the chCorwinges and logistical burdens of cognitively demanding tasks (Arbaugh, Lannin, Jones, & Park-Rogers., 2006; Keiser & Lambdin, 1996; Manouchehri & Goodman, 1998). Typically adaptations lower the cognitive demand of tasks by removing their ambiguous or difficult features (Doyle & Carter, 1998; Stein, Grover, & Henningsen), though teachers have been shown to adapt tasks spontaneously in ways that build on students' contributions and maintain task complexity (e.g., Lampert, 2001). Recent work in mathematics and science education has described teachers who make adaptations to tasks that are productive, in that they maintain coherence with the design of the curriculum, they maintain the cognitive demand of tasks, and they are responsive to the classroom context (Author, 2009, 2011; Author, in press; Brown & Edelson, 2003; Drake & Sherin, 2009; Roth McDuffie & Mather, 2009).

Methods

The Connected Mathematics Project Curriculum Materials

The Connected Mathematics Project (CMP) materials (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2006) emphasize student exploration in chCorwinging tasks as a means of developing mathematical content and they are comprised largely of tasks that are designed to be implemented at a high level of cognitive demand. Tasks in CMP also frequently require substantial amounts of student justification and explanation. Each CMP unit is divided into four to five *investigations*, each of which consists of a task situation or context that serves to organize the investigation. Each investigation begins with an initial problem that focuses on the key mathematical concepts, followed by a number of related problems. The investigations are intended to follow a *launch-explore-summarize* sequence. The launch portion serves to familiarize students with the context and the mathematics as a means of priming student activity in the subsequent exploration, in which students engage with a task that can typically be approached in multiple ways. The summarize portion allows for various solutions to be

compared and for the teacher to emphasize the important mathematical aspects of the tasks (Lappan, Fey, Fitzgerald, Friel, & Phillips., 2004). An important feature of the CMP materials is that mathematical concepts are developed across a sequence of tasks, lending coherence that is typically lacking in U.S. curriculum materials.

The Teachers

The six teachers were selected from a larger video sample. Five of the six teachers had extensive experience with CMP, having used the curriculum materials for a span of five or more years, and all had attended multiple week-long curriculum-specific professional development institutes. The sixth teacher was in her second year of using CMP and had attended a number of professional development workshops, including a week-long institute in Michigan. Two of these four teachers, Audin and Baldeck, were a 6^{th} grade planning team, and the other three, Corwin, Knauff, and Walsh, were an 8^{th} grade planning team, though Corwin was only a part of the team for the school year in which she participated in the study. The sixth teacher, Durst, taught 7^{th} grade in Lakeville, a larger suburban district whose students typically scored lower than those in Brookline and who came from a greater diversity of socioeconomic backgrounds. All six teachers made minimal omissions to the CMP instructional sequences for which they were observed, rarely (if at all) supplemented with new activities. Furthermore, with minimal exceptions, the teachers followed the three-part format for each investigation, as recommended in the teacher resource materials. See Table 1 for a listing of the teachers, their grades, and the units in which they were observed teaching.

Table 1: Teacher, Grades, and Units in Which They Were Observed

Teacher	Grade	Units
Audin	6	Accentuate the Negative, Comparing and Scaling
Baldeck	6	Accentuate the Negative, Comparing and Scaling
Corwin	8	Moving Straight Ahead
Knauff	8	Say It With Symbols
Walsh	8	Say It With Symbols
Durst	7	Comparing and Scaling

Data Collection

The research team created *Unit Sets*, modifying the *teaching set* methodology (Cobb, Zhao, & Dean, 2009), which involves videotaping multiple lessons and using specific events or practices observed in those lessons as the basis of teacher interviews. The pre-unit interviews focused on a teachers' perceptions of the main instructional goals for the unit, the primary chCorwinges they anticipated in terms of enacting the instructional sequences in the unit, and the key tasks and representations that facilitated the learning trajectories within the instructional sequences. The video-stimulated interviews were based on a set of about 10 episodes, ranging from three to 15 minutes in length, selected by the researcher to provoke reflection about instructional sequences in the unit. The researchers generated a series of specific questions around each clip and compiled the questions and the video clips, which was provided to the teacher. The teacher then had an opportunity to preview the clips (usually a week), after which they were interviewed for 60-90 minutes.

Analysis of Task Adaptations

Planned adaptations are characterized in the study by the extent to which they align with the philosophy of the curriculum program, maintain or enhance the cognitive demand and

complexity of tasks, are justified by observations of how students interact with the materials, and ultimately how they provide opportunities for students to engage in mathematical practices. The cognitive demand of tasks was determined by the extent to which ambiguity regarding choice of strategy was maintained and by the extent to which the task emphasized connections to underlying mathematical concepts. For example, if a teacher provided explicit instructions beyond what was in the student text about how to complete a task, it was deemed an adaptation that reduced demand. Conversely, if the adaptation facilitated strategies that became to focus of whole-class discussions in ways that emphasized concepts, it was deemed an adaptation that maintained or enhanced demand. The justifications for adaptations were characterized by the extent to which teachers provided detailed accounts from past enactments to justify their adaptations or whether they simply evaluated a past enactment (i.e., "I tried the task as written in the materials, it didn't go well, so I changed it"). The adaptations were usually identified in the observations of the video data, which gave the interviewer the opportunity to question the teacher about the adaptation and rationale for the adaptation.

Analysis of Classroom Discourse

Building from research on Accountable Talk (Michaels, O'Connor, & Resnick, 2008), the project team separated the teacher discourse codes into two categories that highlighted different ways that teachers attended publicly to student thinking, *teacher probes of students*, and *teacher moves that highlight strategies*. In addition, there was a code for *student explanations*, which was used when students' contributions included evidence to support the steps used to arrive at an answer. The three codes in the *teacher probes* category included *probed student to explain reasoning*, *probed students to confirm/ clarify*, and *elicited comments on strategy*. The *probed student to explain reasoning* code was applied when the teacher asked a student why he or she approached a problem a certain way or asked for justification of a procedure. The *probed students to confirm/ clarify* code was used when the teacher asked a student to confirm if the teacher's stated interpretation of the student' strategy was accurate or when the teacher sought clarification of part of the student's strategy. The *elicited comment on student strategy* code was used when the teacher asked other students what they thought of a particular strategy. The three codes in the *teacher moves that highlight strategies* category include *recalled past strategy*, *expanded on student response*, and *displayed student strategy*. The *recalled past strategy* code was used when the teacher introduced a strategy that had been presented at a minimum of several exchanges earlier and sometimes from much earlier in the lesson. The *expanded on student response* strategy was used when the teacher's recounting of a strategy included new terminology or steps. The *displayed student strategy* was used whenever the teacher publicly displayed a strategy, either by projecting the student's work or by rewriting the strategy on the board.

Results

The Classroom Discourse

The evidence of student engagement was characterized by the frequency with which teachers displayed student strategies. Each teacher displayed student strategies in a third or more of the time, either by displaying the students' version of the strategy on poster paper or projecting it, or by rewriting key steps of the strategy on the board or overhead. These strategies were then discussed, either by the students or, more commonly, by the teacher, usually to highlight a particular concept or procedure. There were substantive differences in the frequency with which students provided explanations for the strategies. Similarly, there were differences in the ways

the teachers elicited explanations associated with student strategies, and in the ways teachers called attention to and organized discussion around student strategies, as shown in Table 2.

There were big differences in the extent to which students offered explanations for their strategies. Audin's and Baldeck's students contributed explanations substantially more often than the students of the other four teachers (two to fifteen times as often). The results in Table 2 show that teacher moves in the category of *teacher probed students* all were strongly associated with the frequency with which students provided explanations, which suggests that these moves were helpful in eliciting student explanations. In these instances the teacher explicitly asked students to provide evidence for their strategies beyond simply describing steps. Similarly, moves coded as *probed student to confirm/ clarify* frequently elicited warrants that were coded as explanations. Moves coded as e*licited comments on strategy* were strongly associated with the probing moves in terms of their frequencies, which might explain the association between the *elicited comments* code and student explanations.

The other teacher move highly associated with student explanations was *teacher recalled past strategy*. This move was used more frequently by Audin and Baldeck and thus its association with student explanations may be confounded with the probing moves. However, this code represents a slightly different phenomenon and stands in contrast with the *teacher expanded student response* code, which also involved teachers broadcasting or explaining a student response or strategy, but which was poorly associated with student explanations. In moves coded as *recalled past strategy*, the teacher re-introduced a strategy that had been presented earlier in the lesson. In this move, the teacher typically highlighted the mathematical features of a strategy or showed how different strategies were related. The temporal distance between the presentation of the student explanation and the teacher recall of that explanation stands in contrast with the more temporally proximate teacher moves coded as *expanded on a student response*, which typically immediately followed a student response and functioned to evaluate and revise that response. The *recalled strategy* showed how the teacher connected student explanations over time and used student explanations strategically to emphasize key mathematical points.

Table 2: Discourse Practices

	Student explanations	Teacher probes of students			Teacher moves that highlight strategies		
		Probed to explain reasoning	Probed to confirm/ clarify	Elicited comments on strategy	Recalled past strategy	Expanded on student strategy	Publicly displayed strategy
Audin	60%*	19%	17%	35%	9%	12%	52%
Baldeck	39%	16%	21%	30%	5%	29%	65%
Corwin	14%	11%	7%	14%	3%	6%	51%
Durst	12%	2%	9%	8%	3%	29%	41%
Walsh	10%	5%	0%	2%	2%	7%	47%
Knauff	4%	0%	6%	2%	0%	23%	37%

*The percentages refer to the number of one-minute segments a code appeared relative to the total number of one-minute segments for that teacher

The *teacher expanded student response* moves provides insight into the differences between the teachers in terms of how they focused discussion on student strategies, especially the four

teachers who most frequently deployed the move. When Durst or Knauff expanded on a student's explanation, they tended to 'hijack' the strategy, explaining it without seeking student input. A third teacher, Baldeck, especially in Comparing and Scaling, discussed the strategies as if she was the student, explaining the mathematics but doing so as if she was recounting the student's thought processes, particularly for unusual strategies in the Orange Juice task. The expansions of the fourth teacher, Audin's, involved adding terminology or concepts to the discussion.

Curriculum Adaptations

There were differences in the extent to which the task adaptations maintained cognitive demand, differences that were associated with the nature of the evidence used to justify the adaptations. In general, those adaptations that maintained cognitive demand were justified by detailed evidence from past adaptations, as noted in Table 3. Audin and Baldeck adapted 16 tasks, of which 11 were determined to have maintained or enhanced the cognitive demand of the tasks. For example, in a unit on integer addition and subtraction, the adaptations focused student attention on connections between various representations, between operations on integers and the absolute values of the integers, and on the connections between student strategies. For example, one adaptation involved having students record and compare the conjectures that emerged from the introductory tasks to the unit. The teachers helped to guide discussions around the conjectures toward recognizing patterns involving addition and subtraction of integers and the underlying operations related to a number's distance from zero (absolute was not yet formalized), with the ultimate goal of building toward establishing sensible and efficient algorithms.

Table 3: Planned Adaptations

	Number of notable adaptations	Impact on cognitive demand/ coherence with design	Nature of evidence for adaptation
Audin	Six	Five maintained or enhanced demand	Five were justified by detailed accounts from past enactments
Baldeck	Ten	Eight maintained or enhanced demand	Six adaptations were justified by detailed accounts from past enactments; four justified by minimal evidence
Corwin	Four	One maintained demand	Justified by minimal evidence or evaluation of students' ability
Durst	Three	All reduced demand	Justified by minimal evidence or evaluation of students' ability
Knauff	Two	All reduced demand	Justified by minimal evidence or evaluation of students' ability
Walsh	Four	All reduced demand	Justified by minimal evidence or evaluation of students' ability

For the other four teachers, only one of the 13 noted adaptations were determined to have maintained or enhance the cognitive demand, while the other 12 reduced the demand of the tasks. In most cases, the reduction in demand resulted from providing explicit procedures to complete the task or from breaking the task into smaller parts, reducing the ambiguity and opportunity to make connections.

Discussion

There was a strong association between eliciting and successfully organizing discussions around student strategies and the tendency to adapt tasks in ways that maintained the cognitive demand of the tasks. Furthermore, the teachers who adapted tasks productively (maintained demand) cited details of student thinking from past enactments to justify those adaptations. A possible explanation for this finding is that teachers with strong mathematical and pedagogical knowledge can facilitate productive discussions as well as productively adapt curriculum materials. While this is plausible, it is important to note that Audin and Baldeck rarely adapted tasks in the initial implementations of the tasks. On average, the adaptations were designed in the third year they implemented the tasks. Both Audin and Baldeck have discussion intensive classes, as indicated in Table 2. In addition to the data presented in Table 2, the practices of these teachers have been observed over a seven-year span and during that time their classroom discourse practices have consistently focused on student thinking. The data in Table 1 indicate two prominent traits for Audin and Baldeck, the teachers who were most able to elicit student explanations: they persistently probed students' thinking, and they more strategically expanded and recalled student strategies to maintain a coherent mathematical thread in the discussion.

The data suggest that a consistent focus on student thinking, as evidenced by the frequency with which students provided explanations, provides opportunities for teachers to develop a deep understanding of how students engage with mathematical ideas in the enactments of demanding tasks. Furthermore, these opportunities translate into productive adaptations by providing a rationale for deciding when to revise a task as well as guidance for how to revise it. The teachers in whose classes the students less frequently provided explanations often decided to revise a task based on the evaluation that the students could not complete it successfully rather than on a more detailed rationale; in those cases, there was little explanation as to why the adaptation would more successfully engage students with mathematical ideas. Instead, the goal was to make it more likely that the students could complete each part of the task, often in isolation of the other parts.

The results of this study have implications for mathematics educators and for districts who adopt Standards-based programs with the goal of transforming instruction. First, discourse practices play a role for teacher learning by providing opportunities for attentive teaching. These discourse practices include not only persistent efforts to probe student reasoning but also skillful expansion and recall of those strategies to make important connections. However, it is the probing practices that open up the opportunities to learn about student thinking and consequently provide objects for teacher attention.

A second implication is that without explicit attempts by teachers to connect their teaching practices to the ways student engage with mathematical ideas, it appears unlikely that teachers will adapt the materials productively. The teachers in this study who primarily evaluated whether students were able to successfully complete a task were also the ones whose adaptations primarily lowered the cognitive demands of the tasks. These teachers did little to elicit and probe student thinking, and consequently had little evidence on which to guide subsequent adaptations of the materials.

The third implication is that the teachers who engaged in the most complex enactments in terms of eliciting and building from student thinking were also able to productively adapt the tasks. This required a messier and more uncertain environment as the teachers needed to productively guide classroom discussions even as they interpreted student strategies and attempted to engage students in interpreting their peer's strategies. At times this was

cumbersome and slower-paced than the classrooms of the other four teachers and created some concerns about content coverage.

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LINEAR ALGEBRA IN NEW ENVIRONMENTS (LINE)

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We report on some results from a multiyear development of new techniques and materials for teaching linear algebra. Our goals were to (a) to create a professional learning community across STEM disciplines, (b) to combine expertise in content and pedagogy in designing effective instructional practice, and (c) to use learning theories to support the conceptual alignment of content and pedagogical goals. In particular, our approach combines the use of domain-specific problems, APOS learning theory, and the development of the professional learning community. This set of practices was developed and deployed across four diverse institutions, in diverse linear algebra courses, and was effective across this diversity. We discuss the development of teaching materials, and present reflections from students and faculty which were shared during and after these "modules" were used in classes.

Keywords: Advanced Mathematical Thinking, Curriculum, Learning Theory, Post-Secondary Education, Problem Solving

Introduction

Linear algebra undergirds a remarkably diverse array of applications, such as computer graphics, network theory, economic and growth modeling, demographics, and cryptology. Indeed, it is this wide array of applications that makes linear algebra courses so popular among both mathematics and non-mathematics majors. Popular courses do not automatically enjoy effective pedagogy, however; improving the teaching and learning of linear algebra is the subject of ongoing research. Based on some of our prior investigations (Martin, Loch, Cooley, Dexter, and Vidakovic, 2010; Vidakovic, Cooley, Martin, and Meagher, 2008), we hypothesized that linear algebra pedagogy can be most effectively advanced through a strategy based on: (a) creating a professional learning community of STEM faculty dedicated to creating classroom experiences drawing from a diversity of application domains; (b) providing collaborative support, drawing on expertise in both content and pedagogy, for designing effective instructional practice; which, together (c) ensure conceptual alignment between content and pedagogical goals through the use of theories of learning and instructional models.

To test our hypothesis, we formed a community of investigation we call **LI**near algebra in **N**ew **E**nvironments (LINE). While LINE rests primarily on the three planks above, we also envision and support local change toward a reflective, collaborative culture of teaching and learning among STEM discipline faculty. Through our approach to linear algebra pedagogy, we model both (a) the integration of theoretical mathematical content, applications, and learning theories, and (b) co-teaching and collaboration among faculty with expertise in a variety of areas including mathematics, computer science (an applications domain) and mathematics education. We posit that this approach provides an enhanced experience for students, allowing them not

only to achieve a deeper understanding of linear algebra but also to develop their ability to reflect on their own learning.

Background and Theoretical Framework

To connect application-specific content, pedagogical practice, and reflection, the LINE approach relies on an instructional methodology based on the Action-Process-Object-Schema (APOS) framework (Asiala, Brown, DeVries, Dubinsky, Mathews & Thomas, 1996). APOS is a well-developed framework for understanding how students develop mathematical concepts; in addition to a model of cognition, it provides an instructional model intended to facilitate student development of richer and more sophisticated understandings of mathematical concepts. The approach has three components: It rests on a *theoretical analysis* (called a genetic decomposition) of what it means to understand a concept and how learners construct new concepts. This leads to the design of *instructional treatment* focused on these mental constructions. Then, as they teach, instructors *gather data*, which is analyzed in the context of the theoretical perspective and used to revise both theory and instruction, as needed.

The faculty in the LINE project, working as a professional learning community (Hamos, Bergin, Maki, Perez, Prival, Rainey, Rowell & VanderPutten, 2009), began the project with a reading seminar in which the faculty jointly examined mathematics learning theories—including APOS and other theories of scientific learning as well as cognitive difficulties related directly to linear algebra (Sierpinska, Drefus & Hillel, 1999) and collaborated in the initial development of modules for linear algebra using approaches based on APOS. The professional learning community continues to foster faculty development through a combination of activities among the different colleges using videoconferencing and working meetings at each other's institutions. Faculty also co-teach the modules, videotape each other while teaching, observe student engagement, and use these videotapes and observations to reflect on the learning and teaching processes in the context of learning theories.

Sample Module

Due to page restrictions, we discuss two representative "modules" to illustrate LINE principles rather than present them. Sample modules can be accessed from the authors. We chose the first one as an application in linear algebra and the second one as a more conceptual, theoretical module. After each module, we share some observations of the students' learning experiences.

There are four colleges participating in line: an urban public university in a major Northeast city with about 15,000 students, a land grant public university in the Northwest with about 30,000 students, a private college in the Midwest with about 2000 students, and an urban public university in a major Southeast city with about 30,000 students. Across these varying settings, though, the modules developed share important commonalities. In accordance with the line perspective, at least two faculty members developed and implemented the modules at each site. The faculty members work collaboratively at each site, and across sites, to develop the modules and integrate them into classes. Most importantly, the modules share a common aim: to introduce topics in ways that bring students to reflect on what they are learning and to actively readjust their understanding as they are exposed to new concepts, building on what they know and using that knowledge to grasp new ideas.

Sample Module I: Dot Product Application

This module was given to students after they had been introduced to the concept of the dot product and its connection to the cosine of the angle between two vectors. The module uses the

dot product to assess if certain texts were written by a particular author by having students create frequency vectors based on "function" words, such as but, an, the, etc. and to find the similarity between these frequency vectors and those frequency vectors from a known original text by the same author in question.

Observations of Student Learning

The whole class discussed the module after it had been completed. Most students were able to successfully decouple the idea of similarity from magnitude. There was some student debate over the question of which vector would be "least similar" to u, with some students holding that –u was least similar (since it was in the opposite direction), and others holding that the perpendicular was least similar. One of the more important misconceptions that emerged from this discussion was that several students talked about using the dot product to measure the angle of *a* vector, rather than the angle between *two* vectors.

Most students successfully identified that two texts with the same frequency vector would be judged as being similar. However, most felt it sufficient to write an original paragraph and claim that, since they had just written it, it could not be by the same author (thereby missing the point that a third person would be evaluating authorship). Two pairs of students gave viable answers: nonsense paragraphs that consisted of nothing but the frequency words.

Sample Module 2: Linear Maps

Whereas many modules were developed by experienced math educators already conversant with the APOS model, this module was developed by a research mathematician in conjunction with the LINE project researchers. He chose linear maps as one of the central themes of his course and proposed a series of problems that students would work on at the start of the study of that material. Line project researchers helped the instructor refine his problems to a set that would be:

- accessible to students in his class prior to formal instruction in this component of the course;
- aligned with the APOS framework for concept development in mathematics; and
- useful to assess student understanding of the material as a guide to instruction.

The instructor participated in the reading seminar online with the other researchers during the summer prior to the course. He then wrote the module reflecting his new understanding of the APOS framework. His initial design goal was to create a series of problems which would scaffold the students' ability to address problems "that require increasing levels of understanding," moving from simple to complex. As we discussed his design, it was not difficult to relate his complexity-based conception to the hierarchy of the APOS framework. The module began with concrete examples concerning linear maps. It then moved on to proofs requiring students to show that a linear map existed between certain vector spaces.

Observations of Student Learning

Some students said that the proof techniques they developed while working on modules carried over into other upper level math courses, such as abstract algebra. Others said the modules helped them think abstractly. Several noted that they liked the concrete examples at the beginning of a module and how they would tie in with the more abstract questions later. They also found the whole-class discussion provided them with significant feedback that comments written on an assignment could not provide. They expressed confidence in their understanding of linear algebra topics and said their knowledge had been enhanced and deepened by the modules.

Students performed quite well on this module, which was the second completed during the semester. As would be expected, the first two questions were correctly answered by most students, any difficulties were with the last two questions. Individual student recollections about the course and this assignment after the end of the semester appeared to match well with their

performance during the course. For example, when asked to identify three important concepts in the course a successful student said, "Linear transformations form the basis for a large portion of linear algebra, and since linear algebra overlaps with a lot of other fields (both with math and outside it), and knowing basics like this are important." When asked whether he could still answer these problems, he responded, "Without my book or another reference, I'm sure I could do 1 and 2, and I could probably start the rest of them. If I were to use references though I'm confident that I could complete all the problems, although it would probably take me longer now than it would have at the end of last semester because I'd have to do some review." He concluded, "The assignments were generally helpful to learning the material."

Conclusion

Our final goal was to ensure conceptual alignment between content and pedagogical goals through the use of theories of learning, with an emphasis on APOS, and instructional models. We believe we substantially met this goal, and that it produced noticeable positive effects on student learning: students, prompted to reflect on their learning, volunteered their beliefs that the modules deepened their conceptual understanding.

This project has been more successful than our previous efforts; whereas before we attempted to create this conceptual alignment by offering parallel courses in math content and learning theory, now we focus on collaboration among faculty. The parallel courses didn't work as well, mainly because students didn't have time in their schedules for an extra class on learning theory. By concentrating on infusing learning theory into instructional practice, we were able to convey some of these ideas implicitly to students. Collaboration between mathematicians and math educators has been extremely valuable in rethinking instruction in upper-level math classes.

This approach appears to be surprisingly universal: the four institutional settings are quite different—in size, location, student population, and curriculum—but this mode of instruction has led to changes in all of them. Finally, this approach to designing instructional material is flexible. Not every module drew equally from the LINE principles; some emphasize applications, some were technologically enhanced, and some were like carefully-structured labs based on a text.

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A CHARACTERIZATION OF TEACHERS' IMPLEMENTATIONS OF A MATHEMATICAL DECISION-MAKING CURRICULUM

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The purpose of this study was to examine teachers' implementation of a specific mathematics curriculum and how their conceptions influenced this implementation. An innovative, powerful tool, known as "teacher logs," was used to examine teachers' implementations of the curriculum; a survey was used to study teachers' conceptions related to their implementations. Results suggest that teachers tend to implement the materials closely to the written text, but there was variety shown both within and among teachers. A relationship was shown between teachers' implementation levels and ***their*** *beliefs about mathematics and their beliefs about students. The results of this study reveal that teachers may or may not stray from the authors' intended curriculum, and whether or not they stray often relates to their conceptions.*

Keywords: Curriculum, Curriculum Analysis, Teacher Beliefs

When new curricula are developed, researchers, policy makers, and teachers often ask, "Do these materials work? Are these materials effective?" As a result, a committee was charged by the National Research Council (NRC) to determine the quality of curriculum evaluations. In their report, *On Evaluating Curricular Effectiveness* (NRC, 2004), the committee defined curricular effectiveness as, "how effective a particular curriculum is, and for whom and under what conditions it is effective" (p. 1). When determining curricular effectiveness, one important part to consider is teachers' faithfulness to the curriculum authors' intentions because if teachers do not implement the materials as the authors intended, they may undermine the effectiveness of the new curriculum. Specifically, implementation fidelity needs to be considered. Holstein, Dietz, and Keene (2010) defined implementation fidelity as "the extent to which a teacher implements a curriculum as the author intended, allowing for adaptation, supplementation, and improvisation by the teacher" (p. 2). The concept of implementation fidelity is not equivalent to teacher quality. That is, low implementation fidelity does not suggest poor teaching. Rather, researchers study implementation fidelity to determine the extent to which the teacher implemented the materials. This information can then be used, in part, to determine curricular effectiveness.

Recent research on implementation fidelity has emerged partly because teachers often transform the written curriculum into a form completely different from the authors' original intentions. One may ask, "Why does this transformation occur?" One explanation is teachers' conceptions (e.g., about math, teaching, students, or curricula) influence their instructional decisions. Therefore, when examining implementation fidelity, it is important to consider teachers' conceptions. We use the word "conception" in the way Lloyd and Wilson (1998) used it: "to refer to a person's general mental structures that encompass knowledge, beliefs, understandings, preferences, and views" (p. 249)

Context and Purpose of the Study

The context for this study is the NSF-funded Mathematics for Decision-Making (MDM) curriculum (Project DRL-0733137). The emphasis of this curriculum is on decision-making using mathematical models based on engineering concepts. The text contains large real-world

problem contexts that require multiple steps to solve and interpretation of the solution. The curriculum is intended for high school seniors who have completed two years of algebra and one year of geometry (or the equivalent).

The purpose of this study was to examine teachers' implementations of the MDM curriculum and how their conceptions influence their implementations. In particular, we studied how closely teachers' enacted curriculum aligned with the authors' intended curriculum. We also investigated teachers' conceptions that could bring their enacted curriculum closer or farther from the authors' intended curriculum. The particular conceptions we considered were teachers' (a) subject matter knowledge, (b) beliefs about mathematics, (c) beliefs about students, (d) beliefs about teaching, and (e) beliefs about curricular materials.

The research questions were as follows:

1. How do teachers' implementations of a mathematical decision-making curriculum align with the authors' intentions?
2. How do teachers' conceptions influence their implementations of a mathematical decision-making curriculum?

Research Background

The NRC (2004) developed a framework to define "what is meant by a scientifically valid evaluation study for reviewing mathematics curriculum effectiveness" (p. 36). It included three major components for curriculum evaluations: "(1) the program materials and design principles; (2) the quality, extent, and means of curricular implementation; and (3) the quality, breadth, type, and distribution of outcomes of student learning over time" (p. 4). Scientifically valid evaluation studies on curricular effectiveness should include all the components of this framework. In the present study, we examined the "Implementation Components" (the second major component) piece of the framework, with the understanding that future studies should include the other components of the framework to determine the overall effectiveness of the MDM curriculum.

One way to study the implementation components of a curriculum is to examine how the curriculum transforms from the written materials to what students actually experience in the classroom. Stein, Remillard, and Smith (2007) provided a framework for studying the transitions of the curriculum from its written form to its enacted form, as shown in Figure 1.

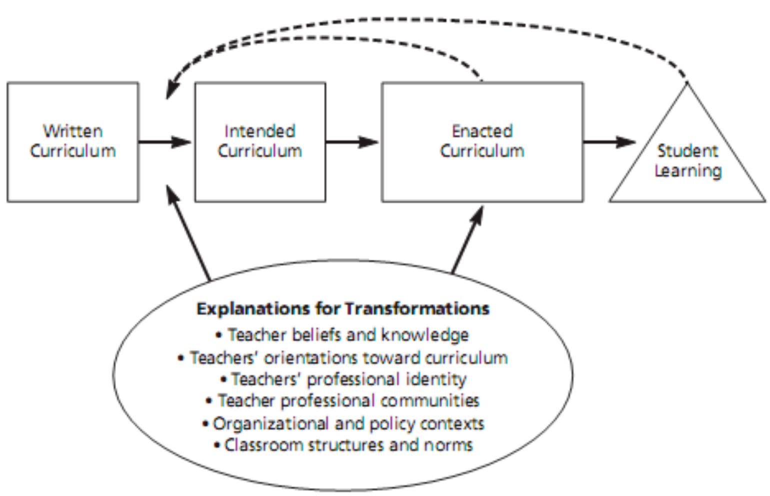

Figure 1: Temporal Phases of Curriculum Use (Stein et al., 2007, p. 322)

The curriculum begins as a written curriculum, which refers to the authors' original printed materials. Then, the curriculum changes into what the teacher intends to teach. This initial transformation occurs due to teachers' conceptions, goals, and contexts, as shown in the oval in Figure 1. Next, the curriculum transforms again into the enacted curriculum, which is the classroom implementation of curriculum materials. The enacted curriculum is also influenced by outside factors. Finally, student learning occurs. Therefore, it is unreasonable to measure the connection between the written curriculum and student achievement without considering the transformations of the curriculum. In other words, to determine whether a curriculum is effective, one must study whether the teacher is using the curriculum as intended by the curriculum authors.

To capture the implementation and to connect it to the authors' intentions, Heck, Chval, Weiss, and Ziebarth (2012) created two constructs: (a) the *mathematical storyline*, which refers to "which topics are addressed, at what depth different topics are addressed, and/or what learning goals and student expectations are pursued" (p. 69) and (b) the *pedagogical storyline*, which includes "which activities are conducted and how they are conducted in terms of openness or guidance, how students are grouped for instructional activities, and whether the patterns of interaction that are designed are actually pursued" (p. 70). These two constructs guided the methodology of this study (described below).

Moreover, one reason the transformation from written curriculum to enacted curriculum occurs is teachers' conceptions. Many conceptions influence teachers' instructional decisions and thus their implementation fidelity. Five conceptions which consistently emerged in the literature as influencing teachers' instructional decisions were examined in this study: (a) subject matter knowledge (e.g., Manouchehri & Goodman, 1998), (b) beliefs about math (e.g., Collopy, 2003), (c) beliefs about teaching (e.g., Stipek, Givvin, Salmon, & MacGyvers, 2001), (d) beliefs about students (e.g., Arbaugh, Lannin, Jones, & Park-Rogers, 2006), and (e) beliefs about curricular materials (e.g., Frykholm, 2004).

Methodology

In this paper, we present the quantitative portion of a mixed-methods study (see Holstein, 2012). To answer the first research question, an innovative tool called "teacher logs" was used. To answer the second research question, a survey was used, where the survey results were compared to the teacher log results. The participants came from the population of high school mathematics teachers who attended an MDM workshop ($N = 232$). Thirteen teachers completed at least one teacher log; 12 teachers completed the survey *and* at least one teacher log.

Teacher Logs

Huntley (2009) stated, "If we truly want to understand the relationships among textbooks, teaching, and student learning, it is critically important to develop curriculum-sensitive measures of implementation" (p. 357). In this study, we used teacher logs (Heck et al., 2012) as our "curriculum-sensitive measures of implementation." Teacher logs were developed by the first author of this paper for five chapters in the MDM curriculum (one teacher log per problem context and approximately three problem contexts per chapter, for a total of 16 logs). Each log contained a list of items for teachers to choose from, where each item illustrated an instructional choice. Every effort was made to include all possible implementation choices. Teachers checked off items that they did during their implementation of the curriculum, where each item (a) supported the authors' intentions, (b) went against the authors' intentions, or (c) was neutral. For example, the MDM curriculum authors included several in-text questions for students to answer themselves, either individually, in groups, or as a class. Therefore, the following item went

against the MDM authors' intentions: "I led a class discussion that covered the information out of in-text questions #1-2 (but did not explicitly ask these questions)." On the other hand, the following item supported the authors' intentions: "Students read and answered in-text questions #1-2 out loud or individually."

Teachers received four scores for each log: (a) a mathematical adherence score, (b) a mathematical focus score, (c) a pedagogical adherence score, and (d) a pedagogical focus score. The two mathematical scores were based on items that either supported or went against the curriculum authors' mathematical storyline; the two pedagogical scores were based on items that either supported or went against the curriculum authors' pedagogical storyline.

The mathematical and pedagogical *adherence* scores refer to how the teacher supported the authors' intentions. These scores were calculated using the following ratio, which yielded a score between 0 and 1:

$$\frac{\text{Number of fidelity elements enacted}}{\text{Total number of fidelity elements}}$$

A high mathematical adherence score means that the teacher addressed most of the content of the curriculum, while a low mathematical adherence score means that the teacher addressed little of the content of the curriculum. A high pedagogical adherence score means that the teacher addressed most of the instructional strategies of the curriculum, while a low pedagogical adherence score means that the teacher addressed little of the instructional strategies.

The mathematical and pedagogical *focus* scores took into account the fidelity elements as well as the items that went against the authors' intentions. These scores were calculated using the following ratio, which yielded a score between 0 and 1:

$$\frac{1 + \frac{\text{Number of fidelity elements enacted}}{\text{Total number of fidelity elements}} - \frac{\text{Number of other elements enacted}}{\text{Total number of other elements}}}{2}$$

A high mathematical focus score means that the teacher addressed more of the content of the MDM curriculum than other content, while a low mathematical focus score means that the teacher addressed less of the content of the MDM curriculum than other content. A high pedagogical focus score means that the teacher addressed more of the instructional strategies of the MDM curriculum than other instructional strategies, while a low pedagogical focus score means that the teacher addressed less of the instructional strategies of the MDM curriculum than other instructional strategies. Notice that the adherence score is one piece of the focus score.

Each teacher's mathematical adherence, mathematical focus, pedagogical adherence, and pedagogical focus scores were averaged. In the end, each teacher had four scores, regardless of how many logs they completed. These scores informed us about teachers' levels of implementation fidelity.

Survey

The purpose of the survey was to gather data on teachers' conceptions. The survey items were adapted from previously validated surveys, where each item measured one of the five conceptions included in this study: (a) beliefs about curriculum (items adapted from Spielman & Lloyd, 2004), (b) beliefs about math (items adapted from Fennema & Sherman, 1976; Perry, Howard, & Tracey, 1999; Suinn, 1988), (c) beliefs about teaching (items adapted from Perry et al., 1999; Spielman & Lloyd, 2004), (d) beliefs about students (items adapted from Fennema & Sherman, 1976; Suinn, 1988), and (e) subject matter knowledge (items heavily adapted from McDiarmid & Wilson, 1991). Note that the "beliefs about curriculum" survey items asked teachers about their beliefs about general curricula and were not specific to the MDM

curriculum; the "beliefs about math" survey items portrayed mathematics on a traditional to nontraditional continuum (Raymond, 1997). All items were on a five-point Likert scale, with the exception of the subject matter knowledge items, which were multiple choice.

Teachers received five scores on the survey (one for each conception). Then, a Spearman's Rank Order correlation was conducted to determine any correlations between the five survey scores and the four teacher log scores.

Results

In this section, we present the results for the two research questions. First, the teacher log results show teachers' implementation fidelity levels during their implementations of the MDM curriculum. Second, the correlations between the teacher log results and the survey results are offered to illustrate teachers' implementation fidelity levels related to their conceptions.

Implementation Fidelity

Averages for the four teacher log scores were calculated for each of the 13 teachers, regardless of the number of logs completed. Table 1 shows the overall results from these logs. In this table, the average score and standard deviation are given for each teacher for each of the four scores. This table also gives the following overall data: (a) the average of the teacher averages, found by averaging the average scores for the 13 teachers; (b) the standard deviation of the teacher averages, found by calculating the standard deviation of the average scores for the 13 teachers; (c) the minimum teacher average score; and (d) the maximum teacher average score.

Table 1: Teacher Log Results

Teacher Number	Number of Completed Logs	Math Adherence		Math Focus		Pedagogical Adherence		Pedagogical Focus	
		Score	SD	Score	SD	Score	SD	Score	SD
T1	12	0.5822	(.1529)	0.7179	(.1238)	0.5451	(.2428)	0.6867	(.1640)
T2	7	0.6638	(.1933)	0.8149	(.0999)	0.7794	(.1969)	0.8897	(.0985)
T3	4	0.7572	(.1478)	0.7328	(.0898)	0.8333	(.2357)	0.8479	(.0851)
T4	1	0.9375	(--)	0.9688	(--)	0.8333	(--)	0.9167	(--)
T5	4	0.9480	(.0392)	0.8832	(.1078)	0.8294	(.2243)	0.8678	(.0884)
T6	1	0.8889	(--)	0.9444	(--)	1	(--)	1	(--)
T7	8	0.7284	(.1466)	0.7749	(.1398)	0.6217	(.2033)	0.6949	(.1829)
T8	7	0.7215	(.1609)	0.7307	(.1434)	0.6643	(.3473)	0.7388	(.2407)
T9	2	0.5729	(.3388)	0.7388	(.2368)	0.5	(.2357)	0.6688	(.0913)
T10	1	0.9444	(--)	0.7103	(--)	0.8889	(--)	0.9444	(--)
T11	4	0.8547	(.1219)	0.8023	(.0916)	0.9069	(.1114)	0.8991	(.0837)
T12	2	0.9444	(.0786)	0.7961	(.1116)	1	(0)	1	(0)
T13	1	0.9444	(--)	0.6627	(--)	0.5556	(--)	0.7465	(--)
Average of Teacher Averages		0.8068		0.7906		0.7660		0.8386	
SD of Teacher Averages		0.1427		0.0925		0.1714		0.1182	
Minimum		0.5729		0.6627		0.5		0.6688	
Maximum		0.9480		0.9688		1		1	

To help make sense of these data, consider two teachers in Table 1 (recall that all scores were between 0 and 1). First, T1 adhered to only about half of the mathematical and pedagogical

intentions (mathematical adherence score of 0.5822 and pedagogical adherence score of 0.5451), but this teacher tended to use the MDM content and instructional strategies more than *other* content or instructional strategies (mathematical focus score of 0.7179 and pedagogical focus score of 0.6867). Second, T3 had relatively high scores, but the pedagogical scores (0.8333 and 0.8479) were higher than the mathematical scores (0.7572 and 0.7328). Thus, this teacher tended to be slightly less faithful mathematically than pedagogically.

When looking at the overall averages, the pedagogical adherence had the lowest average (0.766); this means that teachers did not necessarily use the instructional strategies intended by the authors and that they only used approximately three quarters of the instructional strategies presented on the teacher logs. However, the pedagogical focus score had the highest average (0.8386); thus, teachers used *other* instructional strategies less than they used the strategies intended by the MDM curriculum authors. The individual teachers' standard deviation values show variety within teachers; the overall minimum, maximum, and standard deviation values show that there was variety among teachers.

Conceptions Related to Teachers' Implementation Fidelity

Each teacher received a score for each construct based on the five conceptions (beliefs about curriculum, beliefs about math, beliefs about teaching, beliefs about students, and subject matter knowledge). From these data, a Spearman's Rank Order correlation was conducted to examine correlations between the teachers' implementation fidelity scores on the teacher logs and their conception scores on the survey. Table 2 shows the results from this analysis, including the correlation coefficients and the p-values (in parentheses).

Table 2: Correlations between Implementation Fidelity and Conceptions

	Beliefs about Curriculum	Beliefs about Math	Beliefs about Teaching	Beliefs about Students	Subject Matter Knowledge
Mathematical Adherence	.194 (.545)	.629** (.028)	.072 (.824)	.127 (.694)	.051 (.875)
Mathematical Focus	.111 (.730)	.004 (.991)	-.046 (.888)	.077 (.811)	-.462 (.130)
Pedagogical Adherence	.301 (.342)	.327 (.299)	-.231 (.471)	.659** (.020)	-.268 (.399)
Pedagogical Focus	.426 (.167)	.357 (.255)	.054 (.866)	.536* (.072)	-.317 (.315)

*$p < .1$. **$p < .05$.

The test revealed that there was a statistically significant correlation ($p < .1$) between mathematical adherence and beliefs about math, between pedagogical adherence and beliefs about students, and between pedagogical focus and beliefs about students. All other relationships were not significant.

The results for each conception are discussed here. First, there were no significant correlations between teachers' beliefs about general curricula and their teacher log scores. Second, teachers' beliefs about math were significantly positively correlated only with their mathematical adherence scores. That is, teachers with more nontraditional views of math tended to adhere to the authors' mathematical storyline. Third, teachers' beliefs about students were significantly positively correlated with their pedagogical adherence and focus scores. That is,

teachers with positive views of students and their abilities tended to enact most of the instructional strategies of the MDM curriculum (high pedagogical adherence score) and to enact more of the instructional strategies of the MDM curriculum than *other* strategies (high pedagogical focus score). Fourth, there were no significant correlations between teachers' beliefs about teaching on the survey and their teacher log scores. Fifth, there were no significant correlations between teachers' subject matter knowledge on the survey and their teacher log scores, but lack of subject matter knowledge was nearly negatively correlated with teachers' mathematical focus scores ($p = .13$). This means that teachers who lacked subject matter knowledge sometimes used the MDM content more than other content.

Conclusion

To examine teachers' implementation fidelity to the MDM curriculum, a new tool, teacher logs, were used. These results showed that teachers were generally faithful to the authors' mathematical and pedagogical storylines. This is not surprising as these teachers volunteered to implement the MDM curriculum and to take part in the study. However, there was variety shown both within and among the teachers. These results support previous research on implementation fidelity (e.g., Crawford, Carpenter, Wilson, Schmeister, & McDonald, 2012; Remillard & Bryans, 2004; Stein & Kaufman, 2010), and we extended this research by showing results in the context of the MDM curriculum.

To investigate the connection between teachers' conceptions and their implementation fidelity of the MDM curriculum, a survey was used, where the survey results were compared to the teacher log results. These results showed that teachers' beliefs about math were significantly positively correlated with their mathematical adherence; thus, teachers with nontraditional views of math tended to implement the content of the MDM curriculum—a nontraditional curriculum—with high fidelity. According to previous research, how teachers view the nature of mathematics, its origins, and its usefulness influence their curricular decisions (e.g., Collopy, 2003). Stein et al. (2007) explained that sometimes curricular materials "offer views of mathematics that conflict with those typically held in mainstream culture" (p. 353). Thus, teachers may stray from the written curriculum if the goals do not align with their beliefs about math. The results of this study confirmed that if the goals align, adherence is stronger.

In addition, the results showed that teachers' beliefs about students were significantly positively correlated with their pedagogical adherence and pedagogical focus scores; that is, teachers with positive beliefs about students and their abilities tended to support and to not go against the authors' pedagogical storyline. Alternatively, teachers with negative beliefs about students and their abilities tended to have lower pedagogical adherence and focus scores. Past researchers showed how teachers' beliefs about students play a role in their instructional decisions (e.g., Arbaugh et al., 2006; Remillard & Bryans, 2004). In particular, Arbaugh et al. (2006) found results similar to those in the present study, where teachers with negative beliefs about students and their abilities often strayed from the authors' intentions. However, many teachers in their study with such beliefs strayed from the intended *content* while the MDM teachers primarily strayed from the intended *instructional strategies*. Thus, the results from the present study complement and add to the research connecting teachers' beliefs about students to their implementation fidelity.

The results of this study have implications for many. For example, curriculum developers need to be clear about intentions and provide support for teachers. Teacher professional developers need to be aware of how teachers implement curricula and the conceptions that influence their implementations. Finally, future researchers should utilize the new, innovative

tool from this study: teacher logs. This tool is a powerful way to quantitatively study implementation fidelity.

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TEACHERS' DECISIONS ON TASK ENACTMENT AND OPPORTUNITIES FOR STUDENTS TO LEARN

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Drawing on data from teachers using various elementary mathematics curriculum programs, this study highlights prevalent decisions teachers make to enact the task from the written curriculum and the potential of such decisions for student learning. In doing so, we examine teachers' instructional design decisions, especially those that teachers make when they face things that are not specified in the curriculum and when they see the need to fill in a gap in the teacher's guide in order to enact the task in their classrooms. We also examine whether such decisions can actually create opportunities for students to learn.

Keywords: Curriculum, Instructional Activities and Practices, Curriculum Analysis

Teachers make various *design decisions* when they use curriculum to plan and enact a lesson. First, they need to decide whether to use the task in the curriculum and, if so, how to use it. The curriculum usually includes various kinds of information regarding how to enact the task, such as questions to ask; representations, models, and strategies to use; a set of components of the task; and statements to make. Teachers decide whether to use, modify, or omit each of these provided in the curriculum. We call design decisions involved in this process *fidelity decisions*. Second, teachers need to determine a number of things to enact the task after fidelity decisions are made. Such decisions, which we call *enactment decisions*, need to be made especially when modifications are made or when the curriculum does not address things that may occur in a variety of situations during the enactment of the task. Even though we distinguish the two kinds of design decisions, they are closely related: fidelity decisions affect enactment decisions and vice versa. Sometimes one teacher decision can be both a fidelity and enactment decision (e.g., modifying a question suggested in the curriculum to build on previous learning). It is important to investigate teacher design decisions in terms of the two kinds of decisions and the relationship between them.

As a way to explore teacher design decisions, in this study we examine enactment decisions that teachers make along with fidelity decisions, especially when they face things that are not specified in the curriculum and when they see the need to fill in a gap in the teacher's guide in order to enact the task in their classrooms. We also examine whether such design decisions can actually create and increase opportunities for students to learn, given that teachers take specific actions to help students reach the goals they set for the students. Teacher enactment decisions can maximize the impact of the task on student learning and also can limit the impact, even though teachers have goals for student learning.

Theoretical Foundations

Teachers' ways of using curriculum materials are very diverse (Remillard, 2005). All teacher decisions involved in curriculum use, however, require determining whether to use the curriculum, how to use it, and to what extent. Often the term fidelity of implementation is used to help explain and elaborate on curriculum use. Fidelity of implementation has been investigated

from different perspectives, such as the philosophy and pedagogy of curriculum materials (Chval, Chávez, Reys, & Tarr, 2009), adaptations that maintain curriculum designers' original intent (Huntley, 2012), content coverage and presentation of content (Tarr, McNaught, & Grouws, 2012), the extent of coverage of curriculum materials (Tarr, Chávez, Reys, & Reys, 2006), and literal lessons versus intended lessons (Brown, Pitvorec, Ditto, & Kelso, 2008). Yet, investigations on specific decisions teachers make in individual lessons (e.g., whether to use particular questions and prompts suggested in the curriculum) have rarely been conducted, nor, in particular, how such fidelity decisions shape the enactment of the lesson.

The National Research Council (NRC, 2001) points out that opportunity to learn (OTL) is "the single important predictor of student achievement" (p. 334). Hiebert and Grouws (2007) explain that OTL depends on both teacher and curriculum materials. They further argue that creating moments in classrooms where students learn goes beyond exposing them to subject matter and learning goals. Stein, Remillard, and Smith (2007) argue that curriculum materials can influence students' learning, as they may contain mathematical tasks with cognitive demand that requires reasoning and problem solving. However, whether the task is used as intended depends on the teacher. Stein, Grover, and Henningsen (1996) find that even with tasks that require high cognitive demand, teacher actions can deprive students of opportunities to reason and think mathematically. Also, teachers may eventually change goals and mathematical points of the task (Sleep, 2012). These arguments may indicate that even though both curriculum materials and teachers are significant in creating opportunities for students to learn, the teacher's role seems even more critical. Many factors that can help create opportunities for the student to learn may be present in curriculum materials, and yet they may also lie fallow if not deliberately pursued by teachers during the enactment of lessons. Potential reasons may be partly because teachers make poor fidelity decisions, and partly because teachers need to decide a number of additional things on their own besides those specified in the written materials and tasks.

Research literature is replete with studies that inform us of modifications teachers make when engaging with mathematical tasks (e.g., Brown, 2009; Brown, Pitvorec, Ditto, & Kelso, 2008; Lloyd, 2008; Remillard & Bryans, 2004). Only a few studies, however, attempted to articulate teacher design decisions; even fewer studies examined the connections among modifications teachers make to the task (fidelity decisions), specific things teachers do to enact the task (enactment decisions), and kinds of learning opportunities students may experience (OTL). This study focuses on enactment decisions along with fidelity decisions and the potential of such decisions for student learning; in order to investigate enactment decisions, it is necessary to find out what fidelity decisions are made. In particular, questions that guide our study are: What enactment decisions do teachers make when they decide to use a task from the curriculum? In what ways do such decisions help create or increase OTL?

Methodology

This study is part of a larger study investigating teacher curriculum use – *Improving Curriculum Use for Better Teaching* (ICUBiT) project. The data analyzed in this study were drawn from this larger study.

Teacher participants and curriculum programs. Data were gathered from teachers in grades 3-5 using five different curriculum programs: (a) *Investigations in Number, Data, and Space* (INV), (b) *Everyday Mathematics* (EM), (c) *Math Trailblazers* (MTB), (d) Scott Foresman–Addition Wesley *Mathematics* (SFAW), and (e) *Math in Focus* (MiF). The first three were NSF-funded programs; the fourth was commercially developed; the fifth was originally

from Singapore and has gained popularity in the U.S. over recent years. We used a range of programs because we wanted to investigate teachers' approaches in using curriculum in general, not specific to particular types of curriculum, and what it takes to use curriculum programs effectively, regardless of types of programs. The participant teachers had at least three years of teaching experience and at least two years of using the same curriculum program. We intentionally recruited teachers with such experience because our aim was to investigate teachers' ways of using curriculum materials, not their learning to use curriculum materials, which is typical in novice teachers and those who are beginning to use a new program. This study drew on data from five teachers, one teacher per curriculum.

Data sources. The data we used in this study include classroom observations, teacher interviews (introductory and post-observation), and Curriculum Reading Logs (CRLs). Each teacher completed CRLs for each lesson that was observed: on a copy of the written lesson, the teacher indicated which parts they read as they planned instruction, which parts they planned to use, and which parts that influenced their planning, in yellow, blue, and orange highlighters, respectively. CRLs helped the researchers see plans for instruction and compare written and enacted lessons. Each teacher was observed for three consecutive lessons in each of two rounds. These enacted lessons were videotaped and transcribed. Also, each teacher was asked questions about his/her teaching experience and overall curriculum use at the beginning of the study, and then asked about specific teacher design decisions in the observed lessons after each round of three observations. These interviews were audiotaped and transcribed.

Data analysis. The main part of the data analysis was coding teacher fidelity and enactment decisions to examine the kinds of things teachers did that were not specified in the curriculum and the things that they did differently from the curriculum. We focused on one lesson per teacher, and the content of the observed lessons was primarily number and operations. First, we chunked Written (W) and Enacted (E) tasks using CRLs and videotaped lessons, and created lesson analysis tables that included W- and E-tasks side-by-side. We defined a task as a chunk of activity (including teacher and student activity) aiming at an apparent distinct goal or product. In this way, we examined each lesson in smaller chunks, rather than as a whole. This was because teacher goals, actions, and decisions may vary depending on the task. In each pair of W- and E-tasks, we identified teacher fidelity decisions (task components, questions, statements, representations, strategies, participation structure, time allocation, etc.)—whether each of these was used as recommended in the curriculum, changed, or omitted, or whether any new things were added. Also, we constantly compared the W- and E-tasks to identify enactment decisions by examining what teachers used from the curriculum, what changes they made, and what they did when things were not clearly addressed in the curriculum or when they filled in the W-task in order to enact it. Once each task was analyzed, we compared teacher actions across E-tasks, first within teacher and then between teachers, to develop preliminary codes and refined them as more tasks were analyzed. We did not include tasks that teachers adopted from outside the program they used, because of the focus of the study. Teacher interviews were analyzed to see teacher intention behind their decisions. After examining individual teachers, we searched for patterns in teacher enactment decisions and their potential impact on student learning.

Results

We found prevalent patterns in enactment decisions that the five teachers made. Regardless of the types of programs they used, they exhibited surprisingly similar patterns. W-tasks from EM and SFAW that included minimal directions and guidance regarding how to enact the tasks

and what to do left plenty of room for decisions that EM and SFAW teachers had to make. Even though MTB provided the most detailed guidance for teachers among the five programs, the MTB teacher did a number of things to fill in the gap in the W-tasks to enact them in her classroom. First, we describe the kinds of common enactment decisions and then discuss the potential of these decisions for student learning.

Teacher Enactment Decisions

When teachers used W-tasks to design a lesson, certainly there existed some uncertainty in those tasks. It was found that a number of things were not clearly addressed in the W-tasks, which were up to teachers' own decisions depending on their circumstances. Most of these decisions were intentional with particular goals in mind. The teachers clearly intended to help students get at the mathematics that they thought their students needed to learn from each lesson. To accomplish this goal, they attempted various actions that were not specified in the W-tasks: (a) having an extended launch of the task (including preview of the task, examples, detailed directions, etc.); (b) posing questions and prompts that students need to think about in order to do the task; (c) using additional representations or tools to help students understand something related to the task or to do the task; (d) making directions clear and giving specific guidance to help students complete the task; (e) providing modeling, demonstration, and explanation; (f) tailoring the task in a way that fits the classroom situation (or in the way that the teacher desires); (g) adding a new component to the task (e.g., summary or wrap-up discussion at the end of the task); (h) reviewing previous learning in relation to the task and in response to student thinking; and (i) connecting the task to previous/future learning. Table 1 includes examples of these teacher actions from an E-task of the MTB teacher's lesson. Note that having an extended launch requires many other specific decisions that are included in the table. Also note that these decisions are not mutually exclusive and there are some overlaps among them. In fact, when coding teacher decisions, we found one teacher action could involve multiple decisions. For example, ETA 1 in Table 1 not only helped students think about line of symmetry, but also led to review the concept they learned previously so students could use the concept to do the task.

Each of the five teachers tends to have an extended launch, especially for the main tasks of their lessons. For example, the MTB teacher and EM teacher spend 20 minutes and 25 minutes, respectively, for the launch of one main task they enacted. Interestingly, the W-task from MTB offers very specific guidance, whereas the W-task from EM provides minimal guidance with a bulleted list of three main steps and a couple of additional sentences before and after this list. Yet, both teachers spend a significant portion of their E-task helping students start the task. In such an extended launch, the teachers do a range of things, illustrated in the examples in Table 1. They want to provide very specific directions regarding what to do and how to complete the task. The teachers also ask questions and prompts for students to think about prior to working on the task. In this process, if opportunities come up, the teachers review mathematics concepts and ideas students have already learned that will be used in some way to do the task. They also model or demonstrate strategies, some small steps, or particular ways that are useful or efficient. Sometimes they add new components to the task. For example, the EM teacher provides a preview of the task that helps students see the components and expectations of the task and prepare to start. These are many more details than a curriculum could possibly include. In a sense, curriculum designers assume such teacher decisions and provide potentially necessary information and support for lesson enactment. However, certainly ambiguities exist in each curriculum. The more ambiguities, the more teachers' own decisions are needed to sort out such ambiguities and fill in the curriculum.

Table 1: Teacher Enactment Decisions

Decisions	Example teacher actions (ETA)
Pose questions and prompts students need to think about to do the task	1. How do we find lines of symmetry in general? 2. What else might you trace that would help you to keep track of things? 3. We're going to be using large numbers like in the thousands for addition; anyone have an idea on how we can do this?
Use additional representations/tools	4. If I give you something like this [a base-ten chart]. …… You do not need to use this chart. You can use this chart in your own way. You can do something else than what I have just suggested on the board there.
Make directions clear and give specific guidance to do the task	5. Do you think that there is something about that coat … that would help you be even more efficient than covering the whole coat? Is there something that we know about math that would help us be more efficient than covering up the entire coat? …… We're just going to do the one side, so we'd double it. Good, good.
Provide modeling, demonstration, and explanation	6. Ok, so you can fold your large piece of paper and again it's not going to be exact. You can also eyeball it. … What about if you wanted to, you have your neckline here, what could you, what tool could we use to find out where the middle was there? …… Right, you can measure this from here to here and you might find it's 10 inches and take the middle of, yeah! measure it from here to here.
Tailor the task in a way that fits in the classroom situation (or the way that the teacher desires)	7. We have flats, and then we have skinnies in the bottom. What would you use first? …… with all the bits out and count them up. Is that going to be the most efficient? What are we going to use first? …… This inside of the 2D object. Inside, cover the area. …… First you're going to start off with flats. Then what might we use next?
Add new components to the task	8. I would like to hear from just one or two people, something that you learned today that you didn't know.
Review concept previously learned in relation to the task or student responses	9. So, what is this line called? I am sure you have seen it. We have talked a little bit about it this year. 10. Finding the area, one example would be to find the area of the rug. So what is area? 11. How do we find lines of symmetry in general? …… You can fold it in half.
Connecting to previous/future learning	12. We're going to be doing addition and subtraction, or at least addition with these base 10 pieces. … And, you're going to be able to not just understand how to add or subtract, because you have been doing that for a long time. But we are going to get into some larger numbers. 13. Is there something that we know about math that would help us be more efficient than covering up the entire coat? And I am just going to put in your mind, it's one of the techniques that you use for Joe the Goldfish.

Now we will explain these various enactment decisions by using examples from the MTB teacher's lesson in Table 1. In a W-task from MTB, students are asked to find the area of a coat, using base-ten blocks (bits – ones, skinnies – tens, and flats – hundreds). The context given is that students are asked to find out how much material is needed to cover the front of the coat that will be worn at the school play. Students are expected to discuss specific procedures (e.g., tracing the outline of the coat, and covering half of the front and use that to find the area of the whole thing) and practical points (e.g., the coat should be zipped). The W-task also includes a note that some measurement error can occur. Students are expected to measure the area of their coat in groups of two or three. The written lesson containing this task is for two to three estimated class sessions. The guidance for enactment is provided in much detail. The E-task observed ends before students share the areas of the coats they measured and compare resulting 4-digit numbers (areas), which is done on the following day.

Even with the detailed guidance (see TIMS Project University of Illinois at Chicago, 2008, pp. 36-37), as shown in Table 1, the MTB teacher makes various enactment decisions to use the task in her classroom. She makes prompts very specific to help students work on the task in a desired way. It is efficient to measure only half of the coat and double it, and this strategy was mentioned in the W-task. The teacher not only brings up this strategy, but also makes sure students can use this strategy effectively by connecting a strategy students used in a previous task to this new task (ETA 13); posing questions to encourage students to think about the concept that is useful to do the task (ETA 1); reviewing the concept, line of symmetry, and how to find it (ETAs 9 and 11); providing demonstrations of how to find the line of symmetry (ETA 6); and giving specific directions on how to use the symmetry of a coat to do the task, i.e., find half the area of a coat and double it (ETA 4). She even has students consider a particular measure of the neckline (10 cm) and demonstrates what can be done using the outline of a coat on the board. One issue students may have in the task is keeping track of their record. When they place base-ten blocks to measure the area of a coat, the blocks can be easily moved from the correct place, which can make it hard to accomplish the task. In the W-task, it is recommended that students "trace the outlines of the base-ten pieces on the picture of their coats and figure out the area afterwards" (p. 37). The teacher emphasizes the importance of tracing blocks as students place them and discusses ways of tracing them in a very specific manner. Along with efficiency, she prompts students to consider that they should use flats first and then skinnies and bits (ETA 7). She even comments that, rather than tracing each bit individually, students can place and trace a group of bits and record the total number of those bits. She also encourages students to think about whether to place flats first and recommends that students start along the line of symmetry. These very specific directions and guidance are not clearly addressed or directed in the W-task, even though it does show a figure of outlines around the base-ten blocks, with flats used first along the line of symmetry. The teacher also suggests that students can use a base-ten chart to keep track as they trace blocks (ETA 4), and she demonstrates how to use the chart to record their work in progress and mentions that recording tallies (number of flats, skinnies, and bits traced) on the chart is one way to use this tool to complete the task. Finally, she adds a component that is not included in the W-task. She encourages students to reflect on their exploration by asking them to talk about what they learned in the task (ETA 8).

Opportunities Created for Student Learning

From the teachers' perspective, opportunities to learn may be envisioned in different ways. Even though the five teachers made some common enactment decisions, it seemed that these decisions resulted in different opportunities for student learning: nurturing or constraining OTL.

It is not that a particular teacher always nurtures or constrains OTL, even though there are some patterns in individual teachers. Depending on their focus of the task (they certainly interpret the W-tasks and determine their own goals to achieve by using the tasks) and decisions and actions they make, they create different kinds of OTL and can potentially limit OTL. Likewise, a particular enactment decision (e.g., giving specific directions) does not always nurture or constrain OTL. Depending on how teachers do it, for example, giving specific directions can enable students to focus on the mathematics they learn by removing unnecessary confusion, but it also can limit room for student exploration by eliminating students' struggles that may be important for them to experience. Students are expected to complete the task in an efficient way, and yet formulating procedures for students and equating particular types of problems with particular strategies seem to limit opportunities for students to think and make their decisions and then learn from the results of their decisions. They may experience difficulty or may struggle with their approaches, but this experience makes them learn the importance of certain things, such as keeping track and placing flats first, rather than bits, to measure the area of the coat.

An example of enactment decisions that constrains student learning is from the INV teacher's E-tasks about multiplication and division. This teacher focuses on key words throughout the lessons on multiplication and division by posing questions, giving specific directions, providing explanations, and tailoring the tasks toward finding key words. She tailors the tasks so her students can identify key words to determine which operation is needed, although the same keywords can lead to different operations. She deliberately does this to minimize students' confusion about whether to multiply or divide. However, while giving these specific directions and examples repeatedly, she limits time to discuss the meaning in the story problems (e.g., how to represent the story problems and how to use such representations to make sense of the problem situations and solve them) and the relationship between the two operations. As a result, after two days of using key words, most students were still confused about which operation was needed in which problem situation.

In contrast, the MTB teacher nurtures student learning overall, even though she tries to control and tailor student exploration at times in a way that constrains OTL. She frequently poses questions to invite student thinking and encourage students to consider various concepts and ideas related to the task. Such questions could be funneling and yet she manages to keep them open so that students can remind themselves of previous learning, connect that to the new task they are about to start, and think about approaches to the task. She deliberately suggests the base-ten chart and demonstrates how it can be used, and yet lets students decide whether to use it and how to use it. When trying to give clear directions, the teacher uses student thinking and has students come up with ideas along the way, rather than simply directing them to follow a set of procedures. Also, she adds a component at the end of the E-task that helps students reflect on the overall process. The lesson containing the W-task requires more than one session. The teacher's guide does not indicate where the task could potentially be suspended, let alone how to do so. The teacher decides to have students talk about what they learned and reflect back on their work on the task through this added component. Overall, she guides students in a very specific and organized way to engage students with the mathematics of the task.

Significance

It is surprising that there are still a number of decisions teachers have to make, even enacting W-tasks that include detailed guidance for the enactment (e.g., MTB). These decisions influence learning opportunities for student to experience. The results of this study can be used in teacher

preparation and professional development to help teachers make such design decisions in a way that can nurture student learning. The results can also be used to provide support for teachers to help them make better decisions. For example, the lessons on multiplication and division in INV can address how to guide students about key words and operations in a clearer way than the current edition (e.g., questions to ask, prompts to give, and directions for the task).

Acknowledgements

This paper is based on work supported by the National Science Foundation under grants No. 0918141 and 0918126. Any opinions, findings, conclusions, or recommendations expressed in this paper are those of the authors and do not necessarily reflect the views of the National Science Foundation.

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EVALUATING THE IMPACT OF COMPUTER-BASED AND TRADITIONAL LEARNING ENVIRONMENTS ON STUDENTS' KNOWLEDGE OF ALGEBRA

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We share results from a quasi-experimental study in which we compared achievement between traditional lecture-based and computer-based sections of college algebra on a common multiple choice exam as well as performance on problem solving items. Students in the computer-based group performed better on the final exam and were also more likely to complete the problem solving tasks correctly than students in the traditional courses. However, the computer-based group showed limited ability to interpret an equation and relate it to a contextual situation.

Keywords: Curriculum, Technology, Post-secondary, Algebra & Algebraic Thinking

The use of "emporium" style delivery formats for basic college mathematics courses is spreading in the United States (Twigg, 2011). These courses are held in large computer labs, where the majority of instruction is provided through interaction with a computer program. The software provides examples, explanations, videos, opportunities to practice, and varying degrees of feedback on incorrect solutions. Emporium courses also employ trained instructional assistants who provide one-on-one "just in time" help (Hodges & Brill, 2007). The limited number of empirical studies conducted in these sites indicate positive effects on attendance, pass rates, performance on end-of-course exams, and future enrollment in math courses (Taylor, 2008; Twigg, 2011). Despite theses findings, it is unclear what mathematical knowledge students gain from learning in computer-based environments.

Objectives

The purpose of this paper is to report quantitative findings from a comparative study regarding the impact of computer-based versus face-to-face instruction for college algebra and to analyze students' solution strategies on non-routine mathematics tasks. This research is an important first step towards determining the extent to which these two contrasting learning environments influence mathematical knowledge and the degree to which students in these different environments are able to solve non-routine tasks in mathematics.

According to Twigg (2011), the success of the mathematics emporium model can be attributed to several characteristics, including: (a) students do problems rather than watching a lecturer do the problems, (b) students spend more time on what they do not understand and less time on what they already know, (c) students get timely assistance from the software as well as trained instructional assistants, and (d) students must master a requisite skill before they proceed to a more advanced skill. The characteristics listed above correspond to the broad finding that developing procedural skill requires teaching that incorporates modeling the skill followed by a significant amount of error-free practice on the part of the learner (Hiebert & Grouws, 2007).

At some universities, blended instruction incorporates some of these characteristics without removing the face-to-face lectures, still producing significant learning gains, and in some cases, improved self-efficacy beliefs for students (Hagerty & Smith, 2005; Hagerty, Smith, & Goodwin 2010; Kendricks, 2011; Taylor, 2008). Kodippilli and Senaratne (2008), found that passing rates

were 43% higher in a class that used MyMathLab, a software commonly used in mathematics emporia, than in a class where it was not used.

Despite this reported success, the emporium model raises concerns about the potential lack of attention to conceptual understanding. Erlwanger's (1973) classic description of the elementary student "Benny" serves as a reminder and warning about the kinds of misconceptions that can result from an individualized program that emphasizes correct answers and procedures without attending to students' understanding of mathematical concepts. This study attends to this concern by providing students an opportunity to apply their mathematical knowledge to problems that may seem non-routine and to explain how they think about the mathematical ideas and skills they use to solve the problems.

Methods

Context and Sample

In the Spring of 2012 the mathematics department at a mid-sized suburban university offered twelve sections of Intermediate Algebra. Six sections were held in a computer-based (CB), emporium-style environment and six sections were taught in traditional face-to-face (F2F) classes by adjunct professors who typically teach the course. The students did not know at the time of enrollment whether they would be assigned to the CB or F2F learning environment.

A quasi-experimental matched group design was used to explore the impact of the two settings on student learning, followed by finer grained analyses. As such, the sample includes three levels of participants, determined by data access granted by participants. Level 1 includes all students enrolled in Intermediate Algebra during the Spring of 2012 with an exam score (n_{CB}=134, n_{F2F}=192), Level 2 is made up of a subset of students from each learning environment (n_{CB}=73, n_{F2F}=50) and includes additional student-level predictors, and Level 3 participants are a subset of the Level 2 participants with completed non-routine tasks (n_{CB}=38, n_{F2F}=24).

Data Sources and Analysis

Each student took a common multiple-choice final exam at the conclusion of the semester through their normal means: CB via their computer program and F2F in their classroom. For the Level 1 analysis, a t-test was used to determine if there was a difference in the group means between the CB and F2F sections. To control for initial differences in the two student samples, as well as to explore the factors that influenced student achievement on the final exam between the two treatments, the Level 2 analysis included ANCOVA's with student-level variables for SAT math (prior achievement measure), high school GPA and Algebra II grade, and university GPA.

To determine if there were differences between CB and F2F in how students solved the problem-solving tasks, the Level 3 data include student responses on four problem-solving items. These items were given near the end of the semester for a small amount of extra-credit. They were completed individually, in class, on paper, and were based on major themes of the Intermediate Algebra course: systems of equations and quadratic functions. They were designed to represent various degrees of transfer from the kinds of procedural tasks they solved on the final exam. In this paper, we report on two problems, shown below:

1. Cindy bought 3 burgers and 2 soft drinks for $10.50. At the same restaurant, Sean bought 4 burgers and 3 soft drinks for $14.50. How much does 1 burger cost?
2. For a concert, the income from ticket sales is estimated to depend on ticket price according to the following equation: $t(x) = -25x^2 + 750x$, where x is price per ticket.
    a. Will increasing the price of the ticket always result in greater income? Explain how you know.

b. Find the ticket price where the concert will make the maximum amount of money, according to the equation. (Hirsch, Fey, Hart, Schoen, & Watkins, 2008)

Two members of the research team coded the students' solution strategies and whether the students answered the problem correctly. The two coders reached a 96% agreement on correct versus incorrect and 87% on the strategies the students used.

Level 1: All Students

On average, student achievement on the final exam was higher for students in the CB (μ=70.75, σ=15.29) than for students in the F2F sections (μ=65.49, σ=13.12). This difference was significant (t=-3.33, p" .001), which represented a small effect size (r=.182) (Cohen, 1988).

Level 2: Subset of Students with Additional Predictors

For the students participating in the Level 2 study, there was no significant difference between students enrolled in the F2F section (μ=68.59, σ=12.80) and those in the CB sections (μ=70.60, σ=16.01) (t=-0.74, p=.461). The ANCOVA, using SAT math as a covariate, showed a larger effect size, SAT math score significantly predicted students' final exam score (F=43.93, p<.0001). More interesting, when controlling for SAT math, the effect of the mode of delivery favored CB (F=3.58, p=.06). Furthermore, when the mode of delivery, SAT math, and the interaction between mode of delivery and SAT math were included as predictors, Mode was significantly related to final exam score (F=4.32, p=.040). In addition, Mode*SAT math was a significant predictor of student achievement (F= -2.31, p=.029). These findings suggest that there is a significant relationship between SAT math, the mode of delivery, and the final exam, such that, on average, students with higher SAT scores tended to perform better in the CB setting. Conversely, students with lower SAT scores tended to perform better in the F2F setting.

Level 3: Problem-Solving Tasks

On Task 1, 76% of the CB students answered the problem correctly, whereas only 58% of the F2F students responded with a correct answer. Students used strategies such as guess and check, elimination, or other algebraic strategies. Interestingly, students used different strategies depending on their learning environment (**Table 1**). Students in the F2F classes were more likely to guess and check and less likely to use an elimination strategy than those in the CB classes. These results show that although the CB students were more successful in solving the task, there was less variance in their strategies, whereas F2F students used a greater variety of strategies.

Table 1: Task 1 F2F and CB Percentage of Students' Solution Strategy

	Guess	Elimination	Algebraic	Other
Face-to-Face	46%	21%	16%	16%
Computer Based	27%	60%	12%	0%

The first part of Task 2 asked if increasing the ticket prices would always result in increased income. Of the F2F students, 78% correctly indicated that income would not continue to increase, justifying this conclusion by appealing to the problem context, the negative coefficient in the equation, or by testing values. In comparison, only 58% of the CB students correctly indicated that the income would not continue to increase. All of the F2F students gave a justification for their conclusion, but 18% of the CB students did not give any justification. For the second part, where students had to find the ticket price that would maximize income, only 31% of the F2F students found the correct ticket price, whereas 63% of the CB students found that a ticket price of $15 would maximize income. Students used many different strategies to

answer the second part of the problem (**Table 2**). All 37% of the CB students using the vertex equation obtained the correct answer, and, of the 32% of CB students who solved the problem by testing different values for the ticket price, all but two of them (83%) got the right answer. In contrast, none of the F2F students used the vertex equation and less than half of them found the correct answer by testing ticket prices. This problem reveals that while the CB students were more likely to use an algebraic procedure and to solve the problem correctly, they were less likely to recognize that the income would not continue to rise as the ticket price increased.

Table 2: Task 2 F2F and CB Percentage of Students' Solution Strategy

	Ver tex	Tests Value	Complete Sq./ Factor/Graph	Blank or No	Attempts to
Face-to-Face	0	61%	8	2	4
Computer	3	32%	1	2	0

Significance

As universities across the nation transition to more cost effective means of delivering mathematics instruction, it is important to verify that these methods of delivery are effective, not only in promoting students' procedural fluency but also in providing students with a rich experience that shows them the connections behind algorithmic procedures. This study suggests that while students in emporium-style settings may gain procedural fluency, they may be limited in their ability to interpret mathematical objects and relate them to contextual situations. Given the limited number of empirical studies on the impact of emporium-style math courses, more work is needed to explore not only systemic effects but also the nature of the mathematical knowledge that students gain in these settings.

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AN ANALYSIS OF ALGEBRA CONTENT IN THE FINLAND AND TAIWAN ELEMENTARY SCHOOL MATHEMATICS TEXTBOOKS

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The major purpose of this study was to examine the similarities and differences in the contents of algebra topic in selected elementary school mathematics textbooks of Finland and Taiwan. The mathematics curriculum market leaders of Finland and Taiwan, Laskutaito and Kang Hsuan, were examined using content analysis method. Results show that while both textbook series introduce the early algebra through arithmetic and numerical reasoning, they differ from each other in terms of the use of patterning tasks. Furthermore, there is a significant difference in the representation forms of problems used in these textbooks and the use of contextual problems. Implications related to mathematics textbooks' design and teaching are discussed along with suggestions for future studies.

Keywords: curriculum analysis, algebra and algebraic thinking, elementary school education

Textbooks play a significant role in mathematics classrooms worldwide (Schmidt, McKnight, Valverde, Houang, & Wiley, 1997). They are commonly considered as a source of explanation and exercises for students to learn and for teachers to teach (Reys & Reys, 2006), and an important indicator for the opportunities to learn mathematics (Stein, Remillard, & Smith, 2007). The primary purpose of this study was to examine the similarities and differences in the contents of algebra topic in selected elementary school mathematics textbooks of Finland and Taiwan.

This study is significant in terms of the choice of the target countries, topic focus, and grade levels. Students in Finland and Taiwan are high achievers in PISA 2006 and 2009 (Organisation for Economic Co-operation and Development [OECD], 2010). It is important to find out what opportunities to learn are afforded to the students in these two countries. Furthermore, there is a growing recognition of the important role early algebra learning plays in the overall success with algebra (Common Core State Standards Initiative, 2010). A detailed analysis of the algebra materials in elementary mathematics textbooks from Taiwan and Finland will shed light on the content and characteristics of the mathematics problems used to support the development of early algebra reasoning in these two countries.

Prior Studies on Textbook Analysis

Prior studies on textbook analysis have focused on two primary issues. The first one is the content coverage. This includes learning goals, list of topics, meanings and representations of key concepts, sequence, pacing, and learning trajectories (Charalambous, Delaney, Hsu, & Mesa, 2010; Yang, Reys, & Wu, 2010). The second focus is classifying the textbook exercises and problems by various kinds of constructs and schemes. For example, Charalambous, Delaney, Hsu, and Mesa (2010) examined the fraction constructs (e.g., part-whole, measure, ratio, operator, quotient) used in elementary mathematics textbooks from Cyprus, Ireland, and Taiwan.

Early Algebra Learning

Carraher and Schliemann (2007) identified three entry points for early algebra learning. The first one is through arithmetic and numerical reasoning that includes the field of axioms, such as

commutative, associative properties that result from performing arithmetic operations. The second entry point is arithmetic and quantitative reasoning. The most prominent approach to early algebra learning based on quantitative reasoning is centered on the work developed by Davydov and his colleagues from the Vygotskian perspective (Schmittau, 2005). The third entry point is based on arithmetic and function (Carraher, Martinez, & Schliemann, 2008). Students are introduced to the numerical and geometrical patterns and are asked to predict the next one in informal language. Textbooks in certain parts of the world introduce formal algebraic notation and simple symbolic manipulations in the upper elementary grades. For example, Wong (2005) reported that students in Hong Kong are expected to solve simple equations involving fractions and decimals such as $x - 2.2 = 1.8$ or $y \times 2/3 = 4$ in sixth grade.

Based on the work summarized above, the following two research questions guide the design and analysis of this study:

1. What are the similarities and differences in the types of problems used in the selected textbook series in Finland and Taiwan in terms of content-dependent characteristics such content coverage, sequencing, pacing, and the emphasis in terms of arithmetic, quantitative, functional, or symbolic reasoning?
2. What are the similarities and differences in the types of problems used in these two selected textbook series in terms of content-independent characteristics?

Because of the space limitation, only selected findings will be discussed.

Method

Compulsory education in both Finland and Taiwan shares similar basic structures. Both are 9 years in length; during the first 6 years, a homeroom teacher is in charge of most subjects, while the subject specialists teach different subjects during the last 3 years of schooling. Also, they both have national curriculum guides (Ministry of Education in Finland, 2008; Ministry of Education in Taiwan, 2003).

The selected mathematics textbooks, Laskutaito (Saarelainen, 2006) and Kang Hsuan (KH, Kang Hsuan Educational Publishing Group, 2010), both have the highest market share in Finland and Taiwan, respectively. Both series have one booklet for each semester; thus, there are 12 booklets in each.

This study applied the content analysis method to analyze and code the problems presented in the mathematics textbooks (Yang et al., 2010). The data analysis of this study involved several stages. First, we went through all the problems in the mathematics textbooks and coded each one as to whether it was an algebra problem. The problems include the worked example (solutions were provided) and exercise problems following the worked examples (no solutions were provided) in the student textbooks. Two of the authors did all the coding and compared their results. The reliability was calculated using (shared algebra problems)/(total identified algebra problems) for each textbook (Ding & Li, 2010). The reliability for coding the Finland textbooks was 92%. The reliability for coding the Taiwan textbooks was 95%. The discrepancies between the coders were discussed to reach agreement before data analysis.

Second, we went through all the algebra problems in both textbook series and noted the main characteristics of these problems, in terms of both the content-independent and dependent ones. To confirm our observations, we coded all the algebra problems along these two sets of characteristics. To check the reliability for each of the aforementioned categories of the algebra problems (context vs. non-context, and different forms), all of the algebra problems were randomly selected and coded independently by the first and second authors for each textbook. Both coders discussed any discrepancies among the problems to reach agreement for the final

results. Finally, several themes emerged during these iterations of analyses when comparing similar problems from both textbook series, which will be discussed in the next section.

Selected Findings

The primary focus of the algebra problems in the Finland elementary textbook starts with developing arithmetic and numerical reasoning, and pattern recognition ability. As each operation is introduced, students first learn to write and evaluate expressions for given contextual problems, then solve number sentence problems with the unknown represented with "$\square$," and finally with the letters "x" and "y" in sixth grade. Properties of operations such as commutative laws and order of operations are introduced with concrete examples and plenty of exercises to consolidate the ideas. The design of the Finland elementary math textbook is spiral. Old ideas are revisited briefly before they are used with larger whole numbers or rational numbers with more complex computations. In addition, logical patterning tasks are first introduced at the first grade level and revisited with increased complexity throughout the remainder of the five grades.

Similar to algebra problems in the Finland curriculum, Taiwan textbooks encourage students to learn how to write the expression with the unknown represented with "( )" for given contextual problems, and to learn how to solve them with various algebraic properties first, and then with "x" and "y" in grade six. However, the algebra problems in the Taiwan elementary textbook do not focus on patterning recognition tasks as much as the Finland textbook in the early grades. The Kang Hsuan series does contain exercises that encourage students to find the regularity in the expression for the total number of chairs when the number of tables is "n." This kind of numerical patterning task is not found in the Laskutaito series.

There are a few other marked differences. KH tends to focus on a single unknown quantity, while Laskutaito includes ample exercises that require students to reason with multiple quantities. KH includes far more worked-out examples than Laskutaito. In addition to showing step-by-step instructions, KH provides examples and subquestions that guide students through the reasoning process. For example, KH asks students the following series of questions: "A fruit basket weighs 50 g. How much would it weigh all together after putting in fruit that weighs 100 g, 200 g, 300 g, x g?" After students have plenty of experience in writing expressions with a single operation, they then are given problems that involve multiple operations, for example, "Ji-Hua and three friends went on a picnic. They spent 160 NT dollars to buy bus tickets and x NT dollars to buy snacks. Write an expression to show the amount of money each person should pay. If x = NT 120 dollars, what is one person's share? If x = NT 200 dollars, what is one person's share?" Finally, the textbook illustrates the additive and multiplicative axioms and how they can be used to solve equations such as $y \div 0.8 - 8 = 12$ by first adding 8, and then multiplying by 0.8 to both sides of the equal sign.

Discussion and Implications

This research reports the results of an analysis of elementary mathematics textbooks that are used in Finland and Taiwan. Both textbook series introduce the early algebraic concepts to their students through arithmetic and numerical reasoning, with an emphasis on a conceptual understanding of various principles and properties for the four basic operations. In addition, Laskutaito provides logical patterning tasks throughout all grade levels, while KH includes discussion of numerical patterning tasks that prepares students for a functional approach. The problems in KH tend to focus on one unknown quantity, while Laskutaito provides students an opportunity to reason with multiple quantities. Finally, both textbooks include some worked-out

examples; the solutions in Laskutaito include only computation steps, while those in KH include the reasoning behind those steps.

This difference reflects the different types of algebraic reasoning afforded by each textbook series. The early and systematic approach of algebraic reasoning through arithmetic and numerical reasoning might have contributed to the good performance of Finnish and Taiwanese students in the international assessment. How and to what extent students' algebraic reasoning is influenced by the different curricular emphasis is a topic worthy of future study. For example, will middle school students in Taiwan perform better on the numerical patterning tasks because of their experience with such tasks in the elementary grades? With less explicit reasoning and more variety of representations included in the algebraic problems, teachers and students in Finland are freer to explore and to make connections between concrete and abstract ideas through logical and pattern reasoning than their Taiwanese counterparts. However, such differences in the textbooks' approach may increase the importance of the teacher's knowledge and experience in facilitating students' learning through the given curriculum (Stein et al., 2007). This could potentially lead to more variation in terms of curriculum implementation. Future studies can be conducted to test this conjecture.

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ANALYZING TEACHER INSTRUCTIONAL MOVES AROUND HIGH-LEVEL TASKS: IMPLICATIONS FOR CURRICULUM DESIGN

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Despite the growing research on how teacher instructional moves support or inhibit students in engaging with a task at a high level of cognitive demand, it is unclear how to design teacher materials in ways that support teachers in making instructional moves that bolster student task engagement at a high level. This case study presents an analysis of one teacher's instructional moves as she used the Intensified Algebra curriculum in a double period high school algebra classroom. By analyzing the transcripts of several lesson enactments, we found that this teacher's debrief episodes tend to start out with high-level communication moves but shift into low-level moves by the end of the episode or debrief period. This finding suggests that the teacher materials should be revised to include more information about how to engage in high level communication moves during debrief segments.

Keywords: Curriculum, Instructional Activities and Practices, High School Education

Reform efforts in mathematics education have ushered in changes in the ways that mathematics is taught and learned in classrooms. While students are expected to communicate mathematically about their ideas, discuss and make conjectures, teachers, in turn, are expected to facilitate these discussions among students and ask questions in ways that further students' thinking (NCTM, 2000). Yet, research has consistently demonstrated that teachers' instructional moves can either support or inhibit students' work on a task at a high level of cognitive demand (Boston, 2012; Henningsen & Stein, 1997). Indeed, facilitating students' communication about mathematical ideas and concepts around high-level tasks can be instructionally challenging for teachers as the nature of student thinking is often unpredictable (Henningsen & Stein, 1997). What is less clear are the ways in which mathematics curriculum materials support teachers in maintaining students' work on tasks at a high level during instruction. In this study, we analyze the instructional moves of a teacher using the *Intensified Algebra* (IA) curriculum, a year-long program designed for use in double-period high school algebra classes. In doing so, our aim is to gain insight into ways that teacher materials (e.g., teacher guides) can be designed to support teachers as they facilitate students' communication around and work on tasks at a high level of cognitive demand.

Background

Curriculum materials are a primary tool – a curriculum development artifact that mediates teachers' actions and helps them to accomplish curricular goals - for teachers as they implement a given curriculum program (Brown, 2009). Thus, we assume both teachers and the curriculum materials, including the teacher support materials, interact with and influence each other. Under this view of teacher-curriculum interactions, teachers may either closely adhere to the suggestions in the curriculum materials or choose to make modifications based on various factors in their particular classroom environment, such as students' needs and abilities, experience and familiarity with the curriculum program, and their own beliefs about learning (Brown, 2009). These decisions and the actions that follow may be generally called instructional moves. Given

that teacher materials are generally designed to support teachers in implementing the curriculum and supporting students' work as part of the curriculum, we are particularly interested in the ways in which teachers' instructional moves support students' communication around high-level tasks, and the nature of the instructional support provided in the teacher materials for supporting students' communication at a high-level.

Method

As part of the larger research project around the IA curriculum, six teachers were videotaped and observed across at least 10 lessons over the course of the 2010-2011 school year. All videotaped lessons were transcribed. For the purposes of our current study, we include our analysis of only one of these teachers, Laura. All names are pseudonyms. She was selected as a case study because as we analyzed the videos from the larger project we saw evidence of her using the curriculum materials with a fairly high degree of fidelity during lesson enactments. Furthermore, in an interview with Laura, she indicated that she regularly used the teacher materials (i.e., Advice for Instruction (AFI)) while planning for instruction.

In the first phase of our analysis, we determined levels of cognitive demand that are placed on students as determined by the written task. Using the Stein and Smith (1998) math task framework, we identified whether the lesson contained a high- or low-level task. High level cognitive demand tasks require that students reason about the mathematics and explore multiple solution paths, thus making the instructional work for teachers unpredictable and often challenging (Henningsen & Stein, 1997). Thus, we chose to focus our analysis on those lessons containing high cognitive demand tasks. In the second phase of our analysis, we segmented each lesson transcript into interactional episodes based on the nature of the mathematical task. For example, in the launch portions of the lessons, a teacher may introduce a concept, connect to previous concepts, or introduce a particular task. The episodes allow for an analysis of teacher instructional moves as they relate to the AFI. We then annotated each episode with the information provided in the AFI. In other words, what was the nature of instructional support provided in the AFI for each episode. In the third phase of our analysis, we developed a list of teacher instructional moves that supported student thinking at a high level and those instructional moves that inhibited student thinking at a high level, based on a review of current research on teacher instructional moves, classroom communication, and mathematical tasks. Table 1 below is a list of the codes and their definitions. Due to space constraints, we only include a few codes.

Table 1: Teacher Instructional Moves

Surface-level Instructional Moves		High-level Instructional Moves	
Explicates steps	Teacher specifies explicit procedures <u>prior to investigation</u>, through either: (a) Direct explanation of the procedure or through the use of similar examples (b) Funneling or leading questions prior to investigation.	Elicits reasoning	Teacher asks students for justification of procedures
Takes over	(a) Teacher does part of the	Presses	Teacher presses students

	problem with the whole class, taking the challenge out of the problem (after students have already begun investigation) (b)Teacher finishes an explanation for a student.		for accuracy and clarity in their reasoning
Low accountability	(a) Unclear or incorrect explanations are accepted by Teacher (b) Teacher allows students to supply only procedure when justification is requested	Refers back	Teacher explicitly calls attention to and refers back to shared conclusions or assumptions from a different class session.

The instructional moves have been divided into two categories, surface-level moves and high-level moves, according to whether or not they support students' communication at a high level of cognitive demand or at a surface level. Though surface-level communication moves may not support engagement at a high-level of demand, we do not assume these moves are always negative or necessarily inappropriate. However, noting any patterns or shifts from surface to high-level moves, or high- to surface-level, along with noting to what extent Laura follows the AFI directions, allows us to identify features within the AFI that are less supportive of implementation at a high level of cognitive demand. In the fourth phase of our analysis, both authors independently coded each lesson, compared codes, and resolved any discrepancies through discussion. Once coded, we identified any patterns in the presence and absence of different codes, and compared the final codes with the information provided in the AFI.

Results

Our results suggest several emergent patterns in the data. First, episodes where Laura followed the AFI least—or, in other words, made the most the modifications—had mostly low-level communication moves. In fact, out of the ten episodes in which Laura made modifications, eight of them were coded for low-level communication moves. The type of instructional support in the AFI in these lessons includes some suggested questions to ask students, as well as anticipations of student thinking, however this type of information is not consistently given within or across lessons in the AFI. This suggests that the AFI may not support teachers in making modifications to suit their local conditions in ways that are reflective of the goals of the IA curriculum. However, this may be a professional development issue—how to modify lesson plans when students are confused—than one of redesigning teacher materials.

Second, the debrief episodes tend to start out with high-level moves, such as eliciting reasoning, but shift into low-level moves, such as taking over and holding students to low accountability in their responses, by the end of the episode or debrief period. This suggests that the debrief portions of the support materials may be missing the kind of information teachers need in order to maintain the use of instructional moves that facilitate student communication at a high-level even in the face of student confusion. Though there may be some general ideas to help teachers in this way, and therefore the purview of professional development, it is likely that specific support for a lesson is needed as well.

Finally, in the lessons analyzed, Laura did ask for justification of procedures (i.e., elicits reasoning) but did not ask for explanation of thinking (i.e., elicits explanation). This small but non-trivial distinction between reasoning about a specific procedure versus more general mathematical reasoning may not be clear in the teacher materials. Thus, as teachers modify questions or add in their own, they may not recognize the need for both types of questions, and as such, the materials may need to be revised to address this distinction.

Implications for IA Curriculum Design

Our results suggest important implications for the design of the IA curriculum program. First, many of Laura's low-level moves we identified were examples of holding students accountable for low-level products, such as incorrect responses, one-word or numerical responses when the AFI called for more robust explanations from students. Thus, one implication for the design of the AFI around these lessons is to provide suggestions as to how to support pressing students for clarity. This could take the form of sample student responses for particularly difficult questions. Second, we identified several instances of Laura taking over the thinking around the task. Thus, one implication for the design of the AFI is to include ways to remind teachers not to take over the thinking. This could take the form of placing reminders at key points in which teachers tend to take over. Third, many of Laura's low-level moves also included taking over explanations or the more challenging parts of tasks. Thus, another implication for the design of the AFI is to include ways to support teachers in encouraging students to comment on or argue about other students' ideas. Finally, we identified relatively few instances of Laura linking concepts and strategies used while solving tasks. Thus, another implication for the design of the AFI is to include ways to support teachers in making such links, particularly for lessons with multiple solution paths. This could take the form of diagrams or other representations of multiple solution paths and their links as a means of reminding teachers of the importance of highlighting those links with their students.

To be sure, teachers implement the curriculum through a filter of their conceptions of, and orientations toward, what it means to learn and teach mathematics. Nevertheless, the information gleaned from this analysis can help curriculum developers revisit and refine what is in the IA curriculum materials, specifically the AFI, and determine better ways to facilitate more effective implementation.

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THE ASSOCIATIONS BETWEEN READABILITY MEASURES AND PROBLEM SOLVING IN ALGEBRA

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Solving mathematics story problems requires text comprehension skills. However, previous studies have found few connections between traditional measures of readability and performance on story problems. We hypothesized that recently-developed measures of readability may illuminate associations between text difficulty and problem solving. We used data from 3,394 middle and high school students solving algebra story problems in Cognitive Tutor Algebra. We found that several indicators of readability were negatively associated with incorrect answers. Moreover, indicators of readability were associated with students requesting hints from Cognitive Tutor. These findings are discussed in the context of models of algebra problem solving and previous research on text difficulty in solving story problems.

Keywords: Curriculum, Cognition, Problem Solving

Theoretical Framework

Algebra word problems, or story problems, can provide access to important mathematical concepts. Their concrete, verbal language can connect to students' prior experiences, providing a conceptual foundation on which more advanced skills can be built (Koedinger & Nathan, 2004; Walkington, Sherman, & Petrosino, 2012). However, in order to understand and solve story problems, students must be able to comprehend and interpret the story text (Kintsch & Greeno, 1985). This idea is articulated in Nathan, Kintsch, and Young's (1992) framework for algebra story problem solving, where students develop a mental representation of the actions and relationships in a story problem's text, a *situation model*, as well as a mental representation of the formal mathematics, a *problem model*. The coordination of situation and problem models is a key element of learning from algebra story problems (Walkington et al., 2012).

Based on this framework, it is reasonable to assume that the readability of story problems contributes to their difficulty. However, researchers have found minimal evidence linking the readability with problem solving accuracy (Wiest, 2003). One reason may be the use of traditional measures of readability that assess the text based on the familiarity of the words, the number of syllables, and the number of words per sentence (Graesser, McNamara, & Louwerse, 2012). These measures provide coarse estimates of a text's difficulty, but do not, for instance, capture the *cohesion* of text – the "linguistic glue" that joins the events and ideas, helping readers to understand connections and relationships (Gernsbacher, 1997; McNamara, Graesser, McCarthy, & Cai, in press). For example, the use of consistent verb tense (e.g., past or present) in different sentences can cue the reader that the actions in these sentences occurred during the same time period. Research in elementary mathematics has also suggested that story problems with clear and explicit action (e.g., joining, separating) facilitate problem solving (Carpenter et al., 1999). Given Nathan et al.'s (1992) framework, measures relating to cohesion and explicit action may be a critical component of solving story problems.

In the present study, we examine the texts of a large set of algebra story problems solved by students in a variety of school settings. We argue that some types of story texts may better facilitate the construction of meaningful situation models and their coordination with problem models. We have two research questions: (1) How are different measures of text readability associated with students' accuracy when solving algebra story problems? and (2) How are different measures of text readability associated with students' help-seeking when solving algebra story problems? By investigating these questions, we seek to better understand the previously unclear relationship between text difficulty and solving story problems.

Method

Cognitive Tutor Algebra (CTA) is a computer-based intelligent tutoring system for Algebra I in use in schools across the United States. The tutoring system presents students with algebra problems and provides adaptive hints if the student requests them (Morgan & Ritter, 2002). As students work through the program, detailed log files are created that store students' interactions with the system. We used the CTA log files from 9 high schools and 1 middle school containing $N = 3394$ students with active CTA accounts. These schools were in 10 different states, with 2 rural, 4 urban, and 4 suburban schools. Three schools had 0-33% of students eligible for free/reduced lunch, four had 33-66% eligible, two had 66-100% eligible, and one did not have this information available. Five schools had student populations that were predominately White students, three were predominantly African-American students, and two were predominately Hispanic students. Schools also varied in achievement on standardized mathematics assessments.

Data were collected from the first 8 units in CTA that used story problems with linear functions; CTA is adaptive to student needs, so not all students receive all problems. After demo problems, problems with missing data, and problems with data from fewer than 15 students were omitted, 282 problems remained. On average, each problem had been solved by 398 students ($SD = 321$). Each story problem had several parts and required students to write a symbolic expression stating the linear relationship(s) in the story and fill out a table; some problems also had students construct graphs. See carnegie.learning.com for examples of problems.

In the present study, students' average error rate on each problem and their rate of asking the system for a hint were compiled from the CTA log files. Then the text of the introduction to the story problem was entered into the Coh-Metrix 3.0 software. The introduction can be one or more sentences, and generally describes the slope and intercept terms of the story (see Table 2). Coh-Metrix is a software tool developed in 2002 that provides multiple, varied, and precise measures of readability (e.g., frequency of different types of words like pronouns and verbs), measures of cohesion (e.g., degree of narrative language or time-related cues), in addition to traditional measures of readability (e.g., number of sentences or word syllables; McNamara et al., in press). Coh-Metrix measures were entered into regression models to test if they significantly predicted the percentage of attempts on that problem that were incorrect, or the percentage of attempts in which hints were requested. The regression models controlled for aspects of the story's mathematical structure, including the structure of the linear function(s) (e.g., positive slope/no intercept, negative slope/negative intercept), the type of numbers used (e.g., fractions, large whole numbers), and the unit and section the story problem came from.

Results and Discussion

Accuracy

Table 1 shows only the readability measures from Coh-Metrix that significantly predicted students giving inaccurate responses: Text Easability PC Temporality, and incidence of adverbs, intentional verbs and third person singular pronouns. For the scale on which each of these values was measured by Coh-Metrix (to interpret the B values), see http://cohmetrix.memphis.edu. "Text Easability PC Temporality" measures how many cues about temporality the text contains, and how consistent tense (e.g., present or past) and aspect (perfect or imperfect) are. This captures how well sequences of events that occur in the past, present, or future are made explicit, including cues like "an hour later." Consistent verb tense provides coherence that helps the student connect the information in different sentences (Gernsbacher, 1996). Thus it is not surprising that Table 1 shows that as temporal cohesion increases, incorrect answers decrease (indicated by the negative B value). Related to this is the result in Table 1 that as incidence of adverbs increase, incorrect answers decrease. Adverbs can provide temporal cues, which may help the reader follow the time course of the story. Adverbs can also add descriptive information about how and to what extent actions and events occurred. This information may enrich the story context, providing cues that support situation model construction. Examples of problems with both high and low scores on these measures can be seen below in Table 2.

Table 1: Regression Results for the Impact of Readability on Problem Solving Accuracy

	B	SE(B)	*T*	*p*
(Intercept)	21.153	2.096	10.094	0.0001
Text Easability PC Temporality	-0.030	0.008	-4.042	0.0002
Adverb incidence	-0.045	0.011	-4.176	0.0004
Intentional verbs incidence	-0.026	0.011	-2.290	0.0154
Third person singular pronoun incidence	-0.024	0.011	-2.182	0.0284

Table 2: Sample Problems for Comparison of Readability Scores in Two Measures

Measure	High	Low
Text Easability PC Temporality	We ***work*** for a company that ***makes*** and ***sells*** hot-air balloons. Each hot-air balloon ***sells*** for $3500. Our warehouse ***has*** 20 balloons that ***have already been*** sold but ***have not yet been*** shipped. (Coh-Metrix 98.26 percentile)	We ***are building*** doghouses and ***have already finished*** five of them. A local store ***agrees to buy*** as many as we ***can make*** for $25 apiece. (Coh-Metrix 0.03 percentile)
Adverb Incidence	Many people do not realize that when one issues a false fire alarm the community ***needlessly*** spends thousands of dollars to protect the citizens. The money that gets wasted and ***eventually*** translates into higher taxes for everyone is approximately $4000 per call. This includes the costs to maintain the trucks and equipment the ***very*** expensive insurance and the salaries of the firefighters and other employees. This financial loss to the community is the reason why the fines for ***such*** illegal activity are ***so*** high. (Coh-Metrix score 107.143).	A friend offers us a job selling snow cones. Since he owns the snow cone maker he will let us use it for $50. We will make $0.50 on each snow cone we sell. (Coh-Metrix score 0)

Table 1 also shows that as intentional verbs and third person pronouns in a story problem increase, incorrect answers decrease. Story problems are easier for students to solve when there are purpose-driven, goal-oriented actions like buying a house or cooking a meal. The concrete information provided by intentional verbs may assist the student in developing a more vivid and meaningful situation model. Pronouns also act as facilitators when they have clear referents in the story text (White, 2012). Pronouns provide cues that prompt the student to connect the new information containing the pronoun with the old information containing the referent.

Help-Seeking

Table 3 shows the readability measures from Coh-Metrix that significantly predicted students requesting a hint from Cognitive Tutor. Story problems that had more sentences in them were more likely to promote students asking for hints, rather than attempting the problem; a long story text may be discouraging and increase perceived difficulty (White, 2012). Temporal connectives like "first" and "then" were associated with more hint-seeking, which is unexpected given other temporal coherence cues were negatively associated with inaccurate answers. Students who see a text with multiple temporal cues may be intimidated by the amount of information to process. These students may seek a hint even though the temporal connectives improve situation model construction. Finally, we see that stories with third person singular pronouns are associated with less hint-seeking, which is consistent with the problem-solving accuracy findings.

Table 3: Regression results for the impact of readability on hint requests

	B	SE(B)	*t*	*p*
(Intercept)	2.398	0.868	2.762	0.0018
Sentence count, number of sentences	0.343	0.099	3.470	0.0004
Temporal connectives incidence	0.013	0.005	2.843	0.0042
Third person singular pronoun incidence	-0.014	0.004	-3.282	0.0012

Significance

The purpose of this study was to examine associations between readability measures and solving algebra story problems, using recent advances in software-based text mining. Our results suggest important implications for mathematics teaching and curriculum design. Formulating problems in which the student is cued to connect related events appears to facilitate problem-solving. Avoiding overly lengthy story contexts and overuse of temporal cues may also provide initial access. Future studies could vary the readability levels of problems on these indicators, and examine student learning. These results are different from previous findings on this topic in which traditional, coarse measures of readability have been used. According to the model we presented for algebra story problem-solving (Nathan et al., 1992), our findings suggest that students were better able to solve a story problem if its text supported their construction of a situation model and its coordination with their problem model.

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CROSS-COUNTRY BORDERLAND STUDY OF HIGH SCHOOL MATHEMATICS CURRICULUM IN MEXICO AND USA

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The aim of this comparative study is to examine similarities and differences between Mexico and Texas borderland high school mathematics curriculum. In Mexico, Secretaria de Educación Publica (SEP) is responsible to develop the content, curriculum and schedules at all educational levels. According to the SEP (2012), the mathematics programs contain teachers' goal, blocks distribution, generic skills, and basic competences for disciplinary field. SEP (2012) established that main teachers' goal is to "design meaningful activities that allow students to link previous knowledge with learning objects" (p. 6). The blocks include mathematics topics. Generic skills are students' abilities to understand their environment (local, regional, national or international). Basic mathematics competences include the following: construct and interpret mathematical models; formulate and solve mathematical problems; interpret the results obtained by mathematical procedures and contrast with established models or real situations. On the other side of the borderland, the Texas Education Agency established Texas Essential Knowledge and Skills (TEKS) standards to design high school mathematics curriculum. The TEKS for each high school mathematics course includes basic understandings as well as knowledge and skills, which are subdivided by mathematical process and content standards. According to TEKS (2012), "... the basic understandings of number, operation, and quantitative reasoning; patterns, relationships, and algebraic thinking; geometry; measurement; and probability and statistics are essential foundations for all work in high school mathematics" (p. 1).

The comparative study addresses the following research questions: *(1)* What are the main characteristics (e.g., standards, curriculum, scope and sequence) of high school mathematics curriculum in Mexico? *(2)* What are the main characteristics of high school mathematics curriculum in Texas? *(3)* To what extend high school mathematics curricular in borderland of Mexico and USA are similar and different? The conceptual framework for this study is grounded in the cross-cultural theory. Studies (e.g. Lui & Leung, 2013; Shimizu & Kaur, 2013) mention that understanding and analyzing the similarities and differences of the mathematics curriculum between two countries can facilitate mathematics teaching through cross-national studies. Similarly, this can help mathematics educators to analyze the different programs implemented by educational systems in two countries. The analysis is based on data (e.g., standards, curriculum) drawn from the SEP and TEA resources. The cross–country borderland curriculum analysis table was designed to determine and examine similarities and differences in the scope and sequence of the high school curriculum in Mexico and USA.

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COMPARING STUDENTS' PERCEPTION OF MATHEMATICAL UNDERSTANDING AND STUDENT PERFORMANCE

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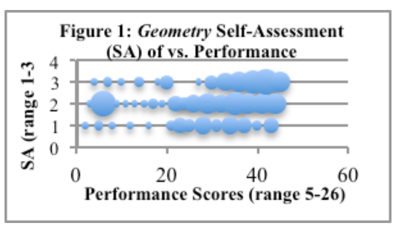
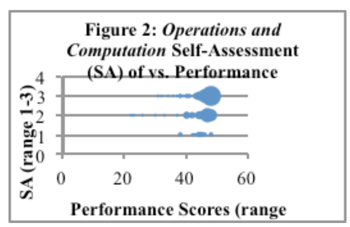
Students' self-perception of ability and understanding in mathematics influences confidence and self-efficacy, which, in turn, influences mathematics learning (Parajes and Miller, 1997; Kilpatrick, 2001; Mengelkamp, 2009; Kleitzman, 2013). Current research has found that educators and researchers underestimate the value of self-assessments as a socio-cognitive tool (Parajes & Miller, 1997). Other research has called for valid measures of self-assessment, not only to evaluate confidence, but also to validate the assessments (Kleitzmann, 2013). Little, if any research, however, evaluates the relationship between curricular self-assessments and student performance on the assessments. The current study evaluates the relationship between student self-reporting of mathematical understanding and performance in mathematics on the end-of-unit tests within the reform mathematics curricula, *Everyday Mathematics*, in order to assess: (1) how students' perceptions of their abilities for a specific curricular goal correlate to their actual performance on the end of unit test, (2) how self-perceptions and performance vary across students, classrooms, and schools, and (3) the validity of the curricular self-assessments, for particular learning goals.

The current study is a part of a larger project that addresses questions about cognitive, instructional, and inferential aspects of validity of assessments embedded in elementary mathematics curricula. Specifically, this study examines the accuracy and other visible features of student responses to evaluate the relationship between students' latent understanding and self-evaluations. Over one thousand samples of student work across three grade levels were scored and analyzed using an innovative scoring method, which includes scoring for accuracy as well as features of student answers that provide evidence of partial understanding.

Initial findings have shown that overall self-reporting of learning goals does not correlate with performance. However, when self-assessments were separated by learning goals, we found different patterns of self-reporting differed among learning goals. For instance, in a grade 4 unit test (n=195), self-assessments scores for *geometry* learning goals were equally distributed across all performance levels (*Figure 1*). However, *operations and computation* showed a stronger relationship between self-reported understanding and performance (*Figure 2*). Analyses across students and the educational implications will be discussed.





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TEACHERS' OPPORTUNITY FOR LEARNING PEDAGOGICAL CONTENT KNOWLEDGE REGARDING INVERSE FUNCTIONS FROM HIGH SCHOOL MATHEMATICS TEXTBOOKS

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Keywords: Curriculum Analysis, Secondary Mathematics, Algebra and Algebraic Thinking

The release of the NCTM 1989 Curriculum Standards led to a shift in mathematics curricula (Remillard, 2005); instead of "speaking *through* teachers" some teacher guides were "speaking *to* teachers" (Remillard, 2000, p. 347). This prompted several studies examining the "extent to which curriculum materials could be designed to be educative for teachers" (Remillard, 2005, p. 231). This study is an exploratory analysis of teachers' opportunities for learning pedagogical content knowledge from current high school mathematics textbooks. Lessons regarding inverse functions were selected from five curricula for this analysis. Inverse functions were identified as the content focus, as this material is often problematic for secondary students (Wilson, Adamson, Cox, & O'Bryan, 2011). Moreover, Wilson and colleagues claim teachers unknowingly mislead students regarding inverse functions because they teach what is "in the textbook" (p. 501). This bold statement motivated a close examination of the information textbooks provide teachers regarding inverse functions.

Remillard (2005) defines *structures* as the "plans, activities, scripts, suggestions, information, explanations and messages" of the curriculum (p.234). This study examines these structures in lessons from the teacher's editions of five high school curricula: Interactive Mathematics Program, Core-Plus Mathematics Program, MATH Connections, Glencoe, and Prentice Hall. These were chosen as a representative sample of current high school curricula and include both integrated and strand-based sequences (i.e., Algebra, Geometry, and Algebra II).

The structures of each lesson were coded by identifying text that aligns with Shulman's (1986) concept of *pedagogical content knowledge*, a notion "which goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge *for teaching*" (p. 9). Within Shulman's concept of PCK, six themes were identified and utilized to code the lessons. These themes are as follows: student pre-conceptions, student conceptions, student misconceptions, importance of multiple representations, use of powerful analogies, and the reorganization of student understanding.

This poster will display the findings from this study. For example, while most of the lessons examined noted possible student misconceptions regarding inverse notation, few of them contained powerful analogies, student pre-conceptions, or multiple representations.

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A MEASUREMENT STUDY OF ENGINEERING, SCIENCE, AND MATHEMATICS ASSESSMENTS FOR ELEMENTARY AND MIDDLE SCHOOL STUDENTS

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Keywords: Measurement, Standards, Curriculum, Assessment and Evaluation

This study describes the ongoing development, scaling, and validation efforts of assessments of elementary and middle school students' understanding in engineering, science, and mathematics. Item response theory models are used to characterize the properties of each assessment and to generate estimates of student's proficiency. The work is part of a larger project focused on increasing student learning in grades 4 – 8 through an engineering design-based integrated approach to STEM instruction and assessment. The assessments fill an important gap in the literature by providing psychometrically sound tools to gauge students' understanding of engineering, science, and mathematics concepts for the specified grades.

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A HISTORICAL EXAMINATION OF THE TREATMENT OF RATIO, RATE AND PROPORTION IN US MATHEMATICS TEXTBOOKS

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This poster is intended to examine, from a historical perspective, changes in the definitions of ratio, rate and proportion and the sequencing of these concepts in textbooks. Three research questions lead the investigation: (a) How are ratio, rate and proportion defined in textbooks? (b) How do textbooks derive properties (e.g., the property to reduce ratio to an equivalent ratio)? (c) What are the curricular sequences of developing proportion through either ratio, rate or other means?

My analysis of the textbooks examined three aspects: definitions, deduction of properties, and curricular sequences. Definitions are considered fundamental in mathematics and mathematics education (Zaslavsky and Shir, 2005) due to the multiple roles they play, for example as a foundation for proofs. As the community of mathematics educators calls for conceptual understanding (e.g, Kilpatrick, Swafford & Findell, 2001), the deduction of the properties and curricular sequences in approaching certain concepts are worthwhile to explore.

An analytical framework was built on both literature and data. Possible definitions of ratio include: (a) number/fraction/percent; (b) division; (c) ordered pairs. Definitions of rate include: (a) percent; (b) a comparison of two quantities with two different units. The definition of proportion is a statement of equality of two ratios. Each textbook was analyzed individually and patterns across the period were observed. Also, I analyzed each textbook's derivation of properties based on the consistency of the process with its given definition. I also examined the curricular sequence through each textbook's set-up of proportions: through either ratio, rate or other means including "per unit rate" and "building a rate table".

Nine elementary textbooks (grade 5 to 8) were selected for inspection based on two criteria: (a) the nine books were distributed relatively evenly from 1833 to 2004; (b) each book had rich content of ratio, rate and proportion. Each page with definitions of the three concepts and each whole chapter or section with a main topic of ratio, rate or proportion was selected for analysis.

Results indicate that the formal definitions of ratio and proportion stay stable across this period but the definition of rate switched from type (a) to (b) around 1960. Across this period, textbooks' explanations of reduction of ratio is consistent with the definition given. Textbooks since the 1990s tend to clarify multiple meanings of ratio without explaining how reduction of ratio works when the meaning shifts. From this perspective, recent textbooks seem less rigorous than older textbooks. Before the 1960s, ratio was the main method of setting up proportions but the explanation was strictly procedural. After the 1960s, rate became more central in setting up proportions within real world contexts.

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CALCULUS TEXTBOOKS' TREATMENT OF THE CONCEPT OF LIMIT AND APOS THEORY

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Although the concept of limit is considered fundamental to the study of advanced mathematics, research suggests that students have limited views and misconceptions about the concept (Tall & Vinner, 1981). Some researchers suggest that students' informal knowledge and intuition can cause misconceptions while other researchers discovered that insufficient development of students' informal knowledge and intuition partially explain their difficulties (Cottrill et al., 1996). Attempts have been made to develop instructional methods to aid students to use the rigorous definition of limit properly (Cottrill et al. 1996). Swinyard and his colleague (2011, 2012) used genetic decomposition, proposed by Cottrill et al. (1996), to guide students through reinventing the formal definition of limit. Cottrill et al. (1996) called this process APOS (Actions, Processes, Objects, Schemas) theory. The results of Swinyard's studies are interesting because providing specialized instruction may be the key to overcoming students' difficulties and misconceptions by filling the gap between informal knowledge and formal mathematics (Gravemeijer & Doorman, 1999). One of the possible obstacles of limited views and misconceptions is knowledge proposed in textbooks (Barbe et al., 2005). This study will illustrate how two popular calculus textbooks, Stewarts and Thomas Calculus, treat the topic and whether exercise problems are designed in a way to use students' intuition and previous knowledge so that students can complete each stage of genetic decomposition and APOS theory.

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INFLUENCE OF FOUNDATIONAL COURSES ON PERSISTENCE IN STEM

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As the world economy becomes increasingly technological, demand for degrees in Scientific, Technical, Engineering, and Mathematical (STEM) fields has grown both among employers and students. But for several decades U.S. students have been graduating with STEM degrees in about half the number desired by industry. Also, about half the students attempting a STEM degree persist to graduation, so any improvement in retention of STEM students will produce a nearly equal proportional reduction in the deficit of candidates reported by industry.

The research question addressed here is *Are student's decisions to leave STEM programs related to their successes or failures in their first college mathematics courses?*

In a study of engineering students, Gardner, et al. (2007) found that the single important factor that predicted graduation, among many considered, was students' grade in their first math class. Other studies of engineering students found engineers the most persistent of all college students (Ohland, et al., 2008) and that women persist at equal rates to men (Lord, et al., 2009). Our study expands the study population to the life and physical sciences and supports and extends some findings, but contradicts others.

We examined the advising records, math course grades, and graduation records of more than 3,700 STEM students in the 2001–2005 freshman cohorts at a large western university. Using logistic regression, we found that grade points earned in the first math class was the most accurate predictor (68%) in a two parameter model of graduation for all STEM disciplines. But contrary to Gardner's results, we found that which math course was first taken is also a strong predictor and that together they account for nearly all of the 71% prediction accuracy. Gender and STEM field pursued each provide a 0.5–1% increase in accuracy, mainly as interactions. We also found interesting differences between men and women outside engineering, especially in the life sciences where women are a majority.

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AN ANALYSIS OF CONCEPTIONS OF INVERSE TRIGONOMETRIC FUNCTIONS

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This project attempts to identify the causes promoting the pedagogical difficulties associated with explanations of trigonometry and inverse trigonometry. Specifically, we seek to analyze how textbook presentation influences teachers' explanations and ultimately students' conceptions of the topic. We present a textbook analysis of conceptions of key ideas associated with inverse trigonometric functions using Balacheff's model of conceptions (Balacheff & Gaudin, 2010), as well as an analysis of a teacher's lectures on the subject and trigonometry and calculus students' interviews on a question addressing this topic

Our research pointed to several different conceptions of angles, trigonometric functions, and inverse trigonometric functions that may help explain the difficulties that trigonometry instructors and their students face with this topic. We found different types of problems that gave way to alternative conceptions and also different means of control that also gave way to alternative conceptions.

These difficulties may be explained in the following way: The conflicting conceptions can be traced back to a division within the textbooks between two representations of regular trigonometric functions. On one hand, these are seen as a direct translation from the x- and y-coordinates of the points at a given angle on the unit circle to the cosine and sine values, respectively, for those angle values. On the other hand, trigonometric relationships are sometimes represented as relationships between the sides of a right triangle. The unit circle method is generally viewed as a more 'advanced method' and one that fits better the changing real world because of the way it incorporates periodicity. When these inverse trigonometric functions are created from these two competing conceptions, fundamental differences arise between the arcsine of the circle method and the 'arcSine' of the ratio method.

This analysis of explanations given by the textbooks highlights the problematic way in which notions of inverse trigonometric functions are presented in the curriculum. This complex state of affairs in the textbooks certainly contributes to the teachers' conceptions of the material, as these same misconceptions were recorded in both the lectures and student interviews.

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NEW STANDARDS, NEW PERSPECTIVES: THE PYTHAGOREAN THEOREM, DISTANCE FORMULA, AND LAW OF COSINES

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The newly introduced Common Core State Standards for Mathematics (CCSSM) promote in-depth student explorations to encourage meaning making within mathematics classroom (Common Core State Standards Initiative 2010). This poster considers the restructuring of the mathematics curriculum by comparing the standards of six states and those included in the CCSSM. In order to effectively compare state standards versus the CCSSM, curricular revisions specifically related to the Pythagorean theorem, the distance formula, and the law of cosines were considered. The state standards for the three topics under consideration for California, Florida, Illinois, New York, Pennsylvania, and Texas were reviewed. These states represent a variety of geographic regions and, according to the U.S. Census, are the six highest populated states representing approximately 40% of the population as of 2010 (U.S. Census Bureau 2010). The researchers identified two primary research questions:

1. In what grades levels are the Pythagorean theorem, distance formula, and law of cosines introduced across states and in the CCSSM?
2. What explicit connections between topics do the authors of state standards and those of the CCSSM recommend?

Although the distance formula is an extension of the Pythagorean theorem in the coordinate plane, the standards for four of the six states considered here introduced these topics over a variety of grades (occasionally over courses that were nonsequential) potentially precluding the opportunity to foster a connection as each subsequent topic was introduced. Further, very few connections between the topics were suggested in any of the standards. Authors of the standards for five of the six states suggested introduction of the law of cosines as a formula to find an unknown side or angle with no mention of the Pythagorean theorem. In Pennsylvania the law of cosines was noted to be a generalization of the Pythagorean theorem.

In contrast, the authors of the CCSSM consistently indicated that teachers should be guiding students to notice and develop relationships between the Pythagorean theorem, distance formula, and law of cosines. As students advance to high school geometry the authors of the CCSSM intended that the concepts introduced with the Pythagorean theorem and distance formula be further developed, eventually culminating in work with the law of cosines. In particular the authors of the CCSSM highlight concept development and demonstration of meaningful connections between topics in the mathematics classroom. Future work will consider state standards and the CCSSM with respect to the mathematical shifts Fluency, Coherence, and Rigor.

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QUESTIONING THE GENDER GAP IN MATHEMATICS: TESTING THE TEST

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For years, researchers have attempted to decipher the factors that contribute to the gender gap in mathematics. A growing body of research has provided evidence to suggest that contextual factors, as opposed to innate gender differences, are more predictive of students' mathematical achievement. Test item response format, conventionality (Gallagher, Holst, McGillicuddy-De Lisi, Morely & Cahalan, 2000), and instruction content (Anglin, Arson & Langer, 2008), in particular, have been shown to differentially influence male and female students' performance. Building on this research, the current study investigates how test item response format and other test-related factors influence the gender gap in mathematics.

Gallagher et al. (2000) showed that, on mathematics problems that demanded creative problem-solving techniques, female test-takers underperformed males. The authors suggested that this difference was a result of students' strategy choice. Anglin et al. (2008) found that mathematics performance in terms of accuracy and participants' ability to formulate unique problem-solving methods improved when students were primed to consider multiple strategies. These studies, in conjunction, suggest a meaningful relationship between strategy choice and accuracy in terms of mathematical achievement.

The current study uses a factorial design to examine the effects of test item response format (open-ended, multiple choice) and strategy priming (multiple, rigid) on participants' performance and choice of strategy (algorithmic, insight) as a function of sex (female, male). Forty-three undergraduates (21 males, 22 females) completed mathematics problems, half of which were open-ended and half of which were multiple choice. Instructional content primed participants' strategy choice by suggesting either multiple strategies or a single method. Students indicated their answer to the problem and their problem-solving strategy. A repeated measures ANOVA with two within-subjects factors, test item response format and strategy prime, and between-subjects factor, sex, revealed a significant interaction between response format and sex, $F(1, 41) = 9.571$, $p=.004$, $\eta 2 = .189$, on accuracy; females performed significantly better on multiple-choice questions than on open-ended problems, whereas males performed equally well on either format. There was also a significant interaction between response format and strategy prime, $F(1, 41) = .24151$, $p<.001$, $\eta 2 = .371$, on strategy choice. Participants used more insightful strategies on multiple-choice problems in the rigid condition than in the multiple strategy condition.

These results suggest that test format might differentially measure certain types of knowledge and, thus, inaccurately represent and characterize the gender gap in mathematics.

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TASK ANALYSIS ACROSS DIFFERENT CONTEXTS IN CALCULUS I

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As part of the National Study of Calculus (Bressoud, Carlson, & Rasmussen, 2012), Tallman et al. analyzed 150 Calculus I exams from 253 submitted (Tallman & Carlson, 2012). They classified the problems they encountered with respect to Orientation, Representations, Format and opportunities to Model, Justify, and Explain. They found that 85% of the assessment items were procedurally focused and only 14% assessed understanding. We extend the work of Tallman and colleagues by using their framework to analyze both exams *and* homework at four community colleges selected by the same study as being successful. We narrow our focus to a few individual colleges, but analyze their curricular materials from across the entire semester. In this way, we reveal a more complete picture of students' opportunities to learn. We sought to answer the following research question: *What are the characteristics of homework problems and exam problems in a college Calculus I course?* In particular, we wanted to see how characteristics of homework and exam problems differ from each other and how characteristics of course problems differ longitudinally across the semester.

Our data thus far comprised 401 homework problems assigned during a semester of Calculus I and 71 exam problems included in three midterms and one final exam from one two-year college instructor. We coded the problems in five dimensions: Orientation, Prompt Representation, Prompt Response, Format, and Opportunities. We modified Tallman et al.'s framework in order to better classify the tasks we had in our data. We plan to apply this system to the data from the other three colleges in our sample.

We found that exam problems are less cognitively demanding than homework problems and use fewer representations. This observation is significant because it complicates the picture found by Tallman et al., who looked only at exam problems; we must also look at homework problems to fully characterize the kind of problems Calculus students encounter. It also raises the question: why are more cognitively demanding problems not included on exams?

We found that the textbook assignments from the chapters about limits and applications of derivatives had many examples of cognitively demanding problems, while, the chapters covering techniques of differentiation and integration, were mostly procedural problems. This reveals a potential orientation of the textbook, but also speaks to the way the instructor uses the textbook, and what skills she thinks are important for students to gain as they are related to different content areas in Calculus I.

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Research Reports

Brief Research Reports

Poster Presentations

YOUNG STUDENTS' EXPLORATIONS OF GROWING PATTERNS: DEVELOPING EARLY FUNCTIONAL THINKING AND AWARENESS OF STRUCTURE

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This study explored the development of early functional thinking in fifteen Kindergarten to Grade 2 students. As part of a teaching intervention that emphasized the co-variation between position numbers and number of items in each position of linear growing patterns, students were given a pre and post interview assessment to document their ability to predict the NEXT, NEAR and FAR terms of a pattern. Results of the interviews indicated that young students have the capability to recognize the relationship between two sets of values. Results also suggest that the recognition of visual and numeric structure in patterns influenced the extent to which students were able to reason recursively or explicitly. Finally, nine of the students developed the "perceptual agility" (Lee, 1996) to perceive the pattern in two ways. A "rows" perception was aligned with recursive reasoning, while a "columns" perception was aligned with explicit reasoning.

Keywords: Elementary School Education, Algebra and Algebraic Thinking

Early Years Algebra

In recent years there has been a shift in the research of mathematics education for young students from a focus on children's arithmetic thinking to the development of algebraic reasoning (Blanton & Kaput, 2002, 2003; Carraher & Schliemann, 2007). Researchers have begun to explore the kinds of learning experiences young students need in the elementary grades in order to develop algebraic thinking prior to formal algebraic instruction in high school. One area of research has been the study of elementary students' understanding of patterns and how this supports the development of functional thinking (Warren & Cooper, 2008).

Past research initially suggested that the connection between working with patterns to developing algebraic reasoning – that is, finding generalizations and expressing these as algebraic rules – is difficult even for students in middle and high school (English & Warren, 1998; Kieran, 1992; Lee & Wheeler, 1987; Orton et al., 1999; Stacey & MacGregor, 1999). One reason for this may be that current patterning curricula tend to emphasize the variation in only one set of data. For example, Figure 1 represents the kind of linear growing pattern found in elementary mathematics textbooks. Students from a young age are taught to focus on the systematic increase of tiles at each subsequent position of the pattern without thinking about the relationship between the tiles and their position in the pattern. Most students would describe and extend this pattern with the rule, "start with three red tiles and add two blue tiles each time". Asking for the 100th term highlights the problem of this method. Relying on this

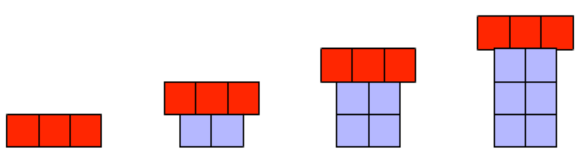

Figure 1: Growing pattern without position cards

recursive reasoning, adding two tiles to the previous term, makes it difficult to devise a general rule that would allow for the prediction of the number of tiles for *any* term of the pattern. Current instruction emphasizes this recursive reasoning by presenting growing patterns without term numbers, or with term numbers acting as labels for each term, but not as numeric quantities that can be used with a rule to determine the number of tiles.

More recently, researchers have been studying the potential of introducing specific pedagogical approaches to assist elementary students in developing algebraic reasoning through working with patterns (Rivera & Becker, 2008). Results of these studies have demonstrated the extent to which working with growing patterns can support students in developing algebraic thinking, particularly functional thinking, when the relationship between the term number and number of items at that term are emphasized during instruction (Beatty, 2007; Moss & McNab, 2011; Warren & Cooper, 2008). As an example, the pattern in Figure 2 represents the equation $y=2x+3$ with the coefficient represented by the blue tiles that increase by 2 at each term, and the constant represented by the red tiles that stay the same. The incorporation of term or position cards serves to help students identify the co-variation between one data set (i.e., the position number of each iteration) and another data set (i.e., the number of tiles at each position). The pattern rule, *number of tiles = term number x2+3* describes this relationship. Recognizing this relationship is referred to as explicit reasoning and allows for the prediction of elements far down the sequence and, ultimately, to identifying a general rule. Developing explicit reasoning is foundational to developing functional thinking, which is the perception of a generalization that relates two sets of objects.

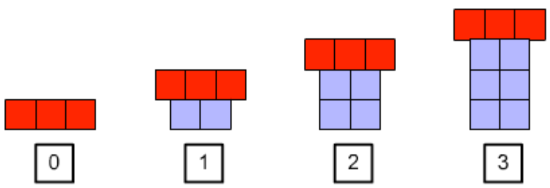

Figure 2: Growing pattern with position cards

Recent studies suggest that another important part of developing algebraic reasoning is the ability to identify mathematical structure (Mulligan et al., 2004; Mulligan & Mitchelmore, 2009). Linear growing patterns have both a numeric and geometric or visual structure. Students who focus on the numeric structure of the pattern in Figure 2 may attend to the fact that the elements at each position increase systematically (grow by 2) or that each element in the pattern is equal to twice the position number an additional three. Growing patterns also have a visual structure. Students who attend to the visual structure of the pattern may perceive it as composed of rows and/or columns of blue tiles with three red tiles on top. It is the development of an awareness of both numeric and visual structure, and identifying these as representations of a functional relationship, that supports algebraic reasoning.

There have been few studies that have looked at the potential to develop functional thinking in primary students (Moss & McNab, 2011; Warren, 2005; Warren & Cooper, 2008). The purpose of this exploratory study was to document how young students (Kindergarten – Grade 2) understand linear growing patterns. Specifically we were interested in the extent to which young students could accurately predict the NEXT (5^{th}), NEAR (10^{th}) and FAR (100^{th}) iteration of a linear growing pattern. Also of interest was whether these students would go beyond recursive reasoning to explicit reasoning by making the connection between the term or position number (the value of the position number) and the number of tiles at each position. We were also interested in how recognition of pattern structure (visual and/or numeric) influenced students' abilities to develop explicit reasoning.

Our research questions were:

1. Can working with linear growing patterns support students in making the connection between the position number and the number of elements at that position (going beyond recursive to explicit reasoning)?
2. Can young students generalize (in this context, can they accurately predict NEAR and FAR iterations of a linear growing pattern)?
3. What role does recognizing numeric or visual structure play in young students' mathematical thinking?

Methods

Setting and Participants

This study took place in four classrooms in two elementary schools (one Kindergarten class, one Grade 1 class and two combined Grade 1 & 2 classes). Five teachers were involved in the study including one Kindergarten teacher, one Grade 1 teacher, two Grade 1 & 2 teachers and one resource teacher. A short 5-lesson teaching intervention was designed to support students in making explicit connections between position numbers and numbers of items in each position, and to generalize by making NEAR and FAR predictions. During the instruction, attention was also paid to the visual and the numeric structures of linear growing patterns.

Data Sources and Analysis

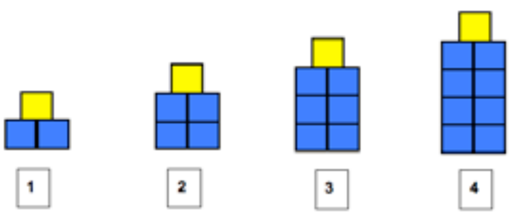

Figure 3: Interview Task Pattern

Individual task-based interviews were administered by the classroom teacher and recorded by the researcher. Fifteen students were interviewed (five Kindergarten, six Grade 1 and four Grade 2). The pre-interviews took place in December, and the post-interviews took place the following May. One of the interview tasks, the focus of this paper, asked children to describe and extend a linear growing pattern (Fig. 3) by building the 5th position. Students were asked to make NEXT (5th and 6th), NEAR (10th) and FAR (100th) predictions of the pattern. Transcriptions of the video recorded interviews were coded with respect to students' abilities to 1) identify predictable growth and constancy, 2) accurately extend the pattern, a NEXT prediction, 3) make accurate predictions down the sequence (NEAR 10th position or FAR 100th position). Transcriptions were also coded with respect to whether students recognized the numeric structure of the pattern, the visual structure of the pattern, and how recognition of structure supported an ability to make accurate generalizations.

Results

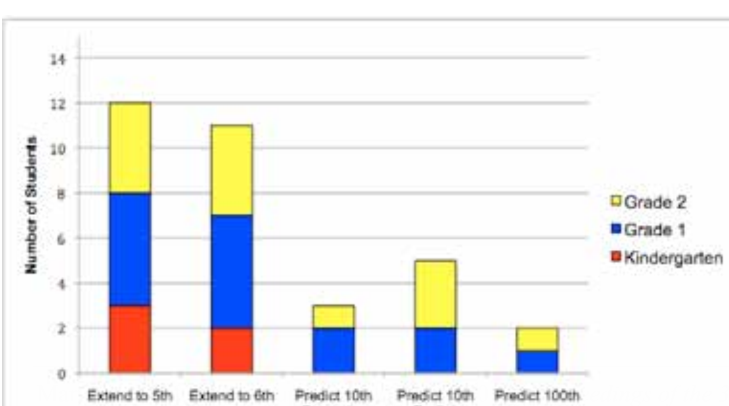

Figure 4: Pre-Interview Results

Pre-Interview Results. Results for the pre-interview are presented in Figure 4. As shown in the graph, most students were able to accurately extend the pattern to the 5th and 6th term. Seven of the students (including all of the Kindergarten students) were unable to make accurate numeric or geometric predictions about NEAR and FAR terms

in the pattern, which suggested difficulty in recognizing the structure of the pattern either visually or numerically. However, eight of the fifteen students made accurate NEAR predictions using either recursive or explicit reasoning, and two students made accurate FAR predictions. We analyzed the thinking of these students to understand their reasoning.

NEXT predictions. Twelve of the fifteen students recognized that the pattern was increasing by two tiles each time, and extended the pattern by adding two more blue tiles, or by counting on from the 8 tiles at position 4. However, none of the Kindergarten and only three of the Grade 1 students could articulate how they knew how to build the 5th position. For example, Jesse (Grade 1) accurately built the 5th position, but stated that he "built a big pattern and all the tiles are blue. Blue, blue, blue, blue, blue, blue, blue, blue, blue, blue and one yellow." Jesse had only been exposed to repeated patterns, which are typically described by articulating the attributes of the pattern that repeat, for example, red, yellow, red, yellow. Although Jesse was accurate in his building, he was unable to quantify the growth of the pattern.

NEAR prediction using recursive reasoning. When predicting the 10th position two Grade 1 and one Grade 2 students employed recursive reasoning to determine the correct number of tiles. These students focused on the numeric structure, the fact that the pattern increased by two blue tiles each time, but when building the 5th term their visual patterns differed from the task pattern. For example, Isabelle (Grade 2) created an incomplete 3x4 array at position 5 using one yellow and ten blue tiles. When asked to predict the number of tiles for the 10th position, she coordinated two internal number lines to skip count by 2s (to represent adding 2 tiles each time) and simultaneously counted up by 1s (to keep track of the position number). She did this without using physical position cards or tiles.

Isabelle: [Pointing to the 5th position card – which had no tiles] 5…ten, 6 would be twelve [still pointing at the 5th position card because there are no other position cards on the table] [shifts her finger to the right of the 5th position card] 7 would be fourteen [shifts her hand to the right again] 8 would be sixteen, um 9 would be eighteen, and 10 would be twenty! Twenty blue and one yellow!

Isabelle incorrectly predicted that the 100th term would have 102 tiles.

NEAR and FAR prediction using explicit reasoning. Two Grade 1 and three Grade 2 students perceived the numeric structure of the pattern and recognized that the value of the tiles was double the value of the position number. They were able to predict the number of tiles at the 10th position using this reasoning, for example "because 10 plus 10 equals 20 blue and then one yellow." Three of these students were unable to make a FAR prediction and we observed that they were unable to accurately build the 5th and 6th position of the pattern (they used the correct number of tiles, but placed them randomly at each of the positions). This suggests that, although they understood the numeric structure of the pattern, these three students did not attend to the visual structure.

Two of these students (one Grade 1 and one Grade 2) built 5th positions that matched the other iterations of the task pattern. These students made accurate FAR predictions. For example, Chloe (Grade 1) explained the relationship she saw between the position card and the number of tiles when predicting positions of the pattern.

Teacher: How did you know there were going to be ten blue tiles at the 5th position?

Chloe: Because the ones on the top [touching the tiles] it's what they equaled. Like 1 plus 1 equals 2 [pointing to position cards and tiles], and 2 plus 2 equals 4, 3 plus 3 equals 6, and 4 plus 4 equals 8. So this one [pointing to the 5th card] would be 10 blue and one yellow. Because you look at the [position] card and that's what it equals.

Teacher: How about at the 100th position?
Chloe: 200 blue and one yellow.
Teacher: What about at the thousandth position?
Chloe: Two thousand blue and one yellow.

Post Interview Results

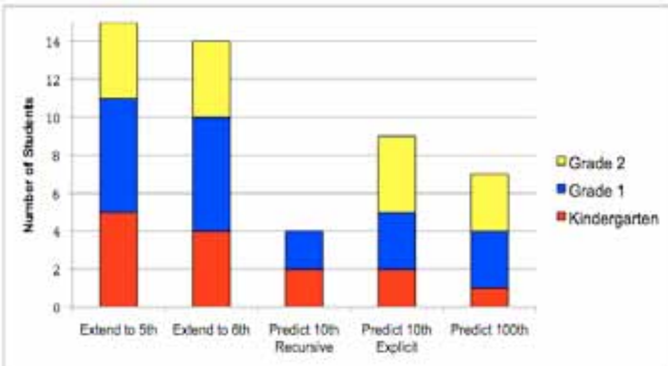

Figure 5: Post Interview Results

Results of the post interviews are shown in Figure 5. As shown in the graph, more students were able to make accurate NEXT, NEAR and FAR predictions during the post interview, and more students used explicit reasoning when making their predictions.

NEXT and NEAR predictions using recursive reasoning. When making NEXT predictions, six of the students demonstrated recursive reasoning that combined an understanding of the numeric structure (increasing by two) with a perception of the visual pattern structure as increasing rows of two tiles. Four of these students continued to use recursive reasoning to make NEAR predictions.

NEXT, NEAR and FAR predictions using recursive *and* explicit reasoning. When making NEXT and NEAR predictions, nine students used both recursive and explicit reasoning based on a shifting perception of the visual structure of the pattern (Figure 6). One perception, the rows perception, demonstrated an understanding that the pattern was increasing by successive increments of 2. This was evident when students justified their predictions either by stating the pattern "goes up by 2 each time" or by skip counting by 2s. Simultaneously these students saw the pattern structure as two columns of tiles with each column composed of a number of tiles equal to the position number. This perception was aligned with explicit reasoning as students justified their predictions by making connections between the position number, and the number of tiles in each column of the pattern. During the interviews, these students went back and forth between these two perceptions.

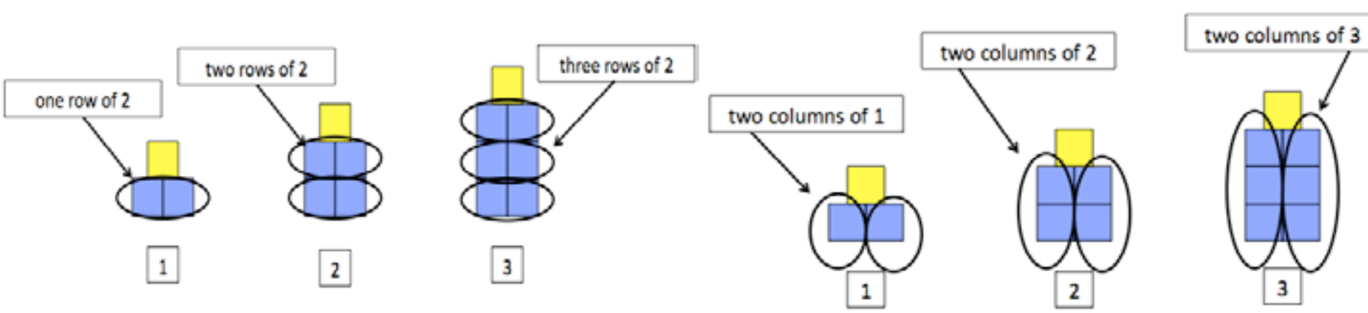

Figure 6: Two Pattern Perceptions

For example, Jesse (Grade 1) demonstrated how he used both the perception of the pattern as growing by one row of 2 tiles at each term, and of the pattern as two columns of tiles (with each

column equal to the value of the position number) to make his predictions and justify his reasoning.

Teacher: [Built the first three positions]. How many am I going to need in the fourth position?
Jesse: Eight blues and one yellow! Because I know that for this one [pointing to the third position] you were going to put six because three plus three equals six [pointing to the two columns of three tiles]. And four plus four [pointing to the position card] is eight [pointing to the two columns of 4 tiles].
Teacher: What would you build in position 5?
Jesse: [Mouthing five plus five] ten!
Jesse then built the 5th position.
Teacher: How did you know how many tiles went in the 5th position?
Jesse: [Jessie uses two fingers to count by 2s from the bottom pair of tiles to the top pair.) Because you go 2,4,6,8,….then 10! Going up by twos!
Teacher: How many blue tiles would you need for the 10th position?
Jesse:[Points to position card 10] Ten plus ten is 20!

Six of the students also used the columns perception when making FAR predictions. For example, Ava (Kindergarten) took a column approach when the teacher asked her to predict how many blue tiles there would be at the 100th position and indicated that "there would be 100 up one side and then another row of 100, so two hundred blue and a yellow."

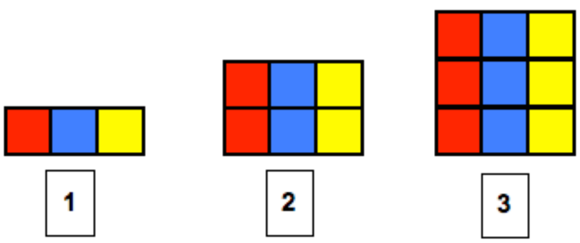

Figure 7: Ava's Pattern

To further probe her thinking, Ava was asked to build a pattern. She constructed a growing pattern with one column of red tiles and a column of blue tiles (Figure 7). The number of tiles in each column was equal to the number on the position card. The teacher then added a third column of yellow tiles and built the pattern to position 3. She asked Ava to predict how many tiles would be needed for the 10th position. Immediately Ava said, "Thirty!"

Teacher: How did you know that?
Ava: Counting by 10s! Ten, twenty, thirty!
Teacher: So if you had four colours, how many tiles would you have at the 10th position?
Ava: [laughs] Forty!
Teacher: What else that can you tell me?
Ava: If there's five colours, then fifty!

Discussion

The results of the pre-interview indicated to us that young students have the potential to work with growing patterns in a meaningful way, and can successfully focus attention on the relationship between two sets of data such as position cards and tiles. The initial analysis of the pre-task results indicated that eight of the fifteen students accurately made NEAR predictions based on their recognition of the numeric and/or visual structure of the pattern. After working through a series of five lessons that specifically focused on identifying the relationship between position number and number of items at the position, and the numeric and physical structure of linear growing patterns, we saw an increase in the number of students who demonstrated an ability to recognize and describe predictable growth, extend a linear growing pattern accurately,

and make NEAR and FAR predictions. During the post-interview thirteen of the fifteen students were able to make NEXT and NEAR predictions, and seven were able to make FAR predictions. This indicates that the initial difficulties some students experienced may have stemmed from a lack of experience, and not from a lack of capability. Their earlier instruction has focused almost exclusively on repeating patterns, which influenced their initial work with growing patterns. This was evidenced by such practices as describing the growing pattern as a repetition of one variable (for example, Jesse's description of the fifth position by articulating the word "blue" ten times).

Although this was an exploratory research project, and the teaching intervention was relatively brief, our results suggest that even very young students may be capable of functional thinking – an idea first posited by Blanton and Kaput (2004). There were some critical components of the teaching sequence that may have supported this capability. During the lessons an emphasis was placed on the relationship between the position number and number of tiles (or other elements) at each position. Students were also asked to make predictions from the NEXT term to NEAR and FAR terms in order to focus on the relationship between the two. Another important component was the use of arrays to represent different growing patterns, and the connection between the position number and number of "groups" in the array. Unlike typical pattern instruction for this age level, the instruction emphasized multiplicative rather than additive thinking, which is necessary for developing an understanding of functional relationships.

Of interest was the "perceptual agility" (Lee, 1996) demonstrated by nine of the fifteen students when making their NEXT, NEAR and FAR predictions. Perceptual agility is the ability to perceive a pattern in multiple ways, and utilize one or more perception as the basis for making generalizations. Because we had incorporated the use of arrays into our instructional design, students developed two ways to consider the growing pattern used during the assessment interview. As the teacher built the pattern for the interview task, she emphasized the series of rows of two tiles. However, the students were able to perceive it as either a pattern of increasing rows of two, or as a pattern of two columns of tiles equal to the position number. The "rows" perception seemed to be connected to recursive reasoning, adding a row of two more tiles each time. Students with this perception considered the variation in one data set and identified the recursive relationship within the pattern. The "columns" perception, on the other hand, seemed to allow students to make the leap to explicit reasoning by considering the pattern as two columns with each column composed of a number of tiles equal to the position number. Students identified the relationship between the position number and the number of elements at each position as evidenced by their ability to make NEAR and FAR predictions without having to rely on "adding two more tiles each time." This was demonstrated by Ava's recognition that at the 10^{th} position of the pattern that she built, if she had 5 colours of tiles there would be 50 tiles. Rather than thinking of ten groups of 5 (rows), she was clearly thinking of 5 groups of 10 (columns) and used that perception in her prediction.

Also of interest is the development of an awareness of visual and numeric structure. This was evident as we analyzed the change in students who had trouble making NEAR and FAR predictions during the pre-interview. Some students could extend the pattern in a way that was visually accurate with two columns of blue tiles and one yellow tile on top, but was not accurate numerically. Other students could make numeric NEAR predictions, but did not represent the visual structure of the pattern accurately, which hampered their ability to make FAR predictions (most of these students predicted 102 tiles would be needed for the 100^{th} term). During the post-interviews, all of the students who made accurate NEAR and FAR predictions had developed an

ability to recognize both the visual and numeric structure of the pattern. This is important because previous research suggests that the ability to recognize both numeric patterns and visual pattern structure are key elements in developing algebraic reasoning (Arcavi, 2003; Berch, 2005; Mulligan & Mitchelmore, 2009; Papic et al., 2011).

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INVESTIGATING INTEGER ADDITION AND SUBTRACTION: A TASK ANALYSIS

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Fifteen elementary and secondary teacher candidates solved sixteen integer addition and subtraction problems during think-aloud interviews. Investigators further probed participants' solution strategies as well as what they noticed first when starting a new problem. Task analyses of participants' solutions led to the creation of two distinct maps detailing the procedural relations among and conceptual elements underlying the six problem types. The task analysis results indicate that participants relied on procedural knowledge and had weak strategic knowledge with little focus on number values in their four main solution strategies. These findings highlight possible areas for instruction to support conceptual understanding.

Keywords: Number Concepts and Operations, Cognition

Purposes of the Study

Students' transition from working with whole numbers to integers in middle school causes cognitive dissonance. They have to reinterpret the meaning of the minus sign (Bofferding, 2010; Vlassis, 2004) and the meanings of addition and subtraction (Bofferding, 2010; Bruno and Martinon, 1999), both of which play a role in how they solve integer problems (Bofferding, 2012). Researchers have explored the relative difficulty of integer problems based on percent correct data and detailed some of students' difficulties in solving the problems (e.g., Murray, 1985); however, their descriptions of students' solution methods are inconsistent, incomplete, focus on a few problem types, and/or pertain to specific contexts. Moreover, the existing studies provide little explanation for how students correctly solved problems or whether their solutions were mathematically rich. Lack of detail regarding students' solutions is troublesome because students can get correct answers to integer problems using incorrect methods (Tatsuoka, 1983).

Developing a classification scheme of integer problems and typical solution paths that students take to solve these problems could unite research efforts in this area. Certain solution paths may draw on procedural thinking while others tap into more complex types of knowledge (e.g., schematic or strategic); such information has important instructional implications. Further, a detailed framework could help teachers develop models of their students' thinking processes and better equip them to adjust their instruction to build on students' knowledge, as was found with Cognitively Guided Instruction (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989).

Research Questions

This paper begins to address these issues by exploring the processes and knowledge involved in solving integer arithmetic problems. Specifically, we explore the following questions:

1) What are the processes through which teacher candidates (who may later teach this topic) solve integer addition and subtraction problems?
2) What knowledge (declarative, procedural, schematic, strategic) do students use to solve the integer problems, and how does this relate to the thinking processes students use?

Theoretical Framework

Negative Number Instruction

One area of insight into how students might solve integer problems comes from the

instruction they receive. Two typical methods for teaching addition and subtraction with negative numbers are movements on the number line and cancellation. Common rules for moving on the number line include the following: when you add, move forward; when you subtract, move backward; if you are adding or subtracting a positive number, face the positive numbers; if you are adding or subtracting a negative number, face the negative numbers (Liebeck, 1990). Given these rules, a student would likely start at whatever number is first in the problem, pretend to face the direction of the second number's sign, and then move up (forward) or down (backward) on the number line or counting sequence.

A common way to think about cancellation is through the use of chips, where 1 positive chip cancels out 1 negative chip. People who use this method might focus on the amounts of the two numbers in the problems. Further, they will need to have rules for dealing with situations when they have to take away more positives or negatives than are originally given (e.g., -3 - -5) (Liebeck, 1990). Other forms of cancellation instruction use different contexts for exploring the canceling operations, such as colored beads on a double abacus (Linchevski & Williams, 1999), cutting and gluing input and output trains (Schwarz, Kohn, & Resnick, 1993), and adding and removing balloons and weights to a hot air balloon (Janvier, 1985); however, the rules remain the same.

Results from instructional and interview studies provide a sample of strategies students use to solve integer problems. For example, to solve -4 + -3, a student might add 4+3=7 and then add a negative sign to get -7 (Bofferding, 2010). Students might also count down to solve problems like 3-5 (Bishop, Lamb, Philipp, Schappelle, & Whitacre, 2011). Tatsuoka (1983) created a system to classify students' response patterns (particularly in terms of their erroneous rules) to a series of integer problems based on possible rules students might use to manipulate the absolute values and signs of the numbers. Although they might not directly match students' thinking, results such as these can be used as a starting place for developing a map of how students solve integer problems.

Task Analyses

Another way researchers have attempted to characterize students' solution processes for solving mathematics problems is through the use of task analysis, "a decomposition of a complex task into a set of constituent subtasks" (Gardner, 1985, p. 157). Recent task analyses – such as the work of Piaget and others – focus on analyzing students' mental processes and conceptions as they engage in tasks (see Resnick, 1976, for a more detailed description of the history). Thus, task analysis is a useful tool in cognitive research (Siegler, 2003) for mapping out the procedures students use and highlighting the conceptual knowledge needed to solve problems (Greeno, 1976).

There are several reasons why task analyses are necessary (especially if modeled after students' solution processes). First, they identify conceptual prerequisites or structures that underlie instructional outcomes (Anderson & Schunn, 2000; Greeno, 1976). For example, Griffin, Case, and Capodilupo (1995) identified that knowledge of number order (such as on a number line) serves as an essential support for other number concepts; they found that when given explicit instruction on the number line, low-performing students caught up to their high-performing peers in mathematics. Task analyses can also be used to determine where students are likely to deviate from correct procedures, which can help teachers target instruction (Anderson & Schunn, 2000). Additionally, from a series of related arithmetic problems, researchers can use task analysis to determine which problems elicit more potential methods and which strategies are most efficient. This information can help instructors identify problems that

might engender better classroom discussions about strategies.

Mathematical Proficiency and Knowledge Types

Not only are the processes through which people solve mathematics problems important but so are the types of knowledge they use in relation to the strands of mathematical proficiency, which is our goal for students in mathematics classes (National Research Council, 2001). The first strand, conceptual understanding, relates to understanding why a procedure works and can help students avoid errors. Further, students with conceptual understanding "understand why a mathematical idea is important and the kinds of contexts in which it is useful" (National Research Council, p. 118). Students with procedural fluency, the second strand, know when and how to use procedures to solve a wide range of problems efficiently and accurately; students who use procedures without conceptual understanding are likely to make errors and lack procedural fluency (National Research Council). Strategic competence, the third strand, involves understanding problems and knowing which strategy is best to use. Further, students with strategic competence understand how problems are related. Adaptive reasoning is the fourth strand, which refers to students' ability to justify their answers, and the final strand, productive disposition, includes "see[ing] sense in mathematics" (National Research Council, p. 131).

The four types of knowledge underlying these interwoven strands require different amounts of cognitive demand. Declarative knowledge is knowledge of facts, which are memorized and recalled. Procedural knowledge requires more cognitive effort and involves knowing the steps for solving a problem. On the other hand, schematic knowledge involves knowing why certain processes or strategies work. Finally, strategic knowledge, or knowing about knowing, includes monitoring one's own knowledge and knowing when a particular strategy is the optimal one to use (Li, Ruiz-Primo, & Shavelson, 2006).

Methods

Participants and Site

Participants in the study included fifteen teacher candidates (8 secondary-focused, 7 elementary-focused) taking courses in a teacher education program at a Midwestern state university. All had taken some mathematics content courses at the undergraduate level. Two of the secondary student participants were male, while the remaining thirteen people were female.

Materials and Data Collection

In order to access the teacher candidates' solution methods as they solved the integer problems, we conducted individual think aloud interviews. Participants were asked to "say everything that you might say to yourself silently or think in your head" as they solved 16 naked number addition and subtraction problems involving integers (see Table 1). Although the use of think-alouds slows processing time when solving problems, they do not change their thinking (Ericsson & Simon, 1993; Ericsson, 2006). Participants started with a practice question to get used to talking out loud while solving the problems. One or two researchers conducted each interview and took notes about participants' solutions. Follow-up questions asked after they solved all of these problems included clarifying questions about their strategies as well as the following: "Which questions were the easiest? Why?" and "When you looked at a new problem, what is the first part you noticed? Which part is most important? Why?" Their work for each problem was collected, and interviews were audio and videotaped, then transcribed.

Data Analysis

To analyze the data, we first developed an initial task analysis of the steps we thought were needed to solve the problems based on accounts of students' strategies in the research and the common ways of teaching integer addition and subtraction. The task analysis was adjusted

throughout the coding process, based on participant responses. Rather than making separate task analysis maps for each procedure students might use, we combined them into a nondeterministic map; therefore, a student could take one of many paths through the map (Greeno, 1976).

During the coding process, we found that determining how people were “starting” the problem, or what they noticed first, was too open to interpretation to be used as the first step in the task analysis. Therefore, we reorganized the analysis based on identifying the problem type as the first step. As we read the transcripts of each participant’s solution steps, we coded each step in their processes based on the final task analysis maps. We double-coded about 20% of the naked number problems with >85% agreement, then discussed differences, and coded another 14 problems with >90% agreement.

Table 1. Integer Problems Solved and Organized by General Problem Type

Addition: Two Negatives	Addition: Negative, Positive	Subtraction: Negative, Positive	Subtraction: Two Negatives	Subtraction: Two Positives
-7+-1	-2+7, -4+6	-3-5, -5-9	-4 - -7, -6 - -9	3-9, 6-8
	-9+2	4 - -5	-8 - -5	
	5+-2	5 - -3, 9 - -2	-8 - -8	

Results

Task Analysis

The final task analysis consists of two maps, showing the six general problem types (those shown in Table 1, as well as Addition: Two Positives), the relations among them, and branches showing the sequential steps participants made when solving problems of these types. Map 1 consists of addition of two positive numbers, subtraction of a positive and negative number, and addition of two negative numbers. These problems are related because problems of one type can be changed into one of the other types using sign rules (e.g., 5+3 is equivalent to 5 - -3, and -2-7 is equivalent to -2+-7). Map 2 consists of subtraction of two positive numbers, addition of a positive and negative number, and subtraction of two negatives. The problems are also related (e.g., 3-9 is equivalent to 3+-9, which is equivalent to -9 - -3). If a person solved -2+7 by identifying the first number as negative, switching the order of the numbers (7+-2), and changing the operation using sign rules (7-2), the map would show the following connections: Addition, Negative and Positive → 1st Number Negative → Commute → Change to Subtracting a Positive → Subtraction, Two Positives → Recall.

Solution Processes by Problem Type

Based on coding from the task analysis, certain problems stand out as related based on the similar ways in which the participants solved the problems. Across participants, the most consistently solved problem type was subtracting a negative number from a positive one (4 - -5, 5 - -2, and 9 - -2). For all three of these problems (except in 1 instance), they identified the second number as negative, changed the problem into adding a positive, and recalled the answer.

Results for subtracting a positive number from a negative one were less consistent. J02 nicely explained how she solved -3-5: “I'll make the five a negative, instead, and so then all I have to do is add the actual numbers together and then just make sure that the sign after that is negative.” Most participants used this reasoning – solving the problem as they would if adding two negatives – for both -3-5 and -5-9. However, two people for -3-5 and five people for -5-9 focused on number order, starting at the negative number and counting down as they would for a normal subtraction problem. One of them thought of this as a hole that was getting deeper.

Subtracting a larger number from a smaller one (3-9 and 6-8) also had fairly consistent responses from participants. The most popular response was to indicate that the larger number was second, subtract the smaller from the larger, and make the answer negative. On 6-8, A05 used zero as a landmark number: "I'll go back 6, then I'll be at 0. Then I'll have 2 more to go, so that's -2." One person for 3-9 and two for 6-8 changed the problems into adding a negative (3+-9 and 6+-8) then either recalled the answer, started at the negative number and counted toward zero, or canceled equal numbers of positives and negatives to get the resulting negative answers.

Although -2+7, -4+6, -9+2, and 5+-2 appear similar because they involve adding a positive and negative number, participants treated -9+2 very differently. For the other three problems, participants typically changed the problems into subtraction by using the commutative property to switch the order if needed (7+-2 and 6+-4, although only a few verbalized this step), changing the problem into subtracting a positive (7-2, 6-4, 5-2), and recalling the answer. Similarly, when solving -4 - -7 and -6 - -9, participants first changed these into adding positive numbers (-4+7 and -6+9) and then followed the same steps.

Changing -9+2 into a subtraction problem was rare because the number with larger absolute value was negative, and in general, the participants did not choose to subtract larger numbers from smaller ones. Instead, on this problem, they either started at -9 and counted up 2 or subtracted 2 from 9 and used the sign of the number with the larger absolute value, reporting the answer -7. Likewise, when solving -8 - -5, students first changed the problem into adding a positive number (-8+5) and followed the same steps as -9+2.

These results present a much more complicated picture of the interrelationships among the problems (see Figure 1) than shown in Table 1. Solid lines indicate problems that are part of the same category from Table 1. Solid arrows indicate problem types that can be changed into each other, and dotted arrows indicate problem types that can be solved using similar processes.

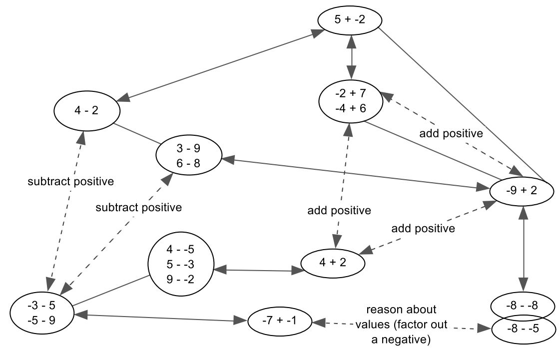

Figure 1: Relations Among Problems

Participants' Solution Process Consistency

Although solutions for the problems were fairly consistent, participants approached the set of problems in four main ways. Five of the participants (A01, A03, A06, O01, R02) solved problems in a way that suggests they preferred to solve problems with *all positive or all negative numbers*, unless it made the problem more complicated. For example, when solving -2+7, they changed the problem into 7-2. In some cases they did not explicitly change -9+2 because they knew the answer or because they knew they could solve 9-2 and add a negative sign. For -3-5 and -5-9 they solved them as addition (-3+-5), adding the absolute value of the numbers and making the answer negative. Also, although problems like -4 - -7 contain two negative numbers, participants still changed them to the form 7-4.

A second group of participants (A02, A05, A04, K01, K02, R01) solved or changed problems so that they could *add or subtract positive numbers*. Therefore, although all of them changed 5+-2 into 5-2, none of them changed -9+2. Further, none of them changed -3-5 or -5-9; instead, they talked about starting at the negative number and going back the positive number

value. The two participants who talked about solving -3-5 and -5-9 by adding the absolute value of numbers and making the answer negative also changed -7+-1 into -7-1 before solving it.

Participant K03 had a different approach than the others; she changed problems so that she could always *add positive and/or negative numbers* because she mainly used a cancellation strategy. She kept -2+7, -4+6, -9+2, and 5+-2 as is and reasoned about the relative number of positives versus negatives to determine what would be left over if each positive cancelled out a negative. Further, she changed 3-9 and 6-8 into 3+-9 and 6+-8 and changed problems like -4 - -7 into -4+7 using sign rules in order to use the same cancellation strategy. For the other problems, she either added two positive numbers (e.g., 4 - -5 became 4+5) or added two negative numbers (e.g., -3-5 became -3+-5, and -7+-1 stayed the same).

Unlike participant K03 and the others, participant J02 primarily focused on starting with the number of larger absolute value. When the number of smaller absolute value was first, she changed the format of the problem. For example, on 6-8, she changed it to 6+-8 and then further changed it to -8+6. However, when solving -2+7, she flipped the numbers to 7+-2 and also changed it to 7-2. This suggests she preferred adding positive numbers, which is confirmed by her statement, “I think adding is easier than subtracting, trying to subtract negative numbers.”

Finally, O02 and J01 did not use a consistent way to solving the problems. For example, they both solved -4+6 by starting at -4, adding the opposite (4) to get to 0, and determining the answer was 2. However, on a similar problem, -2+7, they changed it into 7-2.

Knowledge Types

Overall, participants drew heavily on procedural knowledge when solving the problems; however, they sometimes lacked procedural fluency due to insufficient schematic knowledge. For example, several participants changed problems such as 5 - -3 into addition problems by drawing vertical lines through the minus and negative signs (5 + + 3). When asked how they knew they could do this, they referred to declarative rules rather than conceptual explanations: “When you have two minus signs you can put them together to make a plus sign” (A02) or “You can slash it but then you dash it…cause the minus minus makes a plus” (A06). When asked, “That’s just a rule that you know?” most agreed and restated the rule or indicated that they were told this rule as a child, and one replied, “The theory behind that, I don’t [know].” The one participant who was more successful at explaining how subtracting a negative was similar to adding a positive drew on a description of negative amounts as holes underground, explaining, “Subtracting a negative number, so that's like subtracting part of a hole, which is like adding” (A04).

In limited cases, participants’ lack of schematic or conceptual knowledge led them to change the problems incorrectly (solving -9+2 as 9-2) or use sign rules incorrectly. For example, one participant changed -2+7 into 7-2=5 and answered negative five, claiming, “Because I’m taking away from a negative number,” suggesting she may have thought 7 was negative originally or that because -2 was the first number in the original problem, the answer should be negative.

One of the most striking patterns in participants’ responses was their lack of focus on the values of the numbers, resulting in inefficient methods and suggesting weak strategic knowledge and possibly a lack of conceptual understanding of integers. For -8 - -8, only one person stated that “When you subtract a number from itself you get zero” (A02). All others changed the problem to -8 + 8, and rather than use the additive inverse property, two people further changed the problem to 8-8 before answering 0. Further, although participants solved -7+-1 by solving 7+1 and adding a negative, they never used the same reasoning to solve -8 - -5 as -(8-5).

Discussion and Implications

The results of the task analysis and participant responses illuminate ideas that teachers need to keep in mind when teaching about integers and suggest concepts that students need to know as they learn integers. A prominent way that the participants solved the problems was to change the problems into a different form. Some of them used the phrase "slash and dash" to remember when and how to change the problems, but they had difficulty explaining the meaning behind it. The National Research Council (2001) warns against this type of operating:

> Mnemonic techniques learned by rote may provide connections among ideas that make it easier to perform mathematical operations, but they also may not lead to understanding. These are not the kinds of connects that best promote the acquisition of mathematical proficiency (p. 119).

Rather, if we want students to develop conceptual understanding of integer operations, teachers need to know how to help students meaningfully explain the relation between the operations.

One potential way teachers can help students make these connections is through the use of contexts or the number line. Stephan and Akyuz (2012) successfully used the idea of net worth to help students make sense of losing debt as gaining overall net worth. Further, participants in the study were able to use movements on the number line to add or subtract positive numbers from negative ones. However, they did not use it to add or subtract negative numbers, suggesting that number line instruction needs to address these situations more effectively. By using the number line, the holes context, and cancellation reasoning, some participants broke up numbers in strategic ways, making zero and then adding or subtracting the rest. This method promotes numerical reasoning and works for many problems – especially when changed into the form of adding a positive and negative number – and highlights the importance of zero.

There were a few concepts in particular that participants used frequently and that would likely be important for students to know as they begin to work with integers: relative number values (including number order) versus absolute values and the commutative property. When solving problems, such as -9+2, some participants decided whether the answer would be closer to or farther from zero based on the relative values of the two numbers. On the other hand, when adding a positive and negative number, participants often compared the absolute values of the two numbers to determine if there was enough of the positive number to cancel out the negative number or to move from the negative number through zero. Knowledge of opposites also helped participants solve -8 - -8 as -8+8; however, a stronger focus on number values could have helped them solve the problem more efficiently.

Participants used the commutative property when they changed problems like -4+6 into 6-4; however, they rarely explained this intermediary step (6+-4). In fact, they never explained why they could solve -9+2 as -(9-2) either. When discussing these processes with students, teachers would need to be more explicit about the reasoning behind these steps or students might develop misconceptions about the operations and sign, such as believing -9+2 could be solved as 9-2.

Acknowledgements

This research was funded by a Purdue Research Foundation Year-long Grant.

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CORRESPONDENCE AND COVARIATION: QUANTITIES CHANGING TOGETHER

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Exponential functions are important topic in school algebra and in higher mathematics, but research on students' thinking suggests that understanding exponential growth remains an instructional challenge. This paper reports the results of a small-scale teaching experiment with students who explored exponential functions in the context of two continuously covarying quantities, height and time. We present two major conceptual paths that occurred in the development of an understanding of exponential growth, the covariation view and the correspondence view, and discuss the influence of each perspective on the growth of students' understanding.

Keywords: Algebra and Algebraic Thinking, Learning Trajectories, Middle School Education

Introduction

Exponential functions are an important concept in school algebra, representing a significant transition from middle school mathematics to the more complex ideas students encounter in high school. A focus on the conceptual underpinnings of exponential growth has increased in recent years; for instance, the Common Core State Standards in Mathematics (2010) highlight the need to understand exponential functions in terms of one quantity changing at a constant percent rate per unit interval relative to another. Moreover, these ideas also now appear at the middle school level, both in terms of national standards as well as in middle school curricula (e.g., Lappan et al., 2006).

Implementing this vision, however, may prove challenging given students' documented difficulties in understanding exponentiation. Students struggle to transition from linear representations to exponential representations, to identify what makes data exponential, or to explain what a function such as $f(x) = a^x$ means (Alagic & Palenz, 2006; Weber, 2002). Pre-service teachers have not fared much better, struggling to recognize growth as exponential in nature, to connect a repeated-multiplication understanding to the closed-form equation, and to generalize the rules of exponents (Davis, 2009). In general, understanding exponential growth appears to be challenging for both students and teachers (Davis, 2009; Weber, 2002).

These documented challenges suggest a need to better understand how to foster students' learning about exponential growth, particularly in a way that goes beyond merely highlighting the skills students should acquire. This paper reports on the results of a teaching experiment emphasizing exponential growth in the context of two continuously covarying quantities. We introduce two major conceptual paths that occurred in the development of an understanding of exponential growth, the covariation view and the correspondence view, and discuss the influence of each perspective on the growth of students' understanding.

Background and Theoretical Framework

The Rate-of-Change Perspective

A popular approach to function relies on the correspondence perspective (Smith, 2003), in which a function is viewed as the fixed relationship between the members of two sets. Farenga and Ness (2005) offer a typical correspondence definition of function: "One quantity, y, is a function of another, x, if each value of x has a unique value of y associated with it. We write this as $y = f(x)$, where f is the name of the function" (p. 62). This static view of function underlies much of school mathematics, in which a focus on developing correspondence rules between x and y enables students to solve prediction problems, construct and interpret graphs and tables, and determine roots and missing values.

Smith and Confrey (Smith, 2003; Smith & Confrey, 1994) offer an alternative to the correspondence view, which they call the covariation approach to functional thinking. Here one examines a function in terms of a coordinated change of x- and y-values, where moving operationally from y_m to y_{m+1} is coordinated with movement from x_m to x_{m+1}. Confrey and Smith (1995) have found that students' initial entry into a problem is typically from a covariational perspective; moreover, viewing a function as a representation of the variation of quantities can be a powerful way to promote an understanding of rates of change (Slavit, 1997; Smith & Confrey, 1994).

Castillo-Garsow (2012) describes covariation as the imagining of two quantities changing together; students imagine how one variable changes while imagining changes in the other. Relying on situations that involve quantities that students can manipulate and investigate can foster their abilities to reason flexibly about dynamically changing events (Carlson & Oehrtman, 2005). An approach that leverages covarying quantities may be especially useful in helping students understand exponential growth, as this view is strongly connected to how students think about contexts involving multiplicative relationships (Davis, 2009).

Hypothetical Learning Trajectories

The notion of a *hypothetical learning trajectory* has different meanings among mathematics education researchers. Under Simon's (1995) original description, a hypothetical learning trajectory consists of "the learning goal, the learning activities, and the thinking and learning in which students might engage" (p. 133). Clements and Samara (2004) elaborate on this definition, describing a learning trajectory as a description of children's thinking and learning in a specific mathematical domain connected to a conjectured route through a set of tasks designed to support their movement through the progression of levels of thinking. This paper makes use of a hypothetical learning trajectory for exponential functions (reported elsewhere in Authors, 2013) that is an empirically-supported model of the students' initial concepts and an account of how they changed over time (Confrey et al., 2009). Rather than reporting the trajectory itself, we focus on two conceptual paths represented in the trajectory: the covariation view and the correspondence view. Each path is exemplified through two students' work as they progressed through the teaching experiment.

Methods

The study was situated at a public middle school and consisted of a 12-day teaching experiment with 3 eighth-grade students (ages 13-14) in which the first author was the teacher-researcher. All students were female. The purpose of the teaching experiment was to support students' reasoning about exponential growth through exploring continuously covarying quantities. The sessions focused on the relationship between height and time for an exponentially growing plant called the Jactus; students were able to manipulate the plant's height over time using an interactive computer program called Geogebra. Although this scenario is not realistic,

the context is realizable (Gravemeijer, 1994) in that students could imagine, visualize, and mathematize the relevant quantities. All sessions were videotaped and transcribed.

We assumed that any understanding students might have about exponentiation before entering a teaching experiment would be dependent on an image of repeated multiplication. Building on that conception, our primary goal of instruction was to foster students' understanding of the following set of concepts for an exponential function $y = a \cdot b^x$:

1. The period of time x for the y-value to increase by the growth factor b is constant, regardless of the value of a or b.
2. There is a constant ratio change in y-values for each constant additive change in corresponding x-values.
3. The ratio of the change in y, which can be expressed as $\frac{y_2}{y_1} = b^{x_2 - x_1}$, is always the same for any same Δx and the value of $f(x + \Delta x) / f(x)$ is dependent on Δx.

Data analysis relied on retrospective analysis techniques (Simon et. al, 2010) to characterize students' changing conceptions throughout the course of the teaching experiment. Project team members developed preliminary codes for concepts on the trajectory based on students' talk, gestures, actions, and task responses as evidence of understanding at different stages. The first round of analysis yielded an initial set of codes, which then guided subsequent rounds of analysis in which the project team met as a group to refine and adjust the codes in relationship to one another; this iterative process continued until no new codes emerged. Once coding was complete we chose 20% of the data corpus for independent coding, which yielded an inter-rater reliability rate of 92%. For the purposes of this report, we focus on two students, Uditi and Jill, as their work provided a way to contrast the influence of a correspondence view versus a coordination view on subsequent concept development.

Results

The progression of the students' understanding of exponential growth occurred in three major stages, which we call *pre-functional reasoning*, *covariation reasoning*, and *correspondence reasoning*. The learning trajectory is depicted visually in Figure 1 for brevity's sake (for a complete report of all of the codes, associated tasks, and data examples see Authors, 2013.) Although pre-functional reasoning preceded the development of the covariation and correspondence views, the latter two ways of thinking did not occur in a sequential nature. While all of the students first developed an early, rudimentary coordination between height and time, they then diverged, some focusing on a more sophisticated covariation perspective, some gravitating towards a correspondence view, and others simultaneously developing both. An emergent goal of the teaching experiment was to support students shifting flexibly between these perspectives as needed.

In the following sections we report on two students, Jill and Uditi, as a way to exemplify a focus on correspondence versus a focus on covariation. All of the students readily identified repeated multiplication as the mechanism determining how the Jactus grew, but did not initially connect repeated multiplication in height to the unit of time. In an attempt to encourage coordination of the plant's height with the number of weeks it had been growing, the teacher-researcher introduced a task requiring students to draw the plant's height after 1 week and after 3 weeks if it doubled every week. For example, Jill's drawing indicated the beginning of her coordination of two quantities; both are present in her picture (Figure 2), and Jill explained that the plant's height for Week 4 would be "double the last week; like Week 4 would be 16 inches."

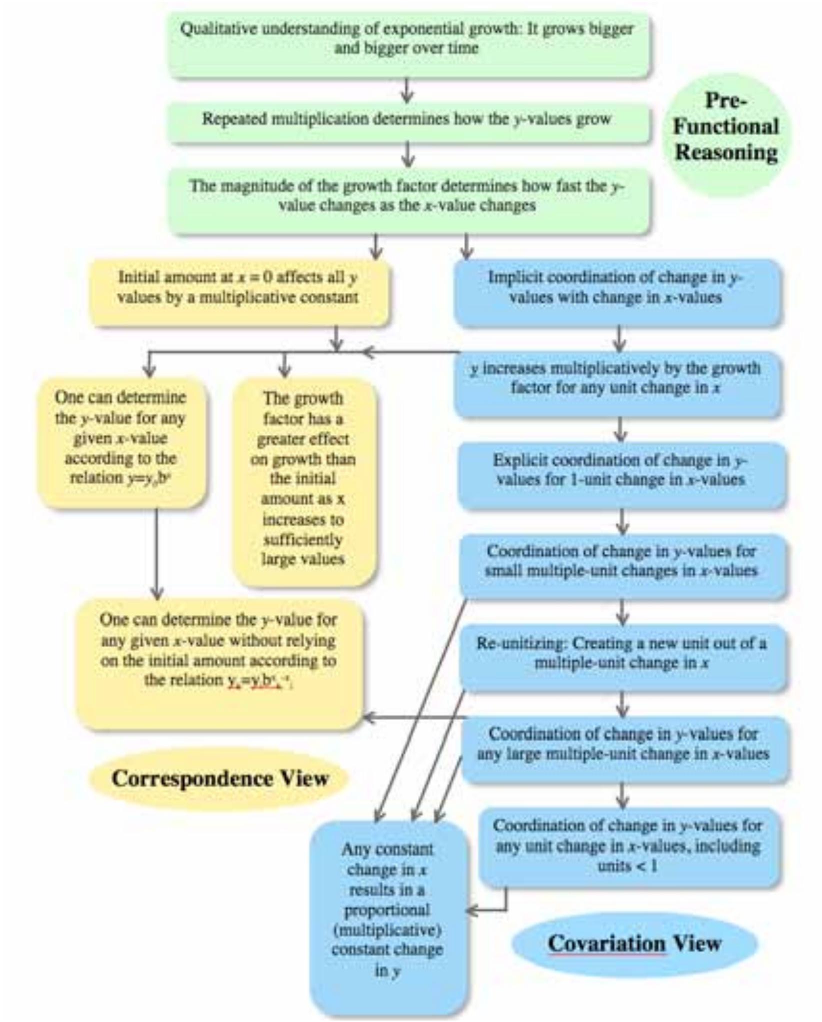

Figure 1: Graphical Representation of Exponential Functions Trajectory

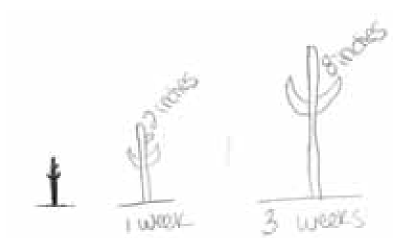

Figure 2: Jill's Drawing of the Growing Jactus

Jill: A Focus on Correspondence

Once Jill determined that the growth factor represented the multiplicative change in height per week, she shifted to a reliance on expressing this relationship algebraically. Jill first developed an algebraic relationship when provided with a data table in which she had to

determine missing height values (Figure 3):

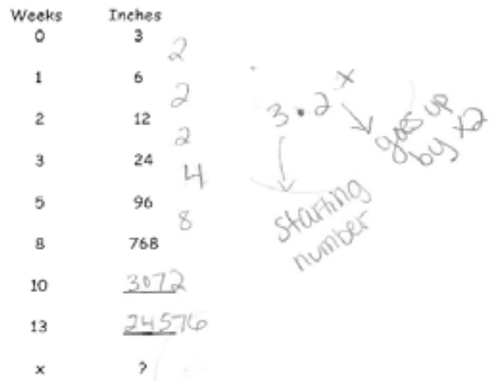

Figure 3: Jill's Strategy for Determining Missing Values in a Data Table

Jill appeared to determine the expression 3•2^x by guessing and checking to find the value of 3. She explained, "You times this number by 2 [justifying 2^x], and then I don't know why the 3 works." Through comparing different doubling plants with various starting heights, Jill began to develop an understanding that in order to determine the correct height at a given week x, she had to multiply 2^x by the initial height, which she called the "starting number": "Well, the starting number of the week 0 is 3, and then do 2 to the x because it's increasing by times 2 each week." In this manner Jill developed a conception of the equation y = ab^x as the "starting number" times "how much it goes up by", where "goes up by" refers to multiplicative growth.

Jill's understanding of the role of the initial height was as the value from which one begins the repeated multiplication process. Once Jill could reliably produce a relation between height and time, she made use of this correspondence rule to determine the growth factor. For instance, when Jill encountered a scenario with only three data points: (0; 1), (20; 1,048,576), and (25; 33,554,432), she explained, "I knew that it would be 1 times something to the x power so I started with 5 [for the missing growth factor] and that was just way too big, and so then I did 3 and it didn't work still, so I did 2 and then it worked and then I plugged in 20 [to check her answer]. So I did 1 times 2 to the 20th power and I got that [pointing to 1,048,576]." Jill's reliance on the correspondence rule shifted her attention to guess-and-check methods to find the missing values that filled in the "slots" in y = ab^x. Although this method is cumbersome, it allows one to avoid taking a ratio to determine the growth factor (for instance, determining the growth factor by taking the ratio of 33,554,432 to 1,048,576, which is 32, and solving the equation x^5 = 32.)

Jill's understanding of the correspondence rule was powerful in that it enabled her to determine missing values; however, it also constrained her thinking. For instance, Jill had to rely on the development of an equation in order to convince herself of the correctness of a growth factor, even when it would be easier to do so by more direct methods. Given three data points for a plant with an unknown growth factor [(8; 19,660.8), (15; 322,122,547.2), and (18; 20,615,843,020.8)], Jill took the plant's height at 8 weeks and multiplied it by different growth factors seven times until she hit upon the correct height value at 15 weeks. Although Jill determined the growth factor was 4 in this manner, she was not convinced of its correctness absent a correspondence rule. Jill therefore divided 19,660.8 by 4 eight times in order to determine the initial height of 0.3 inches. It was only when Jill wrote the equation 0.3 × 4^x that

she could check data points and gain confidence that the growth factor was indeed 4.

The correspondence perspective was so prevalent for Jill that she typically thought in terms of static height and time values rather than imagining how the plant could grow over time; this may be why Jill was unconvinced of her answer of 4 in the problem above. In another example, given a plant that grew 8 times as tall every three weeks, Jill was unable to determine, in absence of an equation, how the plant grew each week. Jill created a sample table of values depicting the plant’s height as 1 inch at 0 weeks, 2 inches at 1 week, 4 inches at 2 weeks, and 8 inches at 3 weeks. Despite the creation of this table, Jill was at a loss to determine the plant’s growth factor, explaining, “I tried to divide 8 by 3 but that didn’t really do anything.” Jill’s restriction to the correspondence view may have hampered her ability to see the growth factor; she may have viewed each entry in the table, such as (1,2), as a static height value, e.g. 2 inches tall at 1 week. A covariation perspective, in contrast, could enable one to also view the point (1, 2) in relationship to the prior point, (0, 1), and see that the plant has doubled in height from 1 inch to 2 inches in the span of 1 week.

Uditi: A Focus on Covariation

Like Jill, Uditi was able to describe the plant’s growth in Figure 2 in a covariational manner: she explained, “In one week it’s going to be 1 inch and in 2 weeks it’s going to be 2 inches, then in 3 weeks it’s going to be 4 and in 4 weeks it’s going to be 8.” When Uditi examined tables of data with jumps in weeks, she initially had to imagine the missing weeks in place. For instance, for Figure 4, Uditi imagined the missing Week 11 in the table: “For the 12 weeks I did times 4 for the answer [to week 11] then I did times 4 for that [to get the height at week 12].” In this manner, Uditi began to coordinate how the height grew with how the weeks grew, keeping track of both and mentally filling in the gaps when necessary.

Eventually Uditi began to formally coordinate the multiplicative growth in inches with the additive growth in weeks. For instance, given a table with a height of 956,593.8 inches at Week 14 and 8,609,344.2 inches at Week 16, Uditi marked a difference of 2 between the successive weeks and a ratio of “×9” between the successive inches. She then described her process: “I divided 8,609,344.2 by 956,593.8 and I got 9. Then I tried to figure out what number times itself = 9 because the difference between 14 and 16 is 2.” In this manner she concluded that the growth factor must be 3, but could not yet generalize her reasoning to any multi-week gap.

Weeks	Inches
0	0.1
2	1.6
5	102.4
9	26,214.4
10	104857.6
12	1677721.6

Figure 4: Uditi’s table of values

In order to encourage this generalization, we introduced tasks with larger gaps between the weeks, such as a plant with an unknown growth factor that is 256 inches at 4 weeks and 1,073,741,824 inches at 15 weeks (Figure 5). The 11-week gap was sufficiently large to discourage Uditi from relying on a mental image of repeated multiplication. Instead, Uditi took the ratio of the height values, which is 4,194,304, and wrote “___^{11} = 4,194,304”. She was then able to determine the missing growth factor that satisfied the equation was 4.

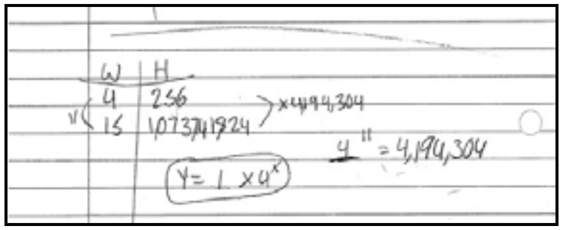

Figure 5: Uditi's strategy for determining the growth factor

In generalizing this approach, Uditi began to coordinate height and time values when $\Delta x < 1$. For instance, given two height values for Week 2.3 and Week 2.4, Uditi could divide them to find the multiplicative ratio, 1.12, and then write "$\_\_\_^{0.1} = 1.12$" in order to determine the growth factor. Tasks asking students to predict how the same plant would grow for different increases in time also fostered the covariation perspective for cases in which $\Delta x < 1$. One such task introduced a plant that tripled each week, asking students to consider how much larger it would grow in 1 day. Uditi responded to the question with "$3^{.14} = 1.17$", explaining, "I divided 1 week into 7 parts, which represents 1 day each and it's .14 of a week." In this manner Uditi was able to make sense of an expression with non-whole number exponent, viewing $3^{.14}$ not just as a static height value, as Jill did, but as an expression of the *growth* in height for 1 day.

Discussion

The expression b^x can both represent a static value (for instance, the height of a plant at time x) and a measure of growth (for instance, how much taller the plant will grow in x amount of time). A correspondence perspective encourages the first conception by emphasizing the direct relationship between height y and time x in the equation $y = ab^x$. Both Jill and Uditi easily understood the expression b^x as a specific height value at a given time. Both students also experienced difficulty in transitioning to an understanding of b^x as a representation of growth in height. We expected this difficulty with Jill, given her focus on correspondence relationships. However, Uditi also found this transition challenging despite her continued emphasis on coordinating multiplicative growth in height with additive growth in time. Uditi's strong early reliance on an image of repeated multiplication may have contributed to this challenge: When conceiving of b^x as a representation of b multiplied by itself x times, it is difficult to think about multiplying a number by itself a fractional amount of times. It is easier to conceive of the x in b^x as a "slot" in which one places the time value, as Jill did.

Uditi's ability to coordinate the ratio of height values with the additive difference in time values played a significant role in supporting the development of algebraic representations. Ultimately, it was easier for Uditi to shift flexibly to a correspondence view, as needed, than it was for Jill to shift to a covariation view. Moreover, Uditi's understanding of the correspondence rules reflected her covariation focus. She understood that when finding the height value at a specific time ($y = ab^{\Delta x}$), she was multiplying the height at week 0 (a) by the multiplicative growth for that time span ($b^{\Delta x}$). In other words, Uditi understood that in order to find the height of the plant, she needed to multiply by the growth factor for every time unit passed. Her reliance on covariation reasoning also enabled her to develop a more sophisticated understanding of the correspondence view in which she no longer needed to start the process from week 0. Instead, she was able to find the height value at a specific time based on the height value at any given week (different than week 0).

In general, a rudimentary level of coordination of growth in height as "doubling each time" with growth in weeks preceded every student's ability to develop correspondence rules of the form $y = f(x)$, which reflects Smith and Confrey's (1994) assertion that students typically approach functional relationships from a covariational perspective first. Our preliminary findings suggest that maintaining an emphasis on covariation may actually assist students' eventual construction and understanding of correspondence rules; therefore, a covariational focus may be the most efficacious in supporting students' understanding of exponential growth.

Acknowledgments

The research was supported in part by the National Science Foundation (NSF) under Award xxx. Any opinions, findings, and conclusions expressed in this material are those of the authors and may not necessarily reflect the views of the NSF.

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SLOPE: A NETWORK OF CONNECTED COMPONENTS

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In this study, we build on previous work on conceptualizations of slope to suggest a framework of slope across the landscape of the mathematics curriculum. Data from multiple studies on students', teachers', and college instructors' conceptualizations of slope are revisited in light of theory on procedural versus conceptual understanding and visual versus analytic interpretations of slope. This synthesis leads to a description of five key slope components. A detailed description of each component is provided and implications for future research are discussed.

Keywords: Algebra and Algebraic Thinking; High School Education; Middle School Education

Introduction

Slope is a key mathematical concept revisited throughout the mathematics curriculum. In addition to being an important mathematical concept of its own right, slope is an important prerequisite concept for advanced mathematical thinking, extending to include the notion of rate of change in precalculus and the concept of derivative in calculus (Carlson, Oehrtman, & Engelke, 2010; Confrey & Smith, 1995; Noble, Nemirovsky, Wright, & Tierney, 2001). Various studies suggest students struggle to understand slope (Barr, 1981; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Lobato & Thanheiser, 2002; Orton, 1984; Stump, 2001b; Teuscher & Reys, 2010; Thompson, 1994). In particular, research shows that students' knowledge of slope does not transfer between problem types (e.g., qualitative versus quantitative and mathematical versus real world settings) and that students do not relate the concepts of slope and rate of change (Hattikudur et. al., 2011; Stump, 2001b; Lobato & Siebert, 2002; Lobato & Thanheiser, 2002; Teuscher & Reys, 2010). Students' difficulties connecting notions of slope are understandable in light of the variety of conceptualizations emphasized in the mathematics curriculum. Moore-Russo and colleagues (Moore-Russo, Conner, & Rugg, 2011; Mudaly & Moore-Russo, 2011; Stanton & Moore-Russo, 2012) have suggested eleven conceptualizations of slope, outlined in Table 1, based on their own research and the earlier work of Sheryl Stump (1999, 2001a, 2001b).

Table 1: Conceptualizations of Slope

Category	Slope as …
Geometric Ratio (G)	Rise over run of a graph of a line; ratio of vertical displacement to horizontal displacement of a line's graph
Algebraic Ratio (A)	Change in y over change in x; ratio with algebraic expressions (often seen as either $\Delta y/\Delta x$ or $(y_2 - y_1)/(x_2 - x_1)$)
Physical Property (P)	Property of line often described using expressions like grade, incline, pitch, steepness, slant, tilt, and "how high a line goes up"
Functional Property (F)	(Constant) rate of change between variables; sometimes seen in responses involving related rates
Parametric Coefficient (PC)	The variable *m* (or its numeric value) found in $y=mx+b$ and $(y_2 - y_1)=m(x_2 - x_1)$
Trigonometric	Property related to the angle a line makes with a horizontal line; tangent of a

Conception (T)	line's angle of inclination/decline; direction component of a vector
Calculus Conception (C)	Limit; derivative; a measure of instantaneous rate of change for any (even nonlinear) functions; tangent line to a curve at a point
Real World Situation (R)	Static, physical or dynamic, functional situation (e.g., wheelchair ramp, distance versus time)
Determining Property (D)	Property that determines if lines are parallel or perpendicular; property can determine a line if a point on the line is also given
Behavior Indicator (B)	Property that indicates increasing/decreasing/horizontal trends of line or amount of increase or decrease; if nonzero, indicates intersection with x-axis
Linear Constant (L)	Constant property independent of representation; unaffected by translation of a line; reference to what makes a line "straight" or the "straightness" of a line

These 11 conceptualizations point to the diversity and complexity of slope in the mathematics curriculum. While past research has investigated the prevalence of each of these conceptualizations among individuals who have completed their K-12 education (Nagle, Moore-Russo, Viglietti, & Martin, *in press*; Moore-Russo, Conner, & Rugg, 2011; Mudaly & Moore-Russo, 2011; Stanton & Moore-Russo, 2012), the focus so far has been an isolated description of the conceptualizations without consideration of how they may fit together to form an individual's network of slope conceptions. In the theoretical contribution provided in this article, we reanalyze the 11 conceptualizations in light of one another and research on understanding. In particular, we combine research on procedural versus conceptual understanding with research on visual and analytic interpretations of slope to draw new connections between the conceptualizations. A review of the relevant research is followed by an application of these ideas to the 11 conceptualizations, resulting in a description of a slope conceptualization network with five underlying components.

Procedural versus Conceptual Knowledge

Since Skemp (1967) first described two distinct bodies of mathematics, termed relational and instrumental, considerable research has focused on what is better known now as procedural versus conceptual understanding. Hiebert and Lefevre (1989) describe procedural knowledge as knowing "how to" and conceptual knowledge as knowing "why". More recently, procedural knowledge was described as knowledge of rules and processes linked to specific problems while conceptual knowledge was said to involve a more flexible understanding of governing rules that can be transferred to various problems and situations (Rittle-Johnson, Siegler, & Alibali, 2001). While researchers have taken different stances on the relationship between these two types of understanding and the appropriate developmental sequence (e.g., conceptual then procedural, procedural then conceptual, or simultaneous development of the two), there seems to be consensus that both play an important role in a student's knowledge of mathematics (Geary, 1994; Halford, 1993; Hiebert & Wearne, 1996; Rittle-Johnson & Alibali, 1999).

Findings suggest many students hold only procedural knowledge of the algorithms and interpretations of slope in specific situations, such as rote application of "change in y over change in x" or "rise over run" to find the slope of a line given two points through which it passes. Although students might be able to identify pairs of lines as parallel or perpendicular based on the coefficients of the x-terms in algebraic representations of the form $y=mx+b$, they often do not understand how to use slope to determine the behavior of a linear graph or the

relationship between nonparallel and nonperpendicular lines. Reiken (2008) described students' understanding of slope as procedural and linked this procedural knowledge to students' view of slope as a number in five contexts: number from formula, number from counting, number in front of x, number as relator, and number as rate of change. While the number as relator (relating x and y) and number as rate of change conceptualizations may suggest some conceptual understanding, the remaining interpretations of slope as a number are procedural (Reiken 2008).

Visual Versus Analytic Interpretations of Slope

Students' strategies for solving mathematical tasks are also of interest to researchers. Lowrie and Clements (2001) distinguished between visual and non-visual (verbal/analytic) approaches to mathematical tasks, and described students' transition to more non-visual emphases throughout a school year. Zaslavsky, Sela, and Leron (2002) described analytic and visual interpretations of slope. Analytic interpretations emphasize slope as a "property of the function" that "does not depend on the function's representation" (Zaslavsky et al., p. 122). Notions of derivatives, difference quotients, and the coefficient m were described as analytic interpretations due to their functional emphasis. By contrast, the visual perspective treats slope as a "property of a line" (p. 122), i.e., as the angle formed between the horizontal axis and the line or as the ratio of the vertical change to the horizontal change (Zaslavsky et al., 2002). This study showed the cognitive conflict that arose in the minds of mathematicians who were asked to find the slope of two lines, one of which was represented on a non-homogenous coordinate system (different scales on the horizontal and vertical axes). Using an analytic perspective, the slope of the lines were equal (m=1). However, a visual perspective focused participants' attention on the angles of inclination formed between the lines and the horizontal axes; angles that appeared to have different measures under the two scaling systems. As a result, the authors argue that an analytic perspective is superior to a visual perspective since it holds under both homogenous and non-homogenous coordinate systems.

Combining Theories: A Slope Conceptualization Network

The theory proposed in this study builds on: (a) previous research related to slope; (b) general theories related to understanding; and (c) data collected for slope research studies conducted from 2007 to 2012. The data referenced included mathematics standards documents as well as observations of and written artifacts collected from a variety of individuals who should have already experienced K-12 instruction in slope, including: practicing middle and high school mathematics teachers; high school graduates enrolled in introductory college calculus courses; college graduates with Bachelor's degrees in mathematics, engineering, childhood education, or mathematics education enrolled in graduate teacher education courses; individuals with Master's degrees in either mathematics or mathematics education enrolled in mathematics education doctoral programs; and post-secondary mathematics instructors. These data were used to revisit examples of the 11 conceptualizations in light of: (a) procedural versus conceptual understanding and (b) analytic versus visual interpretations. The original 11 conceptualizations were scrutinized, with particular attention to their connections. The results of this analysis led to the collapsing of some previously distinct conceptualizations. As a result, the researchers constructed a 2x2 matrix indicating procedural and analytic, procedural and visual, conceptual and analytic, and conceptual and visual dimensions for each identified component of slope. A more detailed description of the identified slope components is in the next section.

Components of the Slope Conceptualization Network

When distinctions were made in terms of procedural versus conceptual understanding and visual versus analytic interpretations, connections and relationships between the previously distinct conceptualizations became apparent. Under this lens, five key slope components emerged: (1) constant ratio, (2) determining property, (3) behavior indicator, (4) trigonometry, and (5) calculus. In addition, real world situations were determined to be possible with each of the five components. A description of the four dimensions associated with each slope component, including relationships to the original 11 conceptualizations, follows.

Constant ratio. Table 2 outlines the constant ratio component. Previously distinct algebraic ratio and geometric ratio conceptualizations merge when visual and analytic interpretations of ratio are considered. A student with a visual approach to slope as a ratio might interpret slope as the ratio of rise over run or the vertical change divided by the horizontal change of two points on a linear graph. A more conceptual, but still visual interpretation would include validating the ratio using the notion of triangle similarity to recognize the constancy of the rise/run ratio. An analytic approach to slope as a ratio involves defining slope as the change in *y* values over the change in *x* values. At the conceptual level, the analytic ratio component recognizes slope as a constant rate of change between two covarying quantities. In addition to combining algebraic and geometric ratios via analytic versus visual distinctions, two other important conceptualizations were absorbed into the constant ratio component by consideration of the conceptual versus procedural approach. Notice that the conceptual level of slope as a ratio includes recognizing slope as a constant ratio for linear functions. In other words, conceptual understanding of slope as a ratio requires recognition that this ratio is independent of the particular location on the graph or input/output pairs that are chosen. Thus, the linear constant conceptualization (see Table 1) was absorbed at the conceptual level of the new constant ratio component. Likewise, a conceptual and analytic understanding of slope as a constant ratio involves recognizing slope as a constant rate of change of two covarying quantities, previously known as functional property. Four previously distinct conceptualizations fit under one umbrella when the procedural versus conceptual emphasis and visual versus analytic interpretations are considered. Note that previous conceptualizations are listed in Table 2 using square brackets.

Table 2: Examples of Slope as a Constant Ratio

	Procedural Emphasis	Conceptual Emphasis
Visual Approach	$R_{v,p}$: rise/run or vertical change/horizontal change [previously geometry ratio]	$R_{v,c}$: similarity of slope triangles yields a constant ratio of rise/run regardless of the position on the graph [previously linear constant]
Analytic Approach	$R_{a,p}$: change in *y* over change in *x*; (y_2-y_1/x_2-x_1) [previously algebraic ratio]	$R_{a,c}$: constant rate of change between two covarying quantities; an equivalence class of ratios and hence a function [previously functional property]

Behavior indicator. The behavior indicator component, outlined in Table 3, relates slope to the increasing, decreasing, or constant behavior of a linear graph (visual approach) or function (analytic approach). Under the analytic approach, students first recognize slope as *m* in the equation of a linear function and then interpret the sign of *m* to indicate the function behavior. This recognition of slope as *m* was previously classified as the parametric coefficient

conceptualization. To move to conceptual understanding of slope as a behavior indicator, students must first be able to interpret slope as a ratio. For a visual approach, students can link the increasing or decreasing behavior of a linear graph to positive or negative values of rise and run, which then leads to a positive or negative ratio of rise/run. In the analytic approach, students use the definition of an increasing or decreasing function to determine the sign of the change in *y*/change in *x* ratio. The constant ratio component appears foundational to a conceptual understanding of slope as a behavior indicator.

Table 3: Examples of Slope as a Behavior Indicator

	Procedural Emphasis	Conceptual Emphasis
Visual Approach	$B_{v,p}$: increasing lines have positive slope; decreasing lines have negative slope; horizontal lines have zero slope	$B_{v,c}$: positive rise corresponds to positive run for an increasing line, yielding a positive slope. For a decreasing line, a negative rise corresponds to a positive run, yielding a negative slope. A horizontal line has zero rise for any run, yielding a zero slope.
Analytic Approach	$B_{a,p}$: value of m in the equation for a linear function (e.g., in $y=mx+b$) indicates whether f is an increasing ($m>0$), decreasing ($m<0$), or constant ($m=0$) linear function [previously parametric coefficient]	$B_{a,c}$: application of the definition of increasing/decreasing/constant functions to explain positive/negative/zero slope, respectively (e.g., f is increasing means that $f(x_1)<f(x_2)$ if $x_1<x_2$, so $[f(x_2)-f(x_1)]/(x_2-x_1)>0$)

Determining property. Table 4 shows that the determining property component involves recognizing that slope can: (a) determine the relationship between lines (e.g., parallel) and (b) indicate that a point and slope determine a unique line. While a procedural emphasis requires that students recognize these relationships, a conceptual emphasis includes understanding the underlying features of the linear graphs or functions that yield these relationships.

Table 4: Examples of Slope as a Determining Property

	Procedural Emphasis	Conceptual Emphasis
Visual Approach	$D_{v,p}$: parallel, coplanar lines have the same slope; perpendicular, coplanar lines (lines that intersect at right angles) have negative reciprocal slopes; slope and a point determine a unique line	$D_{v,c}$: parallel lines have the same vertical change for a set horizontal change (otherwise they would intersect); may be seen in terms of congruent slope triangles
Analytic Approach	$D_{a,p}$: ratio (y_2-y_1/x_2-x_1) is equivalent for parallel lines and results in negative reciprocals for perpendicular lines; slope and a point determine a unique linear equation	$D_{a,c}$: parallel lines have equivalent differences in y values for a set difference in x values, yielding equivalent slope ratios

Trigonometry. The trigonometric component of slope is described in Table 5. Once again, distinguishing between visual and analytic interpretations of slope allowed previously distinct slope conceptualizations to be combined under one category. A focus on steepness, which was previously identified as a physical property conceptualization, can be seen as the manifestation of a visual approach of the trigonometric component. Procedurally, the focus is on determining the angle of inclination, whereas a conceptual focus includes viewing the ratio of rise over run in terms of the opposite and adjacent sides of the right triangle formed by the line and the horizontal. The analytic approach to slope in terms of the trigonometry component involves the procedure of calculating $\tan\theta$. At the conceptual level, a student can relate the angle of inclination to the ratio (y_2-y_1/x_2-x_1), also known as $\tan\theta$.

Table 5: Examples of Slope as a Trigonometric Conception

	Procedural Emphasis	Conceptual Emphasis
Visual Approach	$T_{v,p}$: steepness of a line; slope as the angle of inclination of the line with a horizontal; as a line is rotated about a point, the slope changes [previously physical property]	$T_{v,c}$: the angle of inclination determines the rise/run; a steeper line has a greater rise per given run than a less steep line
Analytic Approach	$T_{a,p}$: slope is calculated as $\tan\theta$, where θ is the angle formed by the graph of the linear equation and an intersecting horizontal line	$T_{a,c}$: the angle of inclination determines the ratio of (y_2-y_1/x_2-x_1), which is equivalent to $\tan\theta$

Calculus. The calculus component, outlined in Table 6, involves interpreting slope of a function at a point. Visually, this is done via the slope of the tangent line to the point, with a conceptual emphasis on using secant line approximations to find the slope. Analytically, the slope is found via the derivative function (either via the limit definition or the various shortcuts for finding the derivative). A conceptual emphasis involves understanding the limit definition as describing the average rate of change over increasingly small intervals. Since the visual approach relies on first understanding slope of a line using the rise/run ratio and the analytic approach relies on first understanding slope of a linear function using the change in *y*/change in *x* ratio, the calculus component of slope appears to be closely linked to the ratio component.

Table 6: Examples of Slope in Calculus

	Procedural Emphasis	Conceptual Emphasis
Visual Approach	$C_{v,p}$: slope of a curve at a point is the slope of the tangent line to the curve at a given point	$C_{v,c}$: visual interpretation of secant lines approaching tangent line to understand slope of a curve at a point as the slope of the tangent line at the point
Analytic Approach	$C_{a,p}$: derivative *f'* is used to calculate slope of the function *f* at a particular point	$C_{a,c}$: interpreting $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$ as the average rate of change over increasingly small intervals

Real world situations and parametric coefficient. One of the original 11 conceptualizations does not appear above. Real world situations can be incorporated with any one (or any combination) of the outlined slope components. For instance, references to the pitch of a roof would incorporate real world situations into the trigonometry component to determine the steepness of the roof (possibly using the angle formed between the roof line and the ceiling of house's interior), while references to speed as the rate of change between distance and time would incorporate real world situations with the constant ratio component to interpret speed as a ratio describing a relationship between covarying distance and time quantities. Table 7 outlines the components of the slope network as they relate to the original 11 conceptualizations.

Table 7: Revised Slope Network Components and Corresponding Conceptualizations

<table>
<tr><th></th><th>Slope Component</th><th>Description</th><th>Underlying Slope Conceptualizations</th></tr>
<tr><td rowspan="5">Real World Situations (span each component)</td><td>Constant Ratio</td><td>Slope viewed as a ratio in visual (rise/run) or analytic (change in y over change in x) form; conceptual understanding extends to explain why linear behavior results in a constant ratio (including functional property for analytic representation)</td><td>Geometric Ratio
Algebraic Ratio
Functional Property
Linear Constant</td></tr>
<tr><td>Trigonometric Conception</td><td>Describes slope in terms of the angle of inclination of a line with a horizontal; conceptual understanding relates steepness to the determination of the tangent of the angle of inclination</td><td>Physical Property
Trigonometric Conception</td></tr>
<tr><td>Behavior Indicator</td><td>Relates slope to the increasing or decreasing behavior of a linear function or graph; links sign of the quantity m with the function or graph's behavior</td><td>Parametric Coefficient
Behavior Indicator</td></tr>
<tr><td>Calculus Conception</td><td>Limit; derivative; a measure of instantaneous rate of change for any (even nonlinear) functions; tangent line to a curve at a point</td><td>Calculus Conception</td></tr>
<tr><td>Determining Property</td><td>Property that determines if lines are parallel or perpendicular; property can determine a line if a point on the line is also given</td><td>Determining Property</td></tr>
</table>

Discussion and Conclusion

The above analysis merges past research on students' understanding of slope to provide a more connected understanding of the components of an individual's network of slope concepts. In particular, the analysis suggests understanding slope can be viewed in light of five components, each with four subcomponents defined by the conceptual versus procedural understanding and analytic versus visual interpretations. While it has been argued in the past that students must build a particular type of understanding of slope (e.g., conceptual or analytic), the network interpretation of slope highlights the importance of procedural and conceptual understanding, as well as visual and analytic interpretations of slope. While Zaslavsky and colleagues (2002) suggested that a visual perspective of slope might be limiting, the network suggests that visual interpretations are an important part of an individual's network of slope components. We would argue that limitations stem from a lack of connections between the various components and subcomponents of slope. Not only should instruction build each of the five slope components, but explicit attention should be given to building connections between

the subcomponents. This means that students must have an opportunity to build and connect slope representations that are analytic, visual, procedural, and conceptual.

A wealth of research supports that slope is a critical topic and that students struggle to build a strong understanding of this important topic. We have contributed to the theory for how slope is understood by synthesizing past research to describe a network of slope components and subcomponents that together build a strong and diverse view of slope. This theory can strengthen future research on slope by tying together the important aspects of understanding slope under one model.

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AGENCY-AS-INFERENCE: TWO SIDES OF THE SAME EPISTEMIC COIN

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This paper attempts to synthesize theory of knowledge objectification (Radford, 2003) with equity research on mathematics education. I suggest mathematical inference as a promising locus for investigating the types of epistemic agency that equity-driven scholars have deemed as vital for student identity and, in turn, participation and learning. I conceptualize students' appropriation of semiotic-cultural artifacts (e.g., algebraic symbols) to objectify their pre- symbolic inferences as conditional on their agency to gradually construct personal meaning for these artifacts. I support this claim with preliminary empirical analysis of a Grade 9 student's participation during an algebraic pattern-finding task.

Keywords: Algebra and Algebraic Thinking, Classroom Discourse, Equity and Diversity

Introduction & Objectives

In this study I suggest the phenomenon of mathematical inference, which is central to cognitivist analyses of learning, as a promising locus for investigating the types of epistemic agency that equity-driven scholars have repeatedly demonstrated are necessary for the development of constructive mathematical identities (Boaler & Greeno, 2000; Nasir & Hand, 2006). I consider epistemic agency and mathematical inference as 'two sides of the same coin'.

The crux of my theoretical argument is based on Luis Radford's (2003) theory of *knowledge objectification* and, in particular, his *semiotic–cultural framework* for the study of students' algebraic reasoning (see below). Through the lens of agency-as-inference, my broader theoretical agenda is to synthesize Radford's approach with equity-oriented perspectives that account for issues related to students' mathematical identity. To support this claim, I have proposed elsewhere a qualification to theory of knowledge objectification that, in contrast to its extant formulation, does not assume classroom homogeneity in opportunities to appropriate mathematical semiotic artifacts (Gutiérrez, 2013). To demonstrate the empirical viability of this emerging approach, this study focuses on algebraic generalization as a specific genre of mathematical inference and applies Radford's framework to video data of a Grade 9 student's participation during an algebraic pattern-finding task.

Theoretical Background - Overview of the Semiotic–Cultural Approach

Luis Radford's (2003, 2008) semiotic–cultural approach was developed in the context of algebraic generalization activity (see Figure 1, below). Specifically, mathematics learning is conceptualized as constructing personal meaning for canonical semiotic artifacts (e.g., algebraic symbols such as the variable "x"). Through consolidation and iteration of these constructions, students appropriate the mathematical semiotic artifacts and, reciprocally, build personal meaning for mathematical content as well as fluency with the disciplinary procedures.

Radford's approach takes into account a vast arsenal of personal and interpersonal resources that students bring to bear in solving mathematical situations, including linguistic devices and mathematical tools. A key construct in Radford's framework is *knowledge objectification*, which is defined as the process of making the objects of knowledge apparent (Radford, 2003).For

example, a mathematics learner, in an attempt to convey a certain aspect of a concrete object, such as its shape or size, will make recourse to a variety of semiotic artifacts such as mathematical symbols and inscriptions, words, gestures, calculators, and so forth. In patterning activity, however, some of the objects of knowledge are general and therefore "cannot be fully exhibited in the concrete world" (Radford, 2008, p. 87). In order to instantiate these ephemeral objects, students must resort instead to personally and culturally available forms such as linguistic, diagrammatic, symbolic, and substantive artifacts as well as the body, which Radford (2003) collectively terms *semiotic means of objectification* (see also Abrahamson, 2009).

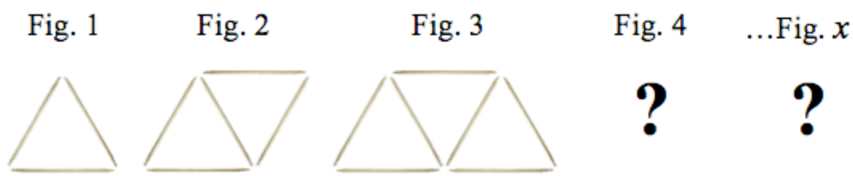

Figure 1: "Toothpicks"— a paradigmatic algebra generalization problem. The task objective is to express U_x the total number of toothpicks in the x^{th} figural extension (i.e., "Fig. x" consists of $2x+1$ toothpicks).

The power of the semiotic–cultural approach is that critical steps within individual learning trajectories can be explained by noting subtle shifts in the subjective function and status of the semiotic artifacts. In particular, mathematics learning in the context of algebraic generalizations can be monitored as subjective transitions along a desired chain of signification, from *factual*, to *contextual*, to *symbolic modes* of reasoning (Gutiérrez, 2010; Radford, 2003).

Factual	Contextual	Symbolic
• Objects and operations are bound to concrete level • E.g., Student utterance: "1 plus 2, 2 plus 3, 3 plus 4…"	• Objects and operations are abstracted and generalized • E.g., "The *figure* plus the *next figure*."	• Objects and operations are expressed through formal symbolism • E.g., "x + (x+1)"

Figure 2: "F-C-S framework applied to the "Toothpicks" problem; see Figure 1, above.

From this perspective, conceptual understanding is viewed as the capacity to flexibly shift across the three semiotic modes, which consequently requires that students assume agency in making these shifts so as to carefully and incrementally construct personal meaning for conventional semiotic artifacts (e.g., the variable "x"). Students' personal acts of generalization are a specific type of mathematical inference; and transitions from one semiotic mode to the next mark *both* conceptual understanding and epistemic agency. That is, agency and conceptual understanding can be co-investigated by interrogating the process and content of students' mathematical inferences (generalizations) within and across the three semiotic modes.

Agency-As-Inference in Algebraic Patterning Activity

I present an example of a classroom problem-solving activity involving the "Toothpicks" problem to illustrate the construct of agency-as-inference. The excerpts that make up this classroom scenario are drawn from my research on equitable participatory design. The specific intervention wherein the data for this study were collected was conducted in a high school

mathematics program for academically at-risk youth. The intervention was intended to implement a classroom participation structure that would facilitate a particular desirable interaction among students and ultimately give rise to authentic engagement and deep learning.

I focus on student Magdalena (Grade 9) and present qualitative data analyses of a series of transcript segments of her contributions to the whole-class discussion. By conducting detailed and sequential analyses of her mathematical behaviors, I aim to elaborate my thesis that a student's mathematical inferences bearing generalization are indicative of epistemic agency.

The transcript segments begin after the teacher, Amil, writes "Fig. 100" on the far right side of the board. Magdalena articulates an arithmetic/recursive strategy in the form Ux+1=Ux+3-1, stating: "if you're adding another triangle, it's another three [toothpicks] but then you subtract one." But she soon realizes the limitations of this strategy for producing figural extensions much further down the line (Fig. 100), because as she states: "you don't know the figure before" (i.e., *U99* is unknown). As a result, she spontaneously proposes the use of a variable "*x*" as a placeholder for a given figure's ordinal position.

Magdalena: You have to do x instead of a number. Because if you use x then it could be any number.

Amil: Ok so you have to use x—we gotta use x. So what is x gonna be?

Magdalena: **X is the amount of triangles**.

Implicit in Magdalena's proposition is the understanding that "*x*" could serve as an indeterminate quantity, thus enabling operation on the figure numbers (ordinal positions) independent of the previous figures in the sequence. Magdalena guides her peers to look for patterned relations with*in* and not just across the figures. So doing, Magdalena leads the class to a closed-explicit solution procedure (U_x=2x+1), first objectified as a contextual generalization and then reformulated as a factual generalization.

Magdalena: I found something. Ok so if you **add the figure number to itself, plus 1**, it will equal the amount of toothpicks.

Amil: Ok figure number to itself. So [indicates Fig. 1] 1 plus 1, plus 1 more? Equals 3. So [indicates Fig. 2], 2 plus 2…

Magdalena: 4, plus 1 is 5.

Amil: …plus 1 equals 5. [Indicates Fig. 3.]

Magdalena: **3 plus 3 is 6, plus 1…7. 4 plus 4…8, plus 1…9. 5 plus 5…10, plus 1…11.**

Amil: Ok how about figure 100? How many [toothpicks] would it have?

Magdalena: **100 plus 100…200, 200 minus—I mean 200 plus 1…201**.

Magdalena's inferential/semiotic trajectory thus far appears to follow the path "S-C-F." That is, Magdalena was operating in the symbolic mode when she proposed the use of the variable "x," which was followed by her objectification of an algebraic generalization in the contextual mode ("add the figure number to itself, plus one"). Amil was able to scaffold her inquiry such that Magdalena went *back* to the factual mode, further grounding her generalizing inferences in actual constituent elements of the source situation (Fig. 1, Fig. 2, Fig. 3, and Fig. 100).

Magdalena further elaborates on her search process and extends her trajectory to "S-C-F-C":

Magdalena: [addressing the class] The thing—see the thing that I did though, I was just looking for things that they all had in common. And **they had the figure number plus another one**.

Ultimately, I diagnose the final "S" of her evolving "S-C-F-C-S" trajectory (see immediately below) as a generalization *to* that mode yet *partially* grounded, because it is Amil and not Magdelena who reformulates her final contextual generalization into symbolic form. I maintain

there was enough conceptual substrate—at the cognitive-semiotic level—for Magdelena to appropriate Amil's reformulation in a way that bore personal meaning.

Amil: "Ok so you said the figure number...1...plus the figure number again, right? What's another way of saying that? Instead of saying the figure number plus the figure number again..."

Magdalena: "I don't know."

Amil: "Ok Uhh let's see. [referring to Figure 2] 2 plus 2 plus 1. What's another way of saying 2 plus 2? Or [indicating Fig. 3] 3 plus 3? [No response from class] How about 2 times the figure number, plus 1? Right?"

Magdalena: "Yeah umm…oh yeah."

Amil: "Ok so that is... [writes "2n+1" on the board] so that's our—" *Magdalena*: "So it could still be—it could still be x! **So it'll be 2 x plus 1**." *Amil*: "Or 2 x plus 1. You can put any letter there."

Magdalena: "Ok."

I conclude that Magdalena's subjective inferences underlying mode-*to*-mode transitions are where, and through which, her epistemic agency is instantiated and achieved (Figure 3).

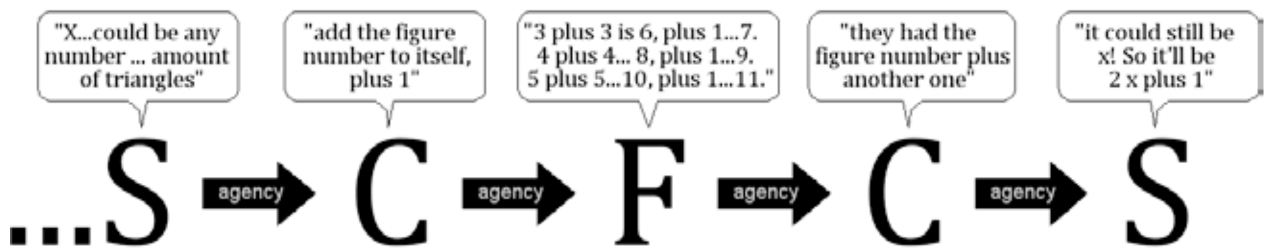

Figure 3: Magdalena's semiotic trajectory. "F," "C," and "S" represent the active semiotic mode of the "text"—the words that are being uttered into the classroom. Whereas semiotic mode is objective and shared in the discursive space, *transitions* between modes are subjective. Arrows "è " represent Magdalena's subjective mode-*to*-mode transitions; for example, "è F" means Magdalena's cognitive-semiotic actions indicate that she indeed generalized *to* the factual mode (Gutiérrez, 2010). It is "in the arrows" where agency-as-inference lives.

Future research will explore the development of agency-as-inference and its impact on students' mathematical identities, specifically for marginalized students whose mathematical understandings are not couched in the mainstream classroom discourse.

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THIRD-GRADERS' GENERALIZATIONS ABOUT EVEN NUMBERS AND ODD NUMBERS: THE IMPACT OF AN EARLY ALGEBRA INTERVENTION

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Third-grade students are participating in a classroom experiment investigating the impact of instruction derived from a research-based Early Algebra Learning Progression. Students completed a pre-assessment item addressing their abilities to justify a conjecture about the sum of three odd numbers. While they were largely unsuccessful on the assessment, through the course of the classroom intervention students are beginning to develop representation-based arguments and provide generalizations using facts about the sum of even numbers and odd numbers.

Keywords: Algebra and Algebraic Thinking, Reasoning and Proof, Elementary School Education

For many students, algebra continues to be a gatekeeper to future academic and employment opportunities. As a result, reform efforts in recent years (e.g., Common Core State Standards Initiative, 2010) have sought to integrate aspects of algebra into elementary and middle school mathematics curricula.

In response to this call, our project's overarching goal is to examine the effectiveness of a longitudinal grades 3-5 early algebra intervention on children's algebra learning and algebra-readiness for middle school. Our intervention—currently in its first year (i.e., grade 3)—is derived from a research-based Early Algebra Learning Progression [EALP] organized around five "big ideas": 1) Generalized Arithmetic, 2) Equations, Expressions, Equality, and Inequality, 3) Functional Thinking, 4) Proportional Reasoning, and 5) Variable (Blanton, Stephens, Gardiner, Isler, & Knuth, 2013).

We focus in this paper on our findings regarding the development of third-grade students' understanding of generalized arithmetic, particularly in the context of even numbers and odd numbers. We share pre-assessment data and representative excerpts from student work and briefly discuss the classroom intervention we believe is contributing to student learning. Posttests will be administered in May 2013 and results will be included in our presentation.

Theoretical Perspective

According to Kaput (2007), the essence of early algebra includes developing and justifying mathematical generalizations in various ways, and reasoning with those generalizations. Blanton (2010) further states that children's work in generalizing arithmetic relationships can also strengthen their abilities to engage in mathematical reasoning and proof in the elementary grades. Carpenter and Levi (2000) provide evidence that students can develop conjectures, representations, and justifications to support their ideas as early as first grade.

Investigating operations on even numbers and odd numbers has been found to engage students in developing and justifying generalizations (Carpenter, Franke, & Levi, 2003). Russell,

Schifter, and Bastable (2011) observed students using four different approaches to justify a conjecture: accepting the claim on authority, trying it out with examples, applying mathematical reasoning based on a visual representation or story context, and proving using algebraic notation and the laws of arithmetic. They further emphasized that employing visual representations or story contexts in making mathematical arguments—i.e., engaging in "representation-based proof"—is an effective route to establishing general claims in elementary classrooms.

It is already known that young children can engage in mathematical reasoning and learn to use sophisticated arguments to justify conjectures about the sums of even numbers and odd numbers (Ball & Bass, 2003; Blanton, 2008). In this paper, we aim to explore third-grade students' capacity for developing conjectures about these types of arithmetic generalizations, and examine how students who have had a year long focus on early algebra (including generalized arithmetic) justify these conjectures in comparison to students who have had more traditional arithmetic-based experiences.

Method

Participants

Participants included 170 third-grade students from two elementary schools in southeastern Massachusetts. The school district in which these schools reside is largely white (91%) and middle class, with 17% of students qualifying for free or reduced lunch. Six classrooms (all from one school) served as experimental sites and 4 classrooms (all from one school) served as control sites.

Data Collection

A one-hour written assessment addressing the range of "big ideas" present in our EALP was administered to all participants prior to the start of the teaching intervention in September 2012. We focus here on one assessment item (see Figure 1) concerning students' justifications related to the sums of even numbers and odd numbers.

> Brian knows that anytime you add three odd numbers, you will always get an odd number. Explain why this is always true.

Figure 1: The Three Odd Number Sum task

An identical posttest will be administered when the third-grade intervention ends in May 2013. Therefore, in this paper, we share results of only the pretest and classroom observation. Posttest results will be shared during the presentation.

Classroom Intervention

Students in the experimental condition are participating in an EALP-based classroom teaching experiment for approximately one hour each week for the majority of one school year. A member of our research team—a former third-grade teacher—is serving as the teacher during these interventions. A typical one-hour lesson consists of a "jumpstart" at the beginning of class to review previously discussed concepts, followed by group work centered on research-based tasks aligned with our EALP.

Two lessons specifically included tasks concerning the sums of even numbers and odd numbers, though students were engaged in conjecturing and justifying more broadly (e.g., in the context of the fundamental properties such as the commutative property of addition) for several weeks prior to this. An example of a task that was used in the intervention is presented in Figure 2.

Jesse is adding two even numbers. Do you think his answer will be an even number or an odd number? Develop a conjecture to describe what you found. Is your conjecture true for any two even numbers you add together? How do you

Figure 2: The Two Even Number Sum task

Data Analysis

A member of the research team coded all the student responses to the written assessment item. Five categories of student responses emerged from the data: *Empirical argument*, *Even/odd generalization*, *Incorrect counter-example*, *Restatement of the argument*, and *Incomplete even/odd generalization*. A response was coded *Empirical argument* when students concluded that the conjecture is true when they tried it out with examples such as "3 + 5 + 7 = 15." A response was coded *Even/odd generalization* when students developed their argument based on what they knew about even and odd numbers, such as "2 odd numbers equal an even number so add an odd number and it turns odd," that included using the previous generalizations about even numbers and odd numbers to produce a generalization in words, pictures, or symbols. *Incorrect counter-example* was used when students stated that the conjecture was not true, possibly because of misreading the question, and justified it with a counter-example such as "3 + 3 = 6, and 6 is an even number, not odd." *Restatement of the argument* was used when students restated the conjecture given in the task. Lastly, *Incomplete even/odd generalization* was used when students made a general but incomplete argument, often missing the final step.

All other responses were coded as either "no response" if a student left the item blank, or "other" if the student response did not fall into any of these categories and was not frequent enough to constitute its own code.

Results and Discussion

The pretest results showed that 21% of students relied on examples to justify the argument, which were coded as *Empirical argument* as explained above. An example from students' written responses is "3 + 5 + 7 = 15, that's an odd number." In contrast, 5% of students provided a generalization about the sums of odd numbers that was coded as *Even/odd generalization*. For instance, one student explained, "It's always true because when you add 2 odd numbers, it makes an even number but if you add a third one, it will be odd." Four percent of student responses were coded as *Incorrect counterexample*. For instance, a student explained "It is not true because if you add 1 + 3, you get 4 and that is an even number." Three percent of students were found to just restate the argument. For instance, one student said "It is true because if you add 3 odd numbers, it will be an odd number." Two percent of students were found to provide a valid but incomplete argument. For example, one student stated "If you add two odd numbers then you`ll get an even number." The final step of the argument, that an even plus an odd is an odd, was missing. Thirty-four percent of all student responses were coded as "no response," and 31% were coded as "other." An example of an "other" code is "It is true because you`re counting with odd numbers." These responses were not found frequently enough to constitute their own code.

Results of the pre-test suggest—not surprisingly—that students were much more likely to produce an empirical argument than provide a general argument. In fact, many students did not provide a response, which might be interpreted as that a majority of them did not have the cognitive tools for thinking about this type of problem. However, our instructional intervention suggests that students can develop ways of thinking that might support a more formal

understanding of proof. At the beginning of adding even and odd number lessons in our EALP-based classroom intervention, students were asked to explore even and odd numbers using their unifix cubes, and were asked to complete a table indicating the "number of pairs created" and "number of cubes left over" for the numbers they explored. The teacher guided the students throughout this activity by asking questions such as "How many pairs of cubes are in the number 6?", "What do you notice in the table?", "What kinds of numbers have no cubes left over after all pairs are made?" The aim was to have students realize the pattern that even numbers have no cubes left over after all pairs are made, and odd numbers have one cube left over. Then, they were given the task of adding two even numbers (see Figure 2). Students first started checking "cases" to see whether the result was an even or odd number. A student said "I think we should try some math facts, like 2 + 2 is 4 [showing with the cubes], so I think even plus an even will be an even." The teacher asked students to develop a conjecture, and think about whether it would work for *all* numbers. Students developed conjectures such as "when you add an even number plus an even number the sum will be even." When asked whether it always works, some students stated that does because "I tried a bunch of examples and it works for all of them." When the teacher asked them *why* it works, they continued their discussion, working in small groups, and justifying with their cubes. As they described it, "When we add even numbers we don't ever start with any leftovers. Everyone has a pair, so we can add them together and everyone will always have a pair." Students' justifications, including representation-based arguments, helped them generalize for all numbers rather than focus on specific numbers. The class continued to explore the sum of two odd numbers, and the sum of an even and an odd number in the following lesson.

Given the appropriate instruction, we see that third-grade students can be successful in developing conjectures, investigating *why* and generalizing. These skills are at the heart of mathematical reasoning, and were found to be attainable for students participating in an EALP-based classroom intervention.

Endnote

The research reported here was supported in part by the National Science Foundation [NSF] under DRK-12 Award # 1219605. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of NSF.

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COORDINATING n+1 LEVELS OF UNITS

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Across multiple mathematical domains, researchers have implicated the critical role of an individual's ability to coordinate units. In this proposal, we describe a theoretically grounded instructional approach for promoting pertinent growth. We present promising results from an implementation of our approach in a ten-week teaching experiment with a sixth-grade student, named Cody. We demonstrate how, over the course of the teaching sessions, Cody progressed from requiring physical activity in order to coordinate two levels of units to being able to coordinate three levels of units (in activity).

Keywords: Number Concepts and Operations, Instructional activities and practices

Significance

Integers, fractions, and algebra present notorious challenges for our students. Recent research on students' ways of operating mathematically indicates a common root cause. Namely, many students lack the ability to coordinate the various levels of units involved in reasoning flexibly within each domain. A growing body of research indicates that the levels of units students coordinate mediate opportunities for learning across several domains (Ellis, 2007; Olive & Çağlayan, 2008; Ulrich, 2012). This introduces a serious problem for mathematics educators because, even by middle school, many students work with only one level of units and few readily work with three levels of units (Norton & Wilkins, 2012).

We address the problem by introducing an instructional approach to promoting growth in the levels of units that students coordinate. The purpose of the underlying study was to test the approach through a teaching experiment with a sixth-grade student. In particular, we tested whether we could provoke reorganization in his ways of operating so that he could assimilate situations involving two-level structures and begin coordinating three levels of units in activity.

Theoretical Framework

Units coordinating refers to the structures and operations students can produce in activity and those students use to assimilate situations involving embedded quantities. Consider the following task: A student is given a short bar, a medium bar, and a long bar and is told that the short bar fits into the medium bar three times and that the medium bar fits into the long bar four times; then the student is asked to determine how many times the small bar fits into the long bar. Table 1 outlines typical responses indicating the number of levels students coordinate.

Table 1: Levels of Units Coordination

	Students' Unit Structures	Students' Reasoning on the Bar Problem
Level 1	Students can take one level of units as given, and may coordinate two levels of units in activity.	Students mentally iterate the short bar, imagining how many times it would fit into the longer bar. This activity might be indicated by head nods or sub-vocal counting.
Level 2	Students can take two levels of units as given, and may coordinate three levels of units in activity.	Students mentally iterate the medium bar four times, with each iteration representing a 3. This activity might be indicated by the student uttering "3, 3, 3, and 3; 12."

Level 3	Students can take three levels of units as given, and can thus flexibly switch between three-level structures.	Students immediately understand that there are four threes in the long bar. This assimilation of the task might be indicated an immediate response of "12", buttressed by an argument that 12 is four 3s.

Our Approach

Starting with a number, say 30, its prime factorization (with multiplicities) determines a maximum number of levels—m+1, where m represents the number of prime factors. For instance, 30=2×3×5 has 3+1=4 possible levels. We can build these four levels in any context that students can assimilate as an embedding structure. For example, we can consider a box of cups, where each cup has a specified number of chips, of a specified value.

Level A: cents
Level B: chips, with relation BA: 2 cents per chip
Level C: cups, with relation CB: 3 chips per cup
Level D: boxes, with relation DC: 5 cups per box

Now we consider all possible permutations of two levels, from among A, B, C, and D. There are 4×3=12 such permutations, and each of them suggests a task. For example, consider DA: "How much is a box worth?" We can also consider permutations of three levels, where the middle level is an intermediate level. Consider ACD: "If you have 12 cents, how many more cups do you need to make a box?" This approach to task design, with the intent of promoting the coordination of higher levels of units, incorporates three principles arising from existing research on students' mathematical ways of operating.

First, if a student has internalized and organized n (1 or 2) levels of units, tasks should challenge the student to coordinate n+1 units in activity (Hackenberg and Lee, 2012). By varying the units students need to coordinate within the same problem setting, our approach provokes students to simultaneously maintain multiple relationships involving n+1 units. Second, media need to provide students with opportunities to physically enact these various coordinations (Clements, 2000). Our approach allows for the use of various manipulatives and contexts, so long as they fit the embedding structure. Finally, students need opportunities to reflect on their activity. Our general task setting is amenable to such reflection because it supports physical activity related to packing and unpacking units. Students can re-present these activities within diagrams and other drawings to illustrate relationships between various units.

Data and Analysis

We conducted a teaching experiment with a sixth-grade student named Cody. During a 15-minute interview with Cody, we determined that he readily engaged in units coordinating tasks but that he operated with only one level of units at a time. This made him an ideal candidate for our study, in which we wanted to engage students in activity that might provoke units coordination with higher levels of units.

Our teaching experiment began on September 13, 2012, during which we affirmed our initial assessment of Cody as a student who had not yet internalized and organized two levels of units. Beginning on September 18, we initiated our approach to promoting that development. Ten more 30-minute teaching sessions followed, charting a progression in Cody's units coordinating activity, and leading up to a key development on November 15.

Session 1: September 18

We used this first session to assess Cody's units coordinating activity within the context of relating cents, chips, cups, and boxes, as described in the approach section. The teacher-researcher began by asking Cody to put three chips in each of five cups, to put these five cups in a box, and to close the box. In the following transcription, "T:" precedes the words [and actions] of the teacher-researcher (first author), and "C:" precedes those of Cody.

T: If a chip is worth two cents, how much money would be in the box?

C: Chips are 2, and there's five. Since there's six in each one, because 2, 4, 6. And there's 5 cups. Six times 5, and it equals 30. [smiles and dances a little]

This approach, of building from the smallest unit, became typical for Cody. He built 2, 4, 6 cents and then considered the value for five of these collections. In a subsequent question, he was asked to determine the number of *chips* in the box, but he did not have an answer. Although the new task would seem no more complicated that the first, chips were a composite unit in this context, and Cody consistently appeared uncomfortable in working with such units. Thus, Cody's actions in this session fit our initial assessment that he could only organize two levels of units in activity, by building up from the smallest unit.

Session 11: November 15

Due to scheduling conflicts, three weeks passed between the tenth teaching session and this one. We engaged Cody in a problem setting involving ten chips in each cup, five cups in each box, and two boxes in a crate. Before the teacher-researcher could begin asking questions, Cody requested a sheet of paper on which he wrote down all of the relations. After Cody quickly answered several two-level questions, the teacher-researcher began the following exchange:

T: If you have 400 chips, how many crates would you have?

C: [stands up and points to the first "crate" and three more imaginary ones] A hundred, 200, 300, 400. Four!

T: Okay. If you had 400 chips, how many boxes would you have?

C: [sweeping his arms across the four (imaginary) crates] Four; two in each one. Eight.

T: Nice! How many cups would you have?

C: [pointing to the first crate and then each of the others, in succession] There's 10, 20, 30, 40! [pulling on the hood of his jacket] I think this is my lucky jacket…. I think this jacket makes me smart.

The session continued with tasks involving three levels of units—the kinds of tasks that had been troublesome for Cody up to this point.

T: If you have 30 chips, how many more cups would you need to make a crate?

C: How much cups… Six, because… Wait, you said 30, so that's three cups. Five plus two, that's seven! Woohoo!

T: If you have five cups, how many more boxes would you need to make a crate?

C: Five cups? Boxes? One.

T: If you have eight cups, how many more chips would you need to make a crate?

C: 20.

If you have one box, how many more chips would you need to make a crate?

C: 50.

If you have one box, how many more cups would you need to make a crate?

C: 5.

T: That *is* your smart jacket.

Conclusions and Implications

At the beginning of the teaching experiment, Cody only assimilated one level of units and had to build higher levels of units through activity, from that first level (and this despite his fluency with multiplication facts and computations). Even when he built two levels of units in activity, he had difficulty in reversing the relationships between the units. Limitations in his ways of operating were especially apparent in responding to questions involving three levels of units, which he could not build even through activity, until the final session.

Session 11 followed a long period of rest, which was preceded by a series of sessions intended to provoke Cody's ways of operating. In such situations, students often have time to subconsciously reorganize their ways of operating, in response to the provocations. Steffe (1991) has attributed this kind of reorganization to a process he calls "metamorphic accommodation." We suggest that Cody's ways of operating underwent a metamorphic accommodation, as indicated by his responses to the three level questions posed in the final session. Leading up to that session, it seems that he had internalized and organized a two-level structure and that he could simultaneously coordinate three levels of units in activity.

If Cody's growth can be attributed to our approach and if its success can be replicated with other students, the implications for middle school mathematics education would be substantial. A growing body of research indicates that units coordination plays a vital role in much of K-12 curriculum (Ellis, 2007; Hackenberg & Lee, 2012; Norton & Wilkins, 2012; Olive & Çağlayan, 2008; Steffe & Olive, 2010; Ulrich, 2012). In particular, students need to coordinate three levels of units in order to meaningfully address much of the middle school curriculum, and with the new *Common Core State Standards* (2010), this demand might very well extend into the upper elementary grades (Norton & Boyce, 2013). Thus, approaches to provoking and supporting students' units coordinating activity require sustained investigation.

Acknowledgements

This research was supported by a grant from the National Science Foundation (#1118571).

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FUNCTION KNOWLEDGE OF 12TH GRADE STUDENTS IN THE UNITED STATES

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This study uses items from the 2005 and 2009 NAEP Grade 12 mathematics assessment to describe student performance on function concepts. The conceptual framework for the study integrates several concepts of the function to account for their relevance beyond algebra, including such areas as geometry and statistics. It further provides an overview of nationwide student performance on function items and a comparison between student performance on function items that engage analytic processing of information versus those function items that engage visual processing of information. Item analysis will shed light on relevant challenges that functions pose for students.

Keywords: Algebra and Algebraic Thinking, Curriculum, Assessment and Evaluation

Most research on student understanding of functions has been limited to small studies (Drijvers, Doorman, Boon, Gisbergen, & Gravemeijer, 2007; Vinner & Dreyfus, 1989), and as a consequence we lack a clear sense of student performance on functions at the national level. To address this gap, the present study (still in progress) analyzes nationally representative data from the National Assessment of Educational Progress (NAEP) and thus provides a broader view of U.S. students' mathematical performance on functions.

For the Grade 12 math assessment, NAEP reports overall results to represent nationwide math achievement as well as results broken down by content strand (number and operation; geometry and measurement; algebra; and data analysis, probability, and statistics). It does not, however, break these strands down into further subcategories or provide information about skills or concepts that extend over several content strands. The study identifies those NAEP items related to functions (across several content strands) and analyzes this data to identify what U.S. students know about functions in general as well as how they perform differently on items that elicit a visual or an analytic approach to handling functions.

While NAEP primarily classifies items related to functions as belonging to the algebra strand, the high-school Common Core State Standards for Mathematics (CCSSM) present functions as their own conceptual category. (The other CCSSM categories are number and quantity, algebra, modeling, geometry, and statistics and probability.) This shift in classification highlights the changing thinking about functions, from a view of functions as a subset of algebra to functions as a dynamic concept that intersects with numerous other mathematical ideas, cutting across content traditionally divided into separate courses. For example, functions play a role in statistics (e.g., linear regression), geometry (e.g., area of a square as a function of the length of its side), and in real-world applications (e.g., the modeling of growth, decay, and motion) (Ponte, 2007).

Another important dimension of functions is that they can be represented in a number of ways (e.g., graphical, algebraic, tabular). Research has repeatedly demonstrated that this diversity of representations—and the conceptual complexity of shifting from one form of representation to another—contributes to the challenge students face when handling functions (Carlson & Oehrtman, 2005; Eisenberg, 1991). Despite their cross-content relevance, functions

are typically discussed most explicitly in an algebra context (Doorman & Drijvers, 2011). The result may be that students have a limited concept image of functions; that is, the set of mental representations that they associate with functions does not reflect the breadth of representations they may encounter through their mathematical learning trajectory (Vinner & Dreyfus, 1989).

In addition to providing an overview of nationwide student performance on functions, this study considers how presentation of test items related to functions may affect students' ability to engage with the approaches that are best suited to obtaining a solution. For example, if the most appropriate or accessible approach to a given problem is outside the student's range of experience with functions, they are unlikely to succeed on the item. While NAEP data do not provide access to this level of detail about student thinking, careful attention to overall student performance as well as to item-level characteristics can shed light on the current state of student understanding of functions in the U.S.

Conceptual Framework

The NAEP Grade 12 assessment is based on a conceptual framework that outlines content assessed, sampling characteristics, and the presentation of items, among other features. For the purposes of this study, the NAEP framework is assumed to be adequate in terms of item development, collection, and reporting of data. It is further assumed that the NAEP items used in the study are appropriate measures of students' knowledge of functions. This study defines student understanding of functions as correct responses on function-related items given on the NAEP assessment. The concept of functions is understood as encompassing the idea of input-output assignment (correspondence) as well as the idea of a process of co-variation to arrive at a broader understanding of the function as a dynamic mathematical object that can be presented in different ways (Oehtrman, Carlson, & Thompson, 2008; Ponte, 2007). The distinction between items that appeal primarily to visual or analytic processing draws on research that has demonstrated a gulf between students' visual concept images of functions and their analytic characterizations of functions (Eisenberg, 1991). Thus, the classification of items as "visual" or "analytic" is primarily geared toward anticipating what thinking about functions would most readily be elicited from students in responding.

Method

The study uses 41 function items from the 2005 and 2009 NAEP Grade 12 mathematics assessment, 21 of which are classified as analytic and 20 of which are classified as visual. All items used for the study relate to functions (including function items categorized by NAEP under geometry and statistics). Items are identified as related to functions (a) if the item refers to functions or includes a representation of a function in either the question or the answer choices or (b) if at least one possible approach to solving the problem relies upon knowledge of functions. Next, the items identified as relating to functions are further classified into two categories: those items that engage analytic processing of information (e.g., the problem includes formulas and/or verbal description) versus those items that engage visual processing (graphs and/or tables) of information. Five items were classified in both analytic and visual categories, since the items allowed for both analytic and visual processing of information. ConQuest software will be used to identify students' ability on function concepts for both analytic and visual function items by building several item response theory (IRT) models. Analysis of specific function-related items will focus on identifying particular areas of challenge for students, whether in the presentation (visual versus analytic) or in the function concept that the item draws upon.

Results

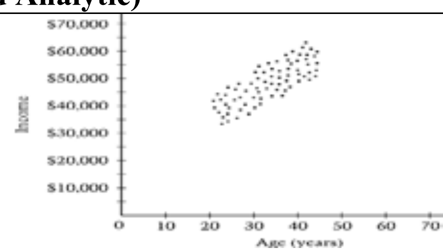
While final analysis of NAEP function items is still underway, the following released NAEP items illustrate how representations may impact students' ability to recognize and manipulate functions. The correct response is marked as (*). Item 1 offers an example of a question that elicits an analytical approach to the concept of functions.

Table 1: Items 1 (Analytic)

Yvonne has studied the cost of tickets over time for her favorite sports team. She has created a model to predict the cost of a ticket in the future. Let *C* represent the cost of a ticket in dollars and *y* represent the number of years in the future. Her model is as follows: $C = 2.50\,y + 13$. Based on this model, how much will the cost of a ticket <u>increase</u> in two years? (**A**) \$5 (**B**) \$8 (**C**) \$13 (**D**) \$18 (**E**) \$26
Student Responses: A 26%* B 7% C 10% D 50% E 5% Omitted 2%

The high percentage of students choosing the incorrect choice D has one obvious explanation in a failure to carefully read the question. The analytic presentation invites computational manipulation (plugging in a number) without significant reflection on the concepts involved.

Table 2: Item 2 (Visual and Analytic)

A random sample of graduates from a particular college program reported their ages and incomes in response to a survey. Each point on the scatterplot represents the age and income of a different graduate. Of the following equations, which best fits the data? (**A**) $y = -1,000x + 15,000$ (**B**) $y = 1,000x$ (**C**) $y = 1,000x + 15,000$ (**D**) $y = 10,000x$ (**E**) $y = 10,000x + 15,000$	
Student Responses: A 4% B 11% C 45%* D 16% E 22% Omitted 2%	

27% of students responded with B or D; both answer choices reflect a failure to visualize the y-intercept of the line of best fit. Because this item draws on both algebra (linear functions) and statistics (scatterplot), it demonstrates the challenge of engaging with functions when the context is not limited to one domain. This item also reflects the conceptual difficulty of crossing from visual to analytical representations of functions.

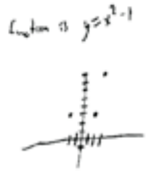
Table 3: Item 3 (Visual) with Sample Responses

The table shows all the ordered pairs (x, y) that define a relation between the variables x and y. Is y a function of x? Give a reason for your answer.	x | y −2 | 3 −1 | 0 0 | −1 1 | 0 2 | 3 3 | 8	**Student Responses:** 53% of students answered the first part of the question correctly. Only 14% of the students gave a valid reason (for each x-value (domain) there is only one associated y-value (range).

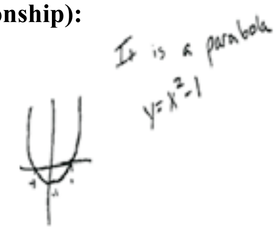
Sample Responses (demonstrating weak or no support for a functional relationship):

Because none of the numbers even relate
A there is no pattern.

C.



B.



In Sample Response A, the student does not recognize the non-linear pattern of the quadratic function as represented in the table, suggesting that he/she has a limited visual concept of functions (one that may only address linear functions). In Sample Response B, the equation $y = x^2 - 1$ does satisfy the ordered pairs on the table, but it is not the only possible function that would do so. Similarly, Sample Response C provides an illustration of one function but fails to recognize that numerous alternative illustrations connecting the ordered pairs exist.

This item asks students to give the definition of a function, but by providing a table of ordered pairs, it invites some consideration of graphical representation. On the one hand, this may have led some students to plot points and recognize a function, resulting in a partially correct response. On the other hand, however, the provision of the table may have also led some students to confuse illustration with justification, a confusion related to incomplete understanding of the essential properties of a functional relationship.

Conclusion

These examples illustrate that there is much to gain from considering the ways in which students engage with function concepts, both in terms of visual or analytic presentation and in terms of the cross-curricular relevance of this concept. As the study progresses, we hope to identify a pattern of performance based on these factors. Already, systematic attention to visual and analytic representations and to the contextual elements of the items (whether they trigger students' existing understanding of functions) has provided a fuller picture of what the nation's students know about functions and where we might best direct our efforts in developing curriculum and instruction.

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USING VIDEOS TO THINK MORE DEEPLY ABOUT HOW CHILDREN LEARN MATHEMATICS

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Videos of children learning math (along with associated commentary) designed for preservice teachers and teachers can be a useful resource to broaden preservice and teachers' knowledge of how children learn mathematics. This study reports the results of an online survey given to 69 undergraduates and 10 graduate students after watching a video of elementary children engaged in the same mathematical activity. Emergent themes from the analysis of the undergraduates responses were: children solve problems in a multiple ways, adults think about mathematics differently than children, and teaching is not telling. Teacher results contrasted with undergraduates in that they demonstrated a greater recognition for the need for manipulatives and richer, classroom based responses. The videos impacted participants' knowledge of how children learn mathematics and their views on teaching children mathematics.

Keywords: Mathematical Knowledge for Teaching, Teacher Education-Inservice/Professional Development, Teacher Education-Preservice, Problem Solving

Recent innovations in mathematics education have included the use of videos with both preservice and in-service teachers. This paper will examine how videos showing children's mathematical thinking have been used with teachers and prospective teachers. Our research project was developed to illustrate children's understanding of mathematics. These videos consist of one-on-one interviews with children. The videos have been edited to show how children learn and think about mathematics. In many cases we combined clips of different children working the same problem to illustrate not only how children think, but also to demonstrate how different children think differently about the same problem. We have added introductory commentary to focus the viewing experience (Sherin & van Es, 2005) on pertinent concepts regarding how children learn mathematics. We have also added commentary at the conclusion of each video clip which is intended to provoke viewers to ask more questions about how children learn mathematics and how to foster children's mathematical learning.

Theoretical Framework

The results reported in this paper are part of a larger study to determine the influence of videos on understanding how children learn mathematics. One of the central questions is how does the viewing of video of children engaging in mathematics activities affect preservice teachers' Mathematics Content Knowledge (MCK). In a prior study, we found that preservice teachers who used written materials about how children learn mathematics compared with a control group who did not use these materials had no significant difference in mathematical scores. However, there were significant differences in efficacy, beliefs, and knowledge of how children learn mathematics (Feikes, Pratt, Hough, 2006). One study (Philipp, et. a., 2007) has shown that videos can impact preservice teachers MCK. However, examining this knowledge with a finer lens it appears that Mathematical Knowledge for Teaching (MKT) is very context bound. This and other studies (Philipp, et al, 2007; Feikes, et. al, 2010) have shown that videos

can impact the Pedagogical Content Knowledge of (PCK) of preservice teachers and subsequent teaching of mathematics to children. Learning from our prior experiences we hypothesize that we must look closely at knowledge of how children learn mathematics.

Videos can be a powerful tool in helping preservice teachers expand their content knowledge (Philips, et al 2007) their Mathematical Knowledge for Teaching (Bell, Wilson, Higgins, & McCoach, 2010) and in developing beliefs that promote developing mathematical understanding (Superfine & Li, 2011). One of the key features of using videos is in helping viewers of the videos *notice* (Jacobs, Lamb & Philipp, 2010) what is important in children's learning of mathematics. What viewers attend to and not attend to is significant in the use of these tools. The videos and especially the narration included at the beginning and end of these videos were designed to help viewers make sense of the video and focus on or notice children' mathematical thinking. Videos have been shown to help teachers modify their focus from pedagogy to students' mathematical thinking (Sherin & van Es, 2005).

Methodology

This smaller study examines and compares the responses of an online survey of two groups of students that was required of both groups as a homework assignment. The first was a group (three sections) of preservice teachers in the second course of a series of three mathematical content courses for elementary teachers. The second was a class of ten practicing teachers taking a graduate mathematics education course on teaching mathematics in the elementary school.

The assignment was to go to a website and watch a video of three second grade students solving the following problem: Mary has 18 marbles. She has blue, green, and yellow marbles. She has 7 blue marbles and 6 green marbles. How many yellow marbles does she have?

After watching the video, the students were required to complete an online survey which required them to submit a written response to five questions. A constant comparative analysis (Strauss & Corbin, 1998) was used to code the open ended survey questions to develop emergent themes. Students were required to put their names on the survey. After some brief commentary, three different students solve the same problem.

In the video the first child selects two numbers from the problem, puts them in column form and either adds or subtracts. She initially wrote 7 - 6 = 1, realized that was not correct and then wrote18 -7 = 11. From the video it is apparent that she knows these solutions were not correct. Without any provocation she begins using the cubes which were next to her the entire time to solve the problem. The second student decides to draw a picture. She colors 8 blue, 7 green and counts the rest to realize she has 6 yellow. The last child does the problem in his head he adds 7 + 6 = 13 and subtracts 13 from 18 to get 5.

Results and Discussion

The following is our initial attempts to analyze the results. We are looking for what did the preservice teachers and teachers learn or perhaps more appropriately glean from watching these videos and reflecting upon. Second we were looking for differences between the two groups.

The first question asked: How did the video help you think more deeply about how children learn mathematics or teaching mathematics to children?

For the undergraduates a common theme that emerged was: children solve problems in multiple ways or have different thought processes in solving the same problem. For example: "I saw that students all use different method… hands on, drawing a picture, and counting up are the three methods that were used in the video."

These results were expected given the nature of the video and the accompanying narration that attempted to direct thinking and noticing. However, an aspect more open to interpretation was the question: "How did the video help you think more deeply about teaching mathematics to children?" The video did not explicitly tell them how to teach and only the last narration suggested how to teach children mathematics. Following are some excerpts that illustrate what preservice teachers took form the video about teaching:

> *Preservice Teacher:* This video really helped me grasp just how important it is to teach multiple ways of thinking and ways of solving problems. Pictures, manipulatives, and mental math are all tactics that children should utilize [sic] and learn at a young age to allow their minds to be able to go back and fourth [sic], depending on the problem, to help them solve the problem.

These responses illustrate that the videos were useful tools in helping students realize that children think about mathematics in different ways. This data also indicates the students were thinking about how they would teach children mathematics. A key finding is the knowledge of how children learn mathematics is the primary emphasis for how they think about teaching.

A second theme that emerged was that preservice teachers appeared to come to the realization that children's thinking is different from how adults or they themselves think about mathematics.

> *Preservice Teacher:* The video made me think more deeply about how children learn mathematics because as an adult we would solve this problem similar to the last young man in the video, but to me an efficient teacher we must understand that all children learn differently. Some children need pictures and manipulatives like this video shows, because they learn at different paces and may not be able to do it how you would do it.

Another theme that emerged was; Teaching is not Telling. Some preservice teachers came to understand that learning takes time and that teachers must be patient and not just tell children the answers and how to solve problems. Other themes that emerged were the importance of manipulatives, the use of pictures, different children solve the same problem differently, and learning mathematics can be exciting.

Similar themes emerged from the graduate students who were all K-6 teachers. However, their responses were richer and often related to their actual practices.

> *Graduate:* There are several ways a child might solve a multi-step problem; manipulatives, drawing pictures, or mental math. There is no correct way to solve it, but the visual aspect is more important some children than to others.

Significant in this response is the knowledge is there is "no correct way" to solve the problem. Often the undergraduates implied that they would have to teach children these different ways. Their intent is the same but their methods of achieving the result are different. Both recognize that children solve problems in different ways but several of the preservice teachers believed they had to directly teach these ways to children.

Teachers were able to elaborate more on what they did not see or what did not work. These teachers seemed focused in developing children's thinking and not just seeing that children solve problems in different ways or even in how to teach problem solving to children.

A final contrasting theme that emerged was that 13 out of 69 or 19% of undergraduate respondents mentioned manipulatives in their response. However, 4 out of 10 or 40% of teachers mentioned manipulatives. While the number of teachers is too small to make valid inferences it does suggest a difference between the two groups. Undergraduates are continually told about the importance of manipulatives and the majority of these students had a mathematical course the semester before where manipulatives were used and emphasized. Teachers, likely have had similar experiences in their undergraduate education and teacher workshops. However, given the realities of the classroom, teachers may not be using manipulatives as much as they feel they should. One teacher said, "I need to make them more available than they are."

Conclusions

Our study and the data collection techniques continue to evolve. We continue to refine the process of making videos, from the location of the camera angle, how to get better sound quality, what narration to include, and how to disseminate the videos on a larger scale.

These courses were all taught by the first author. As this is a design experiment we intend to use the results and the insights we gained in methodology to conduct a larger study at external sites with treatment and control groups. A significant contribution of this study is the responses of the practicing teachers.

Acknowledgments

This paper is based upon work supported, in part, by the National Science Foundation under Grants DUE-1022942

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CONNECTING TO TEACHING PRACTICE IN MATHEMATICS CONTENT COURSES FOR ELEMETARY PRESERVICE TEACHERS

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Our study investigates the use of a particular practice, connecting to teaching practice, employed by mathematics teacher educators in teaching mathematics content courses for elementary preservice teachers. Six mathematics teacher educators, all with experience in teaching content courses for elementary preservice teachers, participated in this study. In an interview setting, the participants described whether, how and why, in their teaching of mathematics content courses, they connect teaching practice to the mathematics content they teach. We first discuss what they do to connect teaching practice to content learning. Then, we discuss participants' rationales for engaging in this particular practice. Overall, we found that practicing components of K-12 teaching, using artifacts of teaching and learning, and modeling are three common strategies that these mathematics teacher educators use to connect preservice teachers' learning to teaching practice in mathematics content courses.

Keywords: Mathematics Teacher Educators, Teacher Education-Preservice, Mathematical Knowledge for Teaching, Instructional Activities and Practices

Introduction

The field of teacher education lacks an evidentiary base for understanding the practices of mathematics teacher educators (MTEs), professionals who work with practicing and/or PSTs to develop and improve mathematics teaching (Jaworski & Wood, 2008). Though there has been some exploratory work in this domain (see Jaworksi & Wood, 2008), currently there has been no synthesis of what MTEs need to know and do in order to support PSTs in developing mathematical knowledge in ways needed for teaching. In this study, we shift the focus of research on mathematics teacher education from what did PSTs learn to what is the work of MTEs as they supported PSTs' learning where "work of MTEs" refers to the practices in which MTEs engage inside the university classroom. Our study takes place in the context of a two-year professional development program for MTEs, and examines the work of six MTEs as they support PSTs' learning of mathematics. Our study is guided by the following question: In what ways do MTEs connect to teaching practice in mathematics content courses for elementary PSTs?

Background

In his work focused on teacher educators, Mason (1998) suggests that the work of MTEs is similar to that of teachers and that, in addition, the work of MTEs involves helping PSTs recognize how to relate what they are learning to teaching. Specifically, he suggests that the work of MTEs involves developing and enhancing different levels of awareness in PSTs, as opposed to simply helping them learn the content that needs to be learned. Attention may be drawn to actions which are being carried out with lesser or greater awareness, so that in the future the learner might become aware of a possibility of choosing actions according to the circumstances (Mason, 1998). PSTs need to be able to engineer instructional situations in which

students experience a shift in their attention where they (i.e., students) become aware of ideas and concepts of which they were previously unaware. Consequently, the work for MTEs is to develop PSTs' understanding of certain mathematical ideas and concepts, and develop PSTs' awareness of how to connect what they are learning to teaching. For example, MTEs must not only develop PSTs' ability to evaluate the transparency of mathematical ideas in representations for themselves as learners, but also support PSTs in recognizing why evaluating the transparency of representations is important for planning lessons and selecting representations that will support the development of student's understandings. Thus, the work of MTEs involves helping PSTs become aware of how what they are learning about mathematics is connected to teaching practice.

Method

To understand MTEs' practice of connecting teaching practice, we employed a grounded theory research design (Strauss & Corbin, 2007) to allow the practices to emerge from MTEs' descriptions of their own teaching practices. Six MTEs, four female and two male, all with experience in teaching content courses for elementary PSTs, participated in this study. All names are pseudonyms. Individual semi-structured interviews were conducted by the authors to probe MTEs' understanding of mathematical knowledge needed for teaching, their perceptions of the nature of their work as MTEs, and the rationales for their course design.

All interviews were transcribed and independently coded by the authors. All discrepancies were resolved through discussion. We conducted three rounds of open coding as part of the analysis process, which involved highlighting transcript segments about MTEs' perceptions of and practices used in teaching, categorizing segments into three broad categories (i.e., MTE practice, conception of knowledge and skills PSTs need, conception of a content course), and then developing related sub-categories, including connecting to teaching practice and assessing PSTs' learning, among others, to provide more detail about the three broader categories. For our current study, we focus only on the results of the coding analysis around the connecting to teaching practice category and related sub-categories.

Results

In this section, we first discuss the ways they connect to teaching practice, and then discuss their rationales for connecting to teaching practice in their content courses. Overall, we found that all six MTEs connect PSTs' learning of mathematics to teaching practice in their content course, albeit to varying degrees. We identified three primary strategies by which MTEs reported connecting to teaching practice: (1) having PSTs practice components of teaching, (2) using artifacts of teaching and learning, and (3) modeling teaching.

What Are the Ways in Which MTEs Connect to Teaching Practice?

Having PSTs practice components of teaching was the most common strategy that MTEs reported using to connect math content learning and teaching practice in their courses. This strategy involved having PSTs "try out" what a teacher might have to manage in the classroom. For example, using questions situated in K-12 teaching practice as a journal prompt for PSTs (e.g., As a teacher, I will use tasks to promote…); acting like a student to create teacher-student interaction situations; eliciting PSTs' experience working with students; having PSTs consider the context of teaching at a specific grade level; considering students' common misconceptions; having PSTs analyze math curriculum standards; and having PSTs explain mathematical

thinking.

Using artifacts of teaching and learning was another common strategy used by the MTE participants. MTEs reported using the following artifacts of teaching and learning to connect content learning and teaching practice: showing videos of students doing math, providing anecdotal stories about currently practicing teachers as well as anecdotes about students' thinking and misconceptions, encouraging PSTs to share stories about students' learning, discussing education policy reforms, analyzing a task from an elementary textbook, and providing opportunities for PSTs to familiarize themselves with manipulatives.

Modeling was another strategy that MTEs reported using to connect PSTs' learning to teaching practice. Two particular ways of modeling were reported: MTEs modeling "good teaching" and not just lecturing, and MTEs modeling how to interact with students.

Rationales for Connecting to Teaching Practice in Their Content Course

In the interviews, MTEs were asked to explain why they connected preservice teacher learning to teaching practice, and how this practice was important for PSTs' development as teachers of mathematics. Due to space constraints, we only discuss rationales from three MTEs.

Nancy: Get them to think deeply about the content they are going to teach. Nancy felt strongly that a math content course should always include an emphasis on teaching practice. She claimed that learning mathematics in the context of teaching practice encourages PSTs to approach math conceptually and to move beyond rote memorization.

> They're (PSTs) in the process of learning it (the mathematics) themselves, so if you put it in the context of how you would teach this to someone else, it forces them to think differently from "oh, I have to memorize this."

Nancy mentioned that her goal is to emphasize process and understanding. The challenge of achieving this goal is to get PSTs to agree on the importance of learning mathematics conceptually. The fact that PSTs will be teachers provides an opportunity for her to engage PSTs in deeper learning. In order to get PSTs to think deeply about the math content, Nancy made explicit the ideas that what they learn is what their future students are going to learn. She also reported using questions, like "what if a kid asks me this?", as a journal prompt to support PSTs in assessing their own understanding. In addition, she reported often acting like a student in order to create situations to motivate PSTs to understand the knowledge required for engaging students with the content.

Edward: I have to teach mathematics to them in a way they will hopefully replicate when they get back out there. Edward's goal for PSTs in his content course is to experience mathematics in an enjoyable and deep way. He believes that it is difficult to engage PSTs' learning of mathematics for understanding, if they do not enjoy learning mathematics. Moreover, he claimed the way we teach PSTs mathematics and the way we want PSTs to teach in the future should be consistent; and for most PSTs, the way that they learned mathematics was not the way we want them to teach their students. So, in his content course, he aimed to teach PSTs mathematics in a way that they will teach in the future.

> It is hard to say that it's okay both teaching the content this way but we want them to do it that way. I've gotta teach it to them in a way that'll hopefully replicate when they get back out there. Many of them haven't done it in that way, so I think just doing it in a way that we hope they would do it.

To achieve this goal, Edward claimed that he had to model a lot how he would teach students. In addition to modeling, Edward reported highlighting aspects of teaching as PSTs practiced components of teaching in class. He provided an example involving explanation. He would find PSTs who provided clear explanations and drew clear pictures during their presentation, and then he would point out the positive features and say: "Wouldn't you want this person to be your teacher?" He believes that highlighting aspects of teaching from PSTs helps them to see teaching standards from their peers. He also reported sharing anecdotes of K-12 teachers' misunderstanding of mathematics content to PSTs in his course.

David: The beauty of mathematics is not gonna teach someone how to be a great teacher. David's approach to teaching content courses is characterized by integrating mathematics and pedagogy. He stated that PSTs need to, at the minimum, know the mathematics that they are going to teach. Learning mathematics content and teaching mathematics should not be separated. He leans towards pedagogy or teaching through the methods. He explained, teaching through methods has the potential to motivate PSTs' learning; also, it provides opportunities to engage with both mathematics content and pedagogy.

> If I'm teaching through the methods, it's a good selling point. Watching children engage in mathematics and seeing how a teacher responds to children is both methods and content. … You have to extract the content out of that to make it meaningful.

Moreover, David pointed out that some of the mathematics knowledge is particularly required for teaching. Without connecting to classroom teaching practice, teaching this kind of knowledge could be mechanical.

> I can teach them why long division works, why different subtraction models. Unless we tie that to actual classroom practice, there might be some type of disconnecting there (the mathematics content PSTs learn and the mathematics they need to teach).

Similar to previous participants, he values the practices of modeling good teaching in a content course. He believes that PSTs take away not only the content but also the way MTEs teach the content.

Conclusion

The purpose of our study is to understand the ways in which MTEs connect PSTs' learning of math to teaching practice in content courses. Through semi-structured interviews, we found that connecting teaching practice to PSTs' content learning was a common practice for these MTEs, but carried out in different ways. Results of our analysis contribute to our limited understanding of the work of MTEs in three ways: 1) provides empirical evidence of the existence of the connecting to teaching practice as part of MTEs' work; 2) provides evidence of the different strategies by which MTEs connect to teaching practice; and 3) elucidates MTEs' rationales for engaging in this particular practice as part of their work.

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INSTRUCTIONAL FACTORS AFFECTING SECOND-GRADERS' CONCEPTUAL AND PROCEDURAL KNOWLEDGE OF PLACE VALUE CONCEPTS

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The benefits of mathematics teaching that focuses on conceptual understanding have been well documented, and previous research has also demonstrated the merits of learning procedures (Rittle-Johnson et al., 2001). Students also need to understand how the two are connected (Hiebert, 1992), but there is no consensus on the most efficient ways to teach for such connected understanding in mathematics. There is some evidence that the order in which lessons are presented has an impact on mathematics learning, and that alternating between concepts and procedures is more effective than introducing concepts before procedures (Rittle-Johnson & Koedinger, 2009). In addition, several have suggested that teachers must make explicit connections between concepts and procedures (e.g., Uttal, 2003), but little systematic research exists to support this claim in early mathematics learning. In the present study, we investigated (a) how the order of presentation of concepts and procedures would impact children's conceptual understanding of place value and procedural execution of addition strategies, and (b) whether the level of explicitness in linking concepts and procedures would moderate these effects.

Eighty-seven ($N = 87$) second-grade students from a suburban school district in Canada were randomly assigned to one of six conditions; each was a different combination of lesson **order** (concepts-first, procedures-first, iterative) and level of **explicitness** during instruction (explicit, not explicit). Students were asked to solve a series of conceptual (e.g., In $\underline{7}1$, how much is the underlined number worth?) and procedural (e.g., 32 + 19) problems before and after instruction.

Two 2 (time) x 2 (explicitness) x 3 (order) repeated measures ANOVAs were performed, with time as the repeated measures factor and explicitness and order as the between group factors. The outcome measures were performance on conceptual tasks and procedural tasks, respectively. For both conceptual and procedural measures, there was a main effect of time (ps < .001), indicating improved performance from pretest to posttest. The analyses also yielded an interaction between explicitness and order for conceptual tasks, $F(2, 81) = 3.230$, $p < .05$, and for procedural tasks, $F(2, 81) = 7.374$, $p < .01$, revealing in both cases that making explicit connections was particularly beneficial for students in the procedures-first group.

These findings suggest that teachers should make explicit connections between concepts and procedures when teaching addition and place value, particularly if students are first introduced to mathematical procedures.

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DO CONCRETE SYMBOLS FACILITATE FRACTION LEARNING?

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A common approach to math instruction is the use of concrete representations to teach abstract concepts. Concrete representations have been described as "contextual, real world examples" (Kaminski et al., in press) and can include physical manipulatives, visual representations, or computer animations (Kaminski et al. in press; McNeil & Uttal, 2009). One area of math instruction in which the use of concrete symbols is particularly pervasive is the teaching of fractions, especially for children with a mathematics learning disability (MLD). However, it is not clear whether the use of concrete materials improves fractions instruction and, if it does, whether it does so equally for all children. For instance, children with MLD or low achievement in mathematics (LA) continue to underperform on fraction items relative to their typically achieving (TA) peers (AUTHOR, 2008). In this longitudinal study we examine whether the use of concrete representations to teach fractions affects performance, and if the effects are comparable across these three groups of children.

Participants were 159 students (seen at Grades 4 and 5), who were classified as having MLD (n=16), LA (n=26) or TA (n=117). We used the *Symbolic Magnitude Comparison of Fractions* (SymMCoF) task to measure children's knowledge of fractions. The SymMCoF includes pairs of fractions represented as visual models, presented with either standard or expanded and contextualized concrete instructions. In each of these two conditions, children identified the larger item in the fraction pairs or could indicate that the pair was equivalent.

Overall, participants were *less* accurate in the concrete versus the standard condition. Using repeated measures ANOVA we found a main effect of condition, $F(1, 156) = 22.081$, $p < .00001$, $\eta^2_p = .124$, with slightly more accurate performance in the standard (70%) versus concrete condition (65%), Cohen's $d = .24$, and a main effect of MLD status $F(2, 156) = 27.023$, $p < .00001$, $\eta^2_p = .257$. Post-hoc comparisons revealed that, in Grade 4, children with MLD performed no differently from their LA peers on both standard ($p = .06$) and concrete visual models ($p = .10$), whereas, in Grade five, children with MLD were less accurate than children with LA on both standard (MLD = 61%, LA = 75%) and concrete visual models (MLD = 55%, LA = 72%; $ps < .003$).

Results from this study indicate that using additional concrete instructions does not improve understanding of fractions during the SymMCoF. In fact, participants were more accurate on items presented under the standard condition. In follow-up analyses, we explored whether working memory or executive function contributed to accuracy levels in the concrete condition.

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CONTRIBUTIONS OF VISUAL REPRESENTATIONS TO STUDENTS' RATIONAL NUMBER REASONING AS THEY SOLVE EQUAL SHARING PROBLEMS

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Keywords: Elementary School Education, Rational Numbers, Instructional activities and practices, Learning Progressions

Empson and Levi (2011) observe that the types of visual representations for rational number reasoning that students encounter in K-6 education "offer few opportunities for children to make connections between fractions and whole numbers" (p. 7). Although Cobb, Yackel, and Wood (1992) warned there is an inherent problem with expecting students to use representations to construct mathematical relationships, Empson and Levi (2011) observe, "learning to create fractional quantities on their own using part-whole and other kinds of representations contributes to children's understanding of fractions" [rational numbers], "because to create workable representations, they need to reason about relationships, such as how the number of parts is related to the whole unit or how the ratio of sharers to shared is related to each person's share" (p. 28). As a classroom community develops ways of symbolizing through the use of visual representations that they create and use, visual representations become "taken-as-shared" symbols by the classroom community for communicating the mathematical relationships that they are constructing—the mathematical relationships that contribute to students' construction of the meaning for rational numbers. This study documents how visual representations provide both support for and information about students' development of rational number reasoning.

Researchers observed third grade students as they solved equal sharing problems in an inquiry classroom setting. Small groups or pairs of students were given equal sharing problems to solve by their teacher. Researchers observed the students' use of visual representations whether self-created or provided by the teacher. Students' use of visual representations was examined for information about how the children were thinking as they solved the problems, how the visual representations supported or did not support their reasoning during the problem solving process, and how students' use of visual representations provided information about what the students understood about rational numbers.

Analysis of the data revealed that the students were most successful when provided visual representations that allowed for ease of partitioning. Students who showed the most progress by the end of the study were able to partition the visual representations of wholes, distribute equal parts, and then recombine the distributed parts into wholes when appropriate. They answered with appropriate mixed numbers and were able to explain their work. Students who made moderate progress either partitioned the visual representations of wholes, distributed the parts equally, answering with an improper faction, or distributed the wholes and only partitioned wholes that could not be distributed, answering with a mixed number. They were not able to relate the two types of answers.

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METAPHORS STUDENTS EXPRESS ABOUT INTEGERS AFTER USING MULTIPLE MODELS

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Keywords: Number Concepts and Operations, Cognition, Learning Theories, Metaphor

Student thinking and skills regarding negative numbers have been extensively studied in order to help students overcome the difficult transition from whole number to negative quantity fluency (e.g., Kilhamn, 2011; Vlassis, 2004), but investigation of the underlying conceptual mechanisms on which we might capitalize have only just begun. Educators and researchers have used many different pedagogical models in their attempts to help students extend their concept of number to negatives. Different models, however, likely activate different conceptual metaphors. Prior research on the conceptual metaphors underlying negative numbers has focused on the metaphors of the mathematics (*object collection*—numbers as things, *measuring stick*—numbers as static lengths, and *motion along a path*—numbers as locations and operations as movements), comparing student with adult use of metaphors, or longitudinal student learning with a teacher who tried to avoid specific models (e.g., use of number line, or chip models) (Chiu, 2001; Kilhamn, 2011; Lakoff & Nunez, 2000). These prior reports of student thinking focused on verbal explanations, including drawings only if students spontaneously created them.

The preliminary data reported here were collected during the piloting of a measure for a larger study from four heterogeneous sections of seventh-grade students (n=95) whose teacher used the *Connected Mathematics Program* (*CMP*). CMP uses a chip model, a number line model, and encourages generalization of rules. The chip model uses the underlying idea of an object collection metaphor, whereas the number line model as promoted in CMP combines a measuring stick and motion along a path metaphor. The research question was: What types of metaphorical and rule-based reasoning did students who experienced multiple models of integers demonstrate? I report results from three items of a written assessment in which I prompted the students who had studied negative numbers to include a drawing in their explanations to students who had not yet studied negative numbers. These items asked questions like *Which number is smaller? What does* ${}^{-}3 - {}^{-}5$ *mean?* and *What does* ${}^{-}152 - {}^{-}85$ *mean?* Written explanations and drawings were coded for use of rules and conceptual metaphors. Preliminary data showed 54% of students used procedures or rules and 88% used conceptual metaphors to explain integers and integer subtraction. Students infrequently used an *object collection* metaphor (25%) or portrayed a number line with a *measuring stick* metaphor (19%). Most students (73%) used a number line representation in ways consistent with the *motion along a path* metaphor, conceiving of numbers as points located on a path. Future research should assess what students learn via each metaphor.

Acknowledgement

Research was funded in part by a Michigan State University College of Education Research Expense Fellowship.

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CHILDREN'S AND ADULTS' USE OF MODELS OF WHOLE NUMBER DIVISION: CONSISTENCY OR VARIABILITY?

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Does children's knowledge about whole number division influence their learning of fraction division? Division by fractions is a notoriously difficult topic in arithmetic, and both children (e.g., Siegler, Thompson, & Schneider, 2011) and adults (e.g., Ma, 1999) struggle to represent its conceptual structure. One potential barrier to a deep understanding of fraction division may be children's intuitive models of whole number division. Children rely primarily on partitive models for whole number division (Fischbein, Deri, Nello & Marino, 1985). However, fraction division is better understood within a quotative model of division. Thus, children's early partitive models of division may pose barriers for understanding fraction division.

The primary aim of this study was to describe the frequency and consistency with which children and adults use partitive or quotative models of division, across a variety of tasks. To address this question, third-graders, fourth-graders, and adults were asked to solve division problems in three contexts: an *object* task (using counting chips to represent a division problem), a *story* task (generate a story to match a division problem), and a *number line* task (demonstrate a division problem using the number line). Each participant solved one division problem in each context, and their models of division (*partitive, quotative, other*, or *no model*) were coded.

Both groups of children as well as adults spontaneously drew on both partitive and quotative models to model whole number division (Grade 3: 45% partitive, 40% quotative, Grade 4: 62% partitive, 24% quotative; Adults, 44% partitive, 33% quotative). Furthermore, most participants in each age group relied on multiple models within a single session. Variability in model use across tasks was examined among participants who were variable in their model use. In the *object context*, most 3rd- and 4th-graders relied on a partitive model, whereas adults were more varied. In the *story context*, 3rd-graders were about equally likely to employ each of the two division models, whereas most 4th-graders and adults used a partitive model. In the *number line* context, the majority of trials in all age groups were classified as quotative.

In contrast to prior work, we found that many children spontaneously used quotative models of division. This is due, in part to the number line context, which was not included in prior studies. However, many current reform curricula introduce and provide children practice with measurement models, and this may support quotative reasoning on the number line task. Given children's spontaneous use of the quotative model in the number line context, and the finding that fraction division is easier to conceptualize within a quotative model (Fischbein et al., 1985), the number line may be particularly useful in fraction division instruction.

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THE ROLE OF UNITS COORDINATION IN CONSTRUCTING AN ADDITIVE RELATIONSHIP BETWEEN SIGNED NUMBERS

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Beginning in Spring 2011, I conducted a semester-long constructivist teaching experiment (Steffe & Ulrich, 2013) with four sixth-grade students in order to investigate how the mathematics of each student, including schemes for unsigned quantities, affected the nature of the additive relationships that the students were able to construct between signed numbers. During the study the two students with the most sophisticated unsigned number schemes independently constructed a strategy that I call *reasoning through zero,* which corresponds to a behavior Peled, Mukhopadhyay, and Resnick (1989) identified as representing (at least) the use of a mental Divided Number Line. More surprising was that the other two students, one of whom had relatively sophisticated ways of operating with whole numbers and fractions, experienced great difficulty in trying to utilize this same type of strategy. Based on empirical and conceptual analysis, I hypothesize that the multiplicative coordination of three levels of units must be interiorized before students can independently develop this strategy. Research (e.g., Steffe & Olive, 2010) indicates that coordinating three levels of units is important across both fractional and algebraic contexts as well, although the coordinations in both of those contexts are multiplicative instead of additive.

Consider the following example of *reasoning through 0* when a student is finding the total displacement from 0 after a trip of -29 and a trip of +37: “I got +8 because I added +29 to that [sketch of -29 trip] and it made it 0 again. Then from 29 to 37 it is 8m...[The 8m] is coming from the 37.” Reasoning through zero involves the ability to partition the second addend (+37) into the additive inverse (+29) of the first addend and its complement (+8), the overall change in value. The reason that the process would be any harder than, say, subtracting 29 from 37, which does not require the coordination of three levels of units, is that the student must maintain an additive comparison with the original reference point as he or she counts past zero in order to recognize the sign of the total change. Therefore, *the partition of the second addend that extends past 0 must maintain its status as a subsequence of the second addend* in order to inherent the sign of the second addend. I believe that the student must form this three-levels-of-units structure as he or she begins to apply the second addend in order to keep track of their position with respect to the beginning of the second addend’s number sequence (the right-hand dotted line) as well as their position with respect to the original reference point. This means that they are not only forming three levels of units in action, but are monitoring and adjusting the three-levels-of-units structure as they continue counting past zero. This monitoring would be quite difficult for a student who has not yet interiorized the three-levels-of-units additive structure.

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MAKING SENSE OF COMPLEX NUMBERS THROUGH THE USE OF STUDENTS' PREVIOUS KNOWLEDGE

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Keywords: Algebra and Algebraic Thinking; Instructional Activities and Practices; Standards; Problem Solving

In conversation with the nature of mathematics is the need to have students make sense of complex numbers. The common core state standards for mathematics (CCSSM) recommend introducing the complex number system towards the end of students' mathematical careers, starting in Algebra Two (CCSSM, 2010). In learning the complex number system, students are also expected to learn how to algebraically manipulate them and solve for complex zeros. Moreover, CCSSM emphasize the importance of students making connections to existing knowledge, but are not clear in *how* to achieve this goal when working with Complex Numbers. Research has shown that incorporating proofs and mathematical task can help facilitate understanding of mathematical concepts (Henningsen & Stein,1997). However, opportunities to use proof are often limited in textbooks (Thompson et al., 2012); when they are presented, students have difficulty producing proofs that teachers would consider satisfactory (Healy & Hoyles, 2000).

The goal of this poster is to present a method to connect students' knowledge of Real and Natural Numbers to Complex Numbers. The lesson asks students in a group discussion to recall the definitions of distributive, associative, and commutative rules. After students report back to the larger group, the teacher introduces the rule of how complex numbers and the idea of *a+bi* is a complex number and how the definition of an imaginary number as i^2=1. Students then are instructed to work through a proof through geometrical representations with their group to reason why the distributive laws hold for complex numbers.

By providing students with more opportunities to work with proofs, students are exposed to multiple representations of the complex numbers allowing them to see different lenses as to how to work with the rules. Specifically, students work with *mindful manipulations* in order to understand the proofs of the distributive properties as well as *implement strategies* to see the connections of how and why the distributive laws hold for complex numbers (Graham et al., 2010). Ultimately, proofs can help scaffold onto students' previous knowledge so that the new knowledge they develop is consistent and meaningful.

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CONTEXTS OF STUDENT-CONSTRUCTED STORIES ABOUT NEGATIVE INTEGERS

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Keywords: Middle School Education, Number Concepts and Operations, Rational Numbers

A holistic understanding of number is abstract and challenging; however, negative integers are particularly difficult because they cannot be physically modeled in the world. Because of this, connecting negative integers to contexts provides a typical pedagogical tool to explain the existence and relevance of negative integers. The *Common Core State Standards for Mathematics* (Council of Chief State School Officers, 2010) suggested uses of contexts include, "temperature above/below zero, elevation above/below sea level, credits/debits, positive/negative electric charge" (p. 43). However, other studies have shown that negative integers may not intuitively emerge from these contexts for students (e.g., Whitacre, Bishop, Lamb, Phillipp, Schappelle, & Lewis, 2012). This poster will describe a study that aims to investigate the ways students connect negative integers to their world. This study used a story-telling approach (Mukhopadhyay, 1997) as a way to determine how students apply context to negative integers.

Six eighth-grade students from the Midwest, after more than two years of instruction on negative integers in school, participated in this study. Following a semi-structured interview protocol, students constructed stories for ten different open number sentences involving negative integers (i.e., 8 – 20 = ☐, -5 + ☐ = 21). After students constructed their stories, they were asked to reflect on how the number sentences connected to their stories.

Findings show that students did not typically utilize the traditionally advocated contexts (e.g., temperature, elevation) in their stories about negative integers. Of the traditional contexts that the students employed, the students utilized the context of credits/debits most frequently. However, the students did not always use the context of credits/debits in ways that connect to the traditional notions of money. This poster will share specific examples of stories that students constructed that are parallel to the conventional credits/debits contexts and are considered to be non-conventional stories. These non-conventional contexts and what they indicate about students' understanding of integers will also be reported. For example, for the number sentence -17 + 12 = ☐, "I want 17 baseball cards. I got 12 baseball cards. I still want 5 cards" represents a nonconventional context provided by a student. These findings have implications for the types of contexts we use with both instruction and research with integers.

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CONCEPTUAL MIS(UNDERSTANDINGS) OF FRACTIONS: FROM AREA MODEL TO MULTIPLE EMBODIMENTS

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Keywords: Rational Numbers, Number Concepts and Operations, Instructional Activities and Practices, Elementary School Education

The idea that area-model approach should take priority in thinking about the teaching and learning of fraction concepts has been emphasized in many elementary textbooks and teaching periodicals (NCTM, 2000). Researchers, however, have questioned the virtue of emphasizing one mode and form of representation of fractions, and advocated the use of multiple embodiment approach (Clements & Lean, 1994). The study was intended to assess knowledge of the unit fractions 1/2, 1/3 and 1/4 of participating end-of-fifth-grade students, and to improve and strengthen their conceptual understandings of the unit fractions.

A paper-and-pencil test was first administered to 40 participants, randomly assigned to two equal-sized groups (Group 1 and Group 2). The Group 1 students were then interviewed and participated in five fractions lessons led by their normal mathematics teacher. Afterwards, all students took a parallel form of the original paper-and-pencil test and were interviewed. The Group 2 students participated in similar fractions lessons, and were re-tested and re-interviewed. About four months later, the students took a parallel form of the paper-and-pencil tests as a retention test. A variety of models including linear, perimeter, discrete, capacity and area models were incorporated into the tests and interview protocol to represent fractions. As to the fractions lessons, there were six distinguishable activities involved during which the students were asked to (a) walk around the boundaries of a large equilateral triangle and a large square that were fixed on the floor, (b) fold paper strips and a piece of rope, (c) pour a full glass of water into identical glasses, (d) share 12 discrete identical blocks, and (e) crouch down and stand up.

Analysis of test results indicated that out of total 28 points, 12 of 40 participating students scored less than or equal to 14 (50%), and 15 scored at least 21 (75%) on the first pre-teaching test; however, only 3 of the students scored less than or equal to 14, and 33 scored at least 21 on post-teaching tests. Before the teaching intervention, many of the students failed to provide correct answers for those tasks where unit fractions were associated with perimeters, capacities, lengths, number lines, and discrete objects, although they were adept at partitioning and shading regional shapes. For example, at the pre-teaching stage, 35% of the students did not locate 1/4 of the way around a square correctly and 45% of them did not locate 1/3 of the way around an equilateral triangle correctly. However, at the post-teaching stage, over 90% of the students were able to respond correctly to these tasks. Students' performances on the retention test were close to their performances on the post-teaching tests, and the students had retained most of the knowledge and concepts that they had learned during the lessons.

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Chapter 4: Geometry and Measurement

Research Reports

Brief Research Reports

Poster Presentations

THE LONGITUDINAL DEVELOPMENT OF UNIT CONCEPTS IN AREA AND VOLUME MEASUREMENT CONTEXTS: A CASE STUDY

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This report describes the longitudinal development of one student, Drew, in terms of his unit concepts in area and volume measurement situations across Grades 2 through 5. Data were collected during individual interviews and an open-response assessment within the context of a four-year longitudinal teaching experiment. Results indicate that Drew's use of unit developed over time with respect to identifying and operating on composite units. In addition, his understanding of the conceptual link between the iteration of a composite unit and the application of multiplication in both area and volume measurement contexts developed gradually. Furthermore, changes in his use of unit in an area measurement context often precipitated changes in his use of unit in a volume measurement context.

Keywords: Measurement, Geometry and Geometrical and Spatial Thinking, Learning Trajectories

The development of children's conceptions of spatial measurement has attracted the attention of researchers. However, much of what is known has been studied using cross sectional studies and has focused on only one domain of measurement in isolation (i.e., Battista & Clements, 1996; Battista, Clements, Arnoff, Battista, & Borrow, 1998; Outhred & Mitchelmore, 2000). Therefore, little is known about how children's sense of unit develops over time and connects across domains of spatial measurement. The purpose of this report is to address this gap in the literature by describing the longitudinal growth of one student, Drew, with respect to his developing sense of unit in the contexts of both area and volume across Grades 2 through 5.

Drew gained sophistication in his thinking with respect to unit across area and volume measurement contexts; however, his longitudinal growth was gradual. By closely studying Drew's evolving sense of unit for both area and volume, we gained insights into how concepts related to unit develop over time and across domains of measurement, especially for students who may struggle to coordinate area and volume.

Theoretical Framework

Our research team conducted a longitudinal study of the development of children's spatial measurement concepts across Grades 2 through 5 from a hierarchic interactionalist perspective (Clements & Sarama, 2007), which is a theoretical framework that synthesizes empiricism, (neo)nativism, and interactionalism. At the core of Sarama and Clements' (2009) theoretical and empirical frameworks are hypothetical learning trajectories (LTs). An LT consists of three components: 1) a specific learning goal in a mathematical content domain, 2) a likely path for

learning, and 3) the instructional tasks specifically designed to engender the mental processes or actions that move students along that path (Sarama & Clements, 2009).

Purpose

The purpose of the longitudinal study was to refine and revise LTs for length, area, and volume measurement. Specifically, we posed the question: How do students develop coherent knowledge and integrated strategies for measurement across the developmental span from Grade 2 through 5? The goal of this paper is to describe how children's concepts of units develop across contexts of area and volume from Grade 2 through 5 by focusing on Drew as a case study.

Methodology

This report focuses on one participant (who is a member of a larger sample of 16 children) at a public school in the Midwest. The data was collected between March of 2008 and May of 2011 within the context of a four-year, longitudinal teaching experiment (Steffe & Thompson, 2000). We used previously developed LTs (Sarama & Clements, 2009) to inform our task development and data analysis as we aimed to investigate children's conceptions of spatial measurement across Grades 2 to 5. Results are drawn from semi-structured interviews of approximately 20 to 30 minutes in length and one open-response assessment. Each interview was videotaped, transcribed, and later analyzed according to the LTs. At the conclusion of the data collection, retrospective analyses were conducted, and longitudinal accounts of growth were generated.

Results and Discussion

During the four-year teaching experiment, the research team posed 55 area tasks within 15 interviews and 48 volume tasks in 11 interviews. This report focuses on an illustrative subset of these tasks, as well as from a 33-item open-response assessment to demonstrate Drew's longitudinal growth across area and volume.

Grade 2: Initial Assessment

In the Spring of Grade 2, the research team administered an open-response assessment designed to elicit his ways of identifying and operating on units across area and volume contexts. For one of the area items, Drew was shown an image of a rectangle printed next to a square unit and asked to draw how it would look if he had a whole bunch of those squares to cover the rectangle. He drew 10 individual squares along two rows of the top of the rectangle and said, "just like that to fill it up" (Figure 1). When asked how many he thought he would need to cover the rectangle, he said "20." Drew did not need to complete his drawing; he was able to imagine repeating square units of area to cover the rectangle.

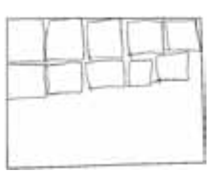


Figure 1: Drew's Drawing

Figure 2: Inch Cube and 2 x 3 x 2 in. Prism

To get a sense of Drew's attention to unit in the context of volume measurement, the interviewer gave him an inch cube and a 2 x 3 x 2 in. rectangular prism and asked how many cubes would be as big as the (2 x 3 x 2) block of wood (Figure 2). Drew, tapped the inch cube on the visible squares on all of the faces of the rectangular prism (turning it to touch all of the

squares on all of the faces) and answered "25." Although he had both the cube and the rectangular prism in his hands, he attended to squares on the faces rather than cubes in his count.

In Grade 2, Drew's unit concepts in an area measurement context were more sophisticated than his unit concepts in volume measurement. His representation (drawing) for the tiling of a rectangle had some errors of alignment, his rows contained the same number of units, and he drew individual approximately the same size squares one-by-one (Figure 1). This suggests that Drew was thinking about repeating individual units of area (without considering composite units). In the context of volume measurement, however, the salient attribute of the rectangular prism was the squares on the faces. Although he correctly identified the square as the appropriate unit for area, he did not seem to identify the cube as the unit of measure for volume on the initial assessment. Drew's nascent sense of unit observed here on the initial assessment evolved to become more sophisticated in Grade 3 in the contexts of both area and volume measurement.

Grade 3

During the Spring of Grade 3, Drew was presented with a 2 x 2 in. square (Figure 3) and asked to find the area. Drew said, "I can't solve the area because in our math boxes in our math books, there's like boxes so we count the boxes." He was then given a 3 x 4 in. rectangle and asked what kind of tool he needed to be able to find the area. He said he needed "a ruler, I think." He placed the ruler along the 4-in. side and noted it was 10 cm. He then placed the ruler along the adjacent side and used the numbered tick marks on the ruler to guide his drawing of tick marks (Figure 4). Drew explained that he used centimeters (as opposed to making use of the given number labels) because the boxes in his math books are about 1 centimeter tall and wide. He explained, "there's a row of boxes between each one, so 10, 20, 30, 40, 50, 60, 70." When asked to draw one of the 70, he drew a row of 10 by extending a single line across the rectangle and used the numbered tick marks on the ruler to guide his placement of segments to subdivide the row into 10 square centimeters. Drew constructed a composite unit and mentally operated on it to determine the area of the 3 x 4 in. rectangle.

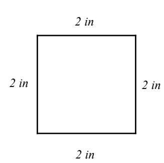


Figure 3: 2 x 2 in. square

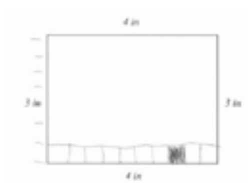


Figure 4: Drew's drawing

Figure 5: Prisms *a*, *b*, and *c*

Also in the Spring of Grade 3, Drew was presented with a task involving unit ratio comparisons and capacity. After comparing prism *a* to *b* and *b* to *c* by pouring water (Figure 5), he said he "figured out a pattern." Drew explained, "it would take two (prism *a*) to get into this one (prism *b*), and two of these (prism *b*) to get into that one (prism *c*)." The interviewer asked, if this (prism *a*) is one, what is this one (prism *b*)? Drew correctly said two. The interviewer then asked, if this (prism *a*) is one, what is this one (prism *c*)? He incorrectly said three. Drew explained that he thought the answer was three "because that (prism *a*) would fill up two (prism *b*); that's (prism *c*) a unit bigger and that would fill up three." Rather than compare the prisms based on reference to a unit, he ranked them in order from smallest to largest by capacity.

In Grade 3, Drew's thinking about unit in area measurement was influenced by the tasks he had seen in his math book that involved counting squares that looked like square centimeters in gridded rectangles. When using the ruler to help him determine the area of the 4 x 3 in. rectangle, he attended to two dimensions and related linear centimeters to square centimeter units of area,

but did not make use of existing number labels (Figure 5). He did not need to draw the entire array in order to operate on composite units (a row of 10) by skip counting; this is the first time we saw Drew think about repeating composite units in a systematic way.

In terms of capacity (Figure 5), his ability to identify and operate on units was limited to situations involving simple doubling. He recognized a one to two ratio for pairs of prisms (a:b is two and b:c is two), but generalized this "pattern" to an ordinal ranking when comparing three prisms (a:b is two and b:c is two, so a:c is three). That is, he did not make ratio comparisons based on reference to a unit when comparing three prisms simultaneously. Drew showed more sophisticated sense of unit in area, attending to composite units, than he did in volume in Grade 3.

Grade 4

Grade 4 was a time for important shifts in his ways of thinking about units for both area and volume. During the Fall of Grade 4 Drew was presented with an image of a 1 x 3 unit rectangle and a rectilinear figure (Figure 6) and asked, "How many grey tiles would fit inside this one (the rectilinear figure)." Drew said, "the top is nine units, and the…all the way across…not like up and down…so you can fit three units on the top…" He discussed first covering the bottom, left and right sides, and then the middle. The interviewer asked him to draw some of the units (Figure 7). Drew then explained, "I kind of just did three across the whole thing and then one, two, three, four, five," pointing to each of the five drawn units in a column. He then traced with his fingers (Figure 8) to indicate the placement of five columns of five units, using the width of his fingers to guide the placement of these columns, saying "one, two, three, four, five…five times five is 25." Once he realized the column structure, he used multiplication to curtail his process of counting by composite units (columns) to structure the rectilinear region.

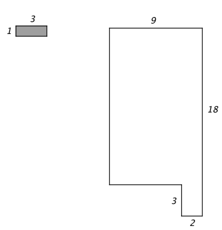


Figure 6: Rectangular Tile Task

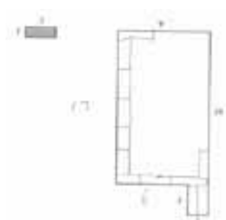


Figure 7: Drew's Drawing

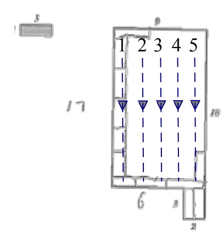
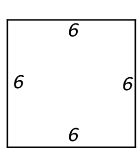


Figure 8: Drew's Finger Tracing

Two months later Drew was provided with an image of a 6 x 6 unit square and asked to determine the area (Figure 9). He traced around the square inside the boundary and explained, "I was trying to do it like painting around the outside first…so that would be like 24 centimeters already…" He went on tracing and explaining "half of 24 is 12…so 24 plus 12 is 36 I think…and then half of 12 would be 6…so 42…and then half of 6 would be 3…so that would be 45...and I think maybe it would cover it all up by then." Figure 10 illustrates Drew's border strategy, or "painting," as indicated by his finger tracing. He imagined square "ring" composites of area units with each successive becoming repeatedly smaller (by half). He was thinking about repeating composites that were the same shape, but not the same size in terms of the number of units.



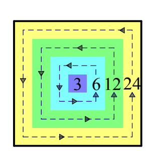


Figure 9: 6 by 6 Unit Square Task

Figure 10: Drew's Border Strategy for Area

One week later Drew again applied his border strategy when comparing a 5 x 8 unit rectangle with a 4 x 9 unit rectangle by area (with linear dimensions indicated by tick marks). He counted around the border of each rectangle to determine the number of square units needed to cover just inside the border and divided this number in half to determine the number of units needed to cover the middle. He explained that this was just one way of finding area and that he could also multiply the length and the width. For Drew, the border strategy and multiplying length by width were two unrelated methods for determining area. Based on his discussion of this task, he did not relate the multiplication of length by width to a structuring of the rectangles by composite units.

During the early Spring of Grade 4 Drew was presented with the images of a net and an open top box (Figure 11). The interviewer explained that the net could be made by cutting along the edges of the box and flattening it and asked, "How many cubes would it take to fill this box? Drew pointed and counted the cubes on the bottom of the net one-by-one in columns in an 'S' pattern (Figure 11). He explained that he needed three "stacks" of 24 or 24 times 3 to fill the box. That is, he operated on a composite unit, a layer, using multiplication to determine the number of cubes that he would need to fill the box.

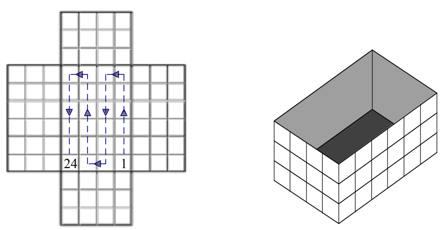


Figure 11: Net and Box

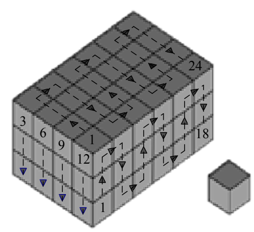


Figure 12: Border Strategy for Volume

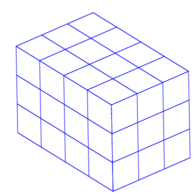


Figure 13: 4 x 3 x 3 Prism

During the same interview, he was presented with an image of a 4 x 6 x 3 unit rectangular prism printed next to a unit cube (Figure 12) and asked, "How many of these blocks (pointing to the unit cube) would it take to make that (pointing to the 4 x 6 x 3)?" Drew counted all of the visible cubes on the front "wall" one-by-one in columns to get 18 cubes. He explained that there were another 18 blocks on the opposite side that he could not see. He then skip counted by threes, saying "three, six, nine, twelve" to determine the number of cubes on the adjacent "wall," and explained that there were 12 more cubes in another "wall" that he could not see. He multiplied 18 by 2 to get 36 and 12 by 2 to get 24 and added 36 + 24 = 60. He then counted the visible squares on the top of the figure one-by-one in rows to get 24. He added the 24 to 60 to find 84. When the interviewer asked him how he got his answer, he explained that he had counted the whole thing, but then decided that he had double counted some of the blocks. He said he thought that he should not have counted the top, so he decided that the first part of his answer, which was 60 was correct; however, he said he could not remember how he got 60. Drew's ways of operating on units in the context of volume changed according to task representation. Without the perceptual support of the net and an open-top box to imagine filling up, Drew attended to individual units on the faces of the rectangular prism rather than composite units (layers).

Five months later, Drew was presented with an image of a gridded 4 x 3 x 3 unit rectangular prism (Figure 13) and the interviewer asked, "How many small cubes like the one shown are in this large block?" Drew said, "I'm going to try a different way," which he clarified to mean that

he was not going to count each unit individually. He counted 12 visible squares on the 4 x 3 face and 9 visible squares on the adjacent 3 x 3 face and multiplied 12 by 9, and when he was asked how he knew to multiply 12 by 9 he said, "I heard of that method kind of…it's usually with tiles, though." The interviewer replied that she had heard of that method for area and asked if he thought the same thing would work here. Drew said, "I think so." When asked if he thought 108 fit there, he said "um…no…it doesn't look like it" and said he thought it looked like it should be less. Drew's application of multiplication here was not connected to a structuring of the rectangular prism by iterating composites (layers).

During Grade 4 Drew exhibited variability in his ways of thinking about units in both area and volume measurement contexts. He operated on composite units using multiplication for some area and volume measurement tasks. This notion of multiplication is repeated addition. Although he operated on a column as a composite unit using multiplication in an area measurement task (Figure 8), he did not consistently relate linear and area units (Kara, Eames, Miller, Cullen, & Barrett, 2011). That is, he did not consistently apply the concept that the length of a side determines the number of area units that fit along the side. As his sense of a composite evolved throughout Grade 4, he generalized his operations on units in an alternative way. For example, he invented the border strategy (Figure 10) as an alternative to multiplying length by width when determining the area of a rectangle. His willingness to suggest that multiplying length by width would give the same answer as his border strategy suggests that he did not seem to see a conceptual link between his use of multiplication to model the mental iteration of a composite unit (as depicted in Figure 8) and the multiplication of length by width.

Drew extended his alterative way of operating on units in area measurement contexts into contexts involving volume during Grade 4. For example, his attention to surface area (Figure 12) seemed to be an extension of his border strategy (Figure 10) from area to volume. At the end of Grade 4, he adapted his multiplication of length by width strategy for area to make sense for him in the context of volume measurement. Instead of multiplying lengths of perpendicular adjacent sides to produce a measure of area, he multiplied surface areas of perpendicular adjacent faces to produce a measure of volume. The variability in Drew's strategy use in Grade 4 (operating on composite, counting around the border, and using multiplication) suggests that his unit concepts were still under development in both area and volume measurement contexts at this time.

Grade 5

Ten months later, during the Fall of Grade 5, Drew was given a 5 x 7 in. rectangle and asked to compare the area of the tile to the area of the larger rectangle. He noted, "5 of these would take up the side rows (pointing to the left and rightmost columns)…and…um…7 of these would take up the longer (pointing to a bottom row of 7)." He then said that the larger rectangle was 35 times bigger than the unit. When asked how he knew that multiplication would tell him how many units would fill the rectangle, he said, "I don't know…I just kinda learned it." The interviewer asked him to draw units, and he drew a column of 5 on the right-hand-side, using the unit tick marks on the ruler to guide his placement of units. Before completing his drawing, he expressed doubt that his answer of 35 obtained by multiplying was correct because he noticed that the corner tiles "took up two spaces." To check, he finished drawing around the border and then covered the interior by drawing the squares by columns. After he finished his drawing he began counting the units one-by-one in the same pattern he described in his tracing in Grade 4 (Figure 15). While counting, Drew suddenly noted that there were five tiles in a column and switched to counting by 5s saying, "5, 10, 15, 20, 25, 30, 35." He realized that counting by columns produced the same answer as multiplying five by seven.

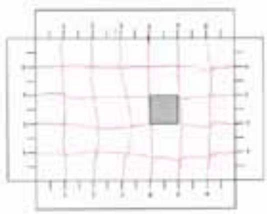


Figure 14: 5 x 7 in. Rectangle

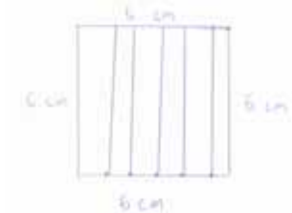


Figure 15: Drew's Rectangle

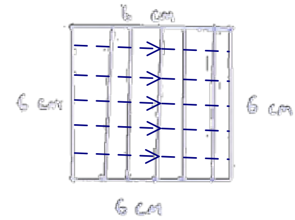


Figure 16: Drew's Gestures

During the Spring of Grade 5, Drew was asked to use a ruler to draw a rectangle with an area of 36 square centimeters. Drew explained that he would have to think backwards from the area formula, and used the ruler to draw the border of a 6 x 6 cm square. After creating his drawing, he said that he thought his square might have too many units. He checked by lining up the ruler with one side and using the numbered tick marks to guide the placement of parallel column line segments (Figure 15). The interviewer interrupted him and asked if he still thought that his square had too many units. Drew gestured with his hand where the parallel row segments would intersect the column segments to form a rectangular array structure of identical square units (Figure 16). He explained that the square has six columns of six square centimeters and six rows of six square centimeters. In the context of this task Drew connected the multiplication of length by width to the structuring of the square with composite units (rows and columns).

Two weeks later, Drew was presented with a wrapped rectangular prism and an inch cube (Figure 17) and he was asked to compare the volume of the inch cube to the volume of prism.

Figure 17: Wrapped Rectangular Prism and Inch Cube

He iterated the inch cube along the length, width, and height of the prism and found that the dimensions were 4 x 3 x 3 in. He said, "I think to find the volume what you're gonna want to do is the width times the height times the length." He noted that 3 times 4 is 12 and 12 times 3 is 36. When asked to draw where 12 would go, he drew a row of 4 on the 4 x 3 in. side and explained that the side would contain three rows of four. For the adjacent 3 x 3 in. side, he drew one unit in the corner and said, "three times three so it would take 9 blocks to fill this side." For the total, he said he would need to do 12 times 9, which the interviewer told him was 108. He then said, "it wouldn't take up 108…I think it will take up 36." To explain his answer, he said "I did 4 times 3 times 3…36…so that's the width times the height times the length." The interviewer drew his attention back to the 4 in x 3 in side by asking how many 12s there were. Drew pointed to the 4 x 3 in. side and explained, "[it would take] 12 blocks to fill that…and then 3 to fill that side…it was actually 9…but the width would be 3 so it would be 3 times 4 times 3...it just goes back to the length times width times height." Although Drew trusted his answer from multiplying length by width by height, he did not clearly connect his application of multiplication by the structuring of the rectangular prism by composite units (layers).

Drew abandoned his border strategy early in Grade 5 because realized that multiplying length by width was the same as operating on a composite unit by skip counting for area (Figure 14). He consistently related linear and area units and used this relationship to develop a drawing strategy for checking his thinking when multiplying length by width. He used the numbered tick marks on a ruler to guide his drawing of parallel row and column line segments to produce a grid so that he could check to see if multiplying length by width produced the correct answer (Figures 15 and 16). He also realized that his adaptation of multiplying length by width for volume (as the multiplication of areas of perpendicular adjacent faces) did not produce the same answer as the multiplication of length by width by height. During Grade 5, Drew showed evidence that he learned that multiplication *does* produce a measure of area or volume; however, he may not yet have understood *why* multiplication produces a measure of area or volume.

Conclusions and Implications

Drew gained sophistication for measuring space more slowly than the other six students in this cohort, perhaps because he sought to systematize his ways of operating on composite units in alternative ways. However, the results of this longitudinal work, which are consistent with prior cross-sectional research (Battista & Clements, 1996; Battista, Clements, Arnoff, Battista, & Borrow, 1998; Outhred & Mitchelmore, 2000), indicate that Drew's understanding of the conceptual link between the iteration of a composite unit and the application of multiplication in both area and volume measurement did increase from Grade 3 to Grade 5. We believe it is important to note that Drew exhibited growth in terms of his use of unit in an area measurement context; and this often precipitated growth in volume measurement. In the current Common Core State Standards for Mathematics, measuring areas of rectangles is introduced and mastered in Grade 3 and measuring volume of rectangular prisms is introduced and mastered in Grade 5. This study adds to the body of research that suggests introducing and mastering these concepts in a single grade level may not be developmentally appropriate for children. Therefore, we recommend providing early experiences for children to think about building and operating on composite units before and during instruction of formulas in a broad time span. Furthermore, Drew's thinking in terms of unit in both area and volume varied in different task representations and use of tools. We suggest that during the instruction of area and volume measurement, tasks should vary in terms of representation.

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HOW ARE GEOMETRIC PROOF PROBLEMS PRESENTED? CONCEPTUALIZING AND MEASURING TEACHERS' RECOGNITION OF THE DIAGRAMMATIC REGISTER

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We describe the development of measures of teachers' recognition of an instructional norm—that proof problems in high school geometry are presented in a diagrammatic register. A first instrument required participants to openly respond to depictions of classroom scenarios in which the norm was breached. A second instrument was a survey that required participants to rate the extent to which they agreed with various explicit statements about instruction. A third instrument capitalized on pros of the other two. We demonstrate how this instrument development process improved our conceptualization of the components included in the diagrammatic register norm.

Keywords: Geometry and Geometrical and Spatial Thinking, Instructional activities and practices, Measurement, Reasoning and Proof

We describe the process of conceptualizing a norm of instruction and measuring its recognition by teachers. The paper illustrates how the process of improving the instruments led to an improved conceptualization of the construct being measured. Weiss and Herbst (2007) had observed that while theorems are often stated in geometry textbooks using a *conceptual* register (referring to mathematical objects by their names, e.g., base angles, diagonals), proof problems often use a *diagrammatic* register, whereby the 'givens' and the 'prove' are stated in terms of specific objects in a diagram (i.e., using their labels). Moreover, diagrams may add information not stated as "givens" that is nonetheless essential to prove the conclusion (these include properties of betweenness, collinearity, concurrency, intersection, orientation, and separation). Our investigation aims at (1) refining the meaning of the statement that proof problems are presented in the diagrammatic register and (2) determining to what extent geometry teachers recognize such presentation as normative.

We build on scholarship that studies social practices and their participants' tacit knowledge (e.g., Bourdieu, 1990; Collins, 2010; Garfinkel & Sacks, 1970). This scholarship offers the notion that practices include regularities that are often tacit even though they are experienced as normative: We call those *norms* and apply that notion to the study of mathematics teaching. Earlier work on the notion of classroom norms (and germane notions such as *cultural script*) has drawn empirical support on detailed analysis of cases of teaching (e.g., Bauersfeld, 1988; Stigler & Hiebert, 1999; Yackel & Cobb, 1996) or on the analysis of practitioners' reactions to cases of instruction (e.g., Jacobs & Morita, 2003; Nachlieli & Herbst, 2009). Scholarship in social psychology (Aarts & Dijkterhuis, 2003; Nolan, et al., 2008) shows examples of how recognition of situational norms can be studied empirically and some of that methodology has percolated to the study of general norms in teaching (Hora & Anderson, 2012). This paper contributes to designing instruments to confirm quantitatively the recognition of norms of instructional situations in mathematics: By instructional situations we mean particular segments of classroom mathematical work that have a taken-as-shared exchange value among the instructional goals at stake in a given course of studies (Herbst, 2006). In this case we focus on the situation of "doing proofs" in which proofs of particular propositions exchange for students' skill at doing proofs.

We focus on the design of instruments to measure recognition of the norm that proof problems are stated in the diagrammatic register. Our initial understanding of what could be meant by *diagrammatic register* was holistic—it referred to relationships between the problem statement and the diagram included. We expected that practitioners would hold that norm tacitly: They would recognize when a proof problem had breached this norm and be able to produce proof problems that complied with the norm, but they would not necessarily be able to say what the norm consists of.

Methods

Initially, we created two kinds of instruments. One, a "tacit norm recognition instrument" (a.k.a., N1), kept the phenomenon at a holistic level, requiring participants to evaluate the work of teacher in classroom scenarios where the teacher posed a proof problem that breached the norm. Scenarios were rendered as image sequences with cartoon characters enacting teacher and students and speech bubbles for their talk. The open ended questions asked participants to describe what they saw happening and to evaluate the teacher's work facilitating the doing of a proof. Four different scenarios were presented to participants (item sets 21002, 21003, 21004, 21006), each of which enacted a breach of the norm in different ways. For example in one scenario the teacher described in writing the figure to which the proof problem referred but did not draw a diagram for it. Participants were told that the scenarios were about doing proofs in geometry but were not told that they included breaches of norms or that there was anything special about the handling of diagrams.

The second instrument, an "explicit norms recognition instrument" (a.k.a., N2), required participants to rate statements that described, in general, possible behaviors of a teacher posing proof problems. Participants had to rate how appropriate and how typical were actions described in statements, such as: "the teacher provides a diagram for students to use while doing a proof". The creation of this instrument gave us a first opportunity of laying out in a more analytic fashion what the "diagrammatic register" norm consisted of. In doing so we proposed five distinct subnorms making up the diagrammatic register. Concisely stated, the initial statements of these subnorms were as follows (construct IDs in brackets): (1) the teacher provides a diagram that has the givens marked when assigning a proof [DP21]; (2) the teacher provides a diagram for students to use while doing the proof [DP24]; (3) the points of the diagram which are needed for the proof (though not necessarily all of the points) are provided and labeled by the teacher [DP36]; (4) the statement of the proof problem uses symbols and labels for the elements of the diagram (e.g., $AB \perp CD$) [DP39]; (5) the diagram the teacher provides accurately represents the concepts at stake in the proof [DP41].

To examine responses to N1 we created a coding scheme that assessed open responses for evidence that participants recognized breaches of the diagrammatic register norm and for evidence of whether participants appraised the teaching negatively. Reliability was established by having two independent coders code through responses and resolve disagreements on a case-by-case basis. To examine internal reliability of constructs in the N2 instrument, we implemented classical item analysis (CIA; see Crocker & Algina, 2006).

Data Sources and Analysis

The N1 instrument was piloted with 50 teachers including 33 that had three or more years of experience teaching geometry (EGT) and 17 teachers without such experience (thus including novices as well as teachers experienced teaching other courses). The N2 instrument was piloted with 44 teachers including 28 EGT and 16 teachers without such experience. Table 1 shows how

many participants recognized the breaching of the diagrammatic register norm in each of the N1 stories. Across items, experienced geometry teachers had higher totals for recognition of the norm. Binomial tests, assuming a null hypothesis that recognition and nonrecognition are equally likely, showed statistically significant results for experienced geometry teachers in story 21002 (p<.05), and 21003 (p<.05), while for these same items the other participants showed no significant difference from chance in their rate of recognition of the norm.

Table 1: Number of Teachers Who Recognized Breach of the Norm

	21002	21003	21004	21006
Exp. geometry teachers (>3 years)	18	18	11	5
Other teachers	7	10	4	3
Total	25	28	15	8

Responses were also coded for the presence of negative appraisals (Martin & White, 2007) of the depicted teaching. We defined the variable "discomfort with the scenario" to capture participants' critiques not sufficiently focused to count as recognitions of the norm. For example, "discomfort" captured entries where participants critiqued that the teacher "did not write anything down." "Discomfort" and "norm recognition" are mutually exclusive—only participants that did not recognize the norm were coded for having discomfort with the scenario. Table 2 shows the aggregate of those participants that either recognized the norm or registered discomfort with the teaching.

Table 2: Aggregate Proportion of Participants that Recognized the Norm and Participants that Registered Some Discomfort with Each Scenario

	21002	21003	21004	21006
Exp. geometry teachers (>3 yrs.)	30	29	17	24
Other teachers	10	14	6	14
Total	40	43	23	38

The data suggests that participants may be reacting negatively to the scenario in ways that are not localized around the breach of the norm and the specific features of the diagrammatic register. Indeed, even among participants that did recognize the norm, that recognition was not always focused on features commensurate with the subnorms identified above. Those results were not surprising on account of our hypothesis that the diagrammatic register norm is tacit: Unlike other norms in "doing proofs"—such as the norm that every statement must be justified by a reason (Nachlieli & Herbst, 2009)—the norm that problems are presented in the diagrammatic register is not explicitly addressed in the work of teaching. Earlier work in ethnomethodology suggested that when such implicit norms are breached, participants engage in repair strategies—ways of signaling that something is amiss—though these strategies do not always include pointing at what was amiss (e.g., reframing the activity as a case of a different situation).

This lack of specific reference to the features of the register within the open responses was the principal motivation that led us to conceive of a new instrument (N3, described below) to

study the diagrammatic register. It seemed that, since the subnorms of the diagrammatic register are tacitly held, testing for these subnorms via open response prompts placed too onerous a reporting burden on the participant. If participants did not comment on specifics of how problems were stated this could mean that participants did not know how to focus their reaction—an explanation that coheres with the hypothesis that the norm is tacit. The other reason that called for developing N3 came from the analysis of pilot data on N2.

Participants made two ratings of each of the N2 statements: An appropriateness rating—this was a 6-valued rating scale, ranging from "Very Inappropriate" to "Very Appropriate"—and a typicality rating—this was a 4-valued rating scale that ranged from "It Never or Hardly Ever Happens" to "It Always or Almost Always Happens". The two kinds of questions aimed at complementary aspects of normativity (appropriateness and frequency) so we examined their internal reliability separately.

Table 3: Reliability for the N2 Constructs.

Construct	Reliability for "Appropriateness" Scale	Reliability for "Typicality" Scale
DP21 [Givens marked]	$\alpha = .78$	$\alpha = .61$
DP24 [Diagram provided]	$\alpha = -.02$	$\alpha = -.06$
DP36 [Labeled points]	$\alpha = .50$	$\alpha = .70$
DP39 [Statement uses labels]	$\alpha = -.37$	$\alpha = .15$
DP41 [Diagram Accuracy]	$r = .40$	$r = .33$

Table 3 shows results of the item analysis for the responses to the N2 item, indicating reliability for some constructs (DP21, DP36, DP41) but not for others (DP24, DP39). DP41 had only two items, so while alpha was not calculated, responses to both items showed moderate correlations. For DP39, items generally performed poorly unless recoded to assume the norm was that all points had to be labeled, and not just the ones relevant for a proof (in which the alphas improved to over .70 for each scale).

Table 4: Correlations with Experience Teaching Geometry for Appropriateness and Typicality

	DP21		DP24		DP36		DP39		DP41	
	App	Typ	App	Typ	App	Typ	App	Typ	App	Typ
Taught >3 years Geometry	-.04	.05	.03	.08	-.36*	-.26[a]	.08	.20	-.09	-.05
Taught ≥ 1 year Geometry	-.27[a]	.12	.02	-.12	-.36*	.19	.07	.23	-.19	.12

[a] *$p < .10$, *$p < .05$, **$p < .01$*

Note: 'App' and 'Typ' represent correlations for the appropriateness and typicality scales, respectively.

Striving to better understand why the constructs performed the way they did, we looked at how participants answered items, by correlating their status of being an experienced geometry teacher (or not), as well as if they had ever taught geometry (or not) with the construct scores.

These results, shown in Table 4, showed few correlations between the constructs and geometry teaching experience. The only statistically significant correlations were negative, suggesting that participants rejected the explicitly stated norms.

Discussion: Revisions Needed to the Instruments

Several issues were raised by the analysis of the data from the first pilot. First, the results of the first pilot indicated that the N1 instrument was insufficiently sharp: It did not elicit commentary from participants that pointed specifically to the features of the diagrammatic register norm. Another issue with the N1 instrument was that some of the discomfort reported by participants could be attributed to ancillary features of the scenarios—e.g., that the teacher had not written the statement of the problem on the board. In revising scenarios for a second pilot we avoided those confounding elements, but our review of the results from the first pilot also raised issues with the N2 instrument. Specifically, the item analysis from N2 led us to question whether recognition of the diagrammatic register subnorms could be assessed with instruments that asked participants to rate general statements. We also realized that in looking for explicit general statements of the subnorms of the diagrammatic register we had failed to include any indications of what mathematical properties are often communicated through the diagram. In the case of developing scenarios to breach the diagrammatic register norm for the N1 instrument, we had assumed that we should choose problems that involved collinearity, separation, concurrency, or betweenness, and that we should present those problems without using diagrams. But we had failed to include any N2 item that explicitly tested for recognition that such properties are normatively telegraphed by the diagram. As a result, we had created some tacit items (for the N1 instrument) that seemed to lose part of their meaning when made into explicit general statements (for the N2 instrument). The logistical and technical difficulties we had run into when examining the pilot data had led us to realize that there were gaps in our conceptualization of what the subnorms really should be. We had the chance to fill those gaps when we conceptualized a new instrument, one that we refer to as N3.

Table 5: Revised Subnorms of Diagrammatics Register After Initial Piloting

Designation	Subnorm statement
Subnorm 1 (New)	The statement of the problem does not make explicit properties of betweenness, intersection, separation, collinearity, or concurrency, which are left for the diagram to communicate.
Subnorm 2 DP21	The teacher provides a diagram for students to use while doing the proof.
Subnorm 3 DP39	The teacher assigns a proof problem with an accompanying diagram where the points needed in the proof are labeled (but not necessarily all points).
Subnorm 4 DP24	The proof problem is stated using symbols and labels for elements of a diagram (e.g., $AB \perp CD$).
Subnorm 5 DP41	When a teacher provides a diagram accompanying a proof problem, the diagram is accurate.

We drew an analogy to an optometrist trying different lenses on a patient to design a novel format for an item that could blend the contextualization virtues of the N1 instrument (where participants handle a specific proof problem) and the analytic virtues of the N2 instrument

(where it is possible to expose participants to many items and thus be able to do an item analysis). The resulting N3 instrument asks participants "which of the two proof problems below is more appropriate for geometry teachers to present to their students" and offers two problems that differ from each other only in regard to their compliance with or breach of one subnorm of the diagrammatic register. The participant answers using a 6-point scale, with options that include "Option A is much more appropriate" (viz., somewhat more appropriate, slightly more appropriate), and same options for Option B. To concentrate on the norms that had been more problematic when developing N2, we created N3 items that tested the subnorms listed in Table 5.

Table 6. Reliability Measures for Revised Instruments for N2 & N3

Assessed Norms	Reliability (α) for N2 in 2nd Pilot	Reliability (α) for N3 in 2nd Pilot
Subnorm 1: The statement of the problem does not make explicit properties of order, separation, collinearity, or concurrency, which are left to the diagram to communicate.	-.10	.64
Subnorm 2: The teacher provides a diagram for students to use while doing the proof.	.69	.80
Subnorm 3: The teacher assigns a proof problem with an accompanying diagram where the points needed in the proof are labeled (but not necessarily all points).	-.28[1] .61[2]	-.12[1] .69[2]
Subnorm 4: The proof problem is stated using symbols and labels for elements of a diagram (e.g., $AB \perp CD$).	.63	.77
Subnorm 5: When a teacher provides a diagram accompanying a proof problem, this diagram is accurate.	.55	.82

[1]*Assumes the norm of "only points relevant to completing the proof are labeled."*
[2]*Assumes the norm of "all points on the diagram are labeled."*

This N3 instrument, along with a revised version of the N2 instrument (whose revisions were a consequence of the further articulation of the subnorms described above), was piloted during three data collection sessions in May and June of 2012. Forty-nine participants completed the revised N2 instrument, while 52 participants completed the N3 instrument. This second round of piloting showed higher validation scores than in the first pilot, as reported in Table 6. Still, while the data showed improvement in the reliabilities for four of the subnorms in the revised N2 compared to its first version, N2 was still not effective assessing recognition of the first subnorm. Overall, while not completely successful, N3 is providing more reliable estimations of participants' recognition of the subnorms that constitute the diagrammatic register. (Incidentally,

note that in the second pilot we did not test for recognition of DP 21, the norm that givens are marked, because that one had achieved acceptable alpha levels in the first pilot.)

The N1 instrument was also revised to sharpen the scenarios. One scenario (21004) was withdrawn and two new ones were included (21005 and 21007). Additionally, the questions in each item set were revised. For each scenario, participants were first asked to describe what they saw happening. Then they were asked to rate, on a 6-point scale, how appropriate the teacher's facilitation of the work on a proof was and they were given a box to explain their rating. Finally they were asked to rate, on a 6-point scale, how appropriate the description of the proof problem assigned to the class was and they were given a box to explain their rating. As in the case of the first pilot, we coded the open response questions for evidence of recognition of a breach of the norm and for evidence of discomfort with the scenario. We defined the aggregates "Norm recognition" (NR) and "Repair" (RP) as follows: For each item set *j*, participants were assigned a 1 for NR(*j*) if at least one of the open responses contained evidence that the participant had noticed a breach in a specific subnorm of the diagrammatic register; participants were assigned a 1 for RP(*j*) if they had a 1 for NR or if in at least one of the open responses they indicated discomfort with the teaching represented in the scenario. Reliability for this coding was very good, as attested by kappa values reported in Table 7. The table also contains basic counts of these variables for the whole sample, along with their significance, assessed using binomial tests against a null hypothesis that recognition or non-recognition of a breach in or a repair to the scenario occurred with equal probability.

Table 7: NR(j) and RP(j) Totals per Session for Second N1 Pilot Study, with Kappa (κ) Statistics for Reliability, and with Binomial Probabilities

Session		NR(*j*)			RP(*j*)		
	n	Total	κ	*p*	Total	κ	*p*
21002	42	32	0.77**	0.0003	37	0.82**	<0.001
21003	39	23	0.78*	0.0686	29	0.77**	<0.001
21005	39	22	0.87*	0.0928	34	0.94**	<0.001
21006	40	17	0.86*	0.0689	30	0.95**	0.001
21007	42	18	0.95*	0.0804	28	0.86**	0.012

**Indicates results that are significant at the .10 level*
***Indicate results that are significant at the .05 level*

These results of the binomial tests reported in Table 7 show that overall participants recognized the breach of the tacit norm in these items at a probability that differed significantly from what would be expected from chance alone. In this respect, item 21002 is set apart from the other breaching items. One reason that could explain this difference is that the classroom story prompt for item 21002 depicted the teacher breaching the tacit norm without an apparent reason for having done so. Whereas in the other items, while the depicted teacher still breaches the norm, in each of these instances there were—by design—mitigating factors suggested by the scenario that could have accounted for the teacher's departure from the norm, such as a teacher's wish to emphasize the conceptual (rather than diagrammatic) statements of particular geometry concepts. The fact that participants offered repairs to the scenario for each of the breach items suggests that even though participants were not as likely to recognize a specific breach in items

21003-21007, they still noticed that something was amiss in some way. Thus, it seems that this set of items is effective detecting teachers' recognition of the diagrammatic register norm.

Conclusion

The development of instruments to measure teachers' recognition of an instructional norm moves our work forward in the study of a phenomenon that is specific to mathematics teaching—i.e., specific to the work of teaching, and specific to the mathematics at play. We contend that this work is also valuable insofar as it illustrates a process of discovery in our field: A dialectics of conceptualization and measurement in the research process that challenges the received wisdom whereby small scale, qualitative, exploratory studies provide sufficient material for theoretical conceptualization while larger scale, quantitative studies are needed only eventually and just to verify or falsify theoretical assertions. Consistent with contemporary views on measurement (e.g., Wilson, 2005), the study shows how the goal of measurement and the analysis of a relatively large set of responses to initial instruments led to further conceptualization of the theoretical construct we wanted to confirm, revise the instruments to actually measure this construct, and obtain results that allow us to report on the extent to which the hypothesized construct was present.

Acknowledgments

Work reported here was done with the support of National Science Foundation grant DRL-0918425 to P. Herbst and D. Chazan. All opinions are those of the authors and do not necessarily represent the views of the Foundation.

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SUPPORTING TEACHERS' AND STUDENTS' KNOWLEDGE OF GEOMETRIC SIMILARITY

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The main goal of the Learning and Teaching Geometry (LTG) project is to build professional development (PD) materials that provide opportunities for teachers to engage in learning about geometric similarity through the use of video cases, in which specific and increasingly complex mathematical ideas are presented within the dynamics of classroom practice. The central component of the LTG materials is the Foundation Module, which includes 30 hours of PD and is intended to provide teachers with a thorough grounding in key mathematical and pedagogical issues related to similarity. Field test data indicate that the Foundation Module supports gains in both teachers' and students' knowledge of similarity.

Keywords: Geometry, Mathematical Knowledge for Teaching, Teacher Education-Inservice/Professional Development, Teacher Knowledge

Introduction

Research has provided emerging evidence that certain characteristics of PD are important for influencing changes in classroom practice and supporting student achievement (Heck, Banilower, Weiss & Rosenberg, 2008; Sample McMeeking, Orsi & Cobb, 2012). For example, programs with a focus on subject matter knowledge and student learning in that subject are more likely to have an impact on student achievement than those focused on more generic topics (Cohen & Hill, 2000). In addition, numerous studies examining the effectiveness of mathematics and science PD have found that teachers who participated in activities that emphasized content knowledge, active learning, and coherence were more likely to report enhanced knowledge along with changes in their teaching practice compared to teachers who did not participate in such programs (Garet, Porter, Desimone, Birmin & Yoon, 2001).

There is strong evidence that best practices for mathematics PD include providing teachers with learning opportunities that are intensive in focus and extensive in duration (Garet et al., 2001) and that are "practice-based"—that is, they enable teachers to examine the mathematical skills and understanding that undergird their classroom curriculum, investigate students' mathematical thinking, and explore instructional practices that support student learning (Ball & Cohen, 1999; Cohen & Hill, 2000). By focusing on developing the understanding, skills, and dispositions that teachers use in daily practice, "practice-based" PD provides a meaningful context for teachers' learning.

Video can be an effective PD tool, as it brings the everyday work of teaching into a professional learning environment. Just as curriculum does not stand alone in classrooms, video does not stand alone in PD. Organizing PD around the study of videos does not, in itself, guarantee significant teacher learning any more than the use of technology in the classroom ensures that students will develop deep understandings. Like technology, videos are only tools; professional developers must help teacher educators use these tools to achieve specific learning goals. Without guidance, such as an analytic framework and explicit tasks, teachers watching videos rarely address subject matter content (Castro, Clark, Jacobs, & Givvin, 2005). Knowing how to observe teaching, what to look for and what to focus on, and how to talk about what one

sees are important skills to be learned (Castro, 2005). Learning these skills helps teachers to more skillfully "see" the subject matter in lessons, discriminate ways that learners comprehend subject matter; identify problematic features; assess student responses; detect, diagnose, and develop instructional responses to student errors; and generally gain more insights into their own practice (Givvin, 2005).

Theoretical Framework for the Learning and Teaching Geometry Project

The main goal of the Learning and Teaching Geometry (LTG) project is to build professional development materials that provide opportunities for teachers to expand their mathematical knowledge for teaching. The LTG PD materials help teachers to meet the challenges of teaching geometric similarity in accordance with the Common Core State Standards, which require defining and applying similarity based on geometric transformations—an approach that is likely to be new to most teachers and students.

The theoretical frame for the project is adapted from the work of Deborah Ball and colleagues (Cohen, Raudenbush & Ball, 2003) that incorporates research on both teaching and learning. As depicted in Figure 1, the content of the video case materials focus on the interactions between the teacher, the content (in this case, similarity tasks), and the students, within the context of an authentic classroom environment. The materials are designed to be used by a teacher educator who is faced with a similar set of relationships: the interactions between the teacher educator, the content (in this case, teaching and learning of geometric similarity), and the teachers with whom he or she works. The LTG Study is based on the premise that both classroom learning and professional learning opportunities are a function of these reciprocal relationships.

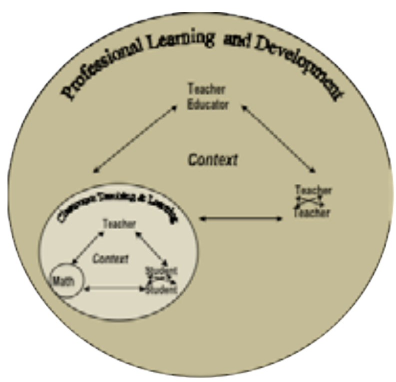

Figure 1: Theoretical Framework (Adapted from Cohen, Raudenbush & Ball, 2003)

The Learning and Teaching Geometry PD Materials

The LTG PD materials are intended for use in the professional development of mathematics teachers serving grades 5-10. Some of the videocases portray student thinking about particular concepts and some videocases portray pedagogical strategies and their impact on students' opportunities to learn (Seago, Driscoll & Jacobs, 2010). The materials follow a learning trajectory that is designed to enrich teachers' MKT as well as their ability to support students' understanding of similarity in alignment with the Common Core State Standards for mathematics. Specifically, they encourage teachers to gain a robust conception of similar figures as part of an infinite family that can be formed by applying one or more geometric transformations. Figure 2 illustrates the mathematical storyline in the Foundation Module, noting the central mathematical focus within and across sessions. The Foundation Module consists of

10, 3-hour sessions that are intended to provide teachers with a thorough grounding in key mathematical and pedagogical issues related to similarity.

Session 1	Session 2	Session 3	Session 4	Session 5	Session 6	Session 7	Session 8	Session 9	Session 10
A dynamic, transformational view of congruence	A dynamic, transformational view of similarity	Relationship between dilation and similarity	Properties of dilation	Preservation of angles & proportional lengths through dilation	Ratios within & between similar figures	Ratios within & between similar figures, Part 2	Connect-ions between similarity and slope & linearity	Area of similar figures	Closure and re-capping of big ideas
Defining Congruence and Similarity			Relationships and Attributes of Similar Figures				Connections		Closure

Figure 2: Mathematical Storyline of the LTG Foundation Module

Each session has at its core one or more digital video clips of a mathematics classroom. These clips are unedited segments selected from classroom footage of un-staged mathematics lessons, representing a range of grade levels, geographic locations and student populations across the United States. The clips offer a window into a variety of issues related to content, student thinking, and pedagogical moves. These clips typically represent a conceptual hurdle or portray some degree of mathematical confusion, based on the expectation that they are likely to provoke inquiry and discussion within the PD setting

Prior to watching a given video clip, teachers' grapple with the same mathematical task(s) the videotaped students tackled. Significant time is devoted to forecasting alternative solutions, comparing solutions, and anticipating student misconceptions. Typical conversations include consideration of: (1) the mathematical skills, procedures, and concepts entailed in the task; (2) the mathematical reasoning and solution strategies (correct and incorrect) that students are likely to apply to the task; (3) the affordances and constraints of different mathematical representations (e.g., pictures, tables, graphs, equations); and (4) instructional moves, such as questions and scaffolds, that might be helpful to support students' learning. Carefully designed focus questions help guide the viewing and analysis of the video clips, and provides teachers' support in unearthing student thinking. The questions and probes that accompany the LTG materials support a deep and substantive examination of classroom interactions.

Field Test Study of the LTG PD materials

Field Test Sites & Implementation

A field test of the LTG Foundation Module was conducted in 8 sites throughout the U.S. by a group of nine experienced facilitators personally recruited by the LTG staff. Most of the facilitators were state- or district-level middle and/or secondary mathematics coordinators, and two were university mathematics or education faculty. The field test was conducted over two academic years (2010-11, 2011-12) and used a treatment and comparison design in order to generate both formative and summative evaluation data related to impacts on teacher and student knowledge. Each facilitator recruited both treatment (87) and comparison (40) teachers. Typically, facilitators asked the treatment teachers to nominate comparison teachers from the same school and grade level.

Measures

To measure the impact of the LTG PD on teachers' mathematical knowledge for teaching in the domain of similarity, Horizon Research, Inc. (HRI) developed three instruments: a multiple-choice assessment, and two assessments embedded within the PD. HRI also developed a student assessment to measure the impact of teachers' participation in the LTG PD on their students' knowledge of similarity.

Teacher multiple-choice assessment. The multiple-choice teacher assessment includes 25 items targeting knowledge in five focus areas: (1) dilation; (2) properties of similarity; (3) ratios and proportions; (4) scaling; and (5) congruence transformations. The items were compiled and modified from released items used in state, national (NAEP), and international (TIMSS) assessment sources. About half of the items are purely content-based; the rest are set in teaching contexts that situate them in the work of teaching geometry. The items were reviewed by the LTG developers to validate them as accurate and appropriate to the content emphasis of the LTG PD materials. Pilot testing conducted as part of the project helped inform revisions to the instrument and demonstrated its sensitivity to teacher engagement with PD based on the LTG materials. The internal consistency of the full multiple-choice assessment was calculated using Cronbach's alpha as 0.81 for the pretest and 0.82 for the posttest.

Embedded assessments. The embedded assessments consist of two tasks that exist within the LTG Foundation Module: a mathematics task and a video analysis task. Both tasks address aspects of MKT, including subject matter content knowledge and pedagogical content knowledge (such as knowledge related to anticipating and interpreting student thinking about similarity). Each task was administered twice during the course of the intervention (first in an early session of the Foundation Module and again in a session near the end of the Foundation Module).

Student multiple-choice assessment. The student assessment contains 20 multiple-choice items and, like the LTG teacher assessment, targets content knowledge in five areas: (1) dilation; (2) properties of similarity; (3) ratios and proportions; (4) scaling; and (5) congruence transformations. Items were drawn from the same sources as the teacher assessment. In addition to review by LTG staff, HRI tested possible items through cognitive interviews with students in grades 7-9. Information gathered from these interviews helped to ensure the items were clear, plausible, and had content validity. Internal consistency reliability of the full student assessment was calculated using Cronbach's alpha as 0.71 for both the pretest and posttest.

Administration and Scoring of Measures

The three teacher knowledge assessments were each administered at two time points to 83 treatment teachers during implementation of the PD. Table 2 identifies the sessions in which the assessments were administered to treatment teachers. The assessments were also administered at two time points to 38 comparison teachers in separate meetings. Facilitators were asked to schedule the administration of assessments to the comparison teacher within the same timeframe as treatment teachers' participation in the PD in order to make data collection from these two groups consistent.

Results

Gains in teachers' geometry content knowledge. Analysis of the teachers' multiple choice assessment scores was conducted using repeated measures ANOVA to test for changes in percent correct scores over time. This approach controls for overall gains in the treatment and comparison teacher groups that may be attributed to learning from a year of teaching practice or from two administrations of the measure.

In the analysis, the main effect of group was non-significant, suggesting that the treatment and comparison teachers were initially comparable. A significant main effect of time (F = 21.09, $p < .05$) demonstrated that, on average, teachers improved from pre- to post-test. A significant time by group interaction term (F = 9.65, $p < .05$) indicated that the change in scores from pre- to post-test varied depending on participation in the treatment or comparison group. The treatment group demonstrated an average gain of 8.73 percentage points from pre- to post-test, whereas the comparison group demonstrated an average gain of 1.68 percentage points (see Table 1). In other words, teachers who experienced the LTG Foundation Module gained, on average, 7.05 more percentage points on their content assessment from pre- to post-test than teachers who did not experience the program. In pooled standard deviation units, the size of this effect was 0.39.

Table 1. Teachers' Geometry Assessment Scores

	Treatment Group (n = 83)		Comparison Group (n = 38)	
	Pre-test	**Post-test**	**Pre-test**	**Post-test**
Mean	**63.66**	**72.39 ***	**65.79**	**67.47**
(sd)	**(17.96)**	**(17.88)**	**(19.48)**	**(17.05)**

* A significant time by group interaction term indicated that treatment teachers demonstrated larger improvements in percent correct scores than comparison teachers (repeated-measures ANOVA; $p < .05$).

Gains in teachers' ability to apply their knowledge to classroom contexts. The mathematics task embedded assessment included three questions about similar rectangles. Questions 1 and 2 asked teachers to provide two different—but correct—methods to solve the same problem. Question 3 asked teachers to provide a method students might use that would lead to an incorrect solution. Scores for each question were analyzed using a Wilcoxon Signed Ranks Test for examining changes from pre- to post-test for each group. The False Discovery Rate adjustment for type I error was used to compensate for multiple comparisons.

Treatment teachers improved in their ability to provide a method, as shown in Table 2. For the treatment group, on average, there was a significant improvement in the scores on Questions 1 and 2 from pre- to post-test (p values < .05). The effect sizes for treatment group gains in scores on these questions were 0.30 and 0.32, respectively. There was not a significant change on Question 3 scores from pre- to post-assessment for treatment teachers. The comparison group, on average, did not demonstrate significant changes in score for any of the three questions.

Table 2. Teachers' Mathematics Task Embedded Assessment Scores

	Treatment Group (n = 82)				Comparison Group (n = 39)			
Question	Pre-test Mean (sd)	Post-test Mean (sd)	Ranks	Effect size	Pre-test Mean (sd)	Post-test Mean (sd)	Ranks	Effect size
1	2.56 (1.42)	3.17 (0.73)	33 + * 36 = 13 –	0.30	2.82 (1.45)	3.18 (1.07)	8 + 28 = 3 –	--
2	1.88 (1.53)	2.73 (1.38)	41 + * 29 = 12 –	0.32	1.95 (1.64)	2.23 (1.40)	16 + 14 = 9 –	--
3	2.73 (1.27)	2.78 (1.12)	27 + 31 = 24 –	--	2.85 (1.31)	2.77 (0.96)	14 + 15 = 10 –	--

[a] Ranks are reported with + indicating the number of teachers showing improvement from pre- to post-test, = indicating teachers with the same score on both administrations, and – indicating teachers with a decline.

- Significant improvement from pre- to post-test scores (Wilcoxon Signed Ranks Tests; p < .05)

The video analysis embedded assessment included three questions, prompting teachers to first interpret, and then apply, a student's approach to solving a similarity problem using dilation. For the comparison group, the Wilcoxon Signed Ranks Test did not reveal significant changes for any of the questions. Table 3 indicates that for the treatment group, on average, there was a significant improvement in the overall scores for all three questions, suggesting improved understanding of mathematical similarity in relation to dilation ($p < .05$ for all questions). The effect sizes for these questions were 0.32, 0.25, and 0.21 respectively.

Table 3. Teachers' Video Analysis Embedded Assessment Scores

	Treatment Group (n = 66) [a]				Comparison Group (n = 39)			
Question	Pre-test Mean (sd)	Post-test Mean (sd)	Ranks	Effect size	Pre-test Mean (sd)	Post-test Mean (sd)	Ranks	Effect size
1	3.89 (2.03)	5.02 (2.21)	39 + * 12 = 15 –	0.32	3.51 (2.61)	4.31 (2.14)	20 + 9 = 10 –	--
2	4.39 (2.55)	5.26 (2.09)	32 + * 16 = 18 –	0.25	3.87 (2.50)	4.38 (2.36)	19 + 5 = 15 –	--
3	2.00 (1.48)	2.56 (1.37)	29 + * 20 = 17 –	0.21	1.97 (1.50)	1.90 (1.50)	16 + 10 = 13 –	--

[a] Data from one field-test site was removed because participants discussed the video analysis task before providing written responses.
* Significant improvement from pre- to post-test scores (Wilcoxon Signed Ranks Tests; $p < .05$)

Gains in students' geometry content knowledge. A three-level hierarchical linear model was used to analyze data collected from students of treatment and comparison group teachers. This approach accounts for the repeated measure and the nested data structure of students in classrooms with the same teacher. A significant effect of time ($t = 5.69$, $p < .05$) indicated that across groups students' scores improved over the course of the year, as would be expected. Adding a time by group interaction term to the analysis revealed a significant difference ($t = 2.31$, $p < .05$) in gains between pre- and post-assessment scores for students of teachers who were engaged with LTG compared to students of teachers who were not involved in the PD (see Table 4). Average pre-post gains for students of treatment group teachers was 6.55 percentage points higher than gains for students of comparison group teachers. The size of this effect was 0.49 pooled standard deviation units.

Table 4. Students' Geometry Assessment Scores

	Treatment Group (n = 162) [a]		Comparison Group (n = 104) [a]	
	Pre-test	**Post-test**	**Pre-test**	**Post-test**
Mean	**36.42**	**45.28 ***	**42.69**	**45.00**
(sd)	**(17.14)**	**(17.15)**	**(17.63)**	**(18.97)**

[a] One treatment student's data and four comparison students' data were not included in the analysis because more than one-third of their item responses were left blank on one or both administrations of the assessment.

* A significant time by group interaction term indicated that students of treatment teachers' demonstrated larger improvements in percent correct scores than students of comparison teachers (repeated-measures ANOVA; $p < 0.05$).

The improvement among students of teachers engaged in LTG appears to have helped these students "catch up" to the students of teachers in the comparison group, whose estimated initial scores were, on average, more than six percentage points higher. In order to explore what changes in performance on the assessment accounted for this improvement, item-level results were examined by comparing the percent of students in the treatment and comparison groups that answered each item correctly on the pre- and post-tests. For 11 of the 20 items, the gain in the percent of students answering the item correctly favored the treatment group by 8 percentage points or more. None of the gains favored the comparison group by more than 5 percentage points. The 11 items with gains favoring the treatment group included all 3 items targeting dilation, 3 of the 4 items targeting congruence transformations, and both items targeting properties of similar figures. In the other two content areas, 2 of 6 items targeting scaling and 1 of 5 targeting ratios favored the treatment group. Although exploratory, these trends suggest that the treatment group students' improvements occurred in the content areas that are especially emphasized in the LTG Foundation Module and, prior to the release of the Common Core State Standards, have been less commonly addressed in middle school curricula (Confrey & Krupa, 2010; Heck, Weiss, & Pasley, 2011).

Discussion

Analyses from the field test study of the LTG PD materials offer evidence of the promise of the Foundation Module for achieving the intended teacher knowledge outcomes, including gains in geometry content knowledge along with the knowledge to apply understandings about content in mathematics instructional practice. These analyses used data from treatment and comparison teacher groups with efforts to ensure some degree of comparability across groups in terms of school and grade level. The analyses also controlled for teachers' initial status on the outcome measures. However, teachers were not randomly assigned to conditions, so factors other than the LTG PD may account in part for the impacts that were found. The field test also provides initial evidence that teachers' engagement in the LTG Foundation Module can lead to improvements in related student knowledge. An important limitation of the field test was the lack of systematic investigation of any changes that may have occurred in participating teachers' classroom geometry instruction, so the link to student outcomes, while promising, remains relatively tenuous.

The empirical evidence supporting the intervention has thus far been based on data in settings where the materials were used within the research and development process. The finalized version of the Foundation Module has not been tested, nor has the complete package of the LTG PD materials (which includes four Extension Modules) been used with any group of teachers at this point in time. Given the intervention's demonstrated potential, a critical next step is to study

the delivery of the finalized version of the intervention by trained, external facilitators, and carefully examining the impact on teachers' knowledge, changes in their instructional practices, and related improvements in their students' knowledge.

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UNDERSTANDING PROSPECTIVE TEACHERS' LEVELS OF GEOMETRIC THOUGHTS: INSIGHTS FROM A DISCURSIVE ANALYSIS

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The study investigates the characteristics of prospective teachers' geometric discourses at the van Hiele model of thinking (1959/1985), using Sfard's (2008) discursive framework. In this report, I align two prospective teachers' pre- and post- van Hiele geometry test (Usiskin, 1982) results with the analyses of their geometric discourses from clinical interviews, to illustrate changes in geometric discourse when a student's test results showed no change in van Hiele levels, and changes in geometric discourse when a student developed her thinking to the next van Hiele level. Revisiting the van Hiele model of thinking, complemented with a discursive lens, helped to understand learning as change in discourses, as prospective teacher develop thinking toward a higher van Hiele level.

Keywords: Geometry and Geometrical and Spatial Thinking, Learning Progressions, Teacher knowledge

The van Hiele model of thinking, known as "the van Hiele levels", was developed by the Dutch educators Pierre and Dina van Hiele (1959/1985). Many researchers have confirmed the usefulness of van Hiele levels when describing the development of students' geometry thinking. However the same researchers often find levels lacking in depth, and they would like a more detailed description of students' levels of thinking. Hoffer's (1981) "Sample Skills and Problems" (p.11) provided a framework that connects the levels of development with five basic skills (e.g., visual skills, verbal skills, drawing skills, etc) that are expected at each van Hiele level. Battista (2007) refined the model with five levels of geometric reasoings. Given my focus on prospective teachers' learning in geometry, I consider the possibility of elaborating the van Hiele model of thinking with a discursive lens in scrutinizing prospective teachers' thinking. I claim that when a student's geometric thinking develops to a higher level, simultaneously there is a development of the student's geometric discourse in discursive terms. If so, the question is, "What additional information does the analysis of geometric discourse provide about prospective teachers' levels of geometric thoughts?"

Theoretical Framework

The van Hiele model continues to be the best-known theoretical account of students' learning of geometric figures and their properties. The model suggests students must progress through a sequence of discrete, qualitatively different levels of geometry thinking. The first four levels in the model are as follows: Level 1, the Visualization level in which students recognize and learn the names of the figures, and figures are judged by their appearance as a whole; Level 2, the Descriptive level, when students begin to recognize figures by their properties or components; Level 3, the Theoretical level, where students begin to form definitions of figures based on their common properties, and understand some proofs; Level 4, the formal logic level, when students understand the meaning of deduction and conduct mathematical proofs using theorems.

My study departs from the van Hiele model, which accepts the basic idea of the levels of geometric thinking and of these main characteristics. In my rendition, thinking becomes a form

of communication, and levels of geometric thinking become levels of geometric discourse. This view of geometric thinking as geometric discourse entails that thinking is communicated through interactions.

Sfard (2008) has proposed that mathematical discourses differ one from another in at least four features: 1) *Word use* (mathematical vocabularies and their use), mathematical words that signify mathematical objects or process; 2) *Routines*, these are well-defined repetitive patterns characteristic of the given mathematical discourse; 3) *Visual mediator*s, these are symbolic artifacts related especially for particular communication; 4) *Endorsed narratives,* any text, spoken or written, which is framed as description of objects , of relations between processes with or by objects, and which is subject to endorsement or rejection, that is, to be labeled as true or false. These features interact with one another in a variety of ways. For example, endorsed narratives contain mathematical vocabularies and provide the context in which those words are used; mathematical routines are apparent in the use of visual mediators and produce narratives.

In what follows, the van Hiele behavioral descriptions at each van Hiele level are reviewed and analyzed with discursive terms (see Wang, 2011). For example, at Level 1, "a child recognizes a rectangle by its form, shape" (the van Hiele quotes), provides information about how a child identifies a figure, what it calls "a rectangle", based on its physical appearance. In discursive terms, the *vocabulary,* "rectangle", signifies a geometric shape has a name, and it is used as a label of the figure. The phrase, "recognizes…by its form, shape" suggests that the direct recognition triggers the decision making, and therefore the *routine* for this course of action is perceptual experience and it is self-evident (i.e., [it is] a rectangle [because I can see it] by its form, shape). The *narrative* is "what is said or described" about the object. The *visual mediators* in this situation could include a drawing or picture of a four-sided figure looking like a rectangle. As described, using the discursive lens not only allowed me to capture what students said, but also what they did when communicating their thinking. Viewing each van Hiele level thinking as its own geometric discourse, and with the help of the discursive lens, I expected to be able to arrive at a refined, "high resolution" picture of the process of geometric thinking.

Method

This report focuses on two participants. Sam and Lulu were prospective teachers, enrolled in a certain mid-west university teacher education program in the United States. They were enrolled in a measurement and geometry course as part of their program requirements. They participated in a pre- and a post-van Hiele Geometry Test (see Usiskin, 1982) as class assignments. One week after Sam and Lulu took the pre- and post-tests, they participated in 90-minutes pre-and post-interviews. The interview tasks are designed to elicit students geometric thinking and they are aligned with van Hiele geometric tests (see Wang, 2011). All the interviews are video recorded and transcribed for further discursive analysis.

I analyzed Sam and Lule's written responses from the van Hiele geometry test results using the test grading methods provided by the Chicago project team (Usiskin, 1982) to get initial information on their levels of geometric thinking. I also analyzed Sam and Lulu's interview transcripts in both what they said and what they did as course of actions during the interviews. In the following section, I share results about Sam and Lulu's test results and their interview analyses to learn more about the process of geometric thinking.

Results

Viewing the van Hiele model with discursive terms sheds light on what vocabularies participants used in describing geometric figures, and how these vocabularies were used in

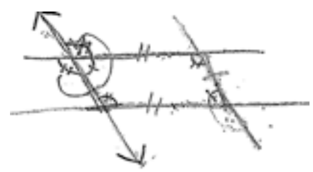
communicating mathematical thinking. The discursive lens helped to bring participants' actions to light, making it possible fro me to differentiate how they substantiated their narratives, either at a meta-level or at an object-level.

Continuity Within A Van Hiele Level: Sam's Case

Sam's van Hiele tests suggested that she was at Level 2 at the pre-test and stayed at Level 2, with no changes in van Hiele levels, but I found changes in her geometric discourses. The analyses of Sam's geometric discourses showed that her use of the word parallelogram changed. When she spoke the word "parallelogram" at the pre-interview, she meant any polygon having pairs of parallel sides, using a definition of parallelogram with only a necessary condition. For example, Sam was asked to draw a parallelogram and then a new parallelogram different from the previous one. Sam's course of action is as follows:

Interviewer: Why is this a parallelogram? [Pointing at the parallelogram]

Sam: I drew it so that this side would be parallel to this side [pointing at the two longer sides of the parallelogram], and this side would be parallel with this side [pointing at the two shorter sides of the parallelogram] Note: Sam drew a parallelogram first, and extended the sides of the parallelogram later.

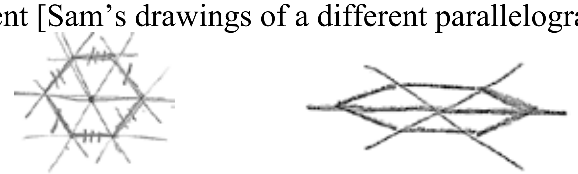


Interviewer: Why is this a parallelogram? [Pointing at the hexagon]

Sam: … because all the sides are parallel to another side.

Interviewer: Why is it a different parallelogram?

Sam: It's different because there are more sides and because the angles are different [Sam's drawings of a different parallelogram].

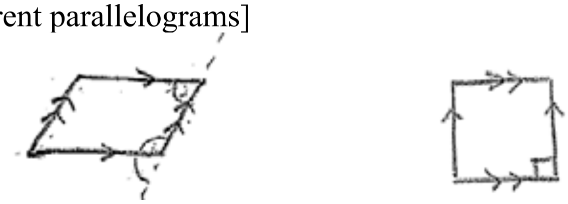


When communicating about parallelograms at the pre-interview, Sam considered two types: 1) "a parallelogram is a figure with all sides being part of parallel line segments (i.e., rhombi, parallelograms, hexagons, octagons, etc.); and 2) "it is a figure that has at least one pair of parallel sides, I think a trapezoid is a parallelogram". Sam did not consider squares and rectangles as parallelograms at the pre-interview.

At the post-interview, the same task was performed and Sam responded to the task with two drawings, a parallelogram and a square.

Interviewer: Why is this a parallelogram? [Pointing at the parallelogram].

Sam: … because it has four sides and each opposing side is parallel to one another. [Sam's drawings of two different parallelograms]

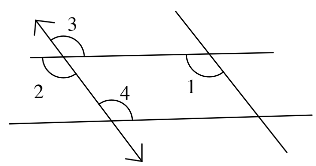


Interviewer: Why is this a parallelogram? [Pointing at the square]
Sam: It's a square, square is also a parallelogram, it has four equal sides and all angles are 90 degrees [Pointing at the square]. It's a different parallelogram from the one I drew because all the angles in this figure are equal.

It is notable that Sam's use of the word parallelogram evolved from pre-interview to post-interview, showing changes in her understanding of parallelograms, with regard to the added necessary conditions "four-sided" figure and "parallel sides". Moreover, Sam's grouping of parallelograms showed changes at the post-interview. She grouped parallelograms and rhombi together because "they all have two sets of parallel sides". She asserted, "rectangles are parallelograms with four right angles", as well as "squares are parallelograms with four right angels and four sides are equal." I argue that, at the post-interview, Sam had a good grasp of the concept of parallelograms in general, but her understanding of the hierarchy of parallelograms was missing, or not clearly demonstrated in the interviews.

There were also changes in Sam's substantiation routines. Substaintiation routines are repetitive patterns charaterising how Sam justifies or proves that naratives she provides are true or false. For example, at the pre-interview, Sam frequently used reflections, rotations and translations in her substantiations of narratives as concrete descriptions of her investigations.

Interviewer: How do you know ∠1 is equal to ∠4?

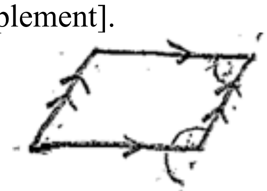


Sam: This angle [∠1] can just be slid over to this position and create ∠2, this line (the one with arrowheads) can be rotated so that ∠2 becomes ∠3. This angle [pointing at ∠3] at this intersection can just slide down and be in this angle's position [pointing at ∠4]. So two angles (∠1 and ∠4) are equal.

In this explanation, Sam used words such as "slide over", "create", "rotated" and "slide down" to indicate a sequence of imaginary movements performed to substantiate the narrative "two opposite angles [∠1 and ∠4] are equivalent". Sam's substantiation was intuitive and visual, and was focused on the descriptions of how lines and angles moved, rather than on the discussions about the results. Sam's routines operated at an object-level at the pre-interview.

In contrast, at the post-interview, Sam was able to use endorsed narratives (i.e., mathematical axioms and propositions) to verify her claims. The following brief substantiation was typical at the post-interview:

Interviewer: How do you know all angles in a parallelogram add up to 360 degrees?
Sam: Because angles on a straight line are going to add up to 180 degrees [Pointing at ∠2 and its complement].



Sam: This angle here is the same as this angle [pointing at ∠2 and its transversal exterior angle] because the parallel lines meet a third line at the angle. By the

> same reason, this angle added to this angle [pointing at ∠1 and ∠2's transversal exterior angle] equals 180 degrees. These two angles [∠1 and ∠2] added up to 180 degrees. For similar reasons, these two [opposite angles of ∠1 and ∠2] added up to 180 degrees. Together they [all four angles of the parallelogram] equal to 360 degrees.

Sam explained why two transversal angles are equivalent, not because you "can see it" as in the pre-interview, but as a result of "two parallel lines meet a third line at the same angle", and reached her conclusion that " they [all four angels of the parallelogram] equal to 360 degrees" by repeating a similar substantiation, "two angles add up to 180 degrees", for two adjacent angles in a parallelogram.

Note that the term "geometric object(s)" refers to all the mathematical objects involved in a particular geometric discourse. In my study, geometric objects discussed are quadrilaterals. The term "substantiation" refers to substantiations at an object-level and at a meta-level. The object-level substantiation emphasizes students' routines in describing the process of how quadrilaterals are investigated. For example, describing static lines, angles and polygons as movable entities under transformations (i.e., rotation, translation and reflection), is a way of substantiation at an object-level. With regard to definitions of different quadrilaterals, routines of substantiation depending on measurement routines to check the sides and angles of quadrilaterals, without thinking about how quadrilaterals are connected, are other examples of an object-level of substantiation. Object-level substantiation is a course of action where the student focuses on what she sees intuitively in explaining geometric objects (i.e., quadrilaterals) during the investigations. In contrast, a meta-level substantiation is a course of action where the student uses endorsed narratives to endorse new narratives. That is, students use mathematical definitions and axioms as results of investigations to construct mathematical proofs.

Two main changes in Sam's geometric discourse are change in word use and change in routines. Sam had developed competence in using definitions to identify and to group polygons with no hierarchy of classification, and had developed some informal deductive reasoning as her geometric thinking moved towards Level 3. Here, I am not trying to contradict the findings from Sam's van Hiele Geometry tests with her interview results, but rather to compile the results and to treat her thinking more explicitly and dynamically. The development of Sam's geoemtric discourse provides evidence of a student's geometric thinking developing continuously within Level 2 and in transition between Level 2 and Level 3, as she became more competent in using definitions to identify polygons, and her routines of substantiation began to operate at a meta-level in using definitions and axioms to construct mathematical proofs.

Continuity Within Two Consecutive Levels: Lulu's Case

Lulu was one of the two prospective teachers of the study who reached Level 4 based on the van Hiele geometry post-test. Lulu was at Level 3 in the pre-test and demonstrated a typical behaviour at this level. So one might ask, "what did Lulu's geometric discourse look like in moving from Level 3 to Level 4?" The detailed analyses of changes in Lulu's discourses are documented in my dissertation (see Wang, 2011). I summarize the changes briefly here. Lulu was at Level 3, coming in with the ability to use definitions to identify and group quadrilaterals, but my analyses of Lulu's geometric discourse show that she did not demonstrate that quadrilaterals were connected in a hierarchy of classifications. For example, in the pre-interview, Lulu's use of the word *parallelogram* referred to two groups: 1) rhombi and parallelograms, characterized by "opposite sides are equal and parallel" and 2) rectangles and squares,

characterized by "opposite sides are equal and parallel, and they all have right angles". Lulu did consider squares as rectangles, but she made no connections between rhombi and squares. At the pre-interview, Lulu performed an object-level of substantiations. For instance, when discussing the angles of a parallelogram, Lulu responded that the opposite angles of a parallelogram were equal. The following conversation took place when she was asked for verification:

Interviewer: How do you know the opposite angles are equal [in a parallelogram]?

Lulu: You mean… prove it to you, that in every case it would be that way?

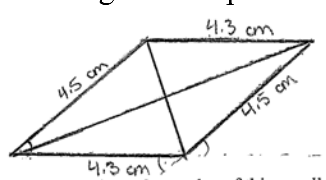
Interviewer: Yeah.

Lulu:I would just measure the angles for you, with a protractor. I've never done a proof before, in this case. I've done a lot of proofs, but not on something like that.

In this conversation, we learned that writing a geometry proof was new to Lulu, but she was aware of the differences between the generality of a mathematical proof and the particularity of checking the measurements of angles in a parallelogram when she asked, "prove it to you, that in every case it would be…?" In a similar scenario, Lulu measured the sides of parallelograms to verify her claim of, "the opposite sides in a parallelogram are equal".

Interviewer: How do you know opposite sides are equal?

Lulu: In this parallelogram? I can measure it. So, this is 4.5 centimetres, this is a little less than 4.5 centimetres [using a ruler to measure one pair of opposite sides]. Right, this is about 4.3. Yeah, it's about the same [measuring another pair of opposite sides].

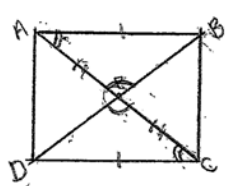


Interviewer: Is it true for every parallelogram?

Lulu: You mean to prove it? Well, I am not sure… but I know it's just a property of parallelogram.

The patterns of measuring sides and angles to verify equal measures were apparent in the pre-interview. In contrast, Lulu was able to demonstrate the hierarchy of classification among quadrilaterals using definitions at the post-interview, and she used propositions and axioms to construct mathematical proofs at that time. For example, Lulu provided a narrative, "diagonals bisect each other in a rectangle", and I asked for substantiation.

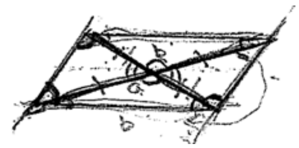
Interviewer: How do you know diagonals bisect each other in this case [pointing at Lulu's drawing, rectangle]?



Lulu: For the same reason as last time. Do you want me to explain again?

Interviewer: When you said, "for the same reason as last time", what do you refer it to?

Lulu: Just, all of it. When you draw diagonals in a rectangle, the diagonals create these congruent triangles. Based on the property, in all parallelograms, diagonals bisect each other. I know that the diagonals bisect each other in rectangles [pointing at the congruent triangles in the parallelogram].



The previous conversation signalled Lulu's ability in making relational transfer, from an endorsed narrative of "in all parallelograms diagonals bisect each other" to justify her claim about the diagonals in a rectangle. Given this opportunity, I asked for a written proof. Lulu's written proof is presented in Figure 1.

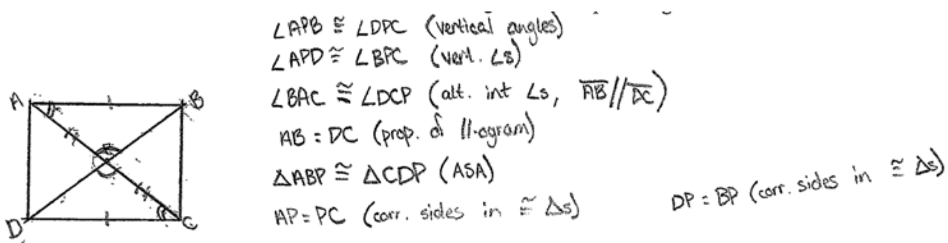


Figure 1: Lulu's Written Proof that Diagonals Bisect Each Other in a Rectangle at the Post-Interview

Lulu's geometric discourse presents a main characteristic of a Level 4 discourse that is absent in Level 3: abstraction of substantiations. Lulu showed Level 4 thinking in using definitions and axioms to construct proofs, and in using algebraic symbols to write a formal mathematical proof. However, at this level we also expect to observe behaviour where students are able to apply inductive reasoning in an unfamiliar situation. In Lulu's case, she was able to apply her knowledge of quadrilaterals to construct mathematical proofs in a familiar situation (e.g., to prove opposite angles or sides are congruent using congruent criterions), having carried out similar proofs in her geometry class for elementary teachers. When Lulu was asked to prove that two definitions of parallelogram were equivalent, she did not finish the proof because the task was new to her, and she did not know how to use the same axioms in a new situation. I argue that Lulu was at the beginning of Level 4 thinking, starting to gain the skills and languages needed for mathematical proof, but needing more practice to move forward to an advanced abstract level. Examining the changes from Lulu's Level 3 geometric discourse at the pre-interview to Level 4 at the post-interview, we observe movement from Level 3 thinking to Level 4 in developing a geometry discourse, in a continuous progression instead of jumps.

In addition, comparing Lulu's geometric discourse at Level 3 with Sam's at Level 2, we find similarities between Lulu's geometric discourse at the pre-interview and Sam's at the post-interview. Both shared familiarity with using definitions in identifying and grouping quadrilaterals, and were able to reason at an object-level. This observation also indicates the continuity of learning in transitioning between two consecutive levels, from Level 2 to Level 3.

Conclusions

The discursive framework provided opportunities to examine students' thinking in greater detail at each van Hiele level. It helped to analyse what students said (i.e. narratives and word use) about different parallelograms and their properties, and what they did (i.e., routines) when asked for substantiations. A careful analysis of students' mathematical word use sheds light on how words are used and whether the words are used correctly. Discursive routines do not determine students' actions, but only constrain what they can reasonably say or do in a given

situation, as negotiated conventions. However, discursive routines offer valuable information about what students do to make conjectures and justifications in a geometric discourse. I find it very useful to examine the details of students' routines of identifying, measuring, defining and justifying when working on a task about geometric figures and their properties, where the roles of definitions are demonstrated at the first three van Hiele levels. I also find it revealing to see the details of students' geometric reasoning across van Hiele levels through the development of geometric discourses.

Battista (2007) argues about the validity of the reasoning, which involves the accuracy and precision of students' identifications, conceptions, explanations, justifications, and points out that "there is a lack of distinction between *type of reasoning* and *qualitatively different levels in the development of reasoning*" (p.853) throughout van Hiele studies. For instance, a student used direct recognition as a type of reasoning that is strictly based on intuition, and used the same type of reasoning to refer to a period of development of geometric thinking when the student's thinking was dominated by direct recognition. One challenge regarding the van Hiele model is to sort out the levels related to type of reasoning and/or the levels of reasoning; and of course, "the devil is in the details" (p.854). Viewing geometric thinking as geometric discourse, Sfard's discursive framework takes a greater consideration of the details.

Acknowledgment

I am grateful to my two advisors during my graduate study, Dr. Glenda Lappan from Michigan State University and Dr. Anna Sfard from University of Haifa for their ongoing support and advice.

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MATHEMATICAL PROOF TOOLS: SUPPORTING THE INTRODUCTION TO FORMAL PROOF

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In this study, conceptual tools that worked to support teachers and their students in the introduction to formal proof in geometry are introduced. The tools provided teachers with a means for scaffolding proof. Study teachers reported positive outcomes from using these tools with their secondary students. Additional studies that explore the use of the tools with instructors in varying contexts are warranted to determine how they can be used to improve the teaching and learning of proof in other classrooms, even potentially in content areas other than geometry.

Keywords: Reasoning and Proof, Geometry, Curriculum, Mathematical Knowledge for Teaching

Considering the teachers' role in navigating the proof terrain, Herbst (2002) conducted an analysis of what is involved when geometry teachers attempt to engage students in the production of a proof. He argued that alternative ways of engaging students in proving must be found if proving is to play, in the classroom, the same instrumental role for knowing mathematics that it plays in the discipline. Thinking about possible instructional alternatives for the reform-oriented classroom as an opportunity, Herbst (2002) stated: "The mandate to involve students in proving is likely to be met with the development of tools and norms that teachers can use to enable students to prove and to demonstrate that they are indeed proving" (p. 200). Calls for the development of teaching tools are important because symbolic tools can enable, mediate, and shape mathematical thinking (Sfard & McClain, 2002). Mathematical tools, which can be viewed as supports for learning, include physical materials, oral language, written notation, and any tools with which students can think about mathematics (Hiebert et al., 1997).

A primary goal of this paper is to introduce a set of conceptual tools that were developed and piloted in a research project whose aim was to explore alternative strategies for navigating the introduction to formal proof in high school geometry. The research question addressed in this paper is: What tools can be used to support teachers' work of introducing proof?

Theoretical Perspective

Despite recent efforts to increase the role of reasoning and proving in school mathematics, geometry remains a central content area for teaching this important mathematical process (Newton, 2010). However, the paucity of more formal reasoning and proof experiences throughout the school mathematics curriculum contributes to the difficulty of teaching and learning proof even in geometry (Clements, 2003). To add to this challenge, some secondary teachers claim that they do not have strategies for teaching proof and even expressed the belief that you cannot teach someone how to construct a proof (Cirillo, 2011). This belief may be an underlying reason that geometry is commonly thought of as the most difficult portion of school mathematics (Knuth, 2002). The well-known challenges that students have with proof (Senk, 1989) may also contribute to these beliefs.

Much of the work of Herbst and colleagues has focused on classroom interactions and proving in geometry at the secondary level. For example, Herbst et al. (2009) described instances

of student engagement with proof in various geometry courses in a high school. Through this work they unearthed a system of norms that appear to regulate the activity of "doing proofs" in geometry class. The authors contended that a collection of actions related to filling in the two-column form are regulated by norms that express how labor is divided between teacher and students and how time is organized as far as sequence and duration of events. As an example, the first 4 of 25 norms reported by Herbst et al. (2009) are listed below:

> Producing a proof, consists of (1) writing a sequence of steps (each of which consists of a "statement" and "reason"), where (2) the first statement is the assertion of one or more "given" properties of a geometric figure, (3) each other statement asserts a fact about a specific figure using a diagrammatic register and (4) the last step is the assertion of a property identified earlier as the "prove"….(pp. 254-255)

This model of the instructional situation of doing proofs in terms of a system of norms is helpful to those who wish to investigate what it might mean to create a different place for proof in geometry classrooms (Herbst et al., 2009). The documentation of classroom norms is relevant here because it provides a frame for examining the alternative practices supported by the tools used by the teachers in this study. This study builds on the work of Herbst and colleagues by examining possibilities outside of these normative practices.

Methods

To learn more about the challenges that teachers face when cultivating formal proof in their classrooms, a three-year study that made use of qualitative methods of inquiry, was designed. Baseline data, collected in Fall, 2010, included two non-consecutive weeks of classroom observations in one target classroom of six teachers. Beginning Spring, 2011, 20 professional development sessions were designed and implemented to assist the project teachers in: attending to the ways in which they introduced proof; reading the literature and considering alternative practices; and re-considering the discourse in their classrooms when they were teaching proof. In Fall, 2011 and 2012, additional data were collected to observe and understand changes made to the introduction and teaching of proof in geometry. Interviews designed to help the researcher better understand the teachers' evolving beliefs about teaching proof were also conducted.

Findings: The Development and Use of Mathematical Proof Tools

After analyzing the baseline data from Year 1 of the study, it was determined that several skills were implicitly and simultaneously being taught when teachers first introduced proof. Students needed to learn: (a) postulates, definitions, and theorems; (b) how to *use* definitions and theorems to draw conclusions (c) how to work with diagrams (e.g., what can and cannot be assumed); (d) a variety of sub-arguments and negotiated classroom norms for writing them up; and (e) how sub-arguments come together to construct the larger argument. It was through these observations in conjunction with the consideration of a claim that one of the teachers (Mike; all names are pseudonyms) made during his introduction to proof (i.e., that there is no shallow end, but only a deep end of proof) that the Mathematical Proof Tools (MPTs) were developed. Based on the shallow end proof pool metaphor suggested by Mike, I hypothesized that perhaps there was a set of skills and practices that students need in order to develop proofs that could be ramped up over time. The professional development sessions and subsequent observations gave me a way to test that hypothesis. In Spring, 2011, the group of teachers and the research team began meeting for professional development where the developing ideas for the MPTs became a

major focus of alternative teaching approaches. The tools began as sample alternative tasks inspired by Cirillo & Herbst (2011) and evolved into the conceptual tools presented in Table 1.

Table 1: Mathematical Proof Tools (MPTs)

Conceptual Tool	Description	Practices That Scaffold Proving
Understanding Mathematical Objects	This tool connects a mathematical object to a definition, notation, or a diagram.	1) Communicating a mathematical object by making use of spoken or written text
		2) Communicating or reading a mathematical object by making use of marked diagrams
		3) Communicating or reading a mathematical object by making use of symbolic notation
		4) Determining examples and non-examples of mathematical objects
Drawing Conclusions	This tool presents the idea of an open-ended task that leads to conclusions that can be drawn from given statements and/or a diagram.	1) Understanding what can and cannot be assumed from a diagram and recognizing that sometimes diagrams can be misleading
		2) Knowing when and how definitions can be used to draw a conclusion from a statement about a mathematical object
		3) Using combinations of postulates, definitions, and theorems to draw valid conclusions from some given information
Sub-arguments	This tool presents the idea that there are common short sequences of statements and reasons that are used frequently in proofs and that these pieces may appear relatively unchanged from one proof to the next.	1) Understanding what valid conclusions can be drawn from a given statement and how those make a sub-argument (i.e., recognizing and utilizing repeated reasoning)
		2) Recognizing a sub-argument as a branch of a larger proof and how it fits into the proof
		3) Understanding how to write particular sub-arguments using acceptable notation and language (often negotiated with the teacher)
Understanding Theorems	This tool highlights the nature of theorems, their structure, and how they are used.	1) Interpreting a theorem statement to determine the hypotheses and conclusion
		2) Marking a diagram that satisfies a hypothesis
		3) Rewriting a theorem written in words into symbols and vice versa
		4) Understanding that a theorem is not a theorem until it has been proved (using definitions, postulates, or previously proved theorems, lemmas, and propositions) and that one cannot use the conclusions of the theorem itself to prove the conclusions of that theorem (i.e., avoiding circular reasoning)
		6) Determining the theorem proved when presented with a proof
		7) Understanding the connection between logic and a theorem, for example, how to write the contrapositive of a conditional statement and the connection between laws of logic and the hypothesis and conclusion of a mathematical statement

Teachers' Reactions to the MPT Use

Seth reported the following after teaching with the tools in Years 2 and 3:

The students see the patterns and understand the relationships, which ultimately makes this a much better exercise in learning to reason. Breaking it into pieces has made it easier for the students to digest and understand the material and see how it fits together. The evidence is in a greater understanding, drawing better conclusions, and getting better grades on harder proofs! Some students even say, "That was fun!"

Nick reported similar results:

> I'm actually excited to teach proof now, which was not the case before the professional development. Proof is the centerpiece for the most interesting discussions we have in class, and it is no longer a stand-alone topic for me that we don't revisit after Chapter 2.

Nick noted that he keeps the MPTs summary (Table 1) on his desk at all times so that when he plans his lessons, he is always considering which tools he can address.

Discussion and Summary

The conceptual tools described in this paper were developed in response to some of the findings related to the challenges of teaching and learning proof in high school geometry. These tools were intended to scaffold the introduction to proof for the students. Although this study only presents findings of a small group of teachers' results in using the MPTs, these findings are promising because the teachers did more than just use the tools in a casual way. Rather, they saw enough potential in the tool use to develop a new curriculum around them, and they reported strong effects from their use. Additional studies that explore the use of these tools with other teachers in varying contexts are warranted to determine how these tools can be used by teachers to improve the teaching and learning of proof, even potentially in other content areas.

Acknowledgements

This work has been supported by grants from the Knowles Science Teaching Foundation, the National Science Foundation (Grant #0918117), and the University of Delaware's General University Research grant. Any opinions, findings, and conclusions or recommendations expressed in this paper are those of the author and do not necessarily reflect the views of KSTF, NSF, or UD. I thank my colleagues, James Hiebert, Jamie Sutherland, Corey Drake, Beth Herbel-Eisenmann, John A. Pelesko, and the teachers who participated in the project.

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ONE HIGH SCHOOL STUDENT'S DEVELOPMENT OF MATHEMATICAL DISCOURSE ON TRANSLATION

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The aim of this study is to explore one Turkish high school student's learning of the geometric concept of translation using a communicational approach. Our approach formulates learning as occurring through the changes in students' discourses as they interact with mathematical concepts. Our results confirm that the student's learning about translation was closely related to the changes in particular elements of his discourse. We conclude that the framework we used provides rich tools with which to examine stages of learning as well as the elements of discourse that play important roles in mathematical learning.

Keywords: Geometry, Learning Trajectories, High School Education

Introduction

Transformation is one of the significant concepts in learning of geometry that allows students to develop broad concepts like congruence and similarity (Jones, 2002). There are many studies that focus on students' learning on transformational geometry from cognitive perspectives (e.g., Clements, Battista, & Sarama, 2001; Hollebrands, 2003). However, few studies have explored the issue through a socio-cultural approach. This study explores one Turkish high school student's learning of the geometric concept of translation using a communicational approach to cognition, which considers learning as the process of change in students' discourse (Sfard, 2001). This study is part of a larger research that explores students' development of discourse on transformational geometry. For space reasons, we only consider the concept of translation here and we address the following questions: How does a (10^{th} grade) student's discourse on the concept of translation develop? More specifically, how do the elements in his discourse change as he continues to participate in the discourse on translations?

Theoretical Background

In this paper, we use Sfard's (2008) communicational approach as a theoretical framework. According to this lens, thinking is a special form of communication and the individual who is thinking is considered to be self-communicating. Sfard (2008) explains communication as a dialogical endeavor; it can be intra-personal or interpersonal. Learning is about changing the ways we think and how we communicate what we think (Sfard, 2001). Learning here is conceptualized as change in one's discourse and students' thinking can be examined through their discourses (Sfard, 2001). According to Sfard (2008), discourse is a "special type of communication made distinct by its repertoire of admissible actions and the way these actions are paired with re-actions" (p.297). If the content of a discourse is mathematical, Sfard (2008) describes it as a mathematical discourse.

There are four elements of mathematical discourse: word use, routines, visual mediators and narratives (Sfard, 2008). *Word use* is one of the distinctive characteristics of discourse, the vocabulary that learners use. There are four phases in the development of word use: first one is *passive use*, where the learner is "not able use a [mathematical] word in her own speech" (Sfard, 2008, p.181). The other is *routine-driven use*. In this phase, a mathematical word is uttered to

signify the learner's actions on the mathematical object. In *phrase-driven use*, "words become linked to constant phrases rather than with the whole routines" (Sfard, 2008, p.181). Last phase is *object-driven use*, where words signify mathematical objects. *Routines* are repetitive patterns in the actions of learners that are characteristic of the discourse. *Visual mediators* are visible objects that are operated through the communication process, like symbols, graphs, drawings, tables. *Narrative* is any sequence of any utterances indicating the relationships among mathematical objects.

The purpose of this study is to investigate one high school student's development of the concept of translation by using the four elements in his discourse.

Methodology

The study was conducted with one 10th grade student named Okan (a pseudonym) in a medium-sized, urban high school in Turkey. The data for this work was collected through three task-based interviews across six weeks to explore Okan's development of mathematical discourse particular to the geometric concept of translation. Okan was selected as a participant because he was observed in the classroom as an uninterested student but occasionally gave interesting answers to the teacher's questions. Although his grades were low in geometry, the informal interviews with the teacher suggested that his geometrical learning was enhanced throughout the discussions in the classroom. The researchers prepared three different task-based interviews about translations. Each interview lasted about 20 minutes and was video-taped. First interview was conducted before the teacher introduced the topic of translation in the class. Second interview was right after the translation class. The third one was conducted 25 days after the second interview.

Each interview consisted of four types of tasks on the concept of translation. Across the three interviews, each type of task was equivalent to those of the same type in terms of the nature of the translation problem presented and the skills the student needed to use when solving the problem. To ensure equivalence and the validity (Patton, 2001) of the tasks, five experts reviewed the tasks in terms of their parallelism and content. The interviews were conducted in Okan's native language and then translated from Turkish into English by the authors. Data were analyzed in terms of word use, visual mediators, routines and narratives (Sfard, 2008).

Results

We analyzed word use considering the context of the discourse in a qualitative way rather than a quantitative way. Throughout the interviews, passive word use about translation was not observed in Okan's discourse.

Interview I

During this interview, Okan's word use was mainly routine-driven. For example, when we asked him what it means to translate an object, he said that it "is the process of positioning the object from one location to another." In terms of visual mediators, Okan's focus was on the vectors and how he can split them into their corresponding x and y-components. However, his use of these mediators was not consistent. In fact, during the first interview, Okan had no consistent routine with which he translated mathematical objects. He used some strategies for the problems in a haphazard rather than repetitive way. Therefore we could not identify any routines in his discourse at this stage. Okan was also confused about which narrative to endorse regarding translation. Results indicate that, before the teacher introduced translation, Okan considered translation as the process of repositioning the location of an object but did not have robust

understanding of the concept.

Interview II

The second interview was conducted after the teacher introduced the topic of translation. In this interview Okan's word use was predominantly phrase-driven. For example, when we asked him what translation is, he said "translation is when the objects are moved towards particular directions within the coordinate system". There were also some object-driven utterances in his discourse. For example, he said that "Translation is a transformation that enables the placement of objects within the coordinate system".

There were four kinds of routines (three geometric and one algebraic) determined in Okan's second interview. Every time Okan translated a geometric shape and gave examples for translation, he used these two geometric routines (GR1 and GR2): GR1. Translating one vertice of a given geometric object along the components of the vectors visually and then using the property of translation (it preserves size and orientation) to draw the rest of the translated object. GR2. Translating each vertice of a given geometric object along the direction of the vector and carrying the vertices to its translated location and then connecting the vertices of the translated shape using the property of translation.

After Okan translated objects using GR1and GR2, the researchers asked Okan to find the coordinates of each translated vertice. Okan's responses to these questions elicited one geometric (GR3) and one algebraic (AR1) routine: GR3. Finding the coordinates of the vertices visually using the grids provided in the questions. AR1. After using GR3, Okan regularly wanted to double-check his answer algebraically by adding the coordinates of each vertice of the original shape with the corresponding coordinates of the vector.

Vectors were the primary visual mediators in Okan's discourse during the second interview. He considered vectors as objects that move the geometric shapes as indicated by one of his geometric routines (GR2). He also used vectors algebraically, treating them as ordered pairs as indicated by his algebraic routine (AR1). One of the endorsed narratives that Okan used in his words, routines and visual mediators was: "Translation is the placement of objects in the coordinate system".

When we compare Okan's discourse in this interview with his first interview, we see the following changes in the elements of his discourse: (a) His mainly routine-driven word use changed to being mainly phrase-driven, and (b) his haphazard solution strategies changed to consistent routines in his discourse. He became close to objectifying translation as a form of transformation that preserves the size and orientation of any given geometric object.

Interview III

The third interview was conducted 25 days after the second interview. In this interview, Okan's word use was mainly object-driven. When talking about translation, Okan frequently referred to the translated geometric objects as the end states of the translation process. There were also some phrase-driven utterances in his discourse. For example, while Okan was translating a geometric object, he said "When an object is translated, area, circumference, and the distance between the points [of the original shape] do not change".

In this interview, we identified one geometric routine in Okan's discourse. He used this routine in three of the tasks on which he worked. Throughout the tasks, his approach was generally based on geometric rather than algebraic routines. His geometric routine was GR1. In terms of visual mediators, Okan mainly operated with vectors geometrically rather than algebraically as indicated by his routine GR1. One of the most frequently endorsed narratives in Okan's discourse was: "If the object is translated...the distances between the points [vertices of

the original geometric object] do not change."

When we compare Okan's discourse in this interview with his second interview, we explored the following changes in the elements of his discourse: (a) His mainly phrase-driven word use changed to being mainly object-driven, (b) his four different types of routines (GR1, GR2, GR3 and AR1) were reduced and stabilized to one geometric routine (GR1), and (c) his phrase and object-driven utterances consistently treated translation as an object long after Okan's second interview. Accordingly, his endorsed narratives started resembling actual mathematical facts rather than idiosyncratic utterances he endorsed at the very beginning of his learning process.

Conclusion and Discussion

In this study, we explored Okan's learning of the geometric concept of translation, which was conceptualized as the change in his discourse (Sfard, 2008). Note that this framework does not only focus on students' use of language but also takes into account other elements in their discourse such as their actions and visual mediators. Our analysis of Okan's discourse revealed that these elements of discourse are in close relationship with each other as he shifted his routine-based discourse to an object-based one in the context of translations. The learning trajectory Okan went through in the course of the interviews resulted in changes in each element of discourse, indicating that students need to command all the elements of mathematical discourse in order to enhance their thinking (Güçler, 2013).

One critical aspect of mathematical learning is to change the talk about processes of mathematics to the objects of mathematics (Sfard, 2008). Okan's discourse reflected such development as his routine-driven word use gave way to phrase- and object-driven word use. For the concept of translation, this means a shift from considering translation as actions formed on geometrical objects to considering translation as a distinct mathematical object that belongs to a class of other mathematical concepts (i.e., transformations). A detailed analysis of student discourse helps identify the characteristics of students' trajectories and discourse regarding these important conceptual shifts in transformational geometry.

Acknowledgments

This study is supported by the Scientific and Technological Research Council of Turkey (TUBITAK).

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PRESERVICE TEACHERS' MATHEMATICAL KNOWLEDGE OF VOLUME MEASUREMENT

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To theorize about what preservice teacher "know" about volume measurement, this study explored the extent to which four preservice teachers demonstrated their mathematical knowledge through four tasks in an interview. I characterized the preservice teachers' knowledge of the major concepts and processes of volume measurement in three types of knowledge: conceptual, procedural, and contextual. Video and audio data was collected, and it was analyzed based on the coding scheme inductively developed for this research. According to the different measurement contexts, the participants showed different conceptions of volume as a number, capacity, enclosed volume, and weight; their conceptions of volume formulas were represented as a composition of cubic units. Some participants showed incomplete procedural knowledge in using volume units and had misconceptions connected with measurement contexts.

Keywords: Measurement, Teacher Knowledge, Teacher Education-Preservice

Teaching school mathematics requires subject matter knowledge in mathematics, and the knowledge should be based on explicit understandings (Ball, 1991). Teachers' knowledge is important not only for their teaching but also for their students' learning (Baturo & Nason, 1996; Fennema & Franke, 1992). We still have, however, little sense of what exactly it is that preservice teachers know, and how that knowledge is acquired and used. A possible entry point to study preservice teachers' mathematical knowledge is to work from what they "do" know—the mathematical knowledge (and skills) they use and the contexts in which these are understood—so that teacher education programs can provide appropriate support for them to develop a stronger mathematical knowledge.

The purpose of this qualitative study was to explore the mathematical knowledge of volume measurement of preservice teachers who will be certified to teach preK-8 grades. This study is a first step toward understanding how preservice teachers make sense of the concepts of volume in measurement and of volume formulas, an important topic in the elementary mathematics curriculum of spatial measurement (length, area, and volume). This need to study preservice teachers' understandings of volume measurement is also guided by the assumption that "an individual's knowledge of a concept may depend on the context in which the knowledge is elicited" (McNeil & Alibali, 2005, p. 286). The research questions guiding this study were (a) "How do preservice teachers explain the concepts of volume in different measurement contexts?" and (b) "How do preservice teachers understand volume formulas?"

Theoretical Framework

The theoretical framework is presented in three main sections that review the research related to: (a) conceptualizing volume concept with its measurement, (b) spatial reasoning behind volume formula, and (c) characterizing volume measurement knowledge.

Conceptualizing Volume Concept with Its Measurement

Piaget, Inhelder, & Szeminska (1956) found that children could hold three different meanings of the volume concept in measurement: (a) an interior volume (e.g., the number of cubic units that compose an object, or the amount of matter which is contained within the boundary surfaces), (b) a

volume as an occupied space (e.g., the amount of space occupied by an object as a whole in the relation to other objects around it), and (c) a complementary volume (e.g., the volume of the water displaced when an object is submerged). Other studies have revealed other meanings: volume as capacity (e.g., the volume of matter that fills a container as capacity), volume as enclosed volume (e.g., the free space enclosed in a closed surface), volume as a number (e.g., a number obtained from numerical information and by the volume formula), volume as a mental object or quality associated with a body's characteristic of having three dimensions (Sáiz, 2003), and volume as weight (Potari & Spiliotopoulou, 1996). A question is raised, in the end: what makes individuals to develop these different conceptual understandings of volume?

Reasoning Behind Volume Formula

Researchers have reported problems in K-12 students' understanding of measurement related to their memorization of formulas (Barturo & Nason, 1996; Clements & Stephan, 2004). In calculating volume by mathematical multiplication, students need to reason about the meaning of triple multiplication of lengths in three dimensions as logical multiplication of the relations between lengths or areas and volumes, involving relations between units of different powers (Piaget et al., 1956); and they also need to conceptualize the spatial structuring process that underlies numerical procedures (e.g., layer structuring, Battista 2003).

Characterizing Volume Measurement Knowledge: Three Knowledge Types

As a conceptual framework for this study, I characterized measurement knowledge in the sense of conceptual, procedural (drawing on Hiebert & Lefevre, 1986; Baturo & Nason, 1996; Lee and Smith, 2011), and contextual (drawing on McNeil & Alibali, 2005). Conceptual knowledge for volume measurement includes the idea that the volume of an object can be thought of as filling an object with the volume unit of measure, and that the term 'volume' can refer to the number of the volume units contained within an object. Procedural knowledge includes measurement processes such as reading markings on a ruler, counting a number of cubes, structuring an array of cubes, measuring the volume of water displaced when an object is submerged, and calculating volume by using volume formulas in cubic units. Contextual knowledge is the recognition of the need to select a unit (or scale), a tool, or notation that is appropriate to measure different attributes/substances, and to use different measurement strategies that are suitable in different measurement contexts (e.g., an empty shoe box as cubic inch/cm, a juice box as fluid ounce/liter, a plastic container filled with rice as ounce, gallons for gasoline or liters for bottled water). To explore how the understandings of volume measurement are acquired in preservice teachers' minds as mathematical knowledge, we need to hear from them concerning the meaning of volume in the measurement context and the process of using volume formulas to determine volume measurement.

Method

The preservice teachers' knowledge of volume measurement was observed as they answered interview questions and performed four tasks in an interview. Four preservice elementary teachers (Amy, Jane, Kate, and Sue, all pseudonyms) volunteered to participate in this study; all they were students in a mathematics course (mainly focusing on geometry and measurement) required for their elementary (K-8) teacher preparation. Individual interviews were conducted before volume measurement was taught in class. The interview instrument was designed as four volume measurement tasks; and three boxes—all rectangular prisms (given in the elementary textbook as a typical physical/concrete model of the volume formula, $L\times W\times H$)—were presented in the interview tasks: an empty shoebox, a plastic food container filled with rice, and a one-liter juice box. In this study design, the different substances of the three rectangular prisms (i.e., the content of each box) were taken to represent the different measurement contexts. The interviews were audio- and video-recorded, and written work during the interviews was collected. The collected data was analyzed for types of conceptual, procedural, and contextual knowledge.

Results

Conceptual Knowledge of Volume Measurement with Contextual Knowledge

In the first interview task, participants were presented with three boxes—an empty shoebox, a plastic food container filled with rice, and a one-liter juice box—and for each asked how they would think about the volume of the box. Participants described volume as the formula of $L \times W \times H$, the space claimed by the substance in a box, the amount of the substance itself in it, and the free space enclosed in a container; thus, their conceptions about the volume concept could be inferred as (a) a number obtained by the formula (Sáiz, 2003), (b) an interior volume in terms of capacity (Sáiz, 2003), (c) weight (Potari & Spiliotopoulou, 1996), and (d) enclosed volume (Sáiz, 2003), respectively. For example, Sue explained the volumes of the three boxes, as "I think of volume as the space it takes up inside of it" (assigned as enclosed volume); "The amount of space all the rice takes up" (assigned as capacity); and "What's inside of it is taken up by the liquid" (also assigned as capacity), respectively. Based on the participants' different volume descriptions by contextual variability (the substance/content of each box), it is possible to infer the context-dependency of participants' conceptions of volume concept.

In response to the question about what the number obtained by multiplying three numbers in the volume formulas means (Task 3), the responses of all of the participants referred to their original conceptions about the volume of the shoebox. To the question of how many cubes were needed to fill the shoe box (Task 4), Amy, Jane, and Kate answered that the total number of cubes equals the number obtained from the formula; and Sue attempted to visualize a layer structure of cubes first and then came up with her estimate. To check/justify their numbers, instead of filling up the shoebox with inch cubes, Amy and Jane located cubes along the lengths in three dimensions; Kate and Sue constructed layers of cubes. Participants' strategies of checking/justifying their estimates reflect their internalized image of volume formulas and mental models of volume measurement (Battista, 2003).

Procedural Knowledge of Volume Measurement with Contextual Knowledge

When asked to explain how they could know the volumes of three boxes (Task 2), participants' strategies for measuring the volume of each box fell into four main categories: (a) using formula, (b) counting the number of counting standard or nonstandard volume units in the container, (c) weighing the container, and (d) measuring the amount of the space (taken by the substance) within the box. The characteristics of participants' strategies reflected their conceptions of volume observed in Task 1, a number, capacity, weight, and enclosed volume, respectively. The assumption of the relation between their conceptions of volume and measurement context was confirmed by participants' different strategies to figure out the volumes of the three boxes.

For Task 3, actually measuring the volume of the shoebox with a ruler or tape measure, all of the participants used the volume formula (i.e., measuring the lengths in inches by using a ruler, and then calculating its volume by using a formula). Thus indicating that they acquired the procedural knowledge of how to use volume formula in measuring the volume of a rectangular prism. As a type of procedural knowledge, however, their understanding of volume measurement units and notation were incomplete. In calculating the volume of the shoebox in cubic inches, participants showed some common misconceptions about the measurement unit, such as incorrectly using a squared unit, which is for area, not for volume, or incorrectly using a decimal for fractional inch measurements.

Conclusion and Discussion

According to the different measurement contexts, the participating preservice teachers in this study presented different conceptions of volume; the context of volume measurement also affected their ideas of the appropriate measuring strategy of the volume of a rectangular prism. The different measurement contexts demonstrated the participants' incomplete and weak knowledge of volume measurement with some misconceptions. For instance, to figure out the volume of the juice box, both

Amy and Kate, who already measured the volume of the shoebox correctly, weighed the full boxes and wrote down their measures in kilograms (e.g., 1.09 kg). It is possible to infer that they were seeing volume as weight, or not distinguishing between volume and weight, in the context of measuring the volume of the juice box. This observation is consistent with McNeil and Alibali's (2005) comments: "[A]n individual may appear to have knowledge of a particular concept in one context but not in another" (p. 304).

Significance of the Study

This research study reveals pre-service teachers' incomplete knowledge of volume measurement, with its context-dependent nature (McNeil & Alibali, 2005). The concept of volume measurement, with respect to the construction of spatial structure and its quantification by means of volume formulas, is a concept that students in teacher education program must have a chance to understand in a variety of circumstance. This would better equip preservice teachers with the capacity to choose appropriate, flexible measurement strategies in different contexts and thus to have a more complete, solid understanding of volume measurement. The context-dependent nature of volume measurement knowledge presents the issue for further investigation and discussion within the mathematics education community, especially a formative effect of variations-across-contexts on mathematical knowledge change.

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"ALL LINES ARE STRAIGHT": THREE GRADUATE STUDENTS' DIFFICULTIES CONCEPTUALIZING SPHERICAL GEOMETRY CONCEPTS

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Keywords: Geometry and Geometrical and Spatial Thinking, Mathematical Knowledge for Teaching, Teacher Education-Preservice

My interest in Spherical geometry emerged after my first long distance flight experience from Istanbul to Philadelphia. While checking the flying route on the monitor, I observed that the plane was following a curved route, which was headed towards a little North. I wondered the reason of following a curved route and I found it interesting because in my view the route between Istanbul and Philadelphia must be straight since I believed shortest distance between two points can be reached by following a straight line. Therefore, this interest directed me to register two sequential geometry courses that are designed for secondary school mathematics teacher candidates at the University of Georgia. In these courses, I understood that the shortest distance between two points on a surface of a spherical object can be reached by following "great circles". I also observed that my classmates were also having similar difficulties to comprehend spherical geometry concepts. So I decided to do this pilot study for future research to understand graduate students' difficulties and their understanding of spherical geometry concepts.

Spherical (elliptic) geometry is one of the two types of non-Euclidean geometry that focuses on the geometric concepts of sphere or spherical surfaces. According to De Villiers (1996), non-Euclidean geometries were developed by Lobachevsky-Bolyai and Riemann in the nineteenth century. Three mathematics education graduate students were volunteered to investigate the following research questions:

1. How do graduate students understand the *straight* line concept on a spherical plane?
2. How do they calculate the distance between two points given on a spherical object?
3. How do they form a triangle and a rhombus on a spherical object and explore the angle sum of the spherical triangle?

Total five hours of semi-structured interviews were conducted to gather data. The interviews were conducted as teaching experiment type in nature, which means there was a teaching goal of the interviews. I accomplished this teaching goal by probing participants' responses.

This initial study yielded that none of the participants was able to correctly define a spherical line. Two of the participants correctly identified the formula for calculating the distance between two points on a spherical object but because of their misassumption of a spherical line their understanding was incomplete. For the last research question, it was observed that only one participant was able to give a reasonable answer for the angle sum of the spherical triangle. In addition, confusion and experiencing difficulty were observed while participants attempted to solve the given interview tasks. The results of this study suggest that further research is needed on this topic. Also, my personal suggestion is that educators should pay more attention to spherical geometry in high school and in college since in my view learning spherical geometry may help students to expand their geometric understanding and understand the Earth where we live on it in a better way.

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EXPLORING STUDENT THINKING THROUGH ALTERNATIVE GEOMETRY PROOF TASKS

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Keywords: Geometry, Reasoning and Proof, Curriculum, High School Education

Textbooks or other curriculum materials are the main source of mathematical tasks used in classrooms (Stein, Remillard, & Smith, 2007). By extension, curriculum and task selection impact students' opportunities to learn reasoning and proof in geometry, the context of this study. Poor geometry achievement found at the secondary level, particularly in the area of proof, is well documented, and the curriculum has been noted as a primary factor (Clements, 2003).

To address this issue, some researchers have begun to develop alternative tasks and strategies for teaching proof in geometry (see, e.g., Cirillo & Herbst, 2011). The purpose of this study was to explore the following research question: How do students take up these alternative proof tasks towards the end of a high school geometry course? These tasks are considered alternative in the sense that they are not the types of problems that tend to be found in geometry textbooks. More specifically, in the work of proving a proposition in geometry, students are typically provided with the "Given" hypothesis and asked to write a proof of a particular "Prove" statement. Most of the time, they are also provided with the diagram to work the proof (Cirillo & Herbst, 2011).

An example of an *alternative* task used in this study was: "Suppose you conjectured that the diagonals of a rectangle are congruent and your teacher started you off with the diagram on the right [rectangle provided]. Write the 'Given' and 'Prove' statements that you would need to use to prove your conjecture." In May 2011 we sat down with five groups of students (n = 2-4 students per group) enrolled in a high school geometry course to work on a set of tasks. The students were asked to work through the tasks together as we video-recorded them working. Results indicated that students struggled to complete these tasks, and they noted that they were different from those tasks they typically encountered in their textbooks.

Acknowledgement

This project was supported by the Knowles Science Teaching Foundation.

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EXPLORING THE RELATIONSHIPS BETWEEN STUDENT MOON OBSERVATIONS AND SPATIAL-SCIENTIFIC REASONING

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Relationships between student moon observation journaling and sixth-grade students' spatial-scientific reasoning after implementation of an Earth/Space unit were examined. Teachers followed the NASA-based REAL (Realistic Explorations in Astronomical Learning) curriculum. Understanding many aspects of astronomy relies on an individual's ability to use the positions and motions of celestial objects to describe observed phenomena and to make predictions based on these observations (Plummer, Wasko, & Slagle, 2011). These astronomical explanations also require spatial ability, which is linked to performance in both mathematics and science (Black, 2005; Lord & Rupert, 1995). As such, the performance on a test of students' mental models of lunar phases should also correlate to their performance in mathematics. The Lunar Phases Concept Inventory (LPCI) was developed to assess students' mental models of lunar phases (Lindell & Olsen, 2002). These mental models are an important part of making sense of the world around us. Student moon observation journals foster a purposeful link between what students can observe about the world around them and the spatial relationships inherent in these observations. The act of keeping a moon journal is an interdisciplinary experience; it includes science observations as well as skills used in many other professions.

An extensive search was conducted in both the multilevel model (HLM) space and the single level model space. The final model identified for this data set is a single level linear model. The model shows that students performing better on moon observation journals, both in terms of overall score and number of entries, score higher on LPCI post-tests. For every 1 point increase in the overall moon journal score, participants are expected to score 0.18 points or nearly one percentage point higher on the LPCI post-test when holding constant the effects of the other two predictors, LPCI pre-test score and number of moon journal entries. An examination of the quality of moon journal entries demonstrates that students who put more effort into their moon journals appeared to notice more patterns in the appearance and location of the moon in the sky. These patterns additionally relate to their development of spatial skills as they are describing the apparently changing location of celestial objects in relation to their single position on Earth. This study is unique in the purposeful link created between student moon observations and spatial skills. The use of moon journals distinguishes this study further by fostering scientific observation along with skills from across STEM fields and other disciplines. Future work will continue to show a strong link between these improved spatial skills and performance in mathematics and science.

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A HISTORICAL PERSPECTIVE ON GEOMETRY TEXTBOOKS' TREATMENT OF CONGRUENCE

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How do we approach thinking about modern high school geometry textbooks? What are useful perspectives for understanding these presentations of geometric knowledge? A historical perspective can help situate our understanding of current geometry textbooks (Donoghue, 2003). In particular this study explores changes in the idea of congruence in geometry textbooks over time. The more formal treatment of congruence as a mathematical idea has changed, perhaps even evolved, over the history of the discipline (Steiner, 1971; Berlinski, 2013). This study identifies and describes four of these disciplinary perspectives, based on the work of Euclid (Heath translation, 1956), David Hilbert (1899/1938), Felix Klein (1872), and George David Birkhoff (1932), respectively. A coding framework was developed based on these four perspectives to analyze the treatment of congruence in geometry textbooks from the late 19th century into the early 21st century. The textbooks selected are considered representative of the textbooks of that time period and/or are considered to have been in popular use within that time period (Donoghue, 2003). The results of applying this framework suggest four categories of congruence treatment in the textbooks over time, shifting from a treatment almost identical to Euclid to a measurement-based approach, rooted in Birkhoff's ruler and protractor postulates.

This historical framework brings to light a tendency in the modern textbooks examined toward incoherent treatments of congruence, which take up pieces from all four mathematical perspectives rather than aligning with any one. The Common Core State Standards for Mathematics (CCSS-M) root the idea of congruence in rigid transformations, aligning with the Klein perspective (National Governors Association Center for Best Practices and the Council of Chief State School Officers, 2010). In the modern textbooks analyzed, transformations were discussed only in a final chapter; transformations were not fundamental to these textbooks' presentations of congruence and similarity. Given that there were isolated textbooks that adopted a transformations-based approach, and that the modern textbooks analyzed do not, this study raises questions about the potential ramifications and challenges of the CCSS-M treatment of congruence for current and future high school geometry textbooks, curricula, and teaching.

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PROVIDING RESEARCH-BASED GUIDANCE ON GEOMETRY AND MEASURES FOR TEACHERS IN MIDDLE AND HIGH SCHOOLS

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Keywords: Geometry and Geometrical and Spatial Thinking, Measurement, Curriculum

A recent report has suggested that if education is going to embrace evidence-based practice "in a serious way" then the first thing that is needed is better systems "for disseminating the findings of research to teachers on the ground" (Goldacre, 2013, p.16). With this in mind, this poster presents selected elements from a research project that has produced a research-informed resource that brings together, in focused sections, knowledge about mathematics and its teaching and learning in a way that is designed to be useful to teachers working in middle and high schools. The resource consists of a book (Watson, Jones, & Pratt, 2013) and a linked website; see: http://www.nuffieldfoundation.org/key-ideas-teaching-mathematics

The task of producing such a resource was approached through synthesizing a comprehensive range of relevant research about conceptual growth, through education, in key areas of the mathematics curriculum. Aimed at teachers, teacher educators, textbook writers, and curriculum policy makers for mathematics education at the middle and high school level, the resource is structured around seven key mathematical domains: relations between quantities and algebraic expressions; ratio and proportional reasoning; connecting measurement and decimals; spatial and geometrical reasoning; reasoning about data; reasoning about uncertainty; and functional relations between variables. Across these domains, the following are argued to be key powerful mathematical ideas: variable; proportionality; similarity; symmetry; linearity; measure; accuracy; dimensionality; representations; prediction; discrete/ continuous number; transformation; proof.

In geometry education, the research evidence synthesized indicates that teaching needs to attend to two closely-entwined aspects of geometry: the spatial aspects and the aspects that relate to reasoning with geometrical theory. These twin aspects of geometry, it is argued, are not separate, but interlocked. Six themes are identified: spatial relations, 3D geometry, symmetry and transformations, similarity and congruence, geometric diagrams, and geometric proof

In teaching topics in measurement, the synthesis of evidence indicates that the principles of measurement are not straightforward for many learners and may require more attention in school than is sometimes given. If the mathematics of measurement is neglected in teaching, this can store up problems for students when they progress to more advanced topics including graphs of functions, loci and vectors. Six themes are identified: measurement scales, compound measures, geometric measures, measurement and proof, angle measures and estimating measures.

Acknowledgement

The research was funded by the Nuffield Foundation. The paper reflects solely the views of the author.

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ELICITING STUDENTS' CONCEPTIONS OF AREA THROUGH MODELING ACTIVITIES

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Keywords: Modeling, Measurement, Instructional Activities and Practices

The use of Model-Eliciting Activities (MEAs) as an instructional tool in teaching mathematics to both preservice mathematics teachers and K-12 students in general holds great promise. By embedding rich mathematics content into open-ended and realistically complex problems, MEAs provide opportunities for teachers to examine students' thinking while at the same time creating engaging activities for students. According to a models and modeling perspective (Lesh & Doerr, 2003), students' conceptions, or mental models, develop and evolve as these models come in contact with or are evaluated against instructional activities. Model Eliciting Activities are designed specifically to challenge students conceptual models, thereby encouraging testing and revision of those models while at the same time allowing teachers a window through which to view students' conceptions. This study specifically examines the types of thinking revealed by students of widely different levels through one such activity.

In this study we compared middle school and undergraduate students' thinking around the concept of area in the context of the *Pelican* Colony MEA. This activity challenged students to develop a procedure for counting the number of pelican nests in different breeding grounds. Students were given aerial photos and maps of the colony to do so. In working on the challenge, they must determine a method for finding the area of an irregular shape. Students in an undergraduate course on algebra and modeling at a large midwestern university as well as 6th grade students in a suburban middle school participated in the MEA. Audio and video recordings of students' conversations while working the problem, as well as artifacts of students' work were analyzed for evidence of conceptions and misconceptions about area measurement using a comparative case study design.

Despite the wide difference in age and experience between the groups of students, several themes emerged across both group that parallel those outlined by Lehrer, Jaslow and Curtis (2003) and Clements and Sarama (2009). Students in both groups showed evidence of confusion of perimeter and area. Students who eventually developed a method based on covering the area with known units did so at varying degrees of sophistication; some repeatedly traced unit squares while others created arrays. Students also approached the counting of those arrays in both additive and multiplicative ways. These results demonstrate both the pervasive nature of certain conceptions even into college as well as the power of MEAs in eliciting students' mental models.

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SIXTH GRADER'S FRACTION SENSE ON NUMBER LINE TASKS: ARE WE READY FOR THE COMMON CORE?

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Keywords: Rational Numbers, Standards, Measurement

The Common Core State Standards (CCSS) for mathematics featured the number line as a central representational tool in helping students understand that fractions are numbers that expand the number system beyond whole numbers (2010). The number line is featured prominently in second through eighth grade as a means for developing number sense.

Prior research by Bright, Behr, Post, and Wachsmuth (1988), Shaughnessy (2011), and Diezmann and Lowrie (2006) noted students' difficulties with number line tasks that included incorrectly identifying the unit and difficulty with tasks where the number line was partitioned into unequal parts. In light of the implementation of the CCSS, the goal of this research study was to replicate prior studies and to get a snapshot of students' ability to understand the partitioning of a unit by identify missing fractions on a number line and also their understanding of ordering of fractions on the number line. This study examined 71 sixth graders over a period of four weeks. Students were given two worksheets. The first worksheet included tasks in which a number line was partitioned into equal lengths such as thirds or fourths and students were given two whole numbers on the line. Students were asked to identify a missing fraction. For the second worksheet students were asked to plot the fractions 4/5 and 5/4 on a number line and then determine which fraction was closer to "1".

The results show that students had a difficult time finding the missing fraction on a number line especially when zero was not one of the whole numbers given. Only 34% and 35% correctly identified fractions of these types where the partitions were thirds and fourths respectively. It is concerning that only 18% of the 71 sixth graders could correctly answer and explain which fraction, 4/5 or 5/4, was closer to 1 given that the Illustrative Mathematics (2011) team listed this as a 3rd grade assessment item. These results suggest students have gaps in their understanding of fractional number sense especially in the area of ordering fractions on a number line. Given the importance of the number line within the CCSS, the implications of this study suggest that more research is needed that delves deeper into students' understanding of these tasks.

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MIDDLE SCHOOL STUDENTS' MATHEMATICAL COMPREHENSION OF LATITUDE AND LONGITUDE

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Lines of latitude and longitude are part of a geographic coordinate system that is used to determine positions on the globe. These lines are measured in degrees because they correspond directly with the measure of an angle at the center of the earth. Also, instead of designating these measurements as positive and negative, the direction of the angle measure is indicated by the labels N, S, E and W. A full comprehension of global coordinate systems requires some mathematical understanding (spatial, geometric, etc.). The jump from simple grid work to longitude and latitude increases in complexity, as elementary students and adults struggle with the spatial reasoning needed to understand maps and direction (Muir, 1985). Students have an implicit cognitive coordinate system that they associate with physical objects that can become unrecognizable if viewed from a foreign orientation (Just & Carpenter, 1985). A student can become confused about notions of east and west in as it relates to an arbitrary measurement from a position, such as the prime meridian. And if they are to truly understand the motivation behind using this coordinate system to model the earth, students will have to have a fundamental understanding of angles. Students don't always recognize all aspects of angle measure, as a joining of two rays, as a wedged (or space between the rays) or as a turn from one ray to another (Browning et al., 2007). Students tend to struggle in defining what an angle is and what it measures (Keiser, 2004). There is little research available that looks into students' understanding of geographic coordinates in relationship to the mathematical concepts that it's based upon.

Middle school students, in an afterschool program, are participants in an interdisciplinary STEM curriculum that studies lunar and astronomical phenomena (Author). For an hour and a half each week, we studied students that were given inquiry and project-based lessons that that develop their mathematical and scientific reasoning and understanding. One lesson, in particular, has students investigate how to pinpoint exact locations on a globe, discover the relationship between angle measure and lines of latitude and longitude, and use this relationship to inform judgments about the relationship between the global coordinates and the position of the sun in the sky. The responses to questions and drawings were collected from the students who worked on the lesson. Our research of this data will seek to determine what mathematical understanding do middle school students use to determine longitude and latitude.

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ONLINE MATHEMATICS FOR ADULT EDUCATION: A DESIGN EXPERIMENT FOR GEOMETRY LEARNING THROUGH GEOGRAPHY CONTENT INTEGRATION

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Introduction

In the work that we are reporting here we have advanced in the identification of some characteristics of a *Geometry and Geography* course, which is part of an online educative program of high school called *BUNAM* (see http://www.bunam.unam.mx/). This course is offered to an adult population that are conformed by Mexican immigrants, older than 30 years old and that work abroad of their original places where they look for better conditions of life. Adults at the online *BUNAM* program care for having better levels of certification in education and at work.

Research Objective

This work is a first phase of a project of exploration around the impact of a cycle of task design on the improvement of online adult learning of geometry. In this phase we specifically carried out a cognitive analysis of the tasks belonging to the mentioned course, and a classification of students' responses. For the second phase of the project we have planned to upload at the platform of the program some founded modifications of the analyzed tasks to come back to see students performances about it.

Data, Analytical Frame and Some Conclusions

We classified student answers under two taxonomies that were taken into consideration due to their compatibility: the SOLO taxonomy (Structure of the Observed Learning Outcome) developed by Biggs y Collis (1982), adapted to analyse different types of tasks that are used mostly to teach Geography; and the van Hiele model for geometric thinking (1956, 1986), which has been used to describe a progressive thinking of students (Blair, 2004). For example, one of the tasks for students to answer is the following: "Explain what scientific validity had provided the evidence that Aristotle gave on the curved shape of the Earth". We found here different SOLO levels in the students' answers. Finally, this work shows the potential that digital learning management systems (LMS) have for mathematics education research in relation with the analysis and design (and later incorporation) of convenient changes to proposed tasks in online educative programs. It could be observed that students' answers on the mentioned task were mostly based on reading comprehension. Nevertheless, it is important to note that a response that goes further from a repetition of what Aristotle said would mean that students have some notion of what a scientific proof is. In that way they should have to analyze the given material and select the relevant information to explain how scientifically correct are the arguments proposed by Aristotle to confirm that the Earth's shape is spherical. Without discussing what is understood here by scientific proof, we have considered as a right answer the one that combines the two main aspects of observation and theory, because it would be more complete; and in general terms we can say that a scientific proof consists in contrasting through an empirical way a sentence that is part of a theory. (Note: References will be given at the presentation).

CHARACTERIZATIONS OF MEASUREMENT ACROSS TEACHER TEXTBOOKS

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Keywords: Measurement, Curriculum Analysis, Elementary School Education, Teacher Education-Preservice

Studies have shown that young children struggle to understand and master measurement concepts (Outhred & Mitchelmore, 2000; Battista, 2004). One avenue for analyzing the content that students learn is to focus on mathematics textbooks for pre-service teachers (McCrory, 2006). Textbooks for teachers largely define the content and knowledge that a pre-service teacher will pass on to their students. My research question is: How do elementary mathematics textbooks for teachers define measurement concepts? Furthermore, how can these definitions be characterized?

I analyzed 6 elementary mathematics textbooks, from a range of authors, and collected definitions for measurement, length, area, volume, and capacity for each textbook. I examined definitions for a discrete or continuous conception of measurement, as a framework. A discrete conception of measurement refers to measurement in terms of units. A continuous conception refers to the entire distance or space, not in terms of discrete units. The coding scheme consisted of coding definitions as discrete, continuous, or integrated. Definitions were coded as discrete if they used the word *unit*, continuous if they referred to the general space, and integrated if the definition contained references to both unit and general space.

I found that the definitions fell along a spectrum of discrete to continuous. The definitions did not fall neatly into purely discrete or continuous conceptions. For example, one textbook's definition for measurement could be "more" discrete than others. Definition characterizations were also consistent throughout each textbook, meaning that a textbook that provided a discrete definition of measurement typically followed through with a discrete definition of length. Further work should be done to refine this discrete to continuous spectrum. The results of this curriculum analysis are an example of one way to characterize textbook definitions. The results also have implications for why students struggle with measurement, if there are radically different ways of explaining measurement depending on the textbook.

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TEACHERS' KNOWLEDGE AND REASONING WHILE SOLVING MATHEMATICAL KNOWLEDGE FOR TEACHING GEOMETRY ITEMS

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Keywords: Mathematical Knowledge for Teaching, Geometry and Geometrical and Spatial Thinking, Teacher Knowledge

Mathematical Knowledge for Teaching (MKT) (Ball, Thames, & Phelps, 2008) has been heavily studied at the elementary and middle school levels. Researchers have only begun to investigate MKT at the secondary level (Herbst & Kosko, 2012, McCrory, et al., 2012). This current study focuses on teachers' MKT for secondary geometry and examines knowledge in three of the domains identified by Ball et al. (2008), knowledge of content and teaching (KCT), knowledge of content and students (KCS), and specialized content knowledge (SCK). In particular, the study seeks to answer the question: What knowledge and reasoning do teachers use in solving MKT-Geometry problems?

Nine geometry teachers, with 4 to 29 years of experience, were interviewed one-on-one while solving nine MKT-Geometry items, three problems in each domain: KCT, SCK, and KCS. Teachers read questions aloud and talked through their thinking while solving. Percents correct for the items ranged from 11% to 100% and are reflective of difficulty levels from a pilot study (Herbst & Kosko, 2012).

Teacher reasoning was based on pedagogical experiences, knowledge of students, and mathematical knowledge. While each problem can be solved using knowledge of its domain, teachers used extraneous information, both details in the problem and other categories of knowledge, that sometimes led them to incorrect answer choices. For example, in one item, four student solutions are presented for finding the measure of an inscribed angle based on other properties in the diagram. The teachers were asked to identify the incorrect solution, in which the student solves for an unrelated angle, which happens to have the same measure as the inscribed angle. The other three correct solutions provide varying levels of detail. While this item measures SCK and only requires knowledge of circles and angles to be solved, several of the teachers' expectations for explanation in student work led them to pick the wrong choice. The incorrect student work was very detailed, but made a jump in logic by connecting two unrelated angles. One teacher acknowledged that the incorrect answer was incorrect, but despite this, chose the shortest correct answer as incorrect because it was significantly less detailed. In fact, three of the nine teachers chose the shortest answer, because even though it was correct, they did not believe the student provided enough explanation. Full results will be displayed on the poster.

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Research Reports

Brief Research Reports

Poster Presentations

GESTURE AS A WINDOW TO JUSTIFICATION AND PROOF

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The role of the body, particularly gesture, in supporting mathematical reasoning is an emerging area of research in mathematics education. In the present study, we examine undergraduate students providing a justification for a task about a system of alternating gears, which involves concepts of number relating to even/odd patterns. Some participants were directed to perform gestures relevant to alternation and parity before attempting their justification, while others were not. Although these directed actions did not seem to influence the gestures participants used to solve the problem, we found an important relationship between gesture and mathematical reasoning. In particular, certain types of gestures during problem solving were associated with valid justifications. This research provides insight into the link between action and mathematical reasoning, and has implications for supporting students' proof activities.

Keywords: Cognition, Learning Theories, Reasoning and Proof

Learners display their mathematical thinking—and even engage in mathematical reasoning—with their bodies as well as their minds. Recent theoretical work in education and psychology has sought to broaden researchers' and educators' perspectives to address the role of the body in mathematical thinking and learning. In this research, we adopt this perspective to explore the role of learners' actions during proof and justification activities. Specifically, we investigate the connections between learners' *gestures* and their *justifications* as they talk aloud while attempting to solve a problem about an underlying numerical pattern presented in a gear-system task. By investigating how body movements relate to different forms of mathematical reasoning, we can better understand students' thinking and consider novel ways to support the construction of valid mathematical justifications. We contribute to *broadening perspectives on mathematics thinking and learning* by detailing a connection between specific gestures and types of proof that challenges the concept of mathematics as a disembodied system.

Theoretical Framework

Action and Gesture

An embodied cognition perspective highlights reciprocal connections between actions and cognition (e.g., Glenberg, 1997; Goldin-Meadow & Beilock, 2010). Specifically, these theories suggest that actions do not simply externalize the output of cognitive processes, but may also directly influence and cause changes in cognition and learning (e.g., Shapiro, 2011). Because this view postulates that actions and cognition are intrinsically linked, it stands to reason that there may be an association between an individual's actions and his or her performance on tasks requiring insight and

problem solving (e.g., Thomas & Lleras, 2009). Research has shown that actions play an important, and potentially vital, role in the learning and using of mathematics concepts and procedures -- from the spatial bodily orientations that represent early concepts of number (Dehaene, Bossini, & Giraux, 1993) and the use of fingers in early counting (Alibali & DiRusso, 1999; Saxe & Kaplan, 1981), to the concrete and perceptual dimensions of mathematical symbols (Landy & Goldstone, 2007). Action can also be an important way to "ground" (Goldstone & Son, 2005) abstract mathematical ideas in students' experiences. Physically-grounded actions and manipulations of real and virtual objects can help students understand concepts like proportionality (Abrahamson & Howsin, 2010), fractions (Martin, 2009), and algebra (Nathan, Kintsch, & Young, 1992). It is clear that actions contribute to mathematical thinking.

One specific type of action that is of special interest to educators is *gesture* –the spontaneous hand movements that speakers produce as they talk. Recent theoretical work suggests that gestures manifest mental simulations of actions and perceptual states (Hostetter & Alibali, 2008). Even though gestures do not physically manipulate the environment, growing evidence suggests that the experience of producing gestures can directly influence cognition (Alibali & Kita, 2010; Beilock & Goldin-Meadow, 2010; Goldin-Meadow & Beilock, 2010). Gestures that represent mathematical objects may serve an important function of grounding mathematical ideas in bodily form (Alibali & Nathan, 2007; Nathan, 2008), and they may also communicate spatial and relational concepts (Alibali, Nathan, & Fujimori, 2011). Moreover, speakers' gestures may reveal unique aspects of thought that are based in perception and action, and that may be crucial to their reasoning about the ideas they communicate (e.g., Alibali & Nathan, 2012; Chu & Kita, 2011). Thus in mathematics, gestures not only can provide a window into students' thought processes; they can help students to represent and understand key ideas and relationships. This may be especially true in mathematical tasks that involve spatial reasoning.

Mathematical Proof and Justification

In the study presented here, we investigated how gesture is related to the type and quality of mathematical reasoning in the task of creating a justification for a mathematical task involving an underlying even/odd pattern. Our theoretical framework for conceptualizing mathematical proof and justification is based on Harel and Sowder's (1998) work, which states that producing a proof involves removing doubt about the truth of a conjecture, both from oneself and from others. Harel and Sowder further distinguish between three major subsets of proof: external conviction, empirical, and analytical. The first orients around the production of self-satisfying proofs that rely upon external resources, such as textbooks or teachers, which we do not focus on in this study. The second subset, empirical, involves validating conjectures using physical facts or perceptual experiences. Finally, the third subset, analytical, "is one that validates conjectures by means of logical deductions" (p. 258).

We rely upon a specific type of proof in this third subset, *transformational*, which Harel and Sowder define as proof activities that are characterized by generality and abstraction, deliberate mental operations, and image transformations. We also utilized a specific type of proof in the empirical subset, referred to as *perceptual* proofs. These proofs involve mental images similar to transformational proofs, but which Harel and Sowder contrast by noting that perceptual proofs "consist of perceptions and a coordination of perceptions, but lack the ability to transform or to anticipate the results of a transformation" (p. 255). Given that transformational and perceptual proofs both involve images, and given the close ties between gestures and mental images (e.g., McNeill, 1992), Harel and Sowder's proof scheme is particularly well suited to our approach.

Gear-Parity Problems

We examined the justifications participants provided as they attempted to generate and justify a conjecture about a system that follows the underlying numerical pattern of *parity*, which we here instantiate with a system of interlocking gears. Parity in this context refers to the idea that in systems

with an odd number of gears, the final gear turns in the same direction as the initial gear, while in systems with even numbers of gears, the first and last gear turn in opposite directions.

When provided with a static display of a row of gears or asked to imagine such a system, participants typically begin solving the problem by simulating the turning of the gears with their hands (Schwartz & Black, 1996). After producing such rotating gestures, many participants shift to a new approach during which they note that the gears move in an alternating sequence. Past work with both undergraduates and young children has shown that concentrated, accurate simulation of the gear movements predicts the transition to recognizing such *alternation* (Boncoddo, Dixon & Kelley, 2010; Trudeau & Dixon, 2007). Thus, participants' ability to abstract the underlying mathematical relationship in the gear system seems to be related to their actions, and in particular, to the action of repeatedly tracing multiple circles as they think about and solve the problems. When a participant discovers parity, they display an understanding that the direction of movement of the final gear is related to the number of gears in the system.

Thus, reasoning about conjectures involving gear systems are of interest to mathematics education because they manifest the abstract parity rule that is a key component of understanding number systems. Moreover, gear systems are often utilized as a grounding context in mathematics lessons focusing on ratio, proportions, and linear functions (e.g., Lobato & Ellis, 2010; Ellis, 2007). For example, Ellis (2007) utilized gear systems in a teaching experiment on linear functions in order to give students a real-world situation within which to learn about linear relationships. Thus, gear systems embody important mathematical ideas, and examining how students solve such problems provides insight into how teachers can support their thinking about these ideas in the mathematics classroom.

Research Questions

In this study, we focus on the gestures that speakers produce when reasoning about a gear-parity conjecture. We first examine whether the types of gestures participants produced while thinking aloud about the conjecture varied depending on the specific actions they were directed to perform prior to being given the conjecture. Second, we examine whether the types of gestures participants produced during justification of the conjecture were related to the type and quality of the justifications they provided. Thus, our first research question was: Is the nature of students' directed actions *prior* to engaging in proof reasoning related to the nature of the justifications they provide? Our second yet primary research question was: Is the nature of students' gestures *during* proof reasoning related to the nature of the justifications they provide? This research provides insight into the link between action and mathematical reasoning, and has implications for supporting students' proof activities.

Methods

Participants

Participants were 120 undergraduate students enrolled in a psychology course at a large Midwestern university. Their average age was 19.6 years (*SD* = 1.08) and 51% of the participants were female.

Procedure

As part of a larger study, each participant was asked to solve a number of problems, including gear-parity, geometry, and transfer tasks; in this paper we focus only on the gear-parity problems. Participants were asked to think aloud (Ericsson & Simon, 1998) as they solved the problems. Instructions and interviewer prompts were standardized, and tasks were presented on an interactive white board. Participant data was coded separately for justifications and for gestures, as detailed later in this section.

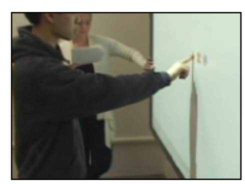
Before solving the gear-parity task, participants were randomly assigned to perform actions that were either relevant or irrelevant to providing a correct justification for the conjecture (see Table 1).

The relevant actions were based on previous studies that have shown that before participants discovered knowledge of the underlying numerical relationship of parity in the gear system, they typically use a strategy called "alternation," in which they abstract the relation that adjacent gears move in alternating directions (Trudeau & Dixon, 2007). In particular, relevant actions involved students tapping back and forth on a screen, to model alternation. For participants who performed relevant actions, we also varied whether participants were told that the relevant actions they performed were related to the gear problem, which we call *projection*. Participants who received projection were explicitly informed that the tapping actions related to the solution to the problem, while other participants were not informed that their actions had any relevance. Finally, half of the participants performed small actions that were entirely within the periphery of their gesture space, while the other half performed larger actions that extended outside the periphery of their gesture space (McNeill, 1992). This factor did not affect participants' success at abstracting the underlying mathematical relationship, so we do not discuss it further in this report.

Justification Coding

A coder blind to experimental condition used the proof categories from Harel and Sowder (1998) to code participants' justifications on the gear task. For our purposes, participants' justifications were coded as *analytical>transformational* if they acknowledged that connected gears form an alternating pattern of motion, and that the turning direction of the final gear is based on whether the total number of gears in the system is even or odd. We also extended the category of *empirical>perceptual* proof to account for a problem-specific nuance, distinguishing between *perceptual with alternation* and *perceptual without alternation*. Perceptual with alternation justifications included an understanding that the gears move in an alternating pattern, but the participant failed to acknowledge parity. Perceptual without alternation justifications included participants making comments that demonstrated a belief that the gears all turned in the same direction or that the last gear would turn in the same direction as the first gear, regardless of the number of gears present. Finally, the justifications of the participants who were unable to come up with a justification for how the gear system would turn were coded as *don't know*.

Table 1: Gear Conjecture and Directed Actions

Conjecture	An unknown number of gears are connected in a chain. You know what direction the first gear turns, how could you figure out what direction the last gear turns? Provide a justification as to why your answer is true.	
Relevant actions	<u>Large Actions</u>: Participant alternates between tapping a blue and yellow diamond placed an arm span apart on the Interactive White Board with their palm. 	<u>Small Actions</u>: Participant alternates between tapping a blue and yellow diamond placed a hands-length apart on the Interactive White Board with their index finger. 
Irrelevant actions	<u>Large Actions</u>: Participant taps only the blue diamond with their palm.	<u>Small Actions</u>: Participant taps only the blue diamond with their index finger.

Gesture Coding

A coder who was blind to experimental condition watched and coded each participant's session without sound in order to classify gestures based on their form while uninfluenced by the verbalizations that accompanied those actions. Participants' gestures were classified into three categories: *rotating* gestures, *ticking* gestures and *other* gestures. These categories were identified as important based on previous studies of undergraduates solving gear-system problems (e.g., Alibali,

Spencer, Knox & Kita, 2011; Schwartz & Black, 1996; Trudeau & Dixon, 2007). *Rotating* gestures depicted one or more gears turning, typically using either one finger or the whole hand. Rotating gestures were further classified as depicting a single gear, multiple gears turning in the same direction, or multiple gears turning in alternate directions. *Ticking* gestures were defined as gestures that displayed a series of ticks, taps, or other discrete movements, typically produced using one finger or the whole hand and moving across space. Gestures that did not fall into either of these categorizes were labeled as "*other*" and descriptions of these gestures were recorded for further future analyses. Other gestures that we noted included beats, single points, and gestures that depicted movements other than rotating or ticking.

Results

Did participants' gestures vary by condition?

We first examined whether participants in the three conditions (relevant action with projection, relevant action without projection, and irrelevant action) varied in the types of gestures that they produced. First, participants were classified by whether they ever produced rotating gestures. Second, participants were then grouped according to whether they ever produced multiple rotating gestures with alternation, or only produced rotating gestures without alternation. As noted above, alternation is of particular importance to the gear-parity conjecture. We expected that more participants in the relevant-action conditions (regardless of whether they also received projection) than in the irrelevant-action condition would produce gestures that manifested alternation. Specifically, we hypothesized that tapping back and forth embodied the alternation relationship found in the gear system, which we expected to be subsequently manifested in the gestures that participants produced during justification. As seen in Table 2 (leftmost column), this was numerically the case. However, the difference across conditions was not significant.

Table 2: Percentage of Types of Gestures Produced by Participants in Each Condition

	Rotating Gestures		
Condition	Any multiple with alternation	Other (no multiple with alternation)	Ticking Gestures
Irrelevant	53	18	20
Relevant + Projection	55	30	8
Relevant	68	23	25

A third coding dimension classified participants in terms of whether they ever produced ticking gestures. We had anticipated that more participants in the relevant-action conditions than in the irrelevant-action condition would produce ticking gestures, since the relevant actions they had performed involved ticking back and forth. Instead, more participants in both the relevant and irrelevant conditions produced ticking gestures than did participants in the relevant-action-plus-projection condition. The difference across conditions in the proportion of participants who produced ticking gestures approached significance, $\chi^2(2, N = 120) = 4.50$, $p = .105$.

Thus, we did not find statistically significant relationships between action condition and patterns of gesture production. However, trends in the data suggest that the relevant actions may have encouraged participants to acknowledge alternation, and that projection may have inhibited ticking gestures during the justification.

Were variations in gesture associated with variations in justifications?

Given that variations in gesture production amongst conditions were not significant, we collapsed across conditions in order to examine whether participants' gestures during problem solving were associated with the quality of the justifications they provided. Figure 1 displays the percent of

participants who provided each type of justification, as a function of the type of gestures participants produced during their explanation. Very few participants produced ticking gestures only ($N = 1$), or ticking gestures with non-alternating rotating gestures ($N = 2$), so these categories are not displayed in the figure. The number of participants in the other categories ranged from 18 to 52.

As seen in Figure 1, participants who produced rotating gestures with alternation (the right two bars) were much more likely to provide valid, transformational justifications than were participants who did not produce such gestures (the left two bars), $\chi^2(1, N = 117) = 18.71, p < .001$ (83% vs. 43%). Importantly, producing rotating gestures *with* alternation was more strongly associated with transformational justifications than was producing rotating gestures *without* alternation (i.e., single circles, or multiple circles that did not alternate direction), $\chi^2(1, N = 78) = 7.13, p = .007$ (comparing the middle two bars to one another, 46% vs. 15%). In addition, participants who produced ticking gestures *along with* rotating gestures with alternation (the rightmost bar) were slightly more likely to provide transformational justifications than were participants who produced rotating gestures with alternation without ticking gestures (the center right bar) (67% vs. 54%). However, this difference was not significant.

Finally, among participants who produced perceptually-based justifications (the two light gray categories), participants who produced rotating gestures with alternation were more likely to incorporate alternation into their justifications than were participants who did not produce such gestures (the left two bars), $\chi^2(1, N = 68) = 18.50, p < .001$.

In sum, producing rotation gestures with alternation was associated with expressing more sophisticated justifications. Participants who produced rotation gestures with alternation were more likely to express transformational justifications, and, if they expressed perceptual justifications, they were more likely to incorporate alternation.

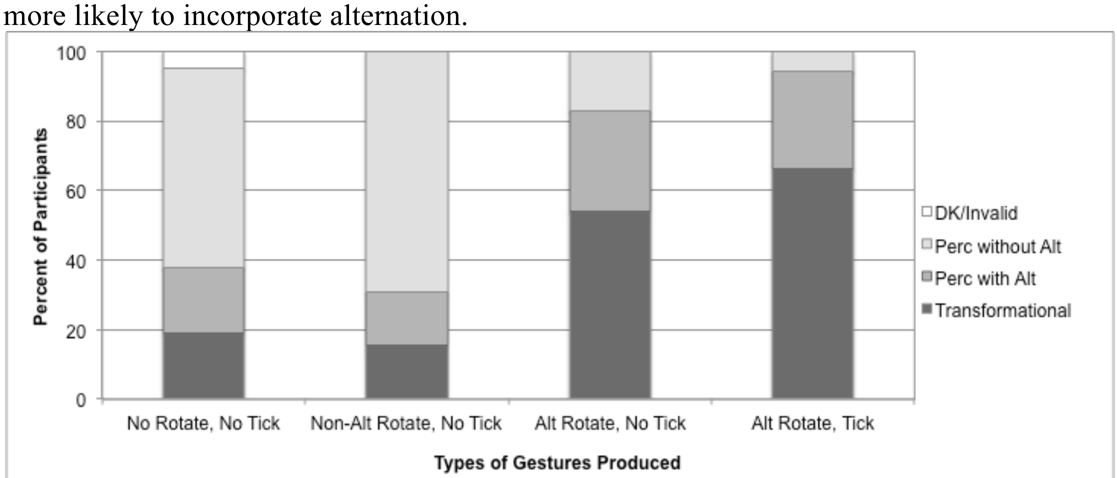


Figure 1: Percent of Participants who Provided Each Type of Justification as a Function of the Types of Gestures Produced

Conclusion and Implications

This study investigated the gestures participants produced when generating and justifying a conjecture about a gear system. Specifically, we found that participants who produced multiple, alternating rotating gestures were the most likely to solve the problem utilizing transformational justifications. This finding aligns with past research indicating that simulation of the gear movements was associated with participants' generating an alternation approach (Boncoddo et al., 2010). Additionally, this finding adds to the body of work that shows that gesture is closely related to, and can perhaps affect, learners' approaches to mathematical justification. The fact that certain, identifiable gestures were closely aligned with valid justifications highlights the importance of

considering both speech and gesture when examining mathematical reasoning. Producing these gestures may be a key aspect of students' understanding and adopting the mathematical insights behind the gear problem.

Participants who produced ticking gestures along with rotating gestures with alternation were most likely to produce transformational justifications, raising the possibility that ticking motions are a potentially useful form of abstraction for the concept of parity. Rotating gestures and ticking gestures may each play different roles in supporting mathematical reasoning, and when used in combination they may be especially powerful. In this light, it is interesting that participants in the relevant-action-with-projection condition were especially *unlikely* to use ticking gestures. The high rate of multiple, alternating gestures, but low rate of ticking gestures in this condition suggests that projection may have encouraged participants to focus too strongly on the concrete movement of the gears, making it more difficult for them to make the mathematical abstraction. In other words, being directed to perform alternating ticking gestures and being explicitly told of their relevance to the gear problem may have made abstraction to the even/odd pattern more difficult, as it focused participants on concrete, perceptual aspects of the situation (e.g., Kaminski, Sloutsky, & Heckler, 2005).

This work has important implications for broadening perspectives on mathematical thinking and learning. While students in mathematics classrooms may or may not be involved in gear tasks specifically, we have shown that certain kinds of gestures can be strongly associated with producing transformational proofs. Thus, it may be beneficial for teachers to encourage students to gesture while reasoning, and to pay close attention to students' gestures to look for key aspects of their reasoning processes and current levels of understanding.

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INDIVIDUAL ADHERENCE TO INQUIRY-ORIENTED NORMS OF DEFINING IN ADVANCED MATHEMATICS

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This study investigates students' individual patterns of adherence to a norm for creating and assessing definitions in an undergraduate real analysis classroom. Our findings indicate that, though all students understood the expectation to define, students differed in their individual adherence to the norm depending upon their perception of the nature of the defining activity and their associated role in the classroom. While some students primarily attributed the expectation to define to the teacher (taken-as-expected), others understood the expectation as truly shared as a means toward classroom learning (taken-as-beneficial). A third group of students internalized the expectation as a means of their membership in the mathematical community (taken-as-meaningful) and were therein willing to challenge the professor's defining choices in light of their values for mathematical defining.

Keywords: Advanced Mathematical Thinking, Reasoning and Proof, Post-Secondary Education

Introduction

For some time, the Emergent Perspective framework (Cobb & Baursfeld, 1995; Yackel & Cobb, 1996) has provided a fruitful set of tools for documenting collective learning phenomena within mathematics classrooms. One of the primary strengths of their framework is a refusal to dichotomize psychological and sociocultural phenomena, instead viewing each as an analytical perspective for which the other provides a necessary context. As a result of this balanced approach, research points to strong connections between collective constructs (classroom norms, sociomathematical norms, or mathematical practices) and individual constructs (mathematical beliefs or conceptions; Yackel & Rasmussen, 2002).

However, the university classroom structure places a larger onus of mathematical learning on students' time outside of class. This indicates that the research community also must consider exactly how classroom norms and practices influence students' mathematical activity outside the classroom. Thus, this paper seeks to address the following questions:

1. In an undergraduate real analysis course characterized by inquiry-oriented norms for defining, what are students' individual patterns of participation in the norms during class meetings and while working on class assignments outside the classroom?
2. How do students' patterns of participation interact with their beliefs about the purpose(s) of shared defining activities and students' role in advanced mathematical activity?

The current investigation draws from classroom observations and interviews with volunteer students from two different inquiry-oriented real analysis classes to answer these questions. These classes, taught by the same professor, adopted an inquiry-oriented approach to defining in the sense that the class treated definitions as "under construction." The class consistently discussed how and why mathematicians define concepts in the way they do. My findings indicate that even after certain mathematical expectations became "taken-as-shared" norms, students display individual differences and shifts in their perception of the source of the expectation: from external, to generalized, to internal. These categories extend the psychological lens of the Emergent Perspective as it relates to students' mathematical activity beyond the classroom.

The Emergent Perspective framework

The Emergent Perspective is especially useful for providing rigorous and actionable means of defining what it means for a collective (members of a class or small group) to "know" something as well as characterizing essential differences between classroom cultures. Drawing from the interactionist perspective of sociology, the framework recognizes that the behavioral regularities of culture are constituted through interpersonal interactions (Blumer, 1969; Prus, 1994). While no one person creates culture, participation in cultural practices simultaneously reinforces those cultural elements. The emergent framework identifies three broad categories of cultural elements pertinent to the mathematics classroom:

1. taken-as-shared expectations for participating in the classroom are *classroom norms*,
2. taken-as-shared criteria by which participation is deemed acceptable are *sociomathematical norms*, and
3. taken-as-shared mathematical activities and associated meanings are *classroom mathematical practices* (Cobb, Stephan, McClain, & Gravemeijer, 2001).

The current study focuses on the first two categories. Examples of classroom norms include explaining and justifying one's reasoning (Cobb, Wood, Yackel, 1993) or presenting proofs and answering questions about them (Fukawa-Connelly, 2012). Examples of sociomathematical norms include mathematical criteria for determining what constitutes acceptable explanation, justification (Yackel & Cobb, 1996), or proof (Fukawa-Connelly, 2012).

Classroom norms begin as expectations for mathematical activity that are endorsed by members of a classroom. Once members of a classroom community: (a) act in accordance with the expectation, (b) respond against breaches of the expectation, and (c) cease to question adherence to the expectation, then that expectation is "taken-as-shared" among that community. Norms that satisfy these criteria can be considered "known" within the classroom not because of their presence or establishment within individual minds, but rather because of their interactively legitimized functionality in classroom activity.

The Individual Lens of the Emergent Perspective Framework

Many researchers link elements of collective understanding to elements of individual understanding such as students' beliefs and community values (Cobb et al., 2001). Fukawa-Connelly (2012) investigated classroom norms in an undergraduate abstract algebra class both via classroom observation and delayed student interviews, and found a large level of agreement between students' descriptions of classroom norms and his analysis of classroom videos. Levenson, Tirosh, and Tsamir (2009) instead noted differences between teacher and student perceptions of classroom norms. However, fewer studies compare classroom mathematical culture to students' individual activity outside the classroom. This is because the Emergent Framework was developed for classroom instructional design (Yackel & Cobb, 1996; Cobb et al., 2001). But keeping with the Emergent Perspective's resistance to dichotomizing individual and collective learning, social constructs must inform or situate the investigation of individual learning. The current investigation employs the Emergent Framework in this alternative way.

One question not clearly addressed in prior literature is, "What is a norm prior to it becoming normative?" By standard definitions it would be a contradiction to say one person introduced a norm because norms must be collectively constituted. I use the term *expectation* for a not-yet-taken-as-shared norm, which is somewhat novel though it appears in the Emergent Perspective literature along with terms such as "obliged" (Cobb et al., 2001; p. 133). If students were "obliged" (a form of "obligated") to act in a certain way, this expresses a mutually understood expectation. Once an expectation is mutually understood, it is taken-as-shared. However, another question that arose from analysis of my data is, "By whom is one expected to act in certain ways?" Even if students act to satisfy an expectation and/or reinforce the expectation to one another, they may attribute the expectation to a particular source. I thus distinguish the source of an expectation (or "locus of expectation") from its "taken-as-shared" status in the classroom.

Reform-oriented norms and student autonomy

The literature on sociomathematical norms consistently points to a link between inquiry-oriented norms and students' sense of autonomy and authority in their own learning (Cobb et al, 2001; McClain & Cobb, 2001; Yackel & Cobb, 1996). When students are allowed to negotiate the criteria by which their

activity is mathematically assessed, they may become a "community of validators" (Yackel & Cobb, 1996) rather than relying fully on the external sources. In such cases, students often display greater intellectual autonomy (McClain & Cobb, 2001), which is a direct goal of many instructional reform efforts (Yackel & Cobb, 1996).

Relevant literature

The current study was conducted in undergraduate, real analysis classrooms and thus within the instructional tradition of "advanced mathematical thinking" (Tall, 1991). Being a proof-oriented course, real analysis is part of mathematics majors' enculturation to the mathematical community and its associated processes (i.e. defining, conjecturing, proving). In the classroom, these processes are expressed and guided by norms of classroom activity. The following literature provides insights into the teaching and learning of these mathematical processes.

Students' use of definition

A survey of the definition literature reveals a recurrent dichotomy between two primary ways in which students reason about categories: either a definition describes a preexistent category or it constitutes the set of all exemplars satisfying its conditions. So, either a category suggests a defining property or the property determines the category members. Several studies (Edwards & Ward, 2008; Alcock & Simpson, 2002) observe that while mathematicians act as though definitions are in in the latter category (*stipulated* definitions), many students instead rely primarily on intuitive notions or prototypes (*extracted* definitions). Mathematicians' emphasis upon reasoning from the formal definition stems from the fact that "appropriate use of the definition means that any correct deductions he makes will be valid for all members of the mathematical category" (Alcock & Simpson, 2002, p. 32).

Engaging students in producing or assessing defining activities

Though a student's untrained use of definitions differs from a mathematician's, several studies indicate that students asked to produce or assess definitions often identify standard values for defining. (Zaslavsky & Shir, 2005; Zazkis & Leikin, 2008). These values include clarity, elegance, non-redundancy, and arbitrarity (non-uniqueness). These studies suggest that defining activities simultaneously reveal and help reorganize students' conceptions about defining.

Methods

As this study intended to relate elements of classroom culture to students' independent work outside of class, I simultaneously developed models of classroom activity and students' individual activity. Toward this end, I gathered data including: detailed notes from all class meetings, bi-weekly professor interviews, and weekly student interviews with a group of 4-6 volunteers per semester (two semesters of data are analyzed).

Professor and instructional context

The professor observed in this study is a tenured mathematician at a large, comprehensive, public university in the Southwestern United States. The professor received multiple teaching awards based on student and colleague nominations, including from her previous real analysis students. The classes met for 75 minutes twice per week for 15 weeks. Professor interview questions generally related to: her intentions and expectations for class sessions, her reflections upon class discussions, and her understanding of students' reasoning (to which she paid a great deal of attention). These interviews informed my emerging model of her instructional practice based on her class activities and teaching actions. The flow of the classroom discourse would best be described as a highly interactive lecture with the professor consistently guiding the conversation (Dawkins & Roh, 2011). Though extended student-to-student conversations were less common, the professor consistently re-voiced and championed students' ideas or played devil's advocate for ideas that she found reflective of commonly held misconceptions.

The classroom norms were reform-oriented in the sense that the professor treated elements of the real analysis theory as "under construction". I identified three types of activities by which she invited students to create and assess definitions: defining (a) *portrayed*, (b) *discussed*, and (c) *enacted*. In *defining portrayed* activities, the professor asked students to compare provided definitions against intuitive ideas

or groups of examples. The group accepted or rejected the definitions based on the emerging shared criteria and examples. In *defining discussed* activities, the professor prompted students to reflect on how and why they defined (see Dawkins, 2012 for a detailed description of one such activity). She invited students to question her and the mathematicians "who wrote the books" to reflect upon definers' intentions. In *defining enacted* activities, students had to produce definitions to turn in or present to the class, often with heavy scaffolding from a previous definition. Through these activities, the professor endorsed the norm that "Students are obliged to create and assess mathematical definitions or conditions within definitions" (or the "norm of defining" for brevity).

Interview participants and analysis

During two semesters of one professor's real analysis course, I solicited a small group (4-6) of volunteers to participate in a sequence of 7-9 weekly interviews regarding their learning in the course. I analyze the activity of seven students (5 from semester 1 and 2 from semester 2), based on the number of participants who provided sufficient data for categorization relative to the research questions. The interviews (a) invited students to reflect on and explain parts of the classroom discussion and (b) documented student work on homework tasks to simulate their independent learning. The professor's *defining discussed* activities aided the study because the first type of interview question naturally extended the classroom dialogue. For instance, after spending a class period on the notion of cluster point without providing a ratified definition, the professor asked whether the class preferred that she provide a definition up front. She drew attention to her intentions in discussing these conceptual issues rather than dispensing pre-formulated knowledge. When I asked questions about this same practice, students appeared to report their reflections about the class dialogue in response to the classroom discussion (or in response to the professor) rather than as an artificial product of the interview environment.

All interviews were audio recorded, transcribed, and coded using a grounded theory open-coding protocol (Strauss & Corbin, 1998) in the NVivo program for qualitative data analysis. I analyzed and coded all cases of students' classroom and interview activity related to the process of mathematical defining. The categories related to a student's: (a) personal concept definitions (PCD), (b) understanding of examples or concepts related to key definitions, (c) ability to produce novel limit definitions (of the form $\lim_{x\to\blacksquare} f(x) = \blacksquare$), (d) choices regarding the professor's defining activities or challenges to her defining choices, (e) perceptions of the values of mathematical defining, and (f) sense of vestment in defining. The emergence of the last three categories motivated the current report. Categories (d) and (f) revealed some students' sense of intellectual autonomy and authority. The students' autonomy often appeared in tandem with particular patterns in their perception of the defining activity as shall be discussed.

Results

Analysis identified three primary patterns of students' adherence to the norm in their individual mathematical activity. These patterns of adherence are not mutually exclusive, being that one student shifted patterns over time, but they differ according to students' (a) understanding of the intent of defining activities, (b) beliefs about the nature of mathematical definitions, and (c) their locus of expectation for the norm of defining. I define *adherence* as a student's individual behavior in relation to a classroom norm, which is the psychological correlate to the collective notion of *participation*. By "behavior," I include cognitive activity in the classroom and isolated activity outside the classroom.

Non-adherence

Vincent was the only participant who clearly displayed a pattern of *non-adherence to the norm of defining*. *Non-adherence* is defined as a willing abstinence from mental or enacted defining despite understanding of the norm of defining. Soon after a lecture in which the class discussed defining function limits without a ratified definition, Vincent and the interviewer (also the author) had the following interchange (March 26):

I: Usually we talk one class period about a subject, and then we define it the next class period […] Why do you think she does that?

> *V*: Maybe cause she wants us to all try and develop our own definition of it, then once we come to the next class, she is going to show us the definition that is accepted by the people who argue over the definition. I don't know… I mainly just wait for her to define it for me.
> *I*: Why is that?
> *V*: Because, I don't know. I guess, lack of motivation to do it on my own because I know she is going to do it. […] When she finally gets us set with a solid definition, sometimes I will look at the other stuff that she was talking about and try to relate 'em and everything. I don't go and find out on my own so much.

Vincent understood the expectation for students to reason about defining, but his perception of the activity kept him from adhering to the norm. Vincent's locus of the expectation was the professor. Though the professor invited students to consider various definitions, Vincent thought of "the definition that is accepted" as a single answer he wanted the teacher to provide. This is more in line with traditional norms of advanced mathematics instruction (Weber, 2004). Vincent located the source of mathematical authority outside of himself, not identifying himself with the "people who argue over the definition." He also expressed lack of motivation or necessity for adhering to the norm. Vincent was the only study participant who had to retake the course.

Peripheral adherence

Over time, Vincent shifted his pattern of adherence to the norm. On April 8, he reported (after a *defining enacted* activity), "She had us try to define in class, which I thought was pretty interesting […] And that was kind of fun, I liked that cause […] I don't really sit there and try to define it because I would rather just look it up, and it was actually different trying to define it and getting pretty close to what was in the book." Vincent consciously noted his shift from *non-adherence* to *peripheral adherence* to the norm. Peripheral adherence is defined as mental or enacted participation in defining as a pedagogical activity toward the end of learning pre-existing mathematical content. Vincent maintained an external sense of authority comparing his defining activity "to what was in the book", but Vincent showed a strong attitudinal shift toward adhering to the class' expectation on his behavior. He later reported that his shift toward independent defining grew from frustrations with having to depend upon his classmates' understanding.

Four of the study participants (Aerith, Celes, Tidus, & Vincent) exhibited *peripheral adherence* to the norm of defining. These students interpreted the defining activity as didactical in nature. Accordingly, they often retained a view that the formal definition was fixed by outside authority, and that they needed to understand those definitions. However, *peripheral definers* generalized the locus of expectation to the classroom. For instance, Tidus described the teachers' intentions in defining portrayed and enacted activities saying: "Instead of just being given information and trying to regurgitate, you're [trying to] see how you got to that point and then it seems to stick in your head more when it's done that way" (Dec 12). Later, he described the class' defining activities saying,

> "We didn't really have anything concrete that we knew without a doubt was true, we were just thrown some ideas, given some actual applications or examples. […] And then all of a sudden we get to the definition, and now it's not just this bland definition sitting on the board. We actually see why it's, why that's being defined, what it's used for, and there's a different appreciation for it. [When the professor poses T/F questions] we are gonna' actually test our and see if we actually understood it or not. And then the homework, of course we get to work out of class and use it some more, try to understand it some more. And then those [expletive] tests to see if we actually got it or not." (Dec 12)

Tidus' comments exemplify peripheral adherence in several key ways. First, he centers his justifications of the activity in terms of improved understanding and recall. He interprets the sequence of classroom activities (discussing, defining, true-false questions, homework, exams) as a connected means of developing and assessing learning. Tidus thus frames (Goffman, 1974) the activity within a didactical structure. Next, though he used a singular pronoun during the first quote ("you") to describe recall, he

also regularly used a plural pronoun ("we") indicating a sense of communal activity and expectation in the classroom. Though the norm was *taken-as-shared,* Tidus' locus of expectation was generalized to the whole class. This differs from Vincent's identification of the professor as the source of expectations. However, *peripheral definers* tended to maintain an external sense of authority, consistent with the didactical frame. Their perceived role in defining is as learners of mathematics known or developed by experts.

Authoritative adoption

Three of the study participants (Cyan, Edgar, & Ronso) displayed *authoritative adoption* of the norm of defining. Authoritative adoption is defined as mental and enacted participation in the norm in accordance with an internal sense of authority in their mathematical activity. These students framed defining as their participation in advanced mathematical activity, implying that they were members of the mathematical community. Authoritative adoption evidenced itself by students' tendency to use the personal pronoun ("I") with respect to their defining activity and their willingness to contest the professor's mathematical claims. Both such behaviors suggest an internalized locus of expectation. All three authoritative adopters explicitly questioned the professor's defining choices based on some meta-mathematical criteria. For example, both Cyan and Ronso expressed dissatisfaction with the professor's choice to include the textbook's requirement that function domains must contain some interval (a, ∞) to have a limit of the form $\lim_{x\to\infty} f(x) = L$. Both students asked why functions defined on the rational numbers or the integers could not have such a limit. They argued instead that the book's definition violates the criterion that mathematical definitions should be sufficiently general within the local body of theory. These students also seemed to properly coordinate between treating definitions as *extracted* while defining and *stipulated* while proving. Like peripheral adherers, adopters said defining helped them learn. However, they acted primarily within a mathematical frame rather than an instructional frame. Adopters were the top-performing study participants in the courses.

Discussion

Summary of categories and constructs

While all students appeared to perceive the shared nature of the norm of defining, students differed in their locus of expectation, sense of authority, and their frame for the defining activity. The non-adherer Vincent located both the expectation and mathematical authority with the professor or other experts. Peripheral adherers like Tidus generalized the locus of expectation to the classroom viewing defining as a shared practice. However, they maintained an external sense of authority consistent with their pedagogical frame for the shared defining activity, which positioned them as learners of the mathematical understandings held by the professor. Authoritative adopters like Ronso or Cyan exhibited an internalized locus of expectation and source of authority, evidenced by their willingness to challenge the validity of the professor's defining choices. Table 1 organizes the characteristics of each category.

To clarify this diversity of adherence to the norm, I offer three new sub-constructs. For non-adherers, the practice is *taken-as-expected* (TAE), meaning the norm's value is extrinsic (imposed) in the sense of being a means toward the goal of satisfying the professor. For peripheral adherers the practice held intrinsic, secondary value, which I call *taken-as-beneficial* (TAB). In the case of TAB, the norm holds intrinsic (personal) value for the individual, but the benefit is derived from the value for learning and is thus secondary. For authoritative adopters, defining held an intrinsic, primary value, and is thus *taken-as-meaningful* (TAM). Once a student feels personally vested in defining as part of their mathematical identity, they intrinsically value the process itself in addition to its other benefits. These constructs are likely inclusive or nested inasmuch as authoritative adopters acknowledged that defining helped them learn (TAB) and understood the professor's expectations. However, intrinsic values generally prove stronger and more durable than extrinsic values suggesting why authoritative adopters displayed more detailed defining activity outside the classroom context. These finer distinctions extend and clarify the Emergent Perspective literature's claims that inquiry-oriented norms help promote students' development of compatible individual mathematical beliefs (Cobb et al., 2001; Yackel & Cobb, 1996).

Table 1: Characterizations of Individual Adherence to the Norm of Defining

Category	*Non-adherers*	*Peripheral adherers*	*Authoritative adopters*
Locus of expectation	Professor	Generalized	Internalized
Source of mathematical authority	Professor/ textbook	Professor/ textbook	Self
Frame for defining	Pedagogical (non-beneficial)	Pedagogical (beneficial)	Mathematical
Individual status of the collective norm	TAE	TAE, TAB	TAE, TAB, TAM

Practical contributions

The university classroom structure assumes that a large onus of learning falls on students' time outside of class (traditionally in a 3:1 ratio). This suggests that understanding classroom activity and the culture therein only accounts for a portion of students' mathematical engagement. It thereby seems important to understand how classroom norms for mathematical practice influence students' mathematical activity beyond the classroom. My findings suggest that the classroom environment strongly influences students' individual activity, but the relationship between collective norms and individual action is more complex than direct correspondence. Factors of affect and identity seemed to strongly influence students' adherence to norms. Vincent's shift in adherence may reveal the positive influence of the classroom environment on his mathematical identity, but further evidence is required in this regard.

I also maintain that advanced mathematics instruction should enculturate students into the mathematical community. Therefore, it is important that students engage in and understand key mathematical practices such as defining. While almost every student reported learning from participating in defining, authoritative adopters also identified values of defining by which they contested the professor's defining activity. As such, they displayed rich understandings of mathematical definitions and expressed personal identification with the community of mathematicians. I hypothesize that these students who internalize the locus of expectation are most likely to adhere to and endorse the norm of defining in their future mathematical activity either as students or teachers (the majority of these students proceeded to graduate studies in mathematics or to secondary mathematics teaching). The ability of students to adhere to such practices beyond the influence of the original locus of expectation (the professor or class) seems a pertinent factor for the "durability of sociomathematical norms" (Van Zoest, Stockero, & Taylor, 2011), and thus warrants further study. The individual sub-constructs to *taken-as-shared* identified in this study should facilitate the continued investigation of such phenomena.

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STRATEGICALLY CHOSEN EXAMPLES LEADING TO PROOF INSIGHT: A CASE STUDY OF A MATHEMATICIAN'S PROVING PROCESS

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Examples play a critical role in the exploration and proving of conjectures. Although proof has been studied extensively, the precise ways in which examples might facilitate successful proofs are not well documented or understood. Working within a larger set of studies that argue for the value of examples in proof-related activity, in this paper we present a case study of one mathematician's work on a conjecture in which his strategic, intentional use of examples led to a proof of that conjecture. By examining his work in detail, we highlight specific mechanisms by which the mathematician's examples led to successful proof production. These mechanisms shed light on precise ways in which examples can directly lead to proof and inform our understanding of the conceptual landscape of the interplay between examples and proof.

Keywords: Advanced Mathematical Thinking, Reasoning and Proof, Post-secondary Education

Introduction

In much of the current literature on teaching proof in school mathematics (e.g., Harel & Sowder, 1998; Stylianides & Stylianides, 2009; Zaslavsky, Nickerson, Stylianides, Kidron, & Winicki, 2012), example-based reasoning is framed as a limited way of thinking that students should avoid. It is true that ways of reasoning about proof that overly rely on examples can be problematic, particularly if examples are perceived by students as valid substitutes for proofs. We posit, however, that these potential limitations of examples have become so prominent in the proof literature that potentially positive roles of examples in proof have not been sufficiently explored. We suggest that the exploration of productive roles of examples in proof offers significant untapped research potential. Given that examples seem to play an essential role in mathematicians' exploration of conjectures and subsequent proof attempts, example-based reasoning strategies can lead to meaningful opportunities for research, illuminating ways in which examples might be used to support students' proof development.

The work presented here builds on recent attempts to learn more about the roles that examples can play in exploring and proving conjectures across a variety of levels (Ellis, et al., 2012). In this paper, we present a case study, examining in detail one mathematician's use of examples in developing a proof of a novel (to him) conjecture. We highlight specific mechanisms through which examples were used as the mathematician formulated a proof. By narrowing our analysis to one mathematician, we can provide significant mathematical detail, allowing for key insights to be shared about how the mathematician leveraged examples in constructing a proof. Through the case study we seek to inform our understanding of the relationship between examples and productive proof-related activity, and we discuss potential implications for the teaching and learning of proof.

Literature Review and Theoretical Framework

Weber (2008) states that, "investigations into the practices of professional mathematicians should have a strong influence on what is taught in mathematics classrooms" (p. 451). A number of other researchers have studied mathematicians' practices under the assumption that their work might provide valuable insight about these practices more generally. For example, Carlson and Bloom (2005) examined mathematicians' problem solving, and Savic (2012) used innovated technology to study what mathematicians did when they reached impasses in their proving processes. In the domain of examples, Epstein and Levy (1995) contend that "Most mathematicians spend a lot of time thinking about and analyzing particular examples," and they go on to note that "It is probably the case that most significant advances in mathematics have arisen from experimentation with examples" (p. 6). Alcock and Inglis (2008) similarly argue that there is often a complex interplay between mathematicians' example-based reasoning activities and their deductive reasoning activities. Several mathematics education researchers have accordingly examined various aspects of the relationship between example-based reasoning and deductive reasoning activities among both mathematicians and mathematics students (e.g., Antonini, 2006; Buchbinder & Zaslavsky, 2009; Ellis, et al., 2012; Iannone, Inglis, Mejia-Ramos, Simpson, & Weber, 2011; Lockwood, et al., 2012, 2013).

The case study presented in this paper builds directly on the findings by Lockwood et al. (2012) in which 219 mathematicians responded to an open ended prompt on a survey: *If you sometimes use examples when exploring a new mathematical conjecture, how do you choose the specific examples you select in order to test or explore the conjecture? What explicit strategies or example characteristics, if any, do you use or consider?* Results of the responses produced a framework for types, uses, and strategies of examples (Lockwood et al., 2012), but here we emphasize one particular use of examples that arose in the mathematicians' responses: proof insight. Fourteen of the mathematicians' responses (5.6%) indicated that they use examples in order to gain insight about how to develop a proof for the conjecture. In addition, in the interview study from which this case study is drawn, there were 23 out of 228 total instances (10.1%) in which mathematicians similarly made statements about examples serving an important role in proving conjectures. While there is evidence that mathematicians might use examples to gain insights into proofs, little is known about the precise mechanisms by which this may happen. We thus build upon such prior work on mathematicians and proving by emphasizing in particular the role of examples as a mathematician produces a proof.

Additionally, the conjecture discussed in this paper is adapted from Alcock & Inglis (2008), in which they discuss different ways in which doctoral students used examples in working with conjectures. Our findings build directly upon their work, as we seek to draw a more in-depth picture of precisely how a mathematician's choice and use of examples helped contribute to his successful proof production.

Methods

The case study presented in this paper is from an interview conducted with a mathematician, Dr. Felton (a pseudonym), as he explored and attempted to prove three mathematical conjectures (due to space, only Conjecture 2 is provided, see Figure 1). Dr. Felton, currently an associate professor in a university mathematics department, received his PhD in mathematics with an emphasis in mathematics education; his area of mathematical expertise is algebra. After working on each conjecture, Dr. Felton was asked clarifying questions about his work. While he was not

given unlimited time (he had approximately 15-20 minutes to explore each conjecture), Dr. Felton was able to sketch of a proof of Conjecture 2 (parts a and b) and Conjecture 3. His work on Conjecture 2 (a task adapted from Alcock & Inglis (2008)) is the subject of this paper.

Conjecture 2
All the numbers below should be assumed to be positive integers.
Definition. An abundant number is an integer n whose divisors add up to more than $2n$.
Definition. A perfect number is an integer n whose divisors add up to exactly $2n$.
Definition. A deficient number is an integer n whose divisors add up to less than $2n$.

Conjecture 2a. A number is abundant if and only if it is a multiple of 6.
Conjecture 2b. If n is deficient, then every divisor of n is deficient.

Figure 1: Conjecture 2a and 2b

All of the interview problems were chosen because they were a) accessible to the mathematicians (regardless of their area of expertise) but not so clearly obvious that they could be proven immediately, and b) accessible to the interviewer, allowing her to follow the mathematicians' work as well as to ask meaningful follow-up questions. While the choice of conjectures did not exactly simulate a mathematician's personal research, the conjectures enabled us to observe what a mathematician might do as he actually explores and attempts to prove conjectures.

The interview was transcribed, and, having identified Dr. Felton's work on Conjecture 2 as involving an illustrative instance of examples leading to proof development, a member of the research team carefully analyzed Dr. Felton's work on Conjecture 2 in particular. The process involved repeated viewings of his work on this conjecture, identifying and characterizing precise ways in which Dr. Felton used specific examples in develop a proof. Findings were discussed and refined during meetings with other research team members.

Results and Discussion

In this section we present a detailed account of Dr. Felton's work on Conjecture 2, including some analytic discussion; in the subsequent Conclusion section we synthesize the results and highlight salient aspects of his proving process. This section is broken into three subsections: *initial exploration*, *targeted exploration*, and *sketching the proof*.

Initial Exploration

Dr. Felton first solved Conjecture 2a, identifying 6 as a counterexample. He stated that he was not familiar with perfect numbers but had recently thought about 6's property that its proper factors sum to itself, which he learned was the definition of perfect. His choice of counterexample here was relevant, as 6 would be a key aspect of his exploration and proof of Conjecture 2b.

After he initially read Conjecture 2b, his reaction was to think it might be true, because "it's hard for me to imagine a counterexample." Having made this assertion, he began a symbolic proof (in line with what Weber and Alcock (2004) call *syntactic* proof production), writing down expressions such as $a*d=b$ and denoting a as having factors $f1, f2, f3$. However, he did not successfully generate a proof and decided that he would try some examples, saying, "This is going to be an interesting one. I feel like I might want to play with a few examples."

The interviewer asked Dr. Felton both why he had initially attempted a symbolic proof, and why he had shifted from that proof attempt to working with specific examples. He explained these decisions in the excerpt below.

Dr. Felton: Well, probably at first I was, well, partly I started with the algebra because…I was already feeling like it was probably true. And I was hoping maybe I'd get lucky and something would jump out at me, the strategy for proving it…And then the real reason why I went after it with examples, not so much that I thought these would be counterexamples, as I thought they would be good test cases, and they'd maybe give me a feel for how, more information as to maybe why this is true.

Since Dr. Felton suspected the conjecture was true, he had hoped that a strategy for a proof would emerge from the algebraic manipulation. Also, he chose specific examples as test cases, hoping they would inform *why* the conjecture might be true. We address what he meant by "test cases" below.

In the following excerpts we follow Dr. Felton's work as he studied a specific example, 12 (6*2=12). What is noteworthy about this work is the care with which he chose this and subsequent examples, and also how he attended to particular mathematical features of the examples. Specifically, he noted that choosing an example that had a perfect factor (such as 6) was judicious because the factor was barely not deficient.

Dr. Felton: I'm curious about 12, now. So, let's see, 2 times 6 equals 12. We know 6 is perfect, which doesn't necessarily make it deficient. It's not deficient…so actually it's a good choice for a potential counterexample, because it's not deficient, but it's not far from being deficient… So, if something's going to work, my, I guess a good way to do it would be to use a perfect factor.

Interviewer: Okay. As opposed to an abundant one?

Dr. Felton: As opposed to an abundant one…'cause then you're going to get tons of stuff which is going to make it harder for you to not make up the difference with the extra.

He would later go on to say that by choosing an example with a factor that was perfect (as opposed to abundant), he was targeting a "test case," or a "boundary case." He described the perfect number as being a boundary situation, "I mean, it's definitely the boundary case here 'cause you have two definitions and perfect is in the middle of the two of them." He later said, "I mean, my emphasis, my focus will probably always be on a boundary case in that kind of a situation… Like what's that critical point when you switch from being one to the other." These comments exemplify Dr. Felton's strategic choice of examples in helping him understand the conjecture. Indeed, Dr. Felton situated the examples he chose within the larger picture of what he was trying to accomplish. We have evidence that by choosing 12 (which has a perfect factor) he was not actually trying to locate a counterexample, but rather to gain insight into the conjecture: "Yeah, I'm mostly, just to kind of play with it. Like, I don't, I really have no belief that there could be a counterexample. But I'm pretty sure this one is true." He acknowledged that he likely would not find a counterexample, and he recognized that while he might not break the conjecture, he could potentially gain insight into the situation, and possibly formulate a proof, by exploring the boundary case 6*2=12.

In working through this example, Dr. Felton wrote out the factors of 12. Below, we see that he made an important observation about this example, noting a specific property, namely that his factor of choice, 6, was exactly half of 12.

Dr. Felton: So 12 has 1, 2, 3, 4, 6. And it also includes 12 in this calculation. Okay, so, so we're looking at this going 1, 2, 3, 6. Um, those guys are giving me the 12, oh, wait a minute. Hold your horses, here. Okay, is that just 'cause it's a half, though?…You're already getting half way there with that one [the 6]. This isn't going to be very helpful when we do, there's

no way this is going to be deficient… because there's already a 12 in that one. This, this conjecture is feeling quite true.

Dr. Felton suspected that the conjecture was true, and his work with the example 12 seemed to have confirmed that, but he became aware of this special property that he identified, and he was cognizant that he perhaps had come upon a special case. We interpret that he noted that for his example e, if a factor f is perfect and is half of e, then because it is perfect all of its factors will sum to f. But f is also already somewhere in the list of e's factors, and this guarantees that the factors of e will already sum to at least twice f (which is e), making e not deficient. This realization that he had chosen an example with a perfect factor that was exactly half of the number led him to another strategic example choice, and he noted "I'm just going to kind of play with one more, see if it's a generalizable phenomenon."

Targeted Exploration

We call this section *targeted exploration*, as his observation about 6*2=12 resulted in intentional examples choices that followed a particular line of inquiry. Based on properties he identified about his initial example described above, he chose 18 (6*3=18) and again paid attention to the factor that was half of his example, in this case 9. Interestingly, in the excerpt below, Dr. Felton made a technical error in his reasoning, because 9 is actually deficient (its proper factors sum to 4, which is less than 9). However, his observation in the excerpt is still insightful, and it played an important role in his subsequent example choice. The point to glean from the following exchange is his observation that if half a number is already not deficient, then the original number will not be deficient.

Dr. Felton: Because as long, it's, if it's [the factor that is half of the original number] not deficient, that guarantees that your number is not deficient, so it's like a contrapositive thing going on. Right, so nine is not deficient, it's going to have, the sum of its factors is going to give you two times nine.

Interviewer: Oh, I see. Yeah. So you're saying it's an even number and half that number is already not deficient…

Dr. Felton: Right. Then the original won't be deficient.

In spite of his error, this was a key observation, and it led him to choose an example with a particular property. The excerpt below shows that his work with 12 and 18 made him want to choose an example, "something like 6 times 11. I want to have a big, like, not much stuff between 6 and the whole number." We see, then, that his initial choice was based on 6 being perfect, and he chose 6*2=12 as a test, or boundary, case. Then, in examining that example he noticed the special property that arose because 6 was half of 12. This led him intentionally to choose an example in which 6 was not half of the original number, and he selected 6*3=18 to see if his observation is a "generalizable phenomenon." He similarly scrutinized this example, and the way in which 9 interacted with 18 made him move on to select an example that did not have "much stuff between 6 and the whole number." Based on this criterion, he chose 6*11=66, and this ended up being the example that would provide the key insight for the proof.

Having justified his next selection of 6*11=66, Dr. Felton then spent time investigating this chosen example, first writing out factors of 6*11=66. Then, as he studied the factors of 66 (see Figure 2), he made the following observation. As his last sentence suggests, his observation about the "duplication of the perfectness of 6" was enough to make him think that he could come up with a proof.

Dr. Felton: It's almost like you get, like a duplication of the perfect-ness of 6 that shows up in this piece here.

Interviewer: Okay, how so?
Dr. Felton: So, so, like this 1, 2, 3 adds up to 6…11, 22, 33 actually adds up to 66. So I'm feeling like I probably ought to be able to prove that this is a true statement.

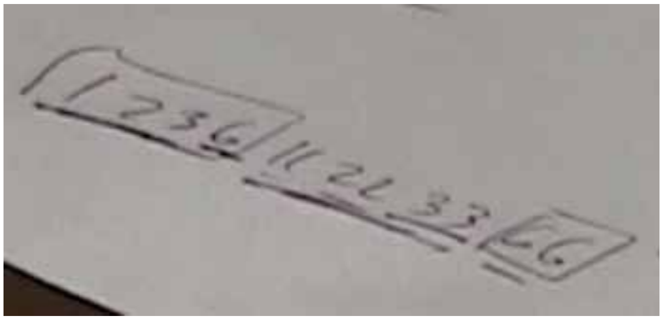

Figure 2: The Perfect-ness of 6 in the Factors of 66

Sketching the Proof

As alluded to previously, Dr. Felton was trying to prove the contrapositive of Conjecture 2b. For the sake of space we briefly summarize his proof, but Figure 3 shows him writing out the sketch of proof and simultaneously referring back specifically to his example in doing so. (In his notation, *b* is the number chosen as the example (*b*=66), *a* is the factor of interest (*a*=6), and *d* is another divisor of *b* (*d*=11)). He was able to prove the contrapositive, arguing that that if a factor of *b*, *a*, is not deficient, then it has factors *f1* through *fk* whose sum is greater than *a.* Then *d*f1* through *d*fk* must also be factors of *b* that are distinct from those factors of *a* (and since *f1* through *fk* are strictly less than *a, d*f1* through *d*fk* must be distinct from *b*). He noted that the sum of *d*f1* through *d*fk* must be greater than or equal to *d*a*, which is itself already a copy of *b*. The sum of *b*'s factors, then, includes *d*f1* through *d*fk,* which is greater or equal to *d*a*, and *b* itself. This is greater than or equal to two copies of *b*, and thus by the definition of deficient, *b* itself cannot be deficient.


Figure 3: Referring to the Example in Writing the Proof

What is most interesting to us is not that he proved the conjecture, but rather the precise role that his example 6*11=66 played in his development of this proof. In reflecting on his proof, he made several statements that highlighted the importance of the example. Specifically, the nature of the multiplication by 11 allowed him to see that certain factors (the multiples of 6) would show up in the complete list of factors. While this is a property that he asserted, "is clearly always going to work out," he acknowledged that the nature of the number 11 made that particularly salient for him. This focus on structure is seen in the following excerpt.

Interviewer: You said something about, maybe, start, being able to start a proof based on that observation you found in that six times eleven…

Dr. Felton: Right, which is kind of what I was thinking of was that the perfect-ness of the, um, 6 is basically copied, replicated by these multiples. Each multiple of the factor of 6. So I've got my 1, 2, 3, 6 here. Right now the 1, 2, 3, adds up to 6. And then multiply each of those by 1, those are also in my list of factors, and I add those up, I get 66. Which, kind of is clearly always going work out…But then when I wrote it down it was actually quite helpful because, I mean, I was also benefited by the choice of 11… Because you multiply by 11, it looks very much like the number you started with before you multiplied by 11… And so, it was much more transparent that the structure on this 11, 22, 33, 66 mirrored the 1, 2, 3, 6.

Conclusion and Implications

In this section we highlight three aspects of Dr. Felton's use of examples as he developed his proof of Conjecture 2b, and we suggest that these specify particular ways in which examples can be leveraged in the proving process. First, Dr. Felton demonstrated that he chose examples in terms of the larger task at hand. He explored a boundary case as a potential counterexample not to attempt to disprove the conjecture, but rather to gain insight into the situation. This activity suggests that he had an understanding of the logical structure of the conjecture, and by looking at an example that would not satisfy the conclusion he was setting himself up for a proof of the contrapositive.

Second, Dr. Felton drew upon particular mathematical properties of his examples, and these led him to strategic subsequent example choices. As described above, the sequence of examples he chose were based on careful mathematical analyses of each example. This activity suggests that Dr. Felton did not choose his examples thoughtlessly, nor did he use his examples merely to check whether the conjecture might be true. Rather he intentionally chose examples, carefully examined their mathematical properties, and used mathematical insight to understand the situation and inform ensuing example selection.

Third, Dr. Felton leveraged the structure of one insightful example in formulating a proof of the conjecture. The nature of the number 11, and the way in which it multiplies, allowed him to recognize in the particular example a key insight about the proof – namely that for all the factors $f1$ through fk of a, $d*f1$ through $d*fk$ would also be factors of b (for $a*b = b$). While he granted that the choice of 11 was fortuitous, he capitalized on this important structure and was able to translate it into the more general argument. As we see in Figure 3 above, he specifically aligned his writing of the proof with the structure of that particular example. This is an instance in which an informed example choice directly affected proof development.

We have previously argued for the value of strategic example choice and progressions of examples, and we have made the case that example structure can play a part in deductive reasoning (Ellis et al., 2012). Additionally, mathematicians have indicated that examples can shed light on proof or lead to proof insight (Lockwood, et al., 2012), but there is not much information on how this occurs. Here, however, we have evidence for how such example-related activity led to a proof in a precise way. Dr. Felton made informed decisions in initial example choice, and he targeted examples with certain mathematical properties. He carefully studied the examples and drew out salient insights from each of them. We do not claim that this is how every conjecture is proved, nor is it how Dr. Felton would even prove every conjecture, nor is it necessarily how people always *ought* to prove conjectures. However, this case study contributes to our understanding of the conceptual landscape of the interplay between examples and proof

production, pointing to some mechanisms by which example-related activity might lead to proof. Dr. Felton's work also offers some potential pedagogical insight. Students might benefit from modeling his careful selection of a progressive sequence of examples, or from leveraging the structure of a specific example, as he did. Additionally, Dr. Felton exhibited particular attitudes toward examples that could benefit students – he was willing to explore with boundary cases, was cautious of overreliance on a special case, and always seemed to keep the larger proving process in mind. Such facility with and attitudes toward examples might provide starting points for meaningful pedagogical interventions that positively frame examples in the context of proof.

Acknowledgments

The authors wish to thank the other members of the IDIOM Project team. The research is supported by the National Science Foundation (NSF) under grant DRL-0814710. The opinions herein are those of the authors and do not necessarily reflect the views of the (NSF).

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VISUALLY GROUPING OPERANDS: PERCEPTUAL FACTORS INFLUENCE ARITHMETIC PERFORMANCE

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Perceptual characteristics of mathematical equations may influence solvers' problem solving. For example, in a study of equations involving addition and multiplication, Landy and Goldstone (2010) showed that participants tended to perform narrowly spaced operations first, suggesting that spacing affects how symbols are grouped for problem solving. Building on this past work, we examined whether perceptual factors affect participants' interpretations of the minus sign. In an experiment with undergraduates, we manipulated the spacing of the operands and the position of the minus sign relative to its neighboring operands. Both the operands' spacing and the position of the minus sign affected performance. These results hold implications for the processing of symbolic representations and for mathematics education.

Keywords: Number Concepts and Operations, Problem Solving

In the abstract symbol system of mathematics, individual numerical and operational symbols are combined into more complex mathematical expressions. These combinations take place in a two-dimensional representational system, in which spatial features may be relevant or irrelevant to formal mathematics. For example, a relevant spatial feature would be the ordering of the digits and operators in a given mathematical equation—because mathematically correct solutions to multi-operation arithmetic problems are reached by following mathematical rules, such as the order of operations. Similarly, the size and spatial location of a superscripted digit to indicate an exponent differentiates its mathematical meaning from the digit used as the base. In contrast to these examples, there is no difference in the formal mathematics of an expression when the horizontal spacing of the symbols is varied; correct application of the order of operations rule should yield the correct answer, regardless of how the equation is horizontally spaced.

However, some recent evidence suggests that, despite its formal irrelevance to mathematical meaning, horizontal spacing affects problem solvers' solutions. Landy and Goldstone (2007a) found that the amount of space surrounding operands influenced participants' judgments of the equality of two mathematical expressions. For example, undergraduate participants were more likely to agree with the statement "Is a + b * c + d = b + a * c + d necessarily true?" when the multiplication operations were more widely spaced than the addition operations (i.e., a + b   *   c + d = b + a   *   d + c), compared to when they were equally spaced. Participants made more errors when spacing was inconsistent with the mathematically correct order of operations (as discussed in Landy & Goldstone, 2007b).

Effects of spacing also are present when people construct mathematical expressions. Evidence from problem transcriptions show that participants implicitly follow spatial patterns when converting equations from word form (e.g., three plus five times two) to number form (e.g., 3 + 5 x 2). Participants spaced multiplication signs more narrowly than plus signs, signifying the precedence of multiplication (Landy & Goldstone, 2007b).

Effects of spacing on performance are also apparent in the time it takes for participants to solve problems (Landy & Goldstone, 2010). Participants solved equation structures of the form

"a + b x c" or "a x b + c" faster and more accurately when the multiplication operation was spaced more narrowly than the addition operation (e.g., a + bxc or axb + c, as compared to a+b x c or a x b+c). Participants displayed higher accuracy on problems with spacing that was consistent with the order of operations rules, and they also processed those equations more quickly.

These findings indicate that, irrespective of its formal irrelevance, the horizontal spacing and proximity of operands and operations can influence people's interpretation and understanding of mathematical expressions. Moreover, it appears that the relationship between horizontal spacing and arithmetic performance is bidirectional. People's knowledge about the order of operations can influence the spacing they use when writing expressions, and spacing can affect the order in which they perform operations when evaluating or solving problems. Furthermore, these findings may have important implications for educational practice. Instructors may be able to use these perceptual features to their advantage in classroom instruction.

In light of these previous findings, the minus sign (i.e., "–") is a particularly interesting case, because it is used not only to represent subtraction, but also to invert the sign of a number. Consequently, the minus sign is sometimes treated differently from other operation symbols; for example, it is "carried" with the associated operands in algebraic manipulations. This can present challenges for students learning algebra (Cangelosi, Madrid, Cooper, Olson, & Hartter, 2013; Demby, 1997; Vlassis, 2004). In addition, as a mathematical operation, subtraction is more difficult than addition (e.g., Das, LeFevre, & Penner-Wilger, 2010). In this study, we investigate whether formally irrelevant perceptual features can influence how the minus sign is interpreted. If perceptual features influence subtraction, as they do addition and multiplication, our research findings may be important for understanding students' difficulties at the transition between arithmetic and algebra.

In the present research, we investigated the effects of perceptual grouping of operands on participants' interpretations of the minus sign. Participants were presented with multi-operation expressions that involved subtraction. The equations were in the format of "a – b + c x d". We tested whether undergraduates' arithmetic accuracy would be enhanced by supportive yet formally irrelevant perceptual cues consistent with order of operations rules or hindered by misleading, formally irrelevant perceptual cues inconsistent with orders of operations rules.

Building on Landy and Goldstone's previous work (2007a, 2007b, 2010), we varied two aspects of spacing: the closeness of *operand spacing* and the lateral *minus sign position*. Before presenting our experimental method, we describe each of these variations and how they might affect performance. In both cases, the variations we employed were subtle, consisting of only one or two spaces in a normal-sized font.

First, we varied the *operand spacing*, specifically for the operators that followed the minus sign. To do so, we removed the spaces between operands and operations for the last three terms, similar to Landy and Goldstone's (2007a) manipulation. The closer proximity of the symbols may lead solvers to perceptually group these terms, as suggested by the Gestalt principles of visual perception (e.g., Werthmeimer, 1923/1938). Evidence from past research (e.g., Landy & Goldstone, 2010) suggests that visually grouping these symbols should give them precedence in problem solving steps. That is, the narrower spacing of these symbols (e.g., 25 – 3+2x5) could create a perceptual group which might lead participants to construe "3+2x5" as the subtrahend, and mistakenly apply the subtraction operation to this entire quantity. This would lead to the error described in Table 1, which we term a "target error". In erring this way, participants incorrectly evaluate the scope of the minus sign, due to the perceptual grouping.

Second, we varied the *minus sign position,* again based on the Gestalt principle of proximity (Wertheimer, 1923/1938). We horizontally shifted the minus sign (see Table 2 for a visual display) to create different perceptual groups. For instance, in the equation "25 – 3 + 2 x 5 = ____", if the minus sign were shifted slightly to the left, (25 – 3 + 2 x 5 = ____), solvers should be more likely to group the remaining symbols "3 + 2 x 5". This, too, may lead participants to produce the target error, because the minus sign may be applied to the entire quantity of "3 + 2 x 5" because of the perceived group. In contrast, if the minus sign were shifted to the right, solvers should be more likely to group the operator with the subsequent subtrahend of 3. Thus, manipulating the lateral position of the minus sign has the potential to affect the quantity participants treat as the subtrahend.

Table 1: Incorrect Solution Strategy Based on Incorrect Perceptual Grouping

Problem	25 – 3 + 2 x 5 = ____	correct answer = 32
Step 1	multiply: 25 – 3 + 10 = ____	correctly following order of operations
Step 2	add: 25 – 13 = ____	Demonstration of a "target error" with incorrect order of operations, potentially based on perceptual grouping of all symbols to the right of the minus sign
Step 3	subtract: = 12	

Although we could have tested many other perceptual features, we chose these two manipulations because of their implications for how the subtrahend is determined. The *minus sign position* manipulation directly alters the proximity of the minus sign to other elements in the equation; this manipulation relies on the proximity of the operator to its operands to generate supportive or misleading perceptual groups. The *operand spacing* manipulation more subtly affects the how the minus sign is drawn into the subtraction operation; the closeness of the latter terms may again prompt a perceptual grouping that influences a solver's interpretation of the scope of the minus sign by affecting the perceptual group that is treated as a subtrahend.

In sum, we hypothesized that perceptual cues inconsistent with the order of operations rules would encourage solvers to incorrectly group operations together and lead to errors in problem solving. Both the mathematically irrelevant spatial features of minus sign position and spacing of the remaining operands could potentially affect how the scope of the minus sign is interpreted. Despite their formal irrelevance, we expected these manipulations to affect participants' solutions to arithmetic problems involving the minus sign.

Methods

Participants

Undergraduate students (N = 92) in introductory psychology at a large Midwestern university participated in exchange for extra credit. Sixteen additional participants were excluded due to a photocopying malfunction leading to faulty stimuli.

Design and Materials

Participants solved arithmetic problems with manipulated perceptual features. In the target problems, subtraction was the first operation presented the equation, followed by addition, and finally multiplication (e.g., 25 – 3 + 2 x 5).

A 2 (*operand spacing*: evenly spaced or closely spaced) x 3 (*minus sign position*: left shift, no shift, or right shift) between-subjects design was used, yielding a total of 6 conditions. Table 2 displays the outcomes of these manipulations. In the evenly spaced problems with no shift, there were two spaces between each operand and the adjacent operator and two spaces around the equal sign. The left-shift condition was created by reducing the space between the first operand and the minus sign by one space and adding a space between the minus sign and the subsequent operand. The opposite was done for a right shift. For the closely-spaced condition , all the spaces after the second operand were removed, which resulted in no spaces between the last two operators and their three surrounding operands, as seen in the first row of Table 2.

Table 2: Example Target Problem with Shift and Grouping Manipulations

	Minus Sign Position		
Operand Spacing	**Left Shift**	**No Shift**	**Right Shift**
Evenly Spaced	25 – 3 + 2 x 5 = ____	25 – 3 + 2 x 5 = ____	25 –3 + 2 x 5 = ____
Closely Spaced	25 – 3+2x5 = ____	25 – 3+2x5 = ____	25 –3+2x5 = ____

Participants received ten target problems and eight control problems of the form "a + b – c x d". The control problems did not afford the target error that the target problems did. The operands in the equations were randomly generated, with the constraint that both the operands and solutions were non-zero positive integers. The problems in each condition were presented in a fixed order and the same operands were used for any given problem across conditions.

On each of the 8.5" x 5.5" (wide) pages of the packet, there were two math problems, separated by a fill-in-the-blank vocabulary question (as seen in standardized tests such as the SAT) as a filler. Size-12 Calibri font was used for all the questions, and the questions were numbered at the left margin. The packets were constructed such that the pages with the target math problems alternated with pages with the control math questions.

Procedure

Participants were tested in groups of up to five students at a time; each completed his or her packet of problems individually. Each participant was given a question packet with the math problems and vocabulary filler questions. Participants had up to thirty minutes to complete the question packet. Once participants finished the problem set, they were asked to provide information regarding their math and reading backgrounds and other relevant demographic information such as year in school and major.

Results

We scored whether each solution on each target problem reflected the target error or not, that is, whether the participant used a mathematically incorrect but perceptually-based group to solve the problem. Thus, careless arithmetic errors (e.g., 3 + 3 = 9) and errors that involved solving from left to right without consideration of order of operations (e.g., 25 – 10 + 3 x 2 = 36), while mathematically incorrect, were coded as No Target Error.

We analyzed the existence of target errors with a mixed effects logistic regression model using the glmmadmb package (Skaug, Fournier, & Nielsen, 2012) in R. Our two factors, minus

sign position and operand spacing, were treated as fixed effects that were allowed to interact, and participant was a random effect.

Figure 2 presents the mean proportion of trials with target errors in each condition. When the operations that followed the minus sign were closely spaced, participants in the left shift condition produced significantly more target errors than participants in the right shift condition, $p = .028$. The left-shift condition did not yield significantly more target errors than the no-shift condition, although the results trend in this direction. Additionally, within the closely-spaced condition, participants in the no-shift condition produced marginally more target errors than participants in the right-shift condition, $p = .059$. However, in the evenly-spaced condition, there was no difference in the proportion of target errors among the left-shift, no-shift, or right-shift conditions.

Focusing on the effects of the position manipulation, when the minus sign was shifted to the left, participants in the closely-spaced condition produced more target errors than participants in the evenly-spaced condition, $p = .026$. Close spacing also led to significantly more target errors than equal spacing when there was no shift, $p = .038$. However, when the equal sign was shifted right, there was no difference between the closely- and evenly-spaced conditions, $p > .05$.

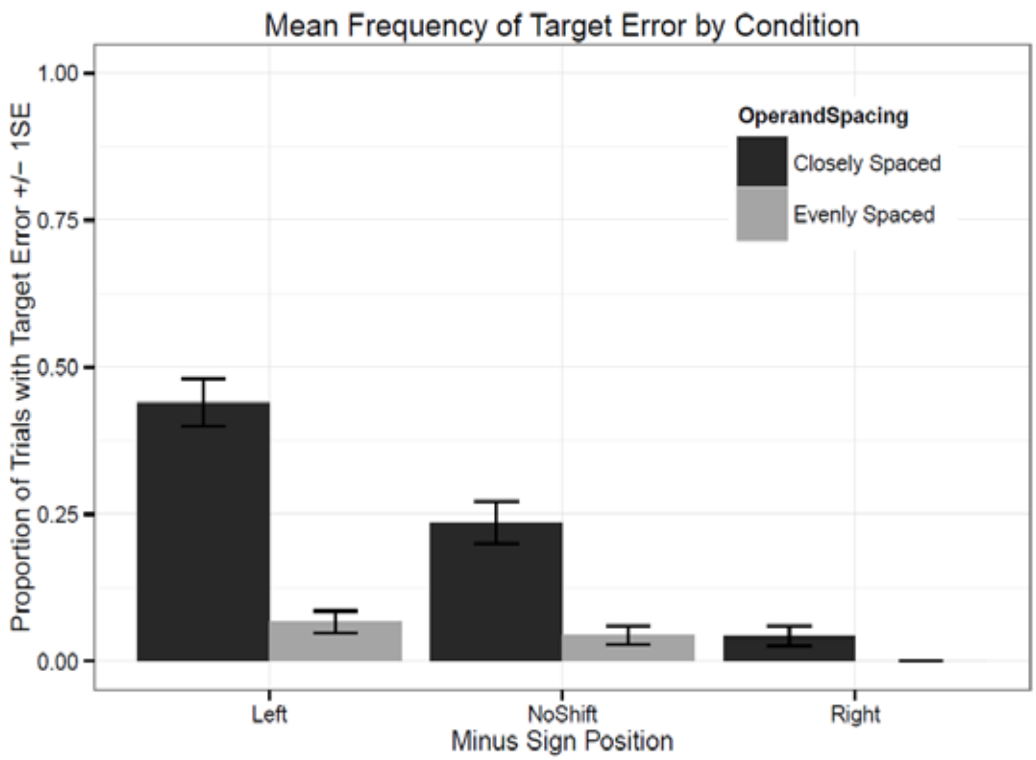

Figure 1: Proportion of Trials with Target Errors by Condition

In summary, target errors were most frequent when the minus sign was shifted left and the operations that followed the minus sign were closely spaced. Closely-spaced symbols were associated with more target errors than evenly-spaced symbols, for both the left shift and no shift conditions. Overall, the proportion of target errors within the closely-spaced condition was affected by the position of the minus sign. On the other hand, target errors were very low when

the operands were evenly spaced, and the proportion of target errors within the evenly-spaced conditions was not affected by the position of the minus sign.

Discussion

This study investigated whether spatial properties of equations would encourage participants to misevaluate the scope of the minus sign. We found that shifting the minus sign to the left and closely spacing the operations that followed the subtraction operation increased the frequency of the target error. This means that participants used the perceptually-based groupings instead of the mathematically-correct grouping, as defined by order of operations rules. Thus, when the manipulations afforded perceptual groups, participants were more likely to perform the perceptually grouped operations first, despite the inconsistency with formal mathematics. The variations we implemented in the position of the minus sign and spacing of the operations did not alter the inherent meaning of the equations. However, the participants had more difficulty accurately processing equations with perceptual features that were inconsistent with the order of operation rules. These findings extend past research by Landy and Goldstone (2010), and they indicate that multiple ways of manipulating perceptual grouping can influence participants' interpretations of mathematical equations and guide their problem solving.

Equations with the final three operands spaced closely elicited more target errors, except when the minus sign was shifted right (e.g., 25   – 3+2x5 = ____). In the right-shift condition, participants seem to have incorporated the minus sign as part of the subtrahend, and this may have protected them from making the target error. This suggests that participants in the right-shift condition were either more likely to perform the multiplication and then go back to the beginning of the equation *or* to realize that the minus sign could be applied to the subsequent operand, making it negative. In contrast, participants in the left-shift and closely-spaced condition, in which the target error was most prevalent, tended to ignore the minus sign (perhaps because it was spatially more distant) and to incorrectly perform the subsequent operation without the minus sign "riding along" (i.e., to calculate "3 + 10" instead of "–3 + 10" in the intermediate step).

The increased frequency of target errors in the left shift condition compared to the right shift condition suggests that perceptual factors may influence whether solvers activate the notion of negative numbers. The proximity of the minus sign to the subsequent operands in the right shift condition may have allowed participants to "attach" the minus sign to the subsequent perceptual group (e.g., in Table 1, "–3+2x5"), so that they viewed the subsequent operand (i.e., 3) as a negative integer. The more distant minus sign in the left shift condition may not have afforded this connection. This possibility is compatible with evidence from Vlassis (2004) suggesting that at the transition from arithmetic to algebra, students expand their understanding from natural numbers to integers, which may incorporate the minus sign to indicate that an integer is negative. However, our findings suggest that, when there is conflicting perceptual information, undergraduates may not automatically activate the notion of subtraction as adding a negative number.

There are many potential directions for future research in this area. One valuable next step would be to conduct a detailed analysis of participants' problem-solving steps, in order to better understand the processes through which perceptual spacing affected performance on these equations. It is possible that participants in the closely-spaced condition were more likely to start their solution process at the end of the expression (i.e., focusing on the grouped operands first).

This would further bolster the claim that perceptually grouped operations are likely to be performed first.

A second important future direction would be to directly investigate the relative prevalence of spatial and semantic information in problem solving. Our results suggest that spatial information may be processed *before* evaluating the semantic values of symbols. We suggest this given the evidence that participants used the perceptual groups to determine the subtrahend, rather than using the scope of the minus sign as defined by the order of operation rules. However, further research that directly investigates the integration of spatial and semantic information is needed.

This work could also be extended by investigating different types of problems, working with younger students who have less expertise at these operations, and considering the effects of math ability and attitudes on susceptibility to the making the target error. More generally, a deeper understanding of the role of the spatial factors is needed for a complete account of students' acquisition of computational and algebraic skills.

Finally, we believe that these data have important implications for educational practices. Perceptual features of mathematical expressions are a relatively little-studied area in mathematics education. However, perceptual features do affect performance, as we have shown here (see also Landy & Goldstone, 2007a, 2010). It could be beneficial to leverage the effects of spatial features on mathematical processing in instructional contexts. For example, spatial manipulations could be used to support students' learning of the order of operations rules, and spatial support could later be faded as students gain proficiency. Similarly, spatial cues based on proximity or other perceptual features (see Wertheimer, 1923/1938) could be implemented to reduce students' misinterpretations of the scope of the minus sign. Although the research here focused on proximity cues that afforded perceptual groups in arithmetic equations, mathematics in general uses a symbol system that incorporates spatial features in its representation. Overall, our findings suggest the need for a deeper consideration of the spatial characteristics of symbolic expressions used in mathematics instruction.

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EFFECTIVE BUT UNDERUSED STRATEGIES FOR PROOF COMPREHENSION

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We present five strategies that mathematics majors can use to improve their proof comprehension. We argue these strategies are effective by presenting qualitative excerpts illustrating the ways in which the employment of these strategies helped four undergraduate students understand the proofs they were reading. Furthermore, we present results of a survey in which the majority of 83 mathematicians indicated they would like students to use these proof reading strategies. Finally, we argue that these strategies are underused by presenting results from another part of the survey, in which the majority of 175 mathematics majors claimed not to employ these strategies.

Keywords: Reasoning and Proof, Advanced Mathematical Thinking, Post-Secondary Education

Introduction

In lamenting the quality of instruction in university mathematics courses, Davis and Hersh (1981) contended that “a typical lecture in advanced mathematics… consists entirely of definition, theorem, proof, definition, theorem, proof, in solemn and unrelieved concatenation” (p. 151). While such a description probably exaggerates the frequency with which proofs are presented in mathematics lectures, it is the case that much of the teaching in advanced mathematics courses consists of students observing the proofs that their professors present to them (e.g., Weber, 2004; Fukawa-Connelly, 2012; Mills, 2011).

To date, research in undergraduate mathematics education on proof reading generally falls into one of two categories. Some researchers have sought to examine the types of arguments that convince mathematics majors of the truth or falsity of mathematical statements. These researchers typically present students with arguments based on different types of evidence (e.g., empirical evidence, perceptual evidence) and ask students to evaluate the persuasiveness of these arguments (e.g., do you find this argument personally convincing?). The researchers use students’ evaluations as a lens to determine their proof schemes (e.g., Harel & Sowder, 1998; Segal, 2000). In a second category of research, researchers present mathematics majors with purported proofs and ask them to determine if the presented arguments are mathematically valid (e.g., Inglis & Alcock, 2012; Selden & Selden, 2003). This research reveals students’ difficulty with validation tasks as well as deficiencies in the processes they use to validate proofs.

In this paper, we introduce a third way to investigate mathematics majors’ proof reading by asking: what strategies can undergraduate students use to better comprehend proofs? The common pedagogical practice of presenting proofs to mathematics majors is based on the assumption that students can learn mathematics from reading them. Both mathematicians and mathematics educators question whether this assumption is true (e.g., Conradie & Firth, 2000; Cowen, 1991; Rowland, 2001). If mathematics majors do not gain understanding from the proofs that they read, it is natural to ask what strategies they can use to improve their comprehension. The purpose of this paper is to address this issue.

Theoretical Perspective

Our theoretical perspective builds upon answers to two related questions found in the literature: why do we present proofs to mathematics majors, and what does it mean for students to understand a proof? Central to the question of how mathematics majors should read proofs is why they are asked to read these proofs in the first place. Mathematicians read proofs published in journals for a variety of reasons, including understanding why theorems are true (de Villiers, 1990; Hanna, 1990) and finding techniques that would be useful for them to prove other theorems in their own research (e.g., Rav, 1999; Weber & Mejia-Ramos, 2011). Many mathematics educators argue that proof should play a similar role in mathematics classrooms. Proofs that merely convince students that a theorem is true are thought to have little pedagogical value (Hanna, 1990; Hersh, 1993), especially as many students are convinced a theorem is true if their teacher or textbook tells them that this is so (Harel & Sowder, 1998). Pedagogical proofs should also provide students with insights such as explanations of *why* the proven result holds true (Hanna, 1990; Hersh, 1993) and other related goals. Although secondary teachers do not appreciate these broader functions of proof (Knuth, 2002), interviews with mathematicians reveal that they present proofs to students for these purposes (Yopp, 2011; Weber, 2012).

Mejia-Ramos et al (2012) put forth a model delineating what it would mean for a student to understand a proof in advanced mathematics. In this model, a proof can be understood either locally as a series of individual deductions, or holistically based upon the ideas or methods that motivate the proof in its entirety. At a local level, understanding a proof would be comprised of (a) knowing the meaning of the terms and statements within the proof, (b) being able to justify how new assertions in a proof followed from previous ones, and (c) understanding how the assumptions and conclusions related to the proof method being used. In this model, the holistic understanding of a proof consists of being able to: (a) provide a summary of the proof that emphasizes its high level goals, (b) apply the methods of the proof in other situations to prove new theorems, (c) break the proof into sub-proofs, and (d) apply the methods of the general proof to a specific example.

As a final note, the ways that proofs are conventionally presented in advanced mathematics mean several dimensions of the Mejia-Ramos et al's proof comprehension model are not transparent. For instance, proofs often contain gaps where the reader has to infer what principles are being used to deduce new statements from previous ones (Weber & Alcock, 2005). The linear deductive manner in which proofs are written often inhibits students from seeing the higher-level ideas of the proof, or how the proof could be broken into its main parts (Leron, 1983). Hence, understanding a proof involves much more than making a literal translation of each statement within the proof, and requires active construction on the part of the reader. In this paper, we describe strategies that mathematics majors can use to facilitate such construction.

Methods

Rationale of this Study

This study employs both qualitative and quantitative methods. First, in a qualitative study, four successful mathematics majors were observed thinking aloud while reading six mathematical proofs. These videotapes were analyzed to identify proof reading strategies that were used by these students to facilitate comprehension. Excerpts from these videotapes illustrate the ways in which these strategies were effective. After these strategies were identified, a quantitative survey study was designed to determine the extent to which: (a) mathematics majors claim to use these strategies, and (b) university mathematics professors desire that their students use these strategies. This survey data provided evidence that these strategies were

mathematically desirable (from the viewpoint of mathematicians), but were not used by many mathematics majors (by their own self-report).

Qualitative Study of Mathematics Students Reading Proofs

Participants. Four seniors who were mathematics majors and prospective teachers agreed to participate in this study. These participants were chosen due to their success in their content-based mathematics courses (suggesting they used productive proof reading strategies), and because we found them to be thoughtful and articulate.

Materials. Participants were asked to read six proofs. For the sake of brevity, they are not included in this report but can be found in Weber & Samkoff (2011). The proofs were chosen such that the mathematical content relied on calculus and basic number theory (so lack of content knowledge would not inhibit proof comprehension) and the proofs employed an interesting technique with which the participants would not have extensive experience. For each proof, we generated comprehension questions using Mejia-Ramos et al's (2012) proof comprehension model described above.

Procedure. Participants met in pairs with the first author for a task-based interview. The first pair of students was assigned the pseudonyms Kevin and Tim, and the second pair was assigned Caleb and Derek. Participants were asked to "think aloud" as they were videotaped reading the proofs. They were given an individual proof and asked to read the proof until they understood it. Once this understanding was attained, the researcher took the proof and asked them to complete proof assessment questions. The interviewer then repeated this process with a new proof. This continued until all six proofs were read. Both pairs of students answered nearly every question correctly (which was uncommon in our proof reading studies), indicating that the pairs of students read the proofs effectively.

Analysis. Since we were looking to identify new strategies, and given that the proof comprehension literature is limited, we used an open-coding scheme in the style of Strauss and Corbin (1990) to identify new strategies. In the first pass through the data, we identified the strategies that were of interest to us. In the second pass, we coded explicitly for each use of those strategies.

Quantitative Survey Study

Survey items. For each strategy that we identified in the qualitative study, we created two survey items. For the mathematics majors, we generated a two choice survey item. Choice A claimed that they frequently used the identified strategy when they read proofs, while Choice B indicated that they did not regularly use the strategy. The survey item asked participants to choose whether they agreed with Choice A, Choice B, or were neutral between the two using a five-point Likert scale. The survey item for the mathematicians was similar, except they were asked if they would prefer that their students used the strategies, not that they would use the strategies themselves. For example, one of the identified strategies consists of trying to prove the theorem before reading its proof. For the students, the two choices in the survey item were:

A. When reading a theorem, I usually try to think about how I would prove the theorem before reading its proof.

B. I do not usually try to prove a theorem before reading its proof. A reason for reading the proof is to see why the theorem is true.

For the mathematicians, the two choices were:

A. When reading a proof of a theorem, I would prefer if mathematics majors think about how they might prove the theorem themselves before reading the proof.

B. I would prefer that mathematics majors not try to prove a theorem themselves before

reading its proof. A reason for reading the proof is to see why the theorem is true.

Participants. The participants in this study were invited to take part in an Internet survey. The participants from this study were conducted from 50 large state universities in the United States. The mathematics department secretaries at these universities were sent an e-mail asking them to forward e-mails to their faculty and their students. Mathematicians were invited to participate in a survey about proof reading with a link to a website for participation. Mathematics majors were sent separate request to participate in a proof reading study with a link to a separate website. Upon reaching the website, participants were asked for demographic information. For mathematicians, one question included whether they had ever taught a proof-oriented course. For students, one question included whether they had ever completed a proof-oriented course. If participants answered no to these questions, their responses were not included for analysis. 175 mathematics majors and 83 mathematicians met this requirement and completed the survey. Because the data were completely anonymous, we do not know which secretaries sent the requests to their faculty or students nor what percentage of faculty or students agreed to participate.

Results

We describe the five strategies that we observed.

Strategy #1: Trying to prove a theorem before reading its proof

For each of the six proofs that Kevin and Tim read, they would first try to prove the theorem before reading its proof. For instance, Kevin and Tim were handed a proof of the claim "$4x^3 - x^4 + 2\sin x = 30$ has no solutions". Immediately after reading the claim, Kevin and Tim began analyzing why the claim was true:

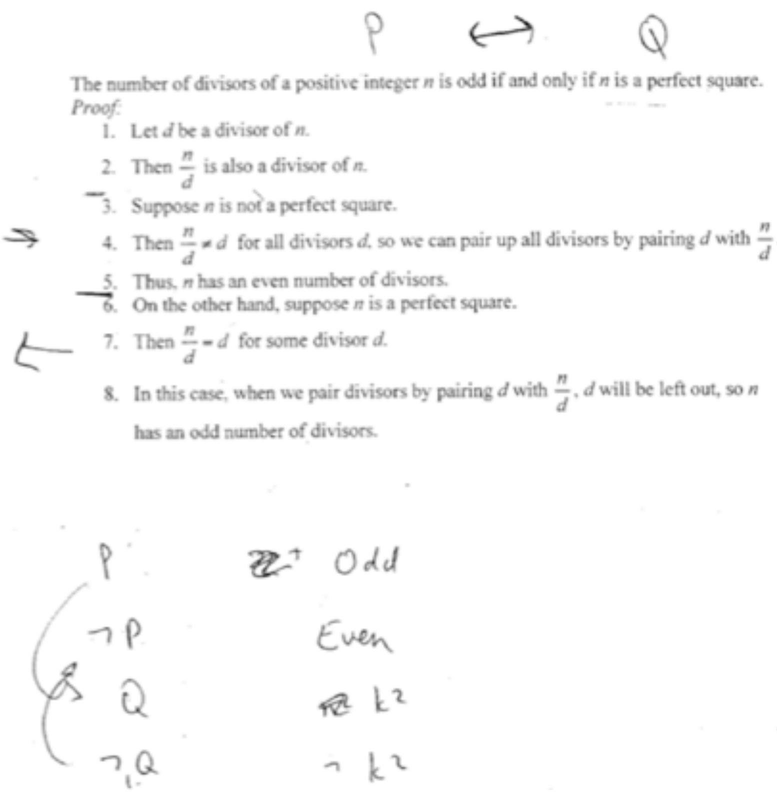
Tim: As x gets really big, it gets dominated by the negative x to the fourth term. And it's a parabola going down basically and it's going to get modulated a little bit.
Kevin: Right. And sine of x is…
Tim: Periodic.
Kevin: It is periodic so that wouldn't really affect it too much… out of the two functions, *f(x)* is the trumping one.
Tim: So in the long run, it's going…
Kevin: It's really *f(x)* that matters.
Tim: And the question is, does it reach 30.

It is important to note that Kevin and Tim were not only trying to understand the statement, they were also trying to understand why the statement was true. While they did not successfully produce a proof, their efforts at this stage appeared to help them comprehend the proof that they read. The proof itself had the same high-level ideas that Kevin and Tim highlighted: finding the bounds of $2\sin x$ and $4x^3 - x^4$. The details in the actual proof, such as using differentiation to find the critical points of $f(x)=4x^3 - x^4$, can be seen as supporting these high-level goals. In this sense, when reading the proof, Kevin and Tim did not view the proof as an aimless series of inferences and calculations, but rather saw the proof as satisfying two high level goals—finding the bounds of $2\sin x$ and $4x^3 - x^4$. Hence, trying to prove the theorem before reading the proof facilitated Tim and Kevin's ability to "structure" the proof (in the sense of Leron, 1983) and provide a summary of it. Other transcripts show how trying to write a proof helped Tim and Kevin understand the proof technique (e.g., proof by contraposition) being employed in the proofs that they read and identifying techniques that were novel to them in the proof that might be useful in other settings.

Using the survey item described in the previous section, we found that 88% of the survey mathematicians chose choice A, desiring students use this strategy, but only 31% of the mathematics majors chose choice A, suggesting most mathematics majors do not use the strategy.

Strategy #2: Comparing the assumptions and conclusions in the proof with the proof technique being used and Strategy #3: Breaking a longer proof into parts or sub-proofs

We illustrate both strategies by showing how Tim and Kevin read the proof presented in Figure 1:



P $\longleftrightarrow$ Q

The number of divisors of a positive integer *n* is odd if and only if *n* is a perfect square.

Proof:

1. Let *d* be a divisor of *n*.
2. Then $\frac{n}{d}$ is also a divisor of *n*.
3. Suppose *n* is not a perfect square.
4. Then $\frac{n}{d} \neq d$ for all divisors *d*, so we can pair up all divisors by pairing *d* with $\frac{n}{d}$.
5. Thus, *n* has an even number of divisors.
6. On the other hand, suppose *n* is a perfect square.
7. Then $\frac{n}{d} = d$ for some divisor *d*.
8. In this case, when we pair divisors by pairing *d* with $\frac{n}{d}$, *d* will be left out, so *n* has an odd number of divisors.

Figure 1: Tim and Kevin's Employment of Strategies 2 and 3

Kevin: From [lines] two to four, it's doing the proof by contradiction. Suppose *n* is not a perfect square. […]
Tim: Suppose *n* is not a perfect square. So you're saying, suppose not this?
Kevin: For which one?
Tim: So you're saying, suppose *n* is not perfect, right, and that's the opposite of the right side.
Kevin: Cause it's dichotomous. If it's not a perfect square, then it's even. So therefore if P implies Q, then not Q implies not P, right? Contrapositive.
Tim: So we're saying this is P [writes P above "the number of divisors of a positive integer n is odd], this is Q [writes Q over "n is a perfect square"].
Kevin: OK sure. So P is a positive integer, right? Positive integer?... Right?... Positive integer?
Tim: Is odd?
Kevin: Right. Sorry. P is odd, right? So then not P would be even. So Q would be a perfect square, k squared. And not Q would not be k^2 or whatever. So here, [referring to lines 3-5 of

Proof 5] it shows not Q implies not P.

Tim: And this part of the proof ends at 5 [Tim draws horizontal dashed between lines 2 and 3 and between lines 5 and 6 to partition the two parts of the proof]

Kevin: So it shows not Q implies not P and therefore P implies Q. So it's proving it forwards. [Tim rights a right arrow next to lines 3, 4, and 5]. It's the forwards way. [Reading line 6] On the other hand, suppose *n* is a perfect square. So now it's going to prove the backwards.

Selden and Selden (2003) and Weber (2010) found that students would often accept a proof of a conditional statement as valid in cases where the proof began by assuming the conclusion of the conditional statement and deduced the antecedent—i.e., the argument proved the converse of the statement. Both research teams took this as evidence that participants did not attend to the proof framework of the proofs that they were reading. The excerpt above illustrates that Kevin and Tim explicitly attend to the assumptions and conclusions of the proof to understand how a valid proof technique is being employed. The surveyed students were presented with these choices:

A. When I read a proof, I first consider is what is being assumed, what is being concluded, and what proof technique is being used.

B. When I read a proof, I first consider how each new statement can be derived from previous statements.

Only 33% of the mathematics majors preferred A, but 64% of the mathematicians preferred A for the analogous item (not presented here for the sake of brevity). In the transcript and figure above, we also see how Kevin and Tim broke the proof into two parts, lines 3 through 5, and lines 6 through 8. This was something that both pairs of students regularly did. In the survey, the mathematics majors were given the following two choices.

A. When I read a long proof, I try to break it into parts or sub-proofs.

B. When I read a long proof, I do not break it into parts but try to understand how each line follows from previous assertions.

Less than half (38%) of the mathematics majors chose A, but 88% of mathematicians chose A for the analogous items.

Strategy #4: Comparing the proof approach to the one's own approach

After reading the proof that "$4x^3 - x^4 + 2\sin x = 30$ has no solutions", Caleb and Derek noted that this is not how they would approach the problem. (The proof proceeded by showing $2\sin x \leq 2$ and $4x^3 - x^4 \leq 27$ for all real x).

Caleb: Definitely not the way I would have gone when I looked at it…

Derek: Not even close.

Caleb: Yeah, I would not have broken it down like that.

Derek: I would not have broken it down like that at all. How I would have solved this, I have no idea.

Caleb: Well when you start putting in trigonometric functions.

Derek: Yeah, that's when it starts getting a little hairy […] I probably would never have ever thought of this but from my background in calculus, it makes sense to me.

Caleb: Yeah, yeah. It's clever [laughs]

Derek: Yeah, it's clever. I never would have dreamed of coming up with something like this.

After reading all six proofs, Caleb and Derek were asked if comparing a proof strategy to one's own approach was something they regularly did. Both agreed that it was. When asked why they did this, Derek said: "it gives you new techniques to solve proofs like that".

The surveyed mathematics majors were given the following choices:

A. When I read a proof, I compare how the method used in the proof compares to the method I would use to prove the theorem.
B. When I read a proof, I try not to consider how I might approach the proof, but focus on what method was used in the actual proof.

Few mathematics majors selected Choice A (26%) but most mathematicians (87%) selected A when given the analogous choices.

Strategy #5: Using an example to understand a confusing inference

When both pairs of students were confused by a statement in number theory, they interpreted the general statement in terms of specific numbers in order to understand what the statement was asserting, and why it was true. For instance, one proof contained the line, "Let d be a divisor of n. Then n/d is also a divisor of n". Tim and Kevin were confused as to why the second claim should be true until Kevin said, "So, say 7 is a divisor of 21. So to get the other factor, it's 3". After this, the pair were satisfied that every product of a number contains two factors and if d was a factor, n/d would be the other factor. The surveyed students read the following choices:

A. When I read a new assertion in a proof, I sometimes check whether that assertion is true with a numerical example.
B. When I read a new assertion in a proof, I try to see how that new assertion is a logical consequence of previous statements. I do not check assertions with specific examples since you cannot prove by example.

A minority (43%) of mathematics majors chose A, but 77% of mathematicians chose A when given the analogous items, indicating most mathematicians would like students to use this strategy.

Discussion

This paper presents five strategies that mathematics majors can use to improve the comprehension of the proofs that they read. For each strategy, we illustrate how it can be useful with an episode of a pair of strong students reading proofs for comprehension. We then presented quantitative data indicating the majority of mathematicians would like their students to use these strategies, but less than half the students agree to regularly using these strategies. This suggests that if mathematics majors could be taught how to apply these strategies and were encouraged to use them, their proof comprehension performance might improve.

There are several caveats worth mentioning. There were only a small number of students (4) reading a small number of proofs (6) in the qualitative study. It is probable that interviewing other strong students or using other proofs would have evinced other strategies as well. Hence, we make no claims that the five strategies that we highlighted are exhaustive. Furthermore, the survey for students was self-report. It is possible that some mathematics majors might not have reported using these strategies, maybe because they were not aware that they did so. If so, explicitly naming and highlighting these strategies may help these students leverage them more effectively.

Finally, the link between identifying strategies and teaching students to use them is not straightforward. The use of these strategies likely relies on students' conceptual knowledge (that may or may not be adequate). Further, these strategies are descriptive and this description might not be detailed enough for students to use them effectively (see Schoenfeld, 1985). We are currently conducting teaching experiments, both to refine these strategies so they are useful to mathematics majors and to find environments and activities that foster the understanding, appreciation, and adoption of these strategies.

Acknowledgments

The research in this paper was funded by a grant from the National Science Foundation (DRL0643734). The views expressed in this paper are not necessarily those of the National Science Foundation.

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PROBLEM POSING AS REFORMULATION AND SENSE-MAKING WITHIN PROBLEM SOLVING

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The study examined problem posing as a form of sense-making that aids the solver's on-going development of goals and purposes throughout problem solving. Drawing from episodes of a Mathematics Education graduate student solving a number array task, our analysis explains: 1. How problem posing evolves from the solver's ongoing interpretations of the problematic situation; and 2. How these posed problems contribute to the solver's problem solving activity.

Keywords: Problem Solving, Learning Theory, Metacognition.

The Co-evolution of Problem Posing and Problem Solving

The study of mathematical problem posing has been an important area of investigation by researchers in mathematics education (Cai, Moyer, Wang, Hwang, Nie & Garber, 2012; English, 1997; Silver, 1994; Silver & Cai, 1996). Having students generate and develop their own problems from particular situations may help them become stronger problem solvers. Proponents of problem posing advocate for its inclusion in the mathematics curriculum for several reasons. These include the view that having students make up their own problems encourages self-reflection that may help students to better reflect on problem situations (NCTM, 2000; Schoenfeld, 1994). Reflection that involves planning potential solution strategies has been associated with effective problem solving in several studies of problem solving (Cifarelli & Cai, 2005). Hence, posing problems is viewed as an activity that may nurture mathematical thinking.

We believe that problem posing occurs throughout problem solving and that solvers use results to monitor the usefulness of current goals, revising or reorganizing their goals and purposes as needed to solve the problem. Problem posing is then a series of transformations of the original problem, with each successive problem posed indicating both progress towards a solution as well as possibilities for action to further expand the scope of the original problem.

Related Research

The exemplary research on problem posing comes from English (English, 1997) and Silver and his colleagues (Silver, 1994; Silver & Cai, 1996). While these studies have added to our knowledge of problem posing, the research is less certain about the roles played by problem posing in problem solving situations. How do the solver's initial problem formulations impact his or her solution activity?

Conversely, how do the student's reflections on carried-out solution activity help him or her re-formulate the current problem, or pose additional problems to solve? According to Brown and Walter (1993),"we need not wait until after we have solved a problem to generate new questions; rather, we are logically obligated to generate a new question or pose a new problem in order to solve a problem in the first place" (Brown & Walter, 1993, p. 114). In this way, problem posing and problem solving are naturally related. Silver (1994) referred to this kind of problem posing as "problem re-formulation [that] occurs within problem solving" (Silver, 1994, p. 19). For example, students engaged in the solution of a problem may generate a result that, upon

reflection, calls into question their prior goals and actions; the ways that students resolve the new question often leads to a reformulation of the original problem which may lead to a solution.

Our research questions were: 1. How does problem posing evolve from the solver's on-going interpretations of the problem situation; and 2. How do these posed problems contribute to the solver's problem solving?

Methodology

The case illustrated in this paper involves Sarah, a Mathematics Education graduate student. We observed Sarah's self-generated questions based on her interpretations of the task that she was presented; and these observations formed the basis for answers to the first research question. We answered the second research question by tracing how her problem posing evolved into sophisticated solving. The data consisted of videotaped protocols, the researchers' field notes, and the student's written work. A transcript of Sarah's verbal responses was generated and protocol analytic techniques were used in the analysis (Cai, 1994).

Results

The analysis of the episode with the graduate student, Sarah, includes her posing and solving of a problem she posed from the array she was given (Figure 1): What is the sum of cell entries in any N by N block of numbers from the array?

Find as many relationships as possible among the numbers.

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

Figure 1: Number Array Task

After exploring the array, Sarah reflected on her initial results and looked for other relationships such as the sums of entries in the square blocks (Figure 2).

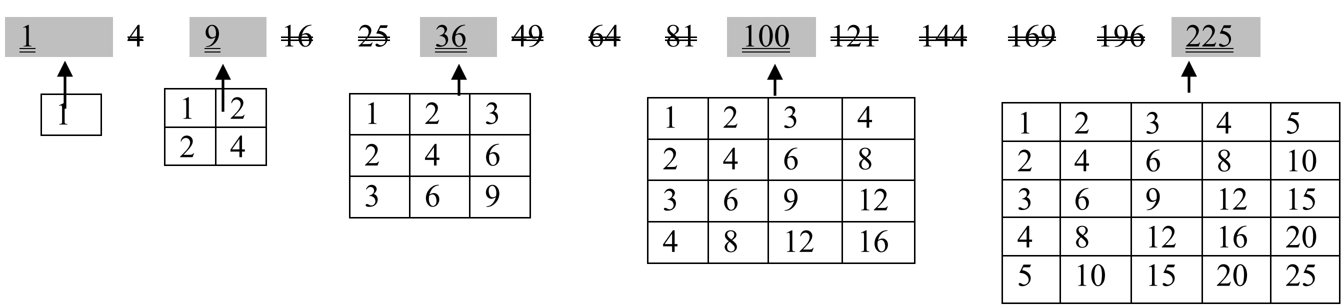

Figure 2: Sarah's Skipping to Find Sums of Block Entries

Sarah: Let's see … *(reflection)* … I was wondering about the square numbers on the diagonal going from left to right. They seem to relate to the sums of these blocks I had earlier *(points to the 2x2, 3x3, 4x4 blocks)*. So, let's check it.

Sarah examined the sum of the entries of each NxN block that contained the square numbers on the diagonal and she developed an informal method to find the sums of the entries of all NxN blocks going down the main diagonal.

Sarah: So, for a 1x1, I get a sum of 1 *(points to the 1 in the upper left corner)*. For a 2x2 *(points to block [1, 2 : 2, 4][i])*, I get a sum of 9 … but what happened to 4? It appears to have been skipped! *(reflection).* Okay, I will write down the sequence of squares of all numbers *(writes the following sequence of square numbers: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225).* So, the first number, 1, tells the sum of the very first matrix, a 1x1. And the first 2x2 has a sum of 9. …. So, I skipped over 4 to get the next sum *(crosses out the 4 in the sequence),* going from 1x1 to a 2x2, a sum of 9. The 4 gets skipped? Interesting!

Sarah has posed a problem: she thinks there may be a relationship between the sequence of square numbers on the diagonal and the successive sums of the entries of NxN blocks. Sarah generalized her 'skip' method to generate the sequence of sums of the entries of all NxN blocks. Sarah then looked to make sense of her method with some further exploration (Figure 3).

Sarah: I wonder why this skipping works? Let's see it another way, for the 6x6, we add the entries in the rows to get 21+42+…+126 = 21(1+2+3+4+5+6) = 21x21 = 441. Do we get 441 by skipping the next 5 in the square sequence? *(Sarah extended her original sequence beyond 225, crossed out the corresponding 'skips,' and got a result of 441 as the next number in the sequence)* (Figure 3). But also, I notice that 21 over here *(points to the factored form 21• (1+2+3+4+5+6))* is the sum of the first 6 numbers in that first row. Yes!

1	2	3	4	5	6	21
2	4	6	8	10	12	42
3	6	9	12	15	18	63
4	8	12	16	20	24	84
5	10	15	20	25	30	105
6	12	18	24	30	36	126

Sum of rows: *21(1+2+3+4+5+6)=21(21)=441*

Skipping in the sequence: Finding the sum for the 6x6 block

225	~~256~~	~~289~~	~~324~~	~~381~~	~~400~~	441
15^2	16^2	17^2	18^2	19^2	20^2	21^2

Figure 3: Sarah's Computation of Sums in a 6x6 Block

Sarah tried her idea on an 8x8 block (Figure 4).

Sarah: To find the sum, I bet you just look at the sum of 1 to N and then square the total. Let's try an 8x8. It would be 1+2+…+8 = 36, I don't know why I am adding these individual numbers since I know the sum is (8x9)/2, and then I take 36^2? That comes out to be 1296. Does it check with my skipping here? I first skip 6 over 21 to get 28^2 for 7x7, and then skip 7 more to get the one for 8x8, so 7 more is 35, and the next one is 36! My algorithm works!

1	2	3	4	5	6	7	8	36
2	4	6	8	10	12	14	16	
3	6	9	12	15	18	21	24	
4	8	12	16	20	24	28	32	
5	10	15	20	25	30	35	40	
6	12	18	24	30	36	42	48	
7	14	21	28	35	42	49	56	
8	16	24	32	40	48	56	64	

Skipping in the sequence: Finding the sum for the 8x8 block

225	~~256~~	~~289~~	~~324~~	~~381~~	~~400~~	441	Skip next 6 for 7x7 case	784	Skip next 7 for 8x8 case	1296
15^2	16^2	17^2	18^2	19^2	20^2	21^2		28^2		36^2

Figure 4: Sarah's Computation of the Sum for the 8x8 Block

Theoretical Implications

Our analysis of the problem posing of Sarah suggests important roles that problem posing can play in solution activity. First, problem posing performed in the solution of a problem helps to both broaden the solver's perspective of the original problem as well as expand its scope. This expansion of scope can further help students engage in unexpected generalizing activity that is rooted in students' own goals and purposes. Second, the findings indicate how solvers make conceptual progress through problem posing and thus highlights the importance of self-generated activity for the purpose of extending one's understanding of the problem.

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Endnote

[i] We use a bracket notation that lists the top-to-bottom rows of the block being considered. For example, the 2x2 is indicated by the sequence [1, 2 : 2, 4] and a 3x3 block is indicated by the sequence [1, 2, 3 : 2, 4, 6 : 3, 6, 9].

EFFECTS OF COLLABORATIVE REVISION ON UNDERGRADUATE STUDENTS' PROOF VALIDATION SKILLS

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Although the ability to determine the validity of a proof is an important skill for mathematicians and mathematics educators alike, there is ample evidence that undergraduate students in a transition to proof course are not proficient at this task. This study employs a teaching intervention called collaborative revision, which refers to the process in which students present a proof they have constructed to their classmates who are encouraged to provide feedback to aid in the revision of the proof. Pre- and post-assessments were administered to students and interviews were conducted to measure students' proof validation skills and to investigate if collaborative revision impacts these skills. Results show that although collaborative revision may not impact students' abilities to identify valid proofs, it does affect the way that students gain conviction about the proof of a statement.

Keywords: Reasoning and Proof, Classroom Discourse

Introduction

This study aims to investigate the impact on argument validation skills of students after participating in a process called collaborative revision in an introduction to proof course. *Collaborative revision* refers to the process in which students present a proof they have written to their classmates and the other students are encouraged to make comments and point out inconsistencies in order to ensure that the proof is valid. Based on feedback from classmates, the student then revises the proof and presents it again, repeating the process until the proof is valid and includes all the relevant details. In this context, this research aims to answer the following research question: To what extent does participation in collaborative revision impact undergraduate students' proof validation skills?

Related Literature

Rav (1999) proposes and Hanna & Barbeau (2009) agree that proofs are of the utmost importance in mathematics, since they, instead of theorems, are the main vehicles in which mathematical knowledge is contained and transferred. Thus, the ability to determine if a given proof is valid becomes an important skill for students. Selden & Selden (2003) highlight that this skill is invaluable for not only future mathematics educators, since they will someday have to evaluate student proofs for assessment purposes, but also for future mathematicians because they will have to examine proofs to learn about new mathematics being produced. Additionally, proof validation is intricately linked to proof construction so students in a transition to proof course need both sets of skills (Selden & Selden, 2003).

Selden and Selden (2003) found that undergraduate students have trouble differentiating between a valid and an invalid proof. In a proof verification task, undergraduate students in a transition to proof course initially judged proofs correctly less than half the time. However, when prompted for reflection about the proofs by the interviewer, the students were able to correctly evaluate proofs at over 80%. The authors conjecture that this may be due to the fact that proof verification skills are not explicitly taught to students and the texts for these courses

include very few proof validation tasks, with those that do exist usually having just a single error to be detected. Alcock & Weber (2005) found similar results with students in an undergraduate real analysis course and write that "this suggests that the ability to validate proofs may be in many students' zone of proximal development and that students' abilities in this regard might improve substantially with relatively little instruction" (pg. 131). Thus, this study aims to determine the effect that making proof validation a part of normal classroom activities will have on students' ability to correctly judge the validity of proofs.

There is much research to support the hypothesis that collaborative learning can greatly enhance student learning about mathematical proof. Yackel & Cobb (1996) note that participation in a community of learners can be a vital part of students' success in mathematics. Additionally, a study by Strickland & Rand (2012) allowed students to submit multiple revisions of proofs in response to teacher feedback and measured the effects on student learning. The teacher comments given were minimal, often just circling a confusing or incorrect passage of the proof, and students were allowed as many revisions as needed. Although the data set was small, on average, students in the revision group did better on the final exam. Proof validation studies (Selden & Selden, 2003; Alcock & Weber, 2005) also show that reflection about written proofs can aid in correctly evaluating the validity of a given proof. Thus, collaborative revision is a way to explore the benefits of combining these proven techniques and this study examines the impact on students' validation skills when using a collaborative revision teaching intervention in an introduction to proof classroom.

Methodology

Context & Participants

This study was enacted at large Midwestern University in courses offered by the Mathematics, Statistics & Computer Science department. The collaborative revision teaching experiment took place in a course correlated with a transition to proof course offered through the Emerging Scholars Program at the university. Students were able to self-enroll in this treatment course and, thus, this is a quasi-experimental study. Since the treatment course was supplemental, the majority of students in the treatment course were concurrently enrolled in the correlated lecture-based transition to proof course, which is the comparison course in this study.

Treatment group participants were drawn from the treatment course in this study and there were 15 students in this group. A comparison group was desired to determine the impacts of the teaching experiment when compared to a lecture-based course. The comparison group solicited students from the comparison course and was comprised of 12 students. Additionally, two students from the comparison group and two students from the treatment group were interviewed to gain insight about what aspects of a proof are convincing to students in each group.

Course Design

Each week students in the treatment course were expected to come to class with a written proof of a statement given by the instructor. At the beginning of class, students were put into small groups (no more than 4 per group) and they discussed the proofs one at a time. During this time, the students were encouraged to carefully read their classmate's proof, writing down things as necessary, and verbalize their impressions of the proofs. Based on the feedback given, the student was responsible for revising the proof and bringing it back in the following week. The treatment course often exposed students to proofs from their classmates, which were neither valid nor written in the most rigorous way. In the context of determining the effects of collaborative revision on student proof validation skills, it was hypothesized that students in this

experimental course would be more proficient at identifying invalid proofs since they would have explicit practice doing this during the course of the semester.

Data Sources

To measure the effects of the collaborative revision on proof validation, a pre and post-assessment was given to all participants. The assessment required students to examine four 'proofs' of a given statement and determine whether each was a valid or invalid proof. The arguments presented to the students were adapted for this study from the proofs given to high school students in Healy and Hoyles (2000) to be appropriate for undergraduates. On the assessment, students were first asked to determine if each proof was valid or invalid, with these terms intentionally undefined to see if students would gain more of an understanding of what a valid proof entails throughout the course of the semester. There were also two more questions on a three-point Likert scale asking students how well they felt they understood each argument and how certain they were about their classification. The same proofs to be validated were given on the pre and post-assessment.

In addition to the assessments, student interviews were conducted with two students in the treatment course and two students in the comparison course. Students were interviewed individually twice during the course of the semester; first shortly after the pre-assessment was administered and again shortly after the post-assessment was administered. The questions asked to students, according to Zazkis and Hazzan (1998), are performance questions, unexpected "why" questions and reflection questions, requiring students to explain why they chose each argument as valid or invalid. During the second interview, students were shown their answers on the pre-assessment and asked to compare to the post-assessment and explain any inconsistencies. This was done to determine what aspects of an argument convinces students, called *student proof schemes* by Harel & Sowder (1998), and if this evolved during this semester.

Data Analysis & Results

Data from the assessments (pre and post) was analyzed to determine the percentage of correct classifications made by each student. Additionally, descriptive statistics were computed for the Likert ratings regarding students' self-reported understanding and certainty about classification for each of the four arguments and independent samples t-tests were done to determine differences between groups (i.e. treatment and comparison). Student interviews were transcribed and analyzed to determine the proof schemes exhibited by each student during the pre and post-interviews.

Consistent with prior proof validation studies (Alcock & Weber, 2005; Selden & Selden, 2003), this study found that students very often incorrectly classified proofs. On the pre-assessment, students in the treatment course were able to classify proofs correctly at an average rate of 60%, while students in the comparison course had a rate of 45%. However, a t-test showed this difference was non-significant ($p = 0.2$). On the post-assessment, the comparison group increased their average correct to 54%, while the treatment group's average remained unchanged. This suggests that the collaborative revision process had little impact on students' proof validation skills. Even though the treatment group did not show an increase in average percentage of proofs correctly classified from the pre to the post-assessment, they did report a significantly better understanding than the comparison group on almost every proof. Additionally, the treatment group reported significantly higher confidence of their classification than the comparison group on two out of four proofs. So, collaborative revision has an impact on students' ability to understand a written proof and their certainty about classifying that proof.

Results from the interviews show that the students interviewed from the treatment group exhibited higher-level proof schemes of Harel & Sowder (1998) from the pre-assessment to the post-assessment, while the proof schemes of students interviewed from the comparison group remained largely unchanged. Table 1 shows the proof schemes that each student interviewed (pseudonyms are used) exhibited during each of the interviews. These results imply that even though collaborative revision has little impact on students' abilities to identify valid proofs, it does impact the proof schemes held by students and how they gain conviction about the proof of a statement.

Table 1: Proof Schemes Exhibited by Students in Each Group During Each Interview

Group	Student	First Interview	Second Interview
Treatment	Stephanie Robert	Ritual Internalized	Transformational Intuitive-Axiomatic
Comparison	James Francine	Empirical Transformational	Transformational Transformational

Acknowledgment

The author acknowledges Dr. Mara Martinez for her supervision and helpful comments on this research project.

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SUPPORTING INDUCTIVE LEARNING TO REDUCE OVERGENERALIZATION

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Human induction, the process of developing abstract generalizations from specific instances, has been shown to promote positive transfer. By comparing induction to telling the formula in a growth pattern problem, this paper presents evidence that inducing mathematical structure also serves as a buffer against negative transfer, or overgeneralization of the ideas beyond their applicable contexts. The second portion of the paper discusses the necessary conditions that must be met to support students in the inductive learning process. A plan to develop a set of principles to inform the design of inductive instructional activities is outlined.

Keywords: Algebra and Algebraic Thinking, Instructional Activities and Practices, Cognition

In algebra, students are challenged to form generalizations that account for many instances. Similarly, human induction, or the process of engaging in generalization (Ellis, 2007), is characterized by the abstraction of a universal idea from its specific instances.

Learning by induction can help people focus on deep structure, which has been shown to lead to positive transfer, or the appropriate use of ideas in new contexts (Gick & Holyoak, 1980). When learning generalizations, it is important that students know the limits of the application of this structure to ensure that they do not overgeneralize ideas, as in the case of negative transfer (Ross, 1987). Research has not explored the ways induction affects negative transfer.

This paper presents a study that investigates whether induction in algebra supports flexible generalization. Individuals who induce a mathematical solution are compared to individuals who are told the solution and practice on the same problems. The second part of this paper addresses the prerequisite instructional conditions that must be met to help people induce structure.

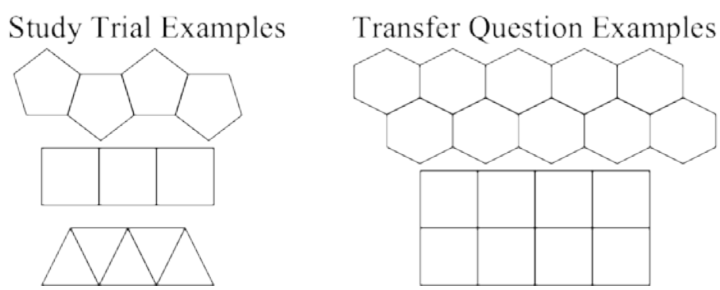

Figure 1: Polygon problem cases (*left*) include different shapes and different numbers of shapes. For the transfer question figures (*right*), the perimeter formula must be adapted.

The Polygon Problem

The learning topic was growth pattern problems – algebra tasks that illustrate the importance of abstraction and the utility of generalizations. These problems directly relate to Common Core Standards (National Governors Association Center for Best Practices, 2010) and are stressed in professional development (e.g. Koellner et al., 2007). The polygon problem asks the solver to find the perimeter of a row of shapes (see Figure 1). The inductive problem-solving approach requires the learner to consider instances to build a rule that accounts for the two key variables: the number of

shapes (n) and the number of sides in each shape (s). One alternative instructional approach is to provide a rule at the start. A form of this algebraic formula is: $(s - 2)n + 2$.

To test whether participants would generalize the formula and transfer to new contexts, the pattern was modified to include two rows of shapes as shown in Figure 1. In these transfer questions the perimeter formula must be adapted to account for the additional shared sides.

Study 1: Experimentally Comparing Induction and Telling

Forty adults (M= 22.75 years) participated in a 45-minute study that consisted of three 18-trial blocks and two transfer questions, as outlined in Table 1. Each block contained problems ranging from 3-10 shapes with 3-6 sides (triangles through hexagons). Each trial was displayed on a computer and participants had unlimited time to determine the perimeter.

Table 1: Overview and Results for Induction and Telling Conditions

	Block 1 Count	Block 2 Treatment	Block 3 Told	Transfer Questions	Transfer Strategy Use	
Induction Condition n = 20	Count 94% correct	**Induce** 89% correct	Told 95% correct	Adapt formula to new patterns:	Correct	11
					Overgeneralize	3
					Other	6
Told Condition n = 20	Count 92% correct	**Told** 94% correct	Told 97% correct		Correct	6
					Overgeneralize	12
					Other	2

Time →

Participants were randomly assigned to condition: Induction or Told. In Block 1, all participants counted the perimeter to become familiar with the problem. In Block 2, Induction participants searched for a formula to use with all shapes and all numbers of shapes. After finding the formula, they continued using it on remaining Block 2 problems. Told participants were shown the perimeter formula with a worked example before Block 2. The variables were defined, but the applicability of the formula was not elaborated. Told participants calculated the perimeter using this formula. For Block 3, all participants received the same formula. For the Told condition, these instructions were identical to Block 2. Induction participants were shown the same presentation the Told condition had seen for Block 2. After Block 3, participants talked aloud during two paper-based transfer questions.

As shown in Table 1, all individuals were quite accurate throughout. There were no significant between-group differences. Searching did not significantly diminish accuracy.

During Block 1 (Counting), Reaction Time (RT) varied directly as a function of perimeter ($R^2 = 0.96$, $p < 0.01$) with no significant between-group differences: as the perimeter increased, participants took longer to count. As this correlation is not significant in later blocks, this serves as a manipulation check, indicating that participants counted during these trials and, notably, did not

naturally induce a formula to solve for the perimeter. In Block 2, searching for a formula took significantly longer than directly applying one. However, after finding the rule, Induction participants' RTs dropped fivefold, from minutes to seconds per problem. By Block 3, all participants used the provided formula and mean RTs did not differ by condition.

In summary, accuracy is not affected by searching, but induction takes significantly longer than using a formula at the outset. The transfer question results illuminate why one would choose to teach inductively, as it appears to be a less efficient strategy.

Participants' responses to the transfer questions fall into three categories: correct adaptation to the new context, overgeneralization (directly applying the formula without any changes), and other (which includes counting). The breakout by instructional condition is shown in Table 1. There is a significant interaction of solution type by instructional treatment ($\chi^2 = 8.87$, $p = 0.01$). Told participants are more likely to overgeneralize and Induction participants are more likely to correctly adapt to the new context.

Discussion of Study 1

The main benefits of induction are seen at transfer. Although Told participants execute the formula efficiently, their knowledge is brittle and they do not develop a sense of the range of examples that it fits. These individuals are more likely to overgeneralize the formula to new situations. Inducing the rule seems to block negative transfer.

However, there are two risks to induction. First, it takes time to induce a pattern. Overall, Induction participants showed the characteristic extreme drop in problem-solving time once they had induced the rule (Schwartz & Black, 1996). Eventually Induction individuals do become fluent users of the formula, as evidenced by shorter reaction times and stable accuracy in the Induction condition after discovering the formula. Being told the formula after engaging in generalization did not hurt, but participants who were told without any opportunity for induction were susceptible to negative transfer. Second, some participants may never induce a useful rule (Gick & Holyoak, 1980). These results highlight a difficulty designing inductive instructional environments: not everyone is guaranteed to discover the phenomenon in question.

Prerequisites to Inductive Learning

Task Orientation

To promote learning through induction, teachers and learning environments must explicitly instruct students to adopt an inductive problem-solving orientation. As seen in Study 1, seeing deep structure does not always happen naturally. Participants did not induce the pattern during the counting trials; they only found the formula when explicitly asked to search for it. Study 1 also demonstrated an important truth with instructional implications: directly telling a formula does not ensure that students will think critically about its meaning. Even when people can fluently apply a formula, they may not understand the underlying structure of the generalization and therefore not have a sense of the scope of situations to which it may apply. Asking students to engage in generalization entails developing clear, explicit instructions to orient them to the task of creating a mathematical expression to account for many instances.

Materials

Even with the right task orientation, students need well-organized materials to support inductive thinking. As a trivial example of poorly chosen materials, if students were only presented with example rows of squares in the polygon problem, it seems unlikely that they would naturally extend their generalization to other polygons without any prompting. By systematically sampling the space of possible rows of shapes, a more comprehensive set of cases would be fruitful in encouraging generalization. Instructional examples chosen to highlight meaningful variation among the set are called contrasting cases. Contrasting cases juxtapose two or more examples that vary on at least one

dimension. For example, in the polygon problem, perimeter varies with respect to two dimensions (*n* and *s*) and changing either variable affects the outcome. When contrasting cases are used in instruction, the students' task is to search for an underlying pattern to explain all cases. This differs from sequential problem solving, where students often treat each question as a separate learning task and fail to notice the commonalities across each instance (Star & Rittle-Johnson, 2009). To help students learn how variables are interrelated, contrasting cases should be used to show mathematical structure instead of merely revealing differences in variables. For example, contrasting cases designed to highlight a ratio relationship must emphasize that the two parts of the ratio are interdependent – when the numerator changes, the denominator must change correspondingly to maintain the underlying ratio. In previous work, contrasting cases have been developed for concepts such as statistical variation (Schwartz & Martin, 2004) and for ratio in the topic of density (Schwartz et al., 2011).

What needs to be included across a set of cases such that students can induce their deep mathematical structure? By systematically varying the presentation of patterns in inductive tasks, research can test hypotheses about which information is necessary and sufficient for learning. These studies involve examining the role of presenting interactions between variables; i.e. for ratios, is it better to hold the denominator constant and vary the numerator or would it be optimal to present cases that show concurrent variation in both dimensions?

Future Work

Data collection has begun to compare several sets of cases designed to help students find ratio relationships among variables in early physics domains. This study investigates the best way to present variation along dimensions in these problems with multiple interrelated factors. Preliminary results suggest that highlighting interactions between variables helps learners discover the importance of mathematizing these relationships. This work is intended to inform practice by creating design principles to aid educators in the construction of inductive learning activities. It is the hope that these principles will encourage widespread adoption of induction activities in math classrooms. Contrasting cases offer a potential way to catalyze subsequent learning because they help students grasp key relational structures. As shown, induction is a powerful way for people to find new-to-them mathematical structure, which in turn, helps them benefit from other instruction and leads to appropriate transfer to new contexts.

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TEACHING FIRST-YEAR MATHEMATICS UNDERGRADUATES TO THINK MATHEMATICALLY

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We present an ongoing design experiment *for the teaching of an undergraduate mathematics course called* Introduction to mathematical thinking. *The goal was to design a rich mathematical environment where students will be able to develop mathematical thinking. Preliminary results suggest that students actively engaged in learning to think mathematically.*

Keywords: Advanced Mathematical Thinking, Reasoning and Proof, Design experiment

In this report we discuss partial results of an ongoing *design experiment* (Brown, 1992; Collins et al., 2004) for the teaching of a first-year undergraduate course called *Introduction to mathematical thinking*, compulsory for students in the Major in Mathematics and Statistics program. These students have a (typically weak) background in college calculus and linear algebra and haven't taken courses that require proving. One of the authors taught the course previously: following the institutions' normal approach, the course focused on proving using algebra and arithmetic as the topics for exemplifying proving techniques and the writing of proofs. This research is mostly motivated by those experiences, where students, although able to write simple proofs, didn't seem to understand why these were necessary or why they were actually *proofs* of anything. The new course was designed with the goal of providing a rich learning environment where student could develop mathematical thinking as defined by model that arises from combining a model of *theoretical thinking* (Sierpinska et al., 2002) with aspects of *metacognition* (Schoenfeld, 1987).

To Think Mathematically

Without ignoring previous contributions to the characterization of mathematical thinking, we consider Sierpinska et al.'s (2002) model of *theoretical thinking* (TT) and extend it to a model of *mathematical thinking* by including aspects of *metacognition*, based on (Schoenfeld, 1987).

Sierpinska et al.'s model is based on Vygotsky's work. In this model, TT is characterized as *reflective* (RT), *systemic* (ST) and *analytic* (AT). The aim of the reflective thinker is not to get things done, rather to reflect on and further investigate a task at hand. RT is about curiosity and challenging one's own mind. ST is thinking about systems of concepts, where meaning is established based on relations with other concepts. The systemic thinker is aware of context and structures and of the conditional character of statements; seeking to uncover implicit assumptions and study logically conceivable cases; concerned with validity and consistency. The analytic thinker is sensitive to symbolic notation, specialized terminology and the structure and logic of mathematical language; the analytic thinker understands the distance between symbol and object.

Among many aspects of metacognition, Schoenfeld (1987) highlights *self-regulation* as having direct implications for mathematics education. The self-regulated problem solver thinks before doing, making sense of the problem at hand. Self-regulation is also about planning and self-monitoring: keeping track of how things are going during a problem solving process. Based on these ideas, we characterize a *mathematical strategist* as someone who efficiently uses self-regulation.

From this perspective, we say that an individual *thinks mathematically*, when in his or her *behavior* we can see evidence of uses of theoretical thinking or mathematical strategies.

Course Design

Based on *design experiment* methodology, we drastically changed the traditional undergraduate mathematics classroom into an *intentional learning environment* where students acted as researchers and as teachers. The curriculum was changed to focus in *depth* on a few *recurrent themes*; a set of activities (in- and out- of class, and online) were created to engage students in mathematical thinking. Dynamic assessment tools were used to monitor progress, focusing on students' uses of mathematical thinking.

Each class, the students worked in small groups (3 up to 5 students) on 2 or 3 activities. These activities included problems easy enough so that students could mathematize them and make conjectures about their solutions but difficult enough so that they couldn't easily (or at all) construct a mathematical proof; problems for which they couldn't provide a definite answer; activities on the notion of truth, definitions, axioms; unpacking statements (Selden & Selden, 1995) and deconstructing proofs. Time was allotted for whole class discussions. The ultimate goals of these activities were to engage students in situations where we believe mathematical strategies, RT and ST are particularly useful, and in oral and written argumentations in which we believe ST and AT strongly come into play. The idea of concepts, techniques, etc. being discussed *on-demand* played an important role in the design of the activities. Approximately every 3 weeks the students had a quiz (4 in total) and there was a final exam at the end of the session – in every case, the students were given time to discuss the assessment problems in groups before engaging in writing individual answers. Also, a web site for the course was set up, including an online assessed activity (*weekly forums*). Students were expected to post questions, problems, solutions, or ideas that *meaningfully* contribute to the ongoing discussion. They could initiate a thread or respond to peers' or teachers' postings. The notion of *meaningful* was constructed through practice. Postings were typically triggered by the in-class activities of the corresponding week.

The course is 13 weeks long, with 2 weekly sessions of 75 min each. There were 28 students enrolled (of which 27 gave consent to the use of their data). A professor and a TA were present every class (we write *teachers* to refer to both of them).

Operationalization of the Model of Mathematical Thinking

To analyze students' thinking, we have operationalized the model for mathematical thinking by making a list of *behaviors* that would provide evidence that the individual is thinking mathematically. This list of behaviors is not exhaustive; it serves as a guideline for identifying and characterizing mathematical thinking in collected data – the data itself and further analysis may add to the list. Examples of these behaviors are: (a) evidence of mathematical strategies: efficient use of self-regulation and self-monitoring, considering different approaches, analyzing the problem (before doing), modeling, constructing examples (simple, pathological); (b) evidence of RT: actively seeking explanation and understanding, valuing one's personal effort and discovery, persistence in dealing with tasks, seeking generalization; (c) evidence of ST: referring to definitions, context and systems of concepts, showing concern for validity, engaging in proving or reasoning; (d) evidence of AT: a concern with the correct use of symbols and language, a concern towards disambiguation and specificity, valuing different representations.

Analysis, Results and Discussion

We consider three sources of data: students' postings in the weekly forums, students' answers to quizzes and final exam, and notes taken by the teachers during class or immediately after class trying to faithfully record what a student had said. Presently, we are using these sources to construct a *mathematical-thinker-profile* for each of the 27 students. Then, students will be classified according to these profiles. In what follows, we discuss preliminary results.

From the first class, students actively and loudly engaged in the (non-graded!) in-class activities and contributed to weekly-forums more than was required, participating in long discussion threads or sharing pertinent resources they found online. The composition of groups was mostly stable through the session and formed during the first classes without any intervention from the teachers. All groups had "leaders" – mathematically stronger than their peers. They would lead the discussions in class and also be the first to post in weekly forums. While the role of the leaders seemed to be to direct the "followers" attention towards meaningful aspects of a problem at hand, the role of the followers was to ask questions and challenge the leaders' approaches. We believe that followers where somehow mimicking the teachers interventions: teachers would listen to what a group was discussing and then challenge their approach with "what if…" type of questions. In our research, we don't account for how the leaders/followers positioning aroused but we do point out that while whole classroom discussions probably reinforced leaders' positioning, the role of followers was validated by the local interventions of the teachers. This structure promoted a research-work approach in each group, and a teaching-learning environment where stronger students will explain mathematical ideas, concepts and procedures to the weaker students. In this context, inside and outside the classroom, students were in control of the knowledge being shared: concepts were discussed on a need-to-know basis responsive to students' needs, typically by leaders, who would provide explanations to their groups but would also go to the blackboard and provide explanations to the whole class; teachers would intervene only when no one in the classroom could provide a mathematically valid concept, solution, justification, etc., to clarify, to provide alternative explanations or strategies, and to *institutionalize knowledge* (Chevallard, 1999).

The analysis of students' postings and answers to quizzes show that students engaged in mathematical thinking as defined by our model. The goal of the analysis is to provide a characterization of students as mathematical thinkers (*profile*) and to account for the development of mathematical thinking features. A full data analysis will be presented at the conference. We now provide a few examples of how data was analyzed and refer to the activities that triggered them. Each quote below comes from a different student.

During the first week, the discussion focused on the meaning of the logical operators "and" and "or", and how their mathematical meanings are different/similar to the uses we give them in everyday language. That week, a student posted: "If we look at the inequality $f(x)g(x)<0$ (where $f(x)$ and $g(x)$ are continuous and real), their products must produce a value that is less than 0. Specifically, $f(x)$ has a negative value <u>and</u> $g(x)$ has a positive value ($f(x)<0$ & $g(x)>0$) <u>or</u> $f(x)$ has a positive value <u>and</u> $g(x)$ has a negative value ($f(x)>0$ & $g(x)<0$)." Our analysis: The student stated the conditions under which the claim is true (ST); he created an example of a complex use of conjunction and disjunction (RT); and he detached symbolism from the objects discussed in class (integers) to create an example about different objects (AT).

In quiz 1, given the 7 bridges of Königsberg, students were asked whether a walk using each bridge exactly once is possible or not. Most groups engaged in an exhaustion process to conclude that the walk was not possible but many engaged in an "efficient" exhaustion (e.g., taking advantage of the symmetry of the problem), even when they had actually gone through every

case, they went back to improve the method (RT). Once the groups found that the desired walk cannot be done, the activity was extended to discussing under which conditions it would be possible. Their postings were a display of mathematical strategies. Some students wrote:

> "We thought that changing small aspects of the problem would make us figure out why [the walk] was not possible in the first place, by making special cases."
>
> "Once we realized [the walk] was not possible we went on to try configurations that did [allow for such a walk]; moving bridges and islands alike. [But then, we] started to look at it less in terms of where to put the bridges and more in terms of how many to put leading from one island to another."

The activity provoked other modes of thinking as well. E.g., showing concern for validity (ST), one student posted, "No, we didn't solve the problem, because we couldn't come up with logical statement[s] that describe exactly why the answer is No."

During the third class, the students were posed the question "what is the sum of the interior angles of a polygon". Without being prompted to, groups engaged first in discussing what a polygon is. This particular case was an overwhelming display of mathematical thinking, which included: discussions about definitions and connections to previous knowledge (RT and ST), discussions on notation and representation (AT) and the construction of simple and of pathological examples and of counterexamples (mathematical strategies).

The design had unexpected consequences: many students reported on a shift of their beliefs about mathematics (another aspect of metacognition that Schoenfeld highlighted as having implications for mathematics education). E.g., a student posted: "Before […], mathematics was simply numbers and formulas to me. I never really gave much thought to how those formulas and laws came to be. […] This class really got me thinking, how were these proven, what did it take to get to that proof, and even if it has been proven, is it really possible to show that a certain formula will work with everything?"

As an example of how students took control of content and knowledge, we recall that the notion of *meaningful post* was constructed through practice. By the fourth week, students were pointing out if a posting was unclear, incorrect, or if they couldn't see its relevance.

Whether students were already mathematical thinkers or if they became so as they engaged in the activities is something we can't answer at this point. Individual cases might be very different. Our conjecture is that in either case, there was *improvement*. Nevertheless, our analysis of data shows that the design provided students with *opportunities* to engage in mathematical thinking; something that doesn't seem to be a commonality in first-year undergraduate mathematics courses – and students did take advantage of these.

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SELF-EFFICACY BELIEFS OF HIGH SCHOOL STUDENTS IN AN ADVANCED COURSE IN PROOFS AND NUMBER THEORY

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In this study, we describe the effect of participating in an advanced course in proofs and number theory on gifted high school students' mathematics self-efficacy. The focus of the course was on learning how to prove theorems, and to that end, the instructor modeled higher order mathematical thinking with the intent of helping students develop mathematical habits of mind. We measured self-efficacy with pre- and post-surveys that included both Likert-scale items and open-ended question to determine ways self-efficacy was affected by course taking and whether effects influenced students' choice of college math courses.

Keywords: Affect, Emotion, Belief, and Attitudes, Reasoning and Proof, Advanced Mathematical Thinking

Introduction

The purpose of this study was to examine mathematical self-efficacy beliefs of high school students enrolled in an advanced mathematics course titled *Proofs and Problems in Number Theory and Algebra* (PPNTA). In this course, students were expected to engage in higher order mathematical thinking, develop skill in constructing proofs and showing counterexamples, participate in intellectual argument with their peers, and work both collaboratively and individually to solve difficult mathematical problems. Although the mathematically gifted students in this study generally have high math self-efficacy, their prior math experiences have emphasized calculation and content-specific objectives. This course was significantly different from the other advanced math courses students have completed because it focused on abstract properties such as elegance in proofs and metamathematics. We were particularly interested in studying self-efficacy in this context and with these students because of the way self-referent beliefs affect how students approach tasks, set goals, and persist when working through challenging tasks (Bandura, 1997).

According to Bandura (1994), self-efficacy refers to the beliefs individuals have about their ability to achieve a goal at a given level of performance. Self-efficacy affects the choices people make, and thus students with high self-efficacy typically make choices that lead to achievement, while students with low self-efficacy may engage in behaviors that undermine their success (Pajares, 1996).

Researchers have determined that gifted and high ability students typically have high self-efficacy and have a more accurate perception of their ability than do students of average aptitude. In our earlier studies (Hendricks & Millman, 2012), we found statistically significant differences in gifted students' pre and post course math self-efficacy. Further, there was less fluctuation in self-efficacy for students who were comfortable with the abstract nature of the course, whereas students who were more comfortable studying concrete concepts said changes in their self-efficacy were dependent on the difficulty of individual topics covered during the course. Though Pajares (1996) explains that gifted students' self-efficacy relies more on ability than performance, our findings did not bear that out. Rather, our results revealed that mathematically gifted students' comfort level with the abstract

nature of the course was a better predictor of self-efficacy than was cognitive ability, which was high and consistent across students in the course.

Methods

PPNTA Course

The course was developed in partnership between Georgia Tech's Center for Education Integrating Science, Math, and Computing (CEISMC) and a public charter high school with an emphasis on math, science, and technology. Funded by Georgia's Race to the Top award from the U.S. Department of Education, the course was taught by a Georgia Tech doctoral student in Industrial and Systems Engineering studying Algorithms, Combinatorics, and Optimization. CEISMC's director (a co-author of this paper and professor of mathematics) collaborated with the course instructor to plan the PPNTA course and made several classroom observations during the semester and delivered lectures. The intent of the 18-week course was to help students discover that mathematics is not based solely on memorization nor is it fundamentally about calculation. To accomplish this, we created an *explore, generalize, prove, think* environment that parallels the culture of that of mathematics researcher. The course included topics such as equivalence relations, topology, and elementary Lie groups—mathematical ideas that go beyond the abstraction of modular arithmetic. Other course topics are basic properties of integers; divisibility and prime numbers; the Fundamental Theorem of Arithmetic; linear Diophantine equations; equivalence relations and their applications; basic properties of polynomials; divisibility of polynomials, divisibility methods, and roots of polynomials; combinatorics; elementary group theory; public key cryptography; and problem solving with computer programming. In studying these topics, students were expected to (a) identify what makes a mathematical proof correct, (b) learn commonly applied proof techniques, (c) develop proficiency in reading and writing mathematics in general and as proofs or counterexamples, and (d) apply problem-solving methods to find solutions and demonstrate the accuracy of their methods.

Participants and Setting

Twenty-eight 12^{th} grade students—18 males and 10 females—were enrolled in the course. Eleven of these students had parental consent and/or assented to be in this study. Of these students 9 completed both the pre- and post-surveys, and our results are based on these students. Six were male, and three were female. Four students were Caucasian (44.4%), 3 were Asian (33.3%), and 1 student each were Hispanic and Middle Eastern.

The study took place at a public charter high school that emphasizes math, science, and technology. Approximately 600 students are enrolled in grades 9 through 12, including 40% Asian, 29% Caucasian, 18% African-American, and 8% Hispanic. The majority of students (74.5%) are in gifted programs. The school offers 17 Advanced Placement (AP) courses, including those in calculus, statistics, physics, biology, chemistry, computer science, and humanities. Our participants were high ability students, and most had completed college-level calculus before enrolling in PPNTA. At the beginning of the course we asked students about their general mathematics capability; 4 said they were *extremely capable*, and 5 said they were *capable*. Five students intended to pursue a university degree in engineering, including 2 double majors in computer science/electrical engineering, 1 in biomedical engineering, one in mechanical engineering, and 1 in general engineering. Two students planned to pursue degrees in computer science, and 2 planned to study chemistry.

Data Collection

Changes in students' mathematical self-efficacy were measured using pre and post self-efficacy scales we developed for an earlier study (Hendricks & Millman, 2012). Items were developed based on Bandura's *Guide for Constructing Self-Efficacy Scales* (2006) and were aligned to course objectives. For example, students rated their confidence in areas such as *identifying fallacious*

reasoning in proofs and *learning different methods to construct proofs*. Following Bandura's guide, pre- and post-survey items related to what students *can do* to measure their perceived capability. On pre-surveys, students also were asked to rank their confidence in achieving each course goal. To clarify, on the pre-survey, students ranked both their current ability for each course goal as well as their confidence that they would be able to reach each goal.

The pretest, given near the beginning of the course, included 13 items related to course goals. For each item, students ranked their self-efficacy on a scale from 0 (low self-efficacy) to 100 (high self-efficacy) for both their current ability for the course goal (*How confident are you that you can already construct valid proofs?)* as well as their confidence they would be able to reach the goal (*How confident are you that you can learn to construct valid proofs?*). The post-survey contained identical items but students only ranked their current ability. The post-survey also contained 11 items that related to students' experience in the course.

Results and Discussion

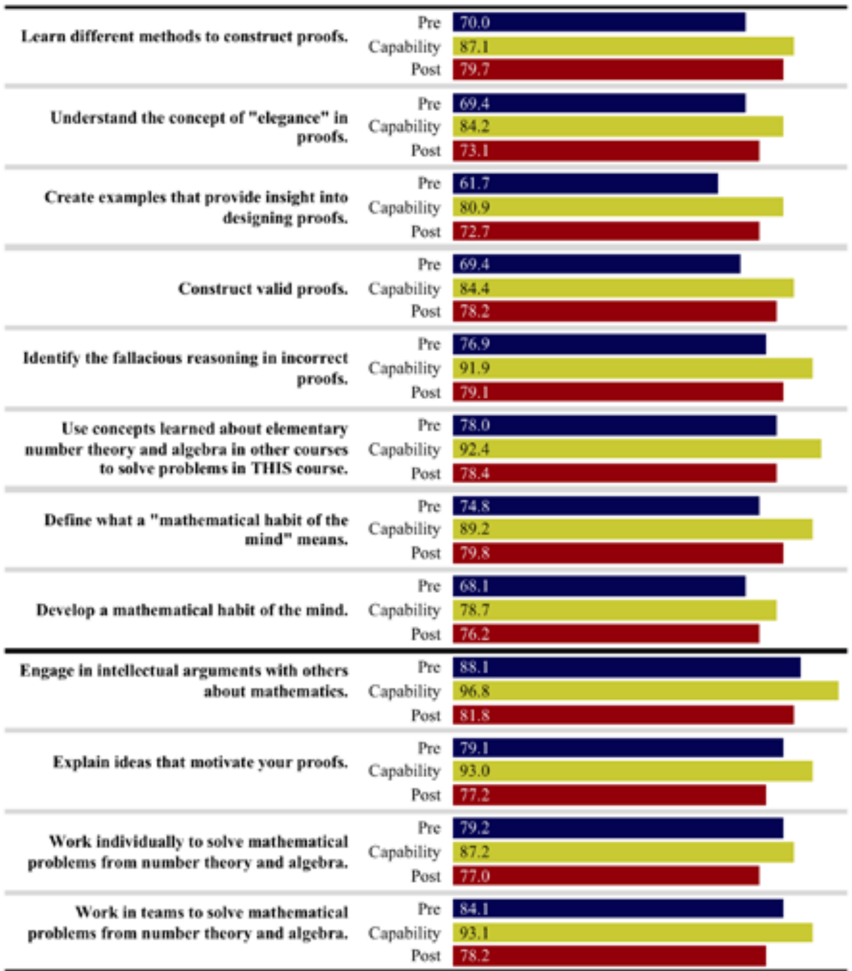
We compared students' self-efficacy beliefs at the beginning of the course (pre self-efficacy), their self-efficacy related to reaching course goals (potential capability), and their self-efficacy at the end of the course (post self-efficacy). Because of the small sample size (n=9), we have provided descriptive statistics only. Students' average pre self-efficacy (SE) across all SE items was 75.5, and their average potential capability was 88.7. However, on the post-survey, average SE was 78.0, which was almost 11 points lower than their estimated potential and just 2.5 points higher than their pre-SE. We saw positive pre-post changes in SE on 8 items, negative changes on 4 items, and no change on 1 item (see Table 1). SE increased most on the items *Create examples that provide insight into designing proofs* (+11) *Learn different methods to construct proofs*. (+9.7), *Construct valid proofs* (+8.8) and *Develop a mathematical habit of the mind* (+8.1). The greatest areas of decrease were on items *Engage in intellectual arguments with others about mathematics* (-6.3) and *Work in teams to solve mathematical problems from number theory and algebra* (-5.9).

Seven of the 9 participants said their mathematics confidence had increased due to their experiences in the course. One student's confidence decreased and another's stayed the same. Students whose confidence increased said they learned to think critically and reason through problems. One student explained, "*I felt like every problem we encountered in this class was a familiar one, but we gained a new insight on how that problem is solved or why a theorem works*." Another wrote, " *I had the chance to examine the reasoning behind the math that I've been using and understand why the formulas and theories work*."

In describing the course a student stated, "I *learned much more about how mathematics works than in any other class. I enjoyed all of the problem sets because they felt more like a challenge rather than another 'problem' to solve*." Other students also described the challenging nature of the course. One student, for example, stated "*It was a bit difficult to follow sometimes because there was a lot to think about all at once but it was cool once I got it*." Another said the course "…*was very frustrating when I fell behind and did not know what was going on, but those moments when I finally [understood] a problem were interesting and fulfilling, maybe even fun*."

Three students (two females and one male) said taking the course sparked their interest in taking similar courses in college. A female student wrote, "*Even though I struggled, I know this kind of math exists and it can help me decide what kind of class I want to take in college*." A male student planning on a double major in chemistry and math wrote, "*The course sparked my interest in proofs and other fields of rigorous mathematics, and I am excited to take more advanced math courses in college*."

Table 1. SE Changes Pre, Capability, & Post



As described by other students in previous semesters, students both enjoyed and were challenged by this course. For example, when asked what she liked best and least about the course, a student said, "*Understanding a problem after working on it for a long time; NOT understanding a problem after working on it for a long time*." Another student wrote, "*I liked it best when I was completing a proof and I'd look back on it and marvel at how elegant and simple it all was in hindsight. I liked it least when I found myself completely stumped by a difficult proof.*"

Though our sample size was small, results of this study are similar to our findings with two other groups of students. Students found the course challenging, they tended to overestimate their capability, but completing the course increased most students' mathematical self-efficacy. Further, a third of the participants in this study, including two females, said their experience in the course had affected their decisions about the math courses they will take in college. As enrollment for the course continues to increase, our goal is to replicate the study with a larger group of students. Further, we plan to track students into college to determine whether they do indeed choose to take advanced math courses based on their experiences in the PPNTA course.

NOTE: This research is Funded by the Georgia Department of Education through the U.S. Department of Education *Race to the Top* grant.

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MATHEMATICAL MODELING PROBLEMS: WHAT AFFECTS TEACHING ASSISTANTS' ABILITY TO PROVIDE FEEDBACK?

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This paper describes opportunities and challenges teaching assistants (TAs) experience when they provide feedback on mathematical modeling problems. Eight TAs were interviewed regarding issues they experience while working for engineering courses in which about 2000 undergraduates create mathematical models to solve realistic problems and communicate their solutions. The opportunities and challenges are associated with the following factors: the process of providing feedback; TAs' perspectives of modeling problems; and professional training to learn how to provide effective feedback on students' solutions to mathematical modeling problems. The specific factors that influence TA's abilities to provide formative feedback are discussed and the use of a reflective practice that enables students to understand what abilities they gain through mathematical modeling activities is suggested.

Keywords: Assessment and Evaluation, Modeling, Problem Solving

Introduction

It has been a challenge to create learning environments where students create, revise, and develop their intuitive mathematical thinking to more refined ways of reasoning (Rasmussen & Blumenfeld, 2007). At the undergraduate level, diverse student populations contribute to the challenge (Holton, 2001). At large research institutions, the critical role teaching assistants (TAs) play in helping large and diverse student groups also adds to the challenge. At Purdue University, TAs help approximately 2000 first-year engineering students each semester solve mathematical modeling problems. By examining the factors that influence TAs' ability to help students solve MEAs, this study aims to answer the following research question: What affects TAs' abilities to provide feedback on students' solutions to mathematical modeling problems?

Theoretical Framework

Model-Eliciting Activities

Model-Eliciting Activities (MEAs) are mathematical problems set in real-world contexts that challenge student teams to create a mathematical model and communicate their mathematical thinking (Lesh, Hoover, & Kelly, 1993). MEAs are used not only in mathematics education but also in other disciplines such as engineering and science education (Hamilton, Lesh, Lester & Brilleslyper, 2008). The six design principles (Lesh et al., 2000) are used to develop MEAs for Purdue University's First-Year Engineering (FYE) courses (Diefes-Dux et al., 2008). Student team work is assessed along four dimensions (Diefes-Dux et al. 2012). The *Mathematical Model* dimension focuses on whether or not the solution addresses the mathematical complexity embedded in the situated problem. The *Modifiability* dimension focuses on the justifications supplied for the mathematics used to solve the problem. The *Re-usability* dimension focuses on

the brief communication of the problem and its solution, and the *Share-ability* dimension addresses whether or not it is easy for the client to follow and apply the solution.

TA Training and Feedback Process

TAs have become integral to the success of introductory classes at universities (Luft et al., 2004). However, little research has inquired into the challenges faced by TAs and their beliefs about teaching (Speer et al., 2005). Owens (2011) noted one of the challenges TAs faced was working beyond the scope of their abilities, which could lead to students' misconceptions. Owens' concern is related to TAs' role in providing formative feedback because their feedback affects how students move forward from misunderstandings. Formative feedback is "information communicated to the learner that is intended to modify the learner's thinking or behavior for the purpose of improving learning" (Shute, 2008, p.1). By learning how to provide formative feedback, TAs will be able to identify student problems, relate their performance to the goal, and provide information that will move students towards that goal (Dale, 2006). Identifying factors that positively influence TAs' ability to provide feedback and creating an environment that is conducive to the TAs' success are essential steps in implementing real world mathematical modeling activities in the classroom.

Methods

In Fall 2012, there were approximately 90 TAs for two FYE courses that worked with a total of 2000 FYE students. Some of TAs' common responsibilities included preparing for class, helping with in-class activities, attending training, and grading. Thirty TAs volunteered for this study. From these, four undergraduate teaching assistants (UTAs) and four graduate teaching assistants (GTAs) were randomly selected to participate in this study.

The eight TAs participated in a semi-structured interview that varied in length from 45 to 90 minutes. The interview questions were divided into seven categories: background information, hiring process, overall experience, thoughts on MEAs, training, expectations and responsibilities, and support and mentoring. These categories were developed after reviewing literature on the implementation and assessment of mathematical modeling activities (Diefes-Dux et al., 2008).

Interviews were audio recorded and transcribed. The research team identified the interviewees' responses in the transcripts that were relevant to MEAs and discussed factors that affected the TAs' abilities to provide feedback to MEAs teams' solutions. These responses were coded as different factors affecting their performance (e.g. unclear grading rubric and the discussion portion of training). The factors were then grouped together based on similarities, which resulted in nine categories of factors that TAs discussed. Three categories were closely related to our research question and discussed by more than half of the TAs.

Results and Discussion

The three main categories were feedback process, training, and their perspectives of MEAs. The UTAs are referred to as Uriah, Ursula, Ulysses, and Udo, and the GTAs are referred to as Gail, Grace, Greg, and Gus. All names used are pseudonyms, but are gender appropriate.

Feedback Process

All of the TAs discussed aspects of the feedback process that had positive and negative effects on their ability to provide feedback. The TAs discussed struggling with the lack of a system to adjust MEA team scores to individual scores, the concept of quantifying subjective work, and the scores on the final response being the only ones that affect students' grades. Gail explained her difficulty in scoring students' work:

> *We're taking a very subjective assignment and trying to turn it into a something that can be quantifiable and I am not saying it can't be quantified, it's just like I said, if I were to look at one team individually I can use the rubric it makes sense, but if I am going to look at this team in comparison to other teams [it doesn't work], which is not suggested, but as a GTA it is hard to not look at compared to how other people are doing.*

Training

All of the TAs discussed various aspects of the MEA training. The two main components of training are the online and face-to-face portions. Most of the TAs found it challenging to solve the MEA online prior to face-to-face training, but rewarding because it enabled them to better understand the potential problems and challenges that the student teams may face. TAs' perceptions of calibrations, which involve comparing their feedback to that of an expert, were mixed with positive and negative feelings. One TA noted that comparing their feedback to the experts helped them better understand their own successes and shortcomings. Other TAs noted that there were either too many calibrations and/or the online training required too much work.

On the other hand, TAs generally found face-to-face training helpful. They indicated the discussion time, the expert's opinions, potential solutions, opportunities to ask questions, and information about providing constructive feedback the most helpful features of this forum. Greg discussed some components he found beneficial about training:

> I think that [calibration] helps a lot because then you know how specific of details they want and you know how well you're doing compared to someone who knows what they're doing. ...you also have your own MEA to write, which is a good idea because then you know ...where they [students]'re gonna have trouble. ...I think [the face-to-face training] is good because then we actually get to ask questions and go into detail about different examples of how to grade ...and then we can actually practice in class.

Some TAs stated that the lecture incorporated into training contained too much content, which made it difficult to stay focused and remember the information. The major criticism that a few TAs had about the training was that it did not cater to varying levels of expertise resulting in repetition, especially for experienced TAs. Uriah proposed the following solution:

> I feel [having two trainings] would be better. The first one put it into like here is generally what goes on in an MEA and the second this is what we are working on. [Then] you're only learning things you need to learn.

TAs' Perspectives of the MEAs

The majority of the TAs discussed their perspectives of MEAs as a pedagogical tool in the FYE courses as well as issues with the implementation of this tool. The TAs who discussed the use of MEAs as a pedagogical tool considered them generally effective. However, some TAs discussed that the FYE students and sometimes even they themselves did not understand the purpose of the MEAs, which contributed to lack of student motivation and/or misunderstandings of TA feedback. Uriah discussed his perspective on the purpose of MEAs:

> I am not sure I fully understand the purpose of MEAs… I know there is a reason they are there… I don't think [students] understand why they are doing [MEAs]…their immediate reaction is well why? What am I gaining from this?…I don't think they understand the purpose behind it, like if there is a greater meaning to it.

The TAs also indicated that unclear language results in students' misunderstanding of the purpose of the assessment dimensions and feedback provided by TAs, but students have positive perspective on MEA once they know the purpose. Grace stated the following regarding MEAs:

> I like the overall intent of MEAs. I think it is actually a good method. And you're walking students through ...how to develop a solution process. How do you implement that? But it's also very frustrating. Just because students you know don't seem to get it at first so, it's brand new, so that's how most learning experiences are. I think it's a decent method.

Conclusion and Implications for Practice

This study is an analysis of TAs perspectives on providing feedback for mathematical modeling activities. Results show specific factors that influence TA's abilities to effectively provide feedback. These findings can be helpful for reforming both current and new programs that implement mathematical modeling or any open-ended problem solving activities.

The current training is the most fundamental component to the success of the TAs. Although MEAs are research-based pedagogical tools that have been heavily researched and developed, there are some miscommunications between the researchers and the practitioners. The TAs commonly complain that students do not understand why they are solving MEAs and some of them do not even understand the learning benefits of MEAs. The learning objectives of MEAs needs to be better communicated in training and the course. It is beneficial to consider implementing a reflective practice that enables the students to try to understand what abilities they gain through a mathematical modeling experience.

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STUDENT MISCONCEPTIONS AND GUESSING ON MULTIPLE-CHOICE EXAMS IN MATHEMATICS

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Multiple-choice assessments can be used to provide rapid and large-scale feedback to students in mathematics. In this study, the nature of student guessing on a multiple-choice exam was used to gain insight into student misconceptions of function. Three students in a university-level College Algebra class were asked in individual interviews about their responses on a multiple-choice exam. Interviews were analyzed using three conceptions of function: operational/structural; APOS (action, process, object, schema) Theory; and concept image. Findings include: a disconnect between algorithms, end-goals, and answers; the failure or lack of algorithms leading to guessing; and the use of incomplete- or pseudo-reasoning to inform guesses.

Keywords: Algebra and Algebraic Thinking, Assessment and Evaluation, Cognition

Introduction

"I don't really know how to do the rest of this, but I think I picked B because I knew I had a minus five there and a plus five there, so they would cancel and my answer wouldn't have any fives."

The preceding is a quote from a student explaining her answer on a multiple choice exam question. As it turned out, B was the correct choice, and the student therefore got credit for answering the question, despite the fact that she did not fully understand the mathematical concept. In mathematics, open-ended questions have been generally preferred to multiple-choice to assess higher-cognitive problems, as one generally receives a richer response from an open-ended question. However, several researchers have noted that since open-ended responses require more time, fewer items can be administered (Hamilton, 1998), and scoring by humans on open-ended responses requires a much greater amount of time on the part of the rater.

The purpose of this qualitative study was to examine the nature of student guessing on multiple choice exams in college algebra. The core research question was "What is the nature of student guessing on multiple choice items?" From this core research question, I developed three sub questions: (a) What are the causes of student guessing? (b) At what point does a student decide to guess? (c) When a student decides to guess, what informs the choice he or she makes?

Theoretical Framework

For this study, I focused my analysis on the concept of function, since there is a large body of existing research on students' conceptions of function. Using Sfard's (1992) operational-structural ontological duality, Dubinsky and Harel's (1992) APOS framework, and Vinner and Dreyfus' (1989) ideas of concept image and concept definition, I analyzed students' conceptions of function as 1) pre-function, operational, or structural, 2) pre-function, action, process, or object, and 3) being aware of contradictions or unaware of contradictions. Using these frameworks, I hoped to gain insight into reasons for student guessing.

Methodology

Participants and Setting

The population of this study was students over the age of 18 enrolled in a university-level College Algebra course. Data was collected from confidence questions on the students' second exam. For each problem, participants were asked to give their level of confidence in their answer using a LIKERT scale, with the statement: "I feel confident in my answer to the question" and responses 1-strongly disagree, 2-disagree, 3-neither agree nor disagree, 4-agree and 5-strongly agree. Three students were chosen for interviews, responding to at least one confidence question with 1-strongly disagree. Students were asked to elaborate on their answers to five questions, some differing between interviews, with topics including: 1) identifying the difference quotient, 2) finding the vertex, 3) identifying the graph of a higher order polynomial, 4) identifying the factored form of a higher order polynomial, and 5) identifying the transformations of a basic function.

Data Analysis

Thematic analysis was performed (Patton, 2002) on the transcripts. Causes of guessing, the moment at which the participant decided to guess, and the information used to inform the guess were identified in initial codings. These were identified as Cause, Moment, and Choice, respectively. Each was coded as operational or structural using Sfard's conception and as action-, process-, or object-level using APOS Theory. Codes developed for evoked concept image were: (a) contradictory, unaware; (b) contradictory, aware; (c) non-contradictory, alternative; (d) non-contradictory; and (e) incomplete. Contradictory was defined as the participant using an idea that is logically incompatible with mathematics. Unaware was defined as a participant not checking the veracity of a claim, and so aware was defined as a participant being aware of the veracity of a claim. Alternative was defined as using knowledge that is not incorrect, but would have proven difficult to provide correct answer in an open-ended response. Incomplete was defined as a participant not possessing a rich enough concept image to give an educated answer.

Findings

In most cases, the students attempted either an operation in the operational/structural sense, or an action in the APOS sense. To capture the similarities between these two, I defined an *algorithm* as any process in which students performed algebraic manipulations or identified algebraic expressions. Examples of algorithms included but were not limited to: 1.) factoring quadratics; 2.) completing the square; and 3.) identifying multiplicities. These operations and actions were intended to produce some kind of result, so I defined an *end-goal* to be a concept or object intended to be found at the end of an algorithm or result of identification. Examples of end-goals include but were not limited to: 1.) the x-intercepts when factoring a quadratic; 2.) the vertex when completing the square to re-write the expression in vertex form; and 3.) the multiplicities of given factors of a polynomial function. In this way, the purpose of finding end-goals was to narrow down the choices to a single answer. Using these constructs of algorithm and end-goals leading to answers, I generated the following themes:

Disconnects Between Algorithms, End-goals, and Answers

For these students, there was at times a disconnect between the algorithms they applied to the problems, the end-goals that they were attempting to find, and the actual answers to the problems. For example, one participant, Emma, attempted to solve for the vertex of a parabola by completing the square. Her end-goal, however, was not to get the expression into vertex form,

but rather to find the x-intercepts. Though she arrived at a method to find the vertex in the roundabout fashion of finding the intercepts and finding the point exactly in between them, it was clear that the disconnect between her algorithm and end-goal caused her to rewrite the expression incorrectly and was then forced to guess. For the same problem, a second participant, Zach, attempted to factor the quadratic expression to find the x-intercepts, but this had no relation to the question of finding the vertex.

In opposition to this, another participant, Zoe, generally had clear connections between her algorithms, end-goals, and the answers to the problems. When she attempted the problems, she applied algorithms for which she understood the end-goal, and she knew that the end-goal was appropriate for answering the problem in most cases. This generally led to Zoe not having to guess on most of the problems, but being able to make choices that followed directly from her work.

Absent or Incorrect Algorithms

In many cases, student-guessing was caused by the lack of an algorithm, or applying an inappropriate algorithm to answer a problem. In both Emma's and Zach's cases, they were unable to properly substitute for $f(x+h)$ in a difference quotient problem, Zach lacking an algorithm and Emma using an incorrect algorithm of substituting $f(x)+h$. For a problem where the students were given a polynomial and asked to identify its factored form, both Zach and Zoe were unable to determine an algorithm that they could use to factor. Zoe, in contrast, used a work-around of multiplying each answer to check against the original polynomial, but she clearly did not have the understanding to answer an open-ended version of the question.

When the participants possessed well-defined algorithms, they were generally able to use deductions to arrive at an answer. Zoe was able to work through the difference quotient quite easily, because she possessed an algorithm for substituting in for $f(x+h)$ that was well-defined in her mind. Similarly, Emma was able to clearly explain how she solved for the factored form of a polynomial using synthetic division, providing an explicit description of the algorithm she followed. Even Zach, who did the most guessing out of all three participants, was able to clearly identify the transformations of a function, because he had a well-defined algorithm of identifying constants in the formula.

Algorithm Failure Led To Guessing

Regardless of the correctness or incorrectness of an algorithm, the participants generally did not begin to guess until their algorithm failed. On both the difference quotient and vertex problems, Emma and Zach both ran into difficulties and could not complete their algorithms. Once this occurred, they both guessed on each problem. When Zach multiplied out all the constant terms in the factored polynomial, that particular algorithm did not serve to narrow down his choices, so he guessed at random without choosing another algorithm to attempt the problem.

When algorithms completed, regardless of whether they were correct or not, the participants were able to make informed choices. In a problem asking students to identify the graph of a fourth-degree, factored polynomial, both Zoe and Emma were able to determine that the polynomial was degree 4. Even though both incorrectly stated that the polynomial had to have three turning points instead of at most three turning points, they were both able to narrow down to a single choice, which was the correct answer. When Zoe answered the vertex problem, even though her algorithm for finding the vertex used an incorrect vertex formula, Zoe was still able to make an informed choice based on her work.

Use of Incomplete- and Pseudo-reasoning

When the participants decided to guess, many times their concept image was either

incomplete or contradictory, regardless of whether the participant was aware of it or not. In this context, *incomplete-reasoning* was defined as the use of reasoning that I would consider "not false," but would prove difficult to apply in an open-ended version of the problem. Some examples of incomplete-reasoning included Emma's claims in the difference quotient problem that the constants involved canceled on top and bottom, and therefore the answer should have no constants. When identifying the graph of a polynomial, both Zoe and Emma used their understanding that the graphs other than their answer choice had too many turning points. While this was not false, they would have had difficulty producing the graph from just the equation using only that knowledge.

To capture when the participants possessed a contradictory concept image, I defined *pseudo-reasoning* as the use of reasoning that I would consider contradictory, with no exploration of the veracity of the reasoning. For the difference quotient, Zach claimed that since the difference quotient had an h in the numerator, which was inside $f(x+h)$, and an h in the denominator, they should "cancel." Zach was almost certainly referring to canceling via division of like terms, although this would seemingly disobey several field axioms as well as any understanding of $f(x+h)$, but he did not feel compelled to determine whether his conjecture made sense or not. When identifying the graph of a higher order polynomial, Zach claimed that since the polynomial was degree four, it was like "two quadratics" multiplied together, so the graph should look like two parabolas that were "connected." This statement seemingly indicated a fundamental error in understanding the nature of functions, even though he was able to choose the correct answer from this reasoning.

Conclusions

In this study, I conducted interviews with three College Algebra students to describe the nature of student-guessing on a multiple-choice mathematics exam. Three subquestions arose out of this core question: (a) What are the causes of student guessing?; (b) At what point does a student decide to guess?; and (c) When a student decides to guess, what informs the choice he or she makes? I discerned that the participants had a disconnect between their algorithms, the end-goals of those algorithms, and the problems they were attempting to answer. The most common cause of guessing was the participant's lacking a correct or using an incorrect algorithm. It was the failure of these algorithms that was the impetus for the participants to guess, even if the algorithm was nonsensical. When the participants decided to guess, they relied on either incomplete- or pseudo-reasoning to narrow their choices and arrive at an answer that at least had some semblance of deduction.

These results have implications in both pedagogy and assessment, as the use and effect of multiple-choice examination has rarely been studied in mathematics. Further research is required to fully understand students' use of algorithms and their associated end-goals, and should be integrated with current research on mathematical reasoning.

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PRESERVICE TEACHERS' CONCEPTIONS OF IDENTIFYING AND DEFINING THE MATHEMATICAL PRACTICES

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This study examines preservice teachers' (PSTs') proposed use of the Mathematical Practices in an algebra task. PSTs in the study struggled to understand in their proposed teaching of the task that the quantity of Mathematical Practices did not necessarily translate to quality of instruction. PSTs described the practices in multiple ways – sometimes in line with the descriptions in the standards documents and sometimes not. The findings suggest the need to continue to flesh out the practices and provide PSTs with experiences to understand, define, and implement the Mathematical Practices.

Keywords: Algebra and Algebraic Thinking, Problem Solving, Standards

Purpose of Study

The Common Core State Standards (CCSS-M) Mathematical Practices (2010) provide a framework for how teachers can engage students in mathematical work. While the CCSS-M document spells out in limited detail how teachers and students might enact the Mathematical Practices, preservice teachers (PSTs) may not have a clear vision of what it means to enact them. We had PSTs explore and explicate how they would attend to the Standards for Mathematical Practices (SMPs) in their teaching of the Skyscraper Windows task (adapted):

> A building is 12 stories high and is covered entirely by windows on all four sides. Each floor has 38 windows on it. The cost of washing the windows is $2.00 for each first-floor window, $2.50 for each second-floor window, $3.00 for each third floor window, and so on. How much will it cost to wash the windows of this building? What if the building is 30 stories tall? *n* stories tall? (Driscoll, 1999, pg. 70)

We explore which SMPs PSTs chose for the task, how their vision of the SMPs compares to the CCSS-M document, and how this work might have implications for future work with PSTs. Our research questions are: 1) How do PSTs identify how they might use the SMPs? and 2) How do PSTs' conceptions compare to the intent of the CCSS-M document?

Theoretical Framework

This study uses a situative perspective, in that learning is "situated in physical and social contexts, social in nature, and distributed across persons and tools" (Putnam & Borko, 2000, p. 12). As PSTs considered the SMPs within the context of their future classroom implementation of the task, they drew on interactions, prior experiences, and engagement in various communities of practice (as K-12 students, college students, PSTs, in practicum, etc.). PSTs' interactions with their environments and other people help determine what and how learning takes place.

Methods

Participants

Participants included 15 undergraduate secondary mathematics PSTs from two mathematics methods courses at two universities in the Midwest. Eleven PSTs had been in their first methods course for no more than two weeks during the time of the study; the remainder was in first methods for no more than four weeks. Both groups had not studied the SMPs prior to this time.

Data Collection

PSTs completed the task individually, read the CCSS-M document (2010), and wrote a reflection on how to include the SMPs. The reflection prompts included PSTs drawing on and using examples from their work to explain how to include the SMPs in teaching this task. During class time, PSTs compared their mathematical solutions to the task with their peers and reflected on how their future students could demonstrate the SMPs.

Data Analysis

Qualitative data analysis occurred during and after data collection to focus and shape the study (Glesne, 2011). We focused our analysis on the PSTs written work, focusing specifically on PSTs' responses to how they intended to implement the SMPs in their instruction. We calculated the frequency of each SMP PSTs chose. We then used open coding (Strauss & Corbin, 1990) to analyze PSTs' written reflections and generated themes within each SMP, examining statements for emerging themes across multiple or all SMPs and comparing the PSTs' statements to the CCSS-M document (2010) in order to identify consistencies and inconsistencies across themes.

Findings

The PSTs selected multiple SMPs for the Skyscraper Windows Task. Here we focus on the two SMPs the PSTs identified most often: *Make sense of problems and persevere in solving them,* and *Model with mathematics.*

Make Sense of Problems and Persevere in Solving Them

Eighty-seven percent of PSTs identified *Make sense of problems and persevere in solving them* as a SMP to include in their teaching of the Skyscraper Windows Task. Below we describe the three emerging themes related to PSTs intended implementation of this SMP.

First theme: get students to identify information. Almost half of the PSTs (47%) shared that students need to identify the important given information to solve the problem. A PST commented that a teacher could focus on this SMP through "having the students break down the givens and ask for-s in the problem." Similarly, another PST noted, "You must understand or have a feel for what the question is actually asking you." These statements align with the CCSS-M document (2010): "[Students] analyze givens, constraints, relationships, and goals." The PSTs attention was on supporting students to make sense of the task and the key information therein.

Second theme: Reduce the initial demand of the task. Thirteen percent of PSTs noted that one way to support students in persevering in solving the task would be to scaffold the task by reducing the initial demand of the task. Instead of having students begin with solving for *n* floors, multiple PSTs suggested beginning with fewer floors:

> The best way I can think to help students persevere in solving this problem is to reduce the initial challenge by reducing the floors in the first example. A little success can go a long way in boosting a student's ability to persevere in the fact of adversity.

The CCSS-M document (2010) highlights that students might "consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution." It is unclear whether the PST meant the teacher should provide an analogous or simpler problem or if the students should be working on their own to recognize a strategy of

working with a smaller-story building before finding the cost of a 30- or *n-story* building.

Third theme: Initiate at the beginning of the task. Thirteen percent of PSTs noted they would engage students in persevering though the problem as students begin working on it. PSTs focused on what students would need to do once a teacher gave them the problem. One PST commented, “It [the SMP] would be used at the beginning of the task. After the students have had the chance to read the task and work on it for a few minutes.” This comment might align with the CCSS-M statement (2010), “Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution.”

Comparison to CCSS-M document. The CCSS-M document (2010) states, “Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends.” PSTs proposed specific prompts to move their students’ thinking forward, but the PSTSs did not identify how the prompts would help students make sense and persevere in problem solving. For instance, PSTs might ask their students to: explain the relationship between the floor number and how much it costs to get the windows cleaned; explain which variables you are using in your equation and why; list how you could calculate the price for the first six floors; and/or explain whether your rule works for even and/or odd number of floors, etc. While these statements align with what the CCSS-M document advocates, there is a disconnect between the questions PSTs might ask to move students’ thinking forward and whether the prompts support their students in demonstrating this SMP.

Model with Mathematics

Eighty-seven percent of the PSTs identified that they would include *Model with mathematics* in their teaching of the Skyscraper Windows task. We describe the three emerging themes on this SMP in more depth.

First theme: Represent a situation with an equation, table, or graph. Over a third of the PSTs (34%) commented that their students would use and make “connections between tables and graphs” when they solved the Skyscraper Windows Task. Additionally, one PST noted, “The goal of the entire problem is to come up with a mathematical expression that represents the problem.” Another PST explained that her students would need to move between two representations: “It will help students visually see and understand the relationship between functions and tables—knowing how to use tables and what an equation should look like depending on the degree of the polynomial.” This response particularly aligns with the CCSS-M document (2010) that notes students should “identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas.” PSTs focused some of their attention related to this SMP on the representations their students might use in solving the task.

Second theme: Real life context helps bridge connection to mathematics. Over a quarter of the PSTs (27%) reflected that the Skyscraper Task included *Modeling with mathematics,* because the task inherently included a “real life” context. One PST reflected, “This is a real life model, so it teaches students how to work with math in the real world. This will help students work with math in the real world.” Another PST noted, “This problem is a real life type of problem so it is modeling with math in and of itself.” The language in the CCSS-M document (2010) could have motivated these statements: “Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace.” Even though a problem uses a real world context, the context itself does not constitute modeling.

Third theme: Use the “follow me” method. Seven percent of responses documented that

Model with mathematics was an instructional strategy where a teacher does the problem first and students follow the teacher's demonstration. A PST suggested: "I could help my students do this problem by modeling similar problems and pointing out how to find solutions." Such language is not present in the CCSS-M (2010) document.

Comparison to CCSS-M document. A key component missing in all PSTs' reflections for this SMP was that their students should be able to assess the strength of their relationships, logic, and conclusions. Instead, PSTs focused only on using graphs and tables and having a real-life context. In addition, PSTs did not concentrate on their students' ability to "analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose" (CCSS-M, 2010).

Discussion & Conclusion

PSTs attempted to explicate how they would include the SMPs in the teaching of the Skyscraper Windows Task. Although the CCSS-M document articulates how students can demonstrate the SMPs, our findings illustrate that PSTs' conceptions of the SMPs do not always align with the CCSS-M document.

PSTs needed to grapple with the fact that the *quantity* of the SMPs in the teaching of a task does not necessarily translate to the *quality* of that instruction. Over a third of the PSTs (33%) stated they would integrate all eight SMPs into teaching this task. While certain tasks can lend themselves to including some mathematical practice more easily than others, many of this study's PSTs seemed to believe that they could enact a SMP based simply on the design of the task: the "CCSS-M mathematical practices sort of fall into place with this problem." Another PST suggested, "I feel as though several of these practices are already embedded in this problem and I don't think a teacher could not use at least a few of these practices simply by presenting the problem." These comments convey that the PSTs felt that SMPs were intrinsic to the Skyscraper Windows task. The PSTs needed to focus their attention instead on how their students might be able to demonstrate behaviors indicative of SMPs within the task implementation. PSTs need opportunities to understand that the SMPs are largely dependent on the teacher, and the teacher needs to provide appropriate scaffolds so students demonstrate them.

PSTs may need multiple opportunities with different contextual activities to better understand the SMPs and how to include them in their instruction. Further discussion with other mathematics educators should focus on continuing to flesh out what the SMPs encompass and working on ways to support PSTs in understanding and implementing them in their classrooms.

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MENTAL MATHEMATICS, OPERATIONS ON FUNCTIONS AND GRAPHS

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This study, part of a larger research program on mental mathematics with objects other than numbers, concerns operations on functions in a graphical environment. It aims at characterizing students' mathematical activity in this mental mathematics environment. The data analysis offers understandings of how students solved the various tasks, highlighting three specific approaches engaged with: algebraic/parametric, graphical/geometric, and numerical/graphical.

Keywords: Instructional Activities and Practices, Algebra and Algebraic Thinking, Problem solving

Context of the Study

Diverse studies show the significant impact of mental mathematics practices with numbers on students' problem solving skills (Butlen & Peizard, 1992; Schoen & Zweng, 1986), on the development of their number sense (Heirdsfield & Cooper, 2004), on their paper-and-pencil skills (Butlen & Peizard, *ibid.*) and on their estimation strategies (Heirdsfield & Cooper, *ibid.*; Schoen & Zweng, *ibid.*). For Butlen and Peizard (*ibid.*), the practice of mental calculations can enable students to develop new ways of doing mathematics and solving arithmetic problems that the traditional paper-and-pencil context rarely affords because it is often focused on techniques that are efficient and do not require other action. Overall, there is agreement that the practice of mental mathematics with numbers enriches students' mathematical work about calculations and numbers. This being so, as Rezat (2011) explains, most if not all studies on mental mathematics focus on numbers/arithmetic. This rouses interest in knowing what teaching mental mathematics with mathematical objects other than numbers might contribute to students' mathematical reasoning. In this study, issues of functions, mainly operations in a graph, are investigated.

Theoretical Grounding of the Study: An Enactivist Frame

Recent work in mental mathematics points to the need for better conceptualizing how students develop mental strategies. Threlfall (2002) insists on the organic emergence and contingency of strategies in relation to the tasks *and* the solver (what he understands, prefers, knows, has experienced with, is confident with; see also Butlen & Peizard, 1992; Rezat, 2011). In mathematics education, the enactivist theory of cognition has been concerned with issues of emergence, adaptation and contingency of learners' mathematical activity (c.f. Maturana & Varela, 1992; Varela, Thompson & Rosch, 1991). Therefore, aspects of the theory (in particular, Varela's concept of *problem-posing*) are used to ground this study to make sense of students' mathematical strategies and activity. For Varela (1996), problem-solving implies that problems are already in the world, waiting to be solved, independent of us as knowers. Varela explains, on the contrary, that we pose or specify the problems we encounter through the meanings we make of the world in which we live, which leads us to recognize things in specific ways. We do not choose or take problems as if they were lying out there, independent of our actions, but we bring them forth. The problems we encounter, the questions we ask, are as much a part of us as they are a part of our environment: they emerge from our interaction with/in it. The problems we solve are implicitly relevant for us as we allow them to be problems for us. If one adheres to this perspective, one cannot assume that instructional properties are present in the tasks presented and that these determine solvers' reactions. Reactions to a task do not reside

inside either the solver or the task: they emerge from the solver's *interaction* with the task, through posing the task. Hence, issues addressed in a task are those that resonate with and emerge from students, although triggered by the task offered. Students *transform* tasks, making them their own, generating their own strategies for solving (their) tasks, as adapted responses, locally tailored to it. It is this dynamic orientation toward the development of strategies that guides this study.

Methodological Considerations, Data Collection and Analysis

Classroom activities/tasks were designed with a grade-11 teacher (covering two 75-minutes sessions for each group), in which students had to operate mentally on functions in a graphical environment: they had to solve tasks without paper-and-pencil or any other aid. For each task, (1) a graph was shown on the board and instructions were given orally; (2) students had 20 seconds to think about the solution; (3) at the signal, students had 10 seconds to write their answer (on a blank Cartesian graph) and hold it up to show the teacher; (4) various students were asked to show/explain their answers. For example, a typical task consisted of showing two functions in the same graph and ask students to add/subtract them (see Figure 1).

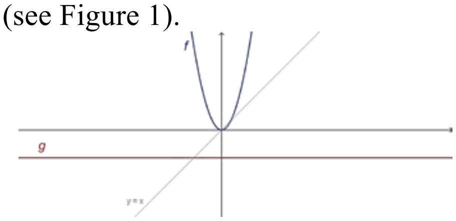

Figure 1: Example of a Graphical Task on Operations on Functions [$f(x) \pm g(x)$]

Tasks were organized in 6 blocks, each composed of 6-10 tasks (consisting of a combination of linear, constant, square-root, rational, step, etc., functions). In block 1, *both* the graph and the algebraic expression of the functions were offered. In block 2, graphs of two functions were given without their algebraic representation. In block 3, students were given the representation of one function and the result of an operation in the same graph. They were instructed to find the function that had been added to or subtracted from the first to obtain the resulting function. Block 4, similar to the second, focused on subtractions. Block 5 differed in that only algebraic expressions of functions were given. These algebraic expressions could not be "directly" computed, like $f(x)=|x|$ or $f(x)=[x]$ with $g(x)=x$ or $g(x)=x^2$. In block 6, students had to add two functions that looked symmetrical in the graph (see e.g. Figure 2 and 4).

Data collection focused on students' strategies, recorded in note form by the PI and a research-assistant. Three types of strategies emerged, which are briefly reported below: algebraic/parametric approach, graphical/geometric approach, graphical/numerical approach.

Findings: Characterization of Strategies

Strategy 1. The Algebraic/Parametric Approach

Even if tasks were proposed in a graphical context, a significant number of students engaged in algebraic-related solving. For tasks where algebraic expressions were not available, students referred to what Duval (1988) calls *significant units* for "reading" the graphical representation of a linear function and offered an interpretation in relation to the algebraic expression. That is, students paid attention to the parameters of the algebraic expression (the *a* and *b* of the linear function $f(x)=ax+b$) to make sense of the graphs and operate. However, again, because the resulting function had to be expressed graphically, they (fluently) explained their answer and strategy algebraically by blending aspects of graphical information. For example, in the following addition task (see Figure 2), where neither function has an algebraic expression attached, students explained that "BOTH FUNCTIONS LOOKED SYMMETRICAL, SO THE 'a' PARAMETER OF EACH LINE WOULD CANCEL OUT, AS WELL AS THE 'b'

AND THUS GIVE x=0”. Students derived algebraic information from the graphs of the functions in order to operate and develop their solution. Even if tasks were offered in a graphical context, students engaged with them algebraically, drawing out algebraic aspects of the functions (expressions, parameters, etc.).

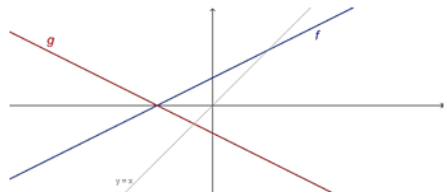

Figure 2: Addition of Function Graphical Task

Strategy 2. The Graphical/Geometric Approach

When facing a function that was not linear (e.g. quadratic, square root, rational, hyperbolic), students assigned a *constantly changing* rate of change/slope to the function they were dealing with. For example, with the addition of a quadratic and a constant function (see Figure 1), the rate of change of the quadratic function was not affected by the addition of a constant function, because a constant function “DID NOT HAD A VARIATION”, hence the slope of the quadratic function “WILL CONTINUE TO VARY IN A CONSTANT WAY”. When students said *constant*, they meant that its appearance was not affected. The resulting function of their addition would have the “SAME RATE OF CHANGE AS THE QUADRATIC FUNCTION” but would simply be “TRANSLATED DOWN” in the graph because the constant function was “NEGATIVE”. Although it is not clear what was meant by this “CONSTANTLY CHANGING” rate of change/slope for nonlinear functions (especially e.g. when they were dealing with $f(x)=1/x$), students developed a way that enabled them to solve tasks and not worry about the variation inherent in the function. In cases where students faced more than one nonlinear function, they began analyzing functions in terms of “parallelism”. For example, in Figure 3 where function g is to be found, some students expressed “EACH FUNCTION WAS PARALLEL TO THE OTHER” and that g had to be a constant function “FOR THE CURVE TO BE TRANSLATED DOWN” and that this constant function was “NEGATIVE FOR BRINGING THE CURVE LOWER”. These students were offering a geometrical interpretation of rate of change/slope, as a property not of the function but of the curve in the graph.

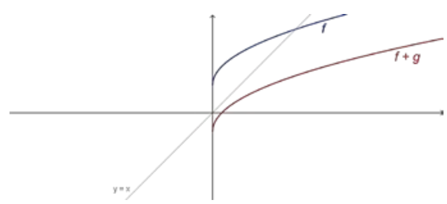

Figure 3: A Task for Which the Parallelism Strategy was Used

Strategy 3. The Graphical/Numerical Approach

A third strategy was to work with specific points in the graphs of functions (related to a pointwise approach, see Even, 1998). Through those points, students obtained exact and approached answers, which they combined to find what the resulting function would look like for *these* points. In Figure 4 students had to find the function resulting from the addition of f and g. In this case, students would fix their attention on specific points: (1) where f crossed the x-axis (x-intercept); (2) where both f and g intersected; (3) where f and g crossed the y-axis (y-intercept); (4) where g crossed the x-axis (x-intercept). For case (1), the operation is an exact calculation, as the addition of the image for f (of length 0) with the one for g results in an image for f+g that is the same as that for g. For case (2), the operation is an approached calculation, as the addition of both image at f and g is the same, and the resulting image is double the value of the intersection point; but a precise location is impossible

without knowing the exact location of the intersecting point in terms of precise length. For case (3), the same approached calculation applies, as both images are added. For case (4), an exact answer is obtained, as in case (1). Here, students mingle both exact and approached calculations to find points for the resulting function.

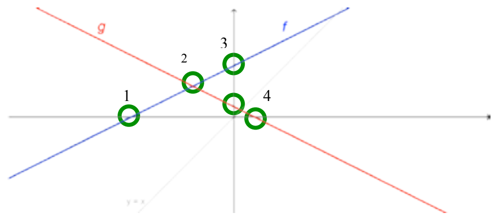

Figure 4: An Addition of Function Task for which Points were Outlined

Discussion of Findings and Final Remarks

These strategies illustrate well students' development *in action* of ways of solving. The strategies reported were enacted on-the-spot, as emergent reactions tailored to the tasks; determined by students' mathematical experience (their past, preferences, habits, etc.) and influenced by tasks (one does not react in the same way to two square-root functions as one does with two linear intersecting ones). Through their entry into the tasks, students *posed* the tasks offered to them, they posed their own problems, making them algebraic, geometric, procedural, etc. Their posing/solving of the tasks generated strategies for finding the resulting functions. Thus an algebraic reading of functions produced an algebraic approach; a graphical reading produced a graphical approach; a numerical/pointwise reading produced a numerical pointwise approach. Through this, students illustrated significant solving capacities, as they were fluent in linking algebraic (symbolic expression), numerical (coordinate points in x or y) and graphical aspects of functions. This seems to contrast with what we know from other studies, as students are frequently reported as experiencing difficulties of many kinds when linking graphs of functions with other representations (e.g. Even, 1998; Hitt, 1998). This underlines some potential of this mental mathematics activity for learning about functions, as it made emerge numerous (and even alternative) ways of conceiving/operating on them. More research is still needed, but this shows important promise to enrich students' experiences with functions.

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PERSISTENCE AND SELF-EFFICACY IN PROVING

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We first discuss our perspective on, and three useful actions in, proof constructions that depend on persistence. Persistence is often important for successful proving because it helps one "explore," including making arguments in directions of unknown value, until one ultimately makes progress. Persistence can be supported by a sense of self-efficacy, which is "a person's belief in his or her ability to succeed in a particular situation" (Bandura, 1995). We then examine the actions of one mathematician when proving a theorem that had previously been given to mid-level undergraduates in a transition-to-proof course. We end with some teaching implications.

Keywords: Advanced Mathematical Thinking, Reasoning and Proof, Beliefs

Introduction

In this paper we first discuss our perspective on proof construction. Then we point out three aspects of proof construction that appear to be difficult to teach. We suggest that the difficulty arises partly from a need for students to have a kind of persistence that depends on students having a feeling of self-efficacy. We next discuss self-efficacy and illustrate the way that self-efficacy and persistence are valuable by examining one mathematician's actual proof construction and note his remarkable persistence. We end with some teaching implications.

An Action Perspective

While we view proofs as texts written in a certain genre (Selden & Selden, in press), we view constructing a proof as a sequence of actions (Selden, McKee, & Selden, 2010). Some of these are physical, such as writing a line of the proof, and some are mental, such as focusing on the conclusion or "unpacking" its meaning. Such actions are taken in response to certain kinds of (autonomously recognized) situations in a partly constructed proof. With practice, the links between some repeatedly occurring proving situations and an individual's resultant actions will become automated, thus reducing the burden on his or her working memory in future proof construction. Many such actions are not recorded in the final written proof.

Three useful actions in proof construction

In several iterations of teaching a second-year university transition-to-proof course in a modified Moore Method way (Mahavier,1999), we have observed proof construction situations calling for one of the following three actions: (a) *Exploring.* In constructing a proof, one may understand both what is to be proved and what is available to use without having any idea of how to proceed. In such situations, one might reasonably try to prove any plausible result, including results of unknown value. However, few students show any evidence of doing this, perhaps lacking confidence in their own ability to use whatever new they might prove. (b) *Reworking an argument in the case of a suspected error or wrong direction.* In constructing a proof, one may come to suspect one has made an error or is arguing in an unhelpful direction. An appropriate response would be to rework part of the argument. However, many students appear to be reluctant to do this. (c) *Validating a completed proof.* Upon completing a proof, one should read

it over carefully for correctness from the top down, checking whether each line follows from what has been said above. Few students appear to do this.

While some student difficulties may depend on a wrong belief about mathematics, on inadequate or incorrect logic, or on a misinterpretation of a definition, failing to take the above three actions in proof construction are not about mathematics per se, but rather about habitually acting appropriately in particular situations. Taking such actions is similar to what Mason and Spence (1999) have called "knowing-to-act in the moment."

Self-efficacy

Self-efficacy is "a person's belief in his or her ability to succeed in a particular situation." "The most effective way of developing a strong sense of self-efficacy is through mastery experiences." Further, performing a task successfully strengthens one's sense of self-efficacy. Short of achieving such success, "Seeing people similar to oneself succeed by sustained effort raises observers' beliefs that they too possess the capabilities to master comparable activities [in order] to succeed." (Bandura, 1994, 1995).

According to Bandura (1994), individuals with a strong sense of self-efficacy: (a) view challenging problems as tasks to be mastered; (b) develop deeper interest in the activities in which they participate; (c) form a stronger sense of commitment to their interests and activities; and (d) recover quickly from setbacks and disappointments. In contrast, people with a weak sense of self-efficacy: (a) avoid challenging tasks; (b) believe that difficult tasks and situations are beyond their capabilities; (c) focus on personal failings and negative outcomes; and (d) quickly lose confidence in personal abilities.

Bandura's ideas "ring true" with our past experiences as mathematicians teaching courses by the Moore Method. Classical Moore Method (advanced undergraduate or graduate) courses are taught from a brief set of notes consisting of definitions, a few requests for examples, statements of major results, and those lesser results needed to prove the major ones. In class meetings, the professor invites individual students to present their original proofs and then briefly comments on errors. Once students are able to successfully prove the first few theorems, they often progress very rapidly in their proving ability, even without apparent explicit teaching, and even when subsequent proofs are more complex. Why should this be? We conjectured then, and also now, that students obtained a sense of self-efficacy from having proved the first few theorems successfully, and this helped them persist in explorations needed to prove subsequent theorems.

Background for One Mathematician's Proof

Below we describe one mathematician's proof of *Theorem 20: If S is a commutative semigroup with no proper ideals, then S is a group.* The needed background information is very small, and we provide it here. A *semigroup* is a nonempty set S with an associative binary operation that we will write multiplicatively as xy for elements x and y of S. *Associativity* means that for all elements x, y, and z of S, $(xy)z = x(yz)$. S is *commutative* means that for all elements x and y of S, $xy = yx$. If A and B are subsets of S, AB is the set of elements ab, where a and b are elements of A and B respectively. In this setting, a nonempty subset I of S is an *ideal* of S provided SI is a subset of I. Such an ideal is called *proper* in S provided it is not all of S. In this commutative setting, S is a *group* if it has two additional properties. First, there must be an *identity* element e of S so that for any element s of S, $es = s$. Second, given any element s of S there must be an *inverse* element s' of S so that $s's = e$.

In proving the above theorem, it is easy to see that if I is an ideal of S, one can conclude I is not proper, that is, $I = S$. What is not so easy is trying to construct an ideal that "looks" different

from *S*, and what that might have to do with producing an identity element *e* of S and inverses, in order to prove that *S* is a group. Since there is nothing else to work on, one must persist in trying to find an ideal of *S* without any idea of whether, or how, that might be helpful. It turns out that some students rather quickly think that if *s* is any element of *S* then $S\{s\}$ (also written *Ss*) might be an ideal of *S*, and hence equal to *S*. Once the idea has been articulated, it is not so hard to prove that *Ss* is an ideal. But how might the set equation $Ss = S$ help in proving that *S* is a group? However, if *t* is also an element of *S*, the above set equation means that there must be an element *x* of *S* so that $xs = t$. That is, the equation $xs = t$ can always be solved for *x*. It turns out that one can use the solvability of this equation in several ways to collect information which, for many students, is of unknown utility. Nevertheless this information, once collected, can be organized to show the existence of an identity element and inverses in *S*. To do this requires both persistence and a willingness to obtain whatever results, in the form of equations, that one can without knowing whether those results will ultimately be helpful.

One Mathematician's Proof Construction

Savic (2012) investigated mathematics professors' naturalistic proving using a set of notes that had previously been used by us with undergraduate students. Several of the mathematicians acknowledged getting "stuck" on *Theorem 20: If S is a commutative semigroup with no proper ideals, then S is a group*. However, none gave up, as most students might, but persisted.

Dr. G's construction of a proof of Theorem 20

Below we describe much of the work of Dr. G in attempting to prove Theorem 20, which he eventually did successfully. Our description is a synopsis of Dr. G's transcript. (Savic, 2012)

As seen below, Dr. G took a remarkably meandering path as he explored how to prove Theorem 20. His various "twists and turns" are indicated in bold typeface. Dr. G started at 7:02 a.m. by considering the statement of Theorem 20, but decided to **think about it** and have breakfast. At 8:07 a.m., he returned from a walk and **realized that gS** (where g is an element of *S*) **is an ideal**, so $gS = S$. He then **thought about inverses** and **struckthrough his entire previous argument**. At 8:09 a.m., he **noted that he needed an identity element** which is not given. At 9:44 a.m., he became suspicious that **Theorem 20 might not be true**, but noted that he had few examples which might show that.

At 9:48 a.m., Dr. G started **"tossing around" the idea** that a [commutative] semigroup with no proper ideals must have an identity, in which case, he could show it is a group. However, he didn't see why *S* should have an identity. He **began to think that translating by a fixed element** would move every element, which would mean there was no identity. Consequently, he **then began to look for a counterexample**. By 9:50 a.m. he neither saw how to prove Theorem 20 nor how to find a counterexample.

He then looked ahead to Question 22, which has three parts that ask whether certain semigroups are isomorphic. He **saw how to answer that and then looked at** *Theorem 21: A minimal idea of a commutative semigroup is a group.* He thought that he could probably prove that, so **he went back to Theorem 20**. By 9:51 a.m. Dr. G **recalled that he had earlier rejected Theorems 3, 9, and 12** of the notes and also did not believe that there are unique minimal ideals. By 9:53 a.m., he recalled that he had not been told any of the theorems were false and **looked at the non-negative integers** under multiplication. He saw that $\{0\}$ is a minimal ideal and noted that the non-negative integers under multiplication do not form a group.

At 9:54 a.m., he **started actually answering Question 22**. By 9:58 a.m., he had answered its three parts correctly. At 9:59 a.m. Dr. G. **took a break to think about Theorem 20** and at 10:08 a.m. he again attempted a proof of it. This time **he saw that for $a \in S$, there is $e \in S$ so that $ae =$**

a and saw that e is "acting like …a right identity on a. Now why does it have to act that way on [an arbitrary] b?" By 10:12 a.m. **he found e' so that $be' = b$**, but that didn't help since he couldn't show that $e = e'$. Then at 10:13 a.m. he **saw that there is an f so that $b = af$**, and then by 10:14 a.m., he **saw that $be = afe = aef = af = b$**. At 10:15 a.m., he **saw that e is the identity** element. By 10:18 a.m., he had **used a similar technique to show S has inverses** and is thus a group.

Perhaps the most important thing about the above description of Dr. G's work is what is *not* there. There is no evidence that Dr. G thought there was anything wrong with having gone in all of those unhelpful directions or with having thought that some theorems were false, that he later discovered were true. What seemed to matter to him was the generation of ideas. If those ideas resulted in errors, one fixed them and learned from them. He exhibited remarkable persistence and a willingness to try argument directions that he clearly didn't know ahead of time would be helpful, and he altered directions when the need arose.

Teaching Implications

It seems to us that in order to take proving actions that require persistence and exploration, a student is likely to need to believe that he or she can personally benefit from his or her persistence or exploration. That calls for a self-efficacy belief, which in turn calls for, perhaps numerous, successes in similar actions.

In order to maximize students' opportunities to experience successes in various aspects of proving, it would be good to have students constructing their own proofs as early and often as possible. There is an aspect of proving that is not usually explicitly taught, but that could be very helpful in facilitating students' early successes. It is based on the relationship between the logical structure of the theorem being proved, the structure of a proof, and the theorems and definitions used in constructing the proof. Selden and Selden (1995) have referred to this as a *proof framework*. In constructing a proof framework, one writes as much as possible of a proof before attempting to generate the ideas needed to complete it. One writes the hypotheses first, leaves a blank space for the body of the argument, and then writes the conclusion. Typically one next "unpacks" the meaning of the conclusion, which allows one to write more of the beginning and end of the proof. One proceeds as far as possible in this way. After a framework has been constructed, a student can then try to generate ideas needed to complete the middle of a proof.

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CULTIVATING LEARNING OPPORTUNITIES FOR JUSTIFICATION IN AN URBAN MIDDLE SCHOOL MATHEMATICS CLASSROOM

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This study explores how whole class discussions across the first four weeks of school provided learning opportunities for justification to students who have little prior experience in this practice. Seven class discussions were analyzed in a 6th grade math class in an urban public school. Preliminary analyses indicate that even brief class discussions offered rich learning opportunities for (1) acceptable modes of justification in the classroom community, (2) what needs to be justified, and (3) the mathematical language for communicating justifications. These findings demonstrate how teachers might cultivate opportunities for justification for all students.

Keywords: Reasoning and Proof, Classroom Discourse, Instructional Activities and Practices

Justification is central to *doing* and *learning* mathematics (Ball, Hoyles, Jahnke, & Movshovitz-Hadar, 2002). This is reflected in the Common Core Standards for Mathematical Practice (2010) that call for students to "construct viable arguments and critique the reasoning of others" across all grade levels. However, a wide range of studies have documented students' challenges with justification (e.g. Balacheff, 1988; Knuth, Choppin, & Bieda, 2009), which may reflect that many US mathematics classrooms offer few opportunities to engage in justification (Stigler & Hiebert, 2009). A major challenge currently facing teachers is how to help *all* students engage in this central mathematical practice when their students may have had few prior learning opportunities to do so. This study explores how a teacher might cultivate learning opportunities for justification in the first weeks of school with students inexperienced in this practice.

Ball and Bass (2003) argue that justification in classrooms rests on two foundations. The first is the *base of public knowledge*. This includes the "taken-as-shared" (Yackel & Cobb, 1996) concepts, procedures, defined terms, and methods of verifying mathematical claims that serve as a starting point for constructing arguments. The notion of taken-as-shared does not imply that all members of the community hold this knowledge; rather, it refers to what is accepted as true without justification within a particular community. The second foundation is *mathematical language*, which they take to mean the symbols, terms, representations, and rules that guide their use in how claims are formed and justified. This includes the central role of definitions in constructing justifications. This distinction between two aspects of justification is similar to Stylianides's (2007) conceptualization of a mathematical argument in which he differentiates between the set of accepted statements, the mode of argumentation and the mode of argumentation representation. These perspectives are helpful in unpacking the types of arguments that students make, as well as what aspects of justification might need to be deliberately cultivated. Ball and Bass cite three aspects of classroom practice that are important for teaching justification, including designing and using tasks that afford opportunities to justify, making records of mathematical knowledge, and making ideas public. Whole class discussions are one classroom structure in which mathematical ideas can be made public. Many scholars have demonstrated how these discussions can support collaborative reasoning in mathematics (e.g. Staples, 2007). Through the close analysis of one middle school classroom during the first four weeks of school, this study addresses the following question: How might a teacher cultivate

justification through whole class discussions in a middle school mathematics classroom?

Methods

The data for this study is drawn from a larger project investigating the teaching and learning of justification in an urban secondary school in an under-resourced community in Northern California. The analyses for this study draw on classroom data from the first four weeks of school in one 6th grade classroom. The focal teacher, Mr. Carrera, was selected based on his commitment to fostering collaborative mathematical inquiry. He identified justification as one of his central goals for *all* students. This 6th grade class was a particularly rich site for exploring the cultivation of justification since the students were all new to the school and Mr. Carrera's teaching approach. Semi-structured interviews established that students' prior experiences in math were typical of many US students (Stigler & Hiebert, 2009): the mathematical authority lay with the teacher and the students' role was to pay attention and practice. There were 26 students in the class (21 Latino/a, 3 Asian-American, 2 African-American). All names are pseudonyms.

These analyses draw on video records of whole class mathematical discussions that spanned the first four weeks of school. These discussions ranged between 12 and 40 minutes long. Sometimes they discussed different solutions to homework problems. Other times the discussions occurred after a warm-up task or after extended work on a more complex task. The analysis of the transcripts of these discussions took place in three phases. First, I identified all episodes of students' attempts to justify a mathematical claim regardless of validity. Since justifications were at times co-constructed by multiple students and even the teacher, each episode was defined as starting with the initial student claim and ending when the teacher and class accepted the emergent justification and the focus of the talk shifted. Second, I coded each episode of attempted justification for whether the teacher (or students) prompted, questioned, praised, critiqued, or otherwise commented on the justification. Third, I coded these instructional moves for if and how they addressed the different aspects of justification described by Ball and Bass (2003). For example, when Juan shared his ideas about 3^2, saying "Three times three", Mr. Carrera asked, "Would you say how you decided?" Juan responded "Because the two here says to write the three two times." Mr. Carrera accepted this, saying "Very good. Exactly." This episode was coded as mode of argumentation, since Mr. Carrera prompted him to say "how you decided" and then accepted a justification that reasoned from a definition.

Emergent Findings

Preliminary analyses suggest that the class discussions provided learning opportunities for justification around the publically accepted knowledge base and the language of justification.

Cultivating Justifications that Make Sense

On the second day of school, the class discussed the following homework problem: *Jackie has 12 square tiles. What are all the different rectangles he could make using his tiles?* Right before this episode, Alonzo drew a 3×4 rectangle and justified this claim through counting by threes. Then Mr. Carrera asked Javier, who used a multiplication strategy for a 2×6 rectangle, how he would approach this. Javier drew three dots above and four to the right of the rectangle.

> *Javier:* 3 times 4. Equals 12. Is the other way to see how many squares there are inside.
> *Teacher:* Does someone have a question for Javier? Javier, why do you multiply? Cause heard Alonzo counting by threes, pointing to them so that made sense to me, but why do you multiply?

Here Mr. Carrera communicates that Javier's statement *that* he multiplied is not an acceptable justification; instead, he needs to justify *why* he multiplied. Furthermore, Mr. Carrera explicitly

communicates his standard for justification when he says "so that made sense to me" about Alonzo's counting by threes. For Mr. Carrera, justifications should make sense. Javier continues.

Javier: It-it could be a faster way.

Teacher: You know why multiplying works though? Let's get some other ideas. Tanisha, do you know why multiplying works?

Tanisha: It works because on the top there are three dots and on the side there are four dots. And basically you just multiply them and get twelve.

Teacher: But why multiply?

Tanisha: Because it's faster.

Teacher: That's for sure, right? If you're good at your times table in your head. But why does multiplying work?

Leo: Cause it equals the number that you multiply.

Teacher: It does give you the right answer, that's for sure. [laughter] Why?

Mr. Carrera does not accept the justification that it's faster or yields the correct answer. Finally, Jorge contributes, "Cause each row's a group. There's three in a group. Three – there's four rows, so there's four groups of three". Mr. Carrera asks him elaborate.

Jorge: Each row is like this. [circles the top row of three squares]. Is a group of three. And there – there are four rows. There's the three-four rows of three. So its – right here is the three of these three. And the four rows are this four. Multiply it because it's four groups of three. Equals twelve.

Teacher: Three groups of four. That's really clear for me.

In accepting this justification, based in the meaning of multiplication, Mr. Carrera offers a learning opportunity for acceptable *modes* of verification as well as *what* needs to be justified.

Cultivating Language for Justification

The discussions also offered learning opportunities for the language of justification. On day three, the class discussed: *What does 2^3 mean?* Some thought it meant 2×3=6; others argued for 2×2×2=8. After several minutes of debate, Mr. Carrera steps in.

Teacher: How do you know? Like how can you tell? Tanisha, you were the original one to say that this one [indicating 2×2×2=8] was right, right? So how do you know?

Tanisha: You know because if you try to do the math, and you're actually writing it up there it would be 8.

Teacher: And you're saying "do the math" means do this? [points to the 2×2×2=8]

Tanisha: Yeah

Teacher: Uh-huh. Not do 2×3 [points to the 2×3]. Yeah. So, does anyone know what this three is called? [points to the 3 in the 2^3]

Mr. Carrera recognizes that the debate about 2^3 cannot be resolved without the definition of an exponent. He then elicits the definition, which allows them to resolve the debate.

Mr. Carrera also frequently emphasized what he calls "technical writing", such as color coding to show connections and labeling diagrams. For example, on day four, Leo presented his ideas for finding the area of 12×9 rectangle. He wrote down 12×9 = 108. Mr. Carrera asked Leo to record "where's the 9 and the 12?" Leo did not seem to know what Mr. Carrera wanted. So Mr. Carrera asked: "Where's the 9 first of all?" Leo used his marker to draw lines that delineated columns of 9 blocks. Mr. Carrera then suggested that he label the top with the 12 and the side with 9. Leo still did not seem to know what to do. Finally, Mr. Carrera used his laser pointer to show exactly where to write the 12 and 9. This episode is a case of providing access to the *mathematical language* necessary to communicate justifications. For statements like 12×9 = 108

to have meaning and be accepted as justifications, the students needed to communicate why 12 and 9. Tools like labeling provided the language to communicate meaning.

Implications and Directions for Future Research

Although these preliminary analyses draw on only a subset of the data, the emerging findings suggest that these whole class discussions were able to offer multiple learning opportunities for justification that ranged from what makes an acceptable justification to the role of definitions and tools for communicating and representing justifications (Ball & Bass, 2003; Stylianides, 2007). Furthermore, these learning opportunities emerged out of tasks that were not designed to elicit general statements and justifications of them. The task of explaining the meaning of 2^3 was not framed as a problem about a general statement or proof; however, the way the teacher enacted the task provided opportunities for students to explore the role of definitions in constructing arguments, thereby increasing the cognitive demand of the task (Stein, Grover & Henningsen, 1996). The fact that the learning opportunities for justification highlighted above were cultivated within relatively short whole class discussions of homework problems is promising. For teachers who may feel less confident in implementing proof-related tasks, these smaller discussions of homework problems might provide a scaffold to begin cultivating opportunities for justification.

Further analyses will explore (1) how these learning opportunities for justification are cultivated and coordinated across the first four weeks of school, (2) how these learning opportunities support students' justifications in discussions and in written work, and (3) how students' engagement in justification looks different at the end of the school year. These analyses will have the potential to inform how teachers can cultivate learning opportunities justification for all students. It will also inform what learning opportunities students might need when they arrive with little prior experience in constructing mathematical arguments.

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CONCEPTUALIZING CCSSM MATHEMATICAL PRACTICES ON TASKS AND EMPIRICAL WORKS

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The Common Core State Standards for Mathematics contains both mathematical content and practices for grades K-12. Many researchers focus on the content, but overlook how the practices are embedded within tasks. Each practice includes a brief description that people interpret multiple ways. This study conceptualizes the practices and uses the framework to code two textbook series on bivariate data tasks to determine if students are exposed to the mathematical practices within the textbooks. Initial results indicate that the tasks provide opportunities for students to attend to precision but not to look for and express regularity in repeated reasoning.

Keywords: Curriculum analysis, Data Analysis and Statistics

Purposes of the Study

The Common Core State Standards for Mathematics (CCSSM) was released in 2010 followed by adoption in 45 states and five territories in the U.S. The CCSSM has the potential to shift the landscape of education in the U.S. particularly with the change of content and sequence (Heck, Weiss, & Pasley, 2011). Most discussions concerning the CCSSM have focused on the content standards (e.g., Conley, Drummond, de Gonzalez, Rooseboom, & Stout, 2011; Schmidt & Houang, 2012). However, one important issue related to the CCSSM – the eight mathematical practices (SMPs), has been overlooked. Some researchers and practitioners have sought to elaborate the practices and provided scenarios to illustrate them (e.g., Koestler et al., 2013). However, there have not been any systematic frameworks to code for the SMPs in curriculum materials and no formal research about that work. In this study, we describe how we conceptualize the practices in curriculum materials and provide our initial results about empirical work using examples from one high school textbook series for the content related to bivariate data.

Perspectives and Coding Scheme

In this study, we look at the nature of tasks found in curriculum materials. That is, we seek to determine if the tasks offer the potential for students to utilize the practices. Admittedly, the tasks might offer potential for students to access the practices, but the practices might not be implemented in the same way in each task (Stein & Lane, 1996). Within this study, we highlight the characteristics of tasks that provide the opportunity for students to implement the practices. The performance expectation framework developed in the TIMSS study (Schmidt et al., 1997) was used to provide examples of tasks that offer the opportunities to look for the eight practices. Following is a summary of the coding scheme for SMPs.

1. Make sense and persevere in solving problems (MP1).

Persevering in solving problems is only observed in practice, not in the task itself. Therefore, the framework focuses on the former component. Particularly, we look to determine if the task requires students to formulate and clarify problems and situations such as to: (a) construct a verbal or symbolic statement or a question in which a mathematical problem goal can be specified, (b) design an appropriate statistical experiment to solve a stated problem or to specify the data and range of data needed.

2. Reason abstractly and quantitatively (MP2).

We consider if the task provides opportunities for students to: (a) make sense of quantities (not merely numbers), (b) reason about the relationship between two quantities, and (c) reason abstractly with symbols and formulas.

3. Construct viable arguments and critique the reasoning of others (MP3).

We consider if the task provides opportunities for students to argue or critique others' arguments. Particularly, we look for performance expectations from the task that ask students to: (a) verify the computational correctness of a solution, or justify a step in the solution, (b) identify information relevant to verify or disprove a conjecture, (c) argue the truth of a conjecture or construct a plausible argument, (d) identify a contradiction (something that is never true), (e) critique a written or spoken mathematical idea, solution, result, or method for solving a problem and the efficiency of the method or similarly critique an algorithm and its efficiency.

4. Model with mathematics (MP4).

We consider if the tasks ask students to: (a) construct a verbal or symbolic statement of a real world or other situations, (b) simplify a real world or other problem situation by selecting aspects and relationships to be captured in a representation modeling the situation, (c) select or construct a mathematical representation of a problem (real-world or other problem situation plus a related question/goal), and (d) develop notations or terminologies to record actions and results of real-world or other mathematizable situations.

5. Use appropriate tools strategically (MP5).

We examine if the task mentions something about the selection of tools used to solve the task or if the task inherently requires students to use a technological tool, but not merely for calculation. In this situation, the technological tool is used to deepen understanding.

6. Attend to precision (MP6).

We determine if the tasks ask students to: calculate, measure, or use specialized terms and symbols. Particularly, performance expectations from the task might ask the student to: (a) use equipment to measure, (b) compute/calculate with or without instruments, (c) graph with scale with or without technology/device, (d) collect data by surveys, samples, measurements, etc., (e) develop or select, using notations, terminologies to record actions and results in dealing with real-world or other mathematizable situations, and (f) describe the characteristics of a formal algorithm or solution procedure.

7. Look for and make use of structure (MP7).

We consider if the task provides opportunities for students to look for a structure such as: (a) fit a curve of given type to a set of data (only if students are not told what kind of curve to fit), (b) classify mathematical objects by implicit criteria (e.g., geometric shapes), (c) predict a number, pattern, outcome, etc., that will result from an operation, procedure or experiment before it is actually performed.

8. Look for and express regularity in repeated reasoning (MP8).

We examine if the task offers students opportunities to abstract from a series of similar situations, a general technique, strategy, or algorithm to use in a class of problems. In particular,

we look to see if some of the performance expectations of the task ask students to: (a) describe the effect of a change in a situation (e.g., the effect on its graph of changing a parameter), (b) develop a formal algorithm for computation or a formal solution procedure for problems of a specified class or type, (c) identify a class of problems for which a formal solution procedure is appropriate, (d) generalize the solution, the strategy, or the algorithm of a specific problem, and (e) abstract the common elements from multiple related situations.

Methods

We examined coding schemes and specific examples from the performance expectation framework of the TIMSS study (Schmidt et al., 1997) and adapted the scheme to fit in the context of bivariate data. In particular, we highlighted the performance expectations providing potential for students to access the practices. We then used the framework to code an investigation from the teacher's edition of the *Core-Plus Mathematics* (CPMP) (Hirsch, Fey, Hart, Schoen, Watkins, Ritsema, et al., 2008) to determine if the practices appeared in the set of tasks and revised the coding scheme. In another round, we independently coded the practices for all the tasks related to bivariate data from three teacher's editions: CPMP series (published by Glencoe McGrawHill, 2008, 2009, 2010), The University of Chicago School Mathematics Project (published by Wright Group McGrawHill, 2008, 2009, 2010) and Holt McDougal Larson (HML) series (published by Holt McDougal, 2012). For reliability, pairwise agreements for all eight practices were more than 70%.

Results

Table 1 summarizes the percentages of tasks that address the mathematical practices across the three series. For the set of bivariate data tasks, the percentages vary by series and by practices. CPMP offers the most tasks attending to the mathematical practices, followed by UCSMP and HLM series. Most of the tasks offered students the potential to *model with mathematics* (MP4) and *attend to precision* (MP6) with the focus on computation and calculation. In contrast, very few tasks offer expectation for students to access *make sense and persevere in solving problems* (MP1) and *look for and express regularity in repeated reasoning* (MP8).

Table 1: Percentages of Tasks that Address the Mathematical Practices

	MP1	MP2	MP3	MP4	MP5	MP6	MP7	MP8
HML (N=122)	3.3	57.4	54.1	76.2	37.7	81.1	12.3	0
UCSMP (N=214)	5.1	76.2	43	86.9	52.8	84.6	36	1.9
CPMP (N=246)	10.6	82.4	83.3	89	44.1	87.8	55.9	7.8

Following is the coding of a task taken from CPMP series.

Make a model bungee jump by attaching a weight to an elastic cord or to a chain of rubber bands.

a. Use your model to collect test data about bungee cord stretch for at least five weights. Record the data in a table and display it as a *scatterplot* on a graph.

b. Use the pattern in your experimental data to predict length of the stretched bungee cord for weights different from those already tested. Then test the accuracy of your predictions.

c. Compare your results to those of others doing the same experiment. What might explain any differences in results? (Hirsch et al., 2008, p. 5)

When solving this task, students have chances to (2a) make sense of quantities such as weight and length and (2b) reason about the relationship between the two quantities; (3b) identify information relevant to verify or disprove a conjecture, (c) argue the truth of a conjecture or construct a plausible argument, and (e) critique a written solution, result; (4b) simplify a real world situation by selecting aspects and relationships to be captured in a representation modeling the situation, (4c) select or construct a mathematical representation of a problem (in this case, a graph); (6a) use equipment to measure, (6c) graph with scale with or without technology/device, (6d) collect data by surveys, samples, measurements, etc.; and (7c) predict a number, pattern, outcome, etc., that will result from an operation, procedure or experiment before it is actually performed.

Discussion

The framework for coding mathematical practices shows to be applicable to code for bivariate data or statistical tasks. The tasks' low presence of *look for and express regularity in repeated reasoning* within tasks is significant because educators may need to modify the tasks if they want to address the mathematical practice. It is conceivable that the variation across textbooks illustrates curriculum developers' different interpretation of how to embed the mathematical practices. In addition, the framework clarifies and operationalizes how the written curriculum embeds the SMPs. Further research needs to examine the framework in different domains. Furthermore, when coding tasks from the textbooks, we look specifically at the teacher's edition to see how the authors expect the tasks to be implemented. This might be different from the tasks as implemented in classroom. Hence, future research needs to develop frameworks for coding tasks in implemented phase.

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THE ROLE OF PRIOR KNOWLEDGE IN ANALOGICAL REASONING ABOUT THE FORMAL DEFINITION OF A LIMIT

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Research has documented that the formal definition is a roadblock for mathematics students. Studies suggest that students' understanding of a crucial relationship, called the *temporal order* of two variables, ε and δ within the formal definition warrants further investigation (Davis & Vinner, 1986). In the definition, ε comes first before δ, and in instruction δ is often said to depend on ε. This case study is part of a larger study investigating the role of prior knowledge in calculus students' understanding of the formal definition. The larger study argues that relevant intuitive knowledge resides in everyday understanding of quality control. Thus in the individual interview, students engaged with an instructional analogy called the Pancake Story, in which the idea of quality control is illustrated in producing pancakes within specified parameters.

This study explores how one student, DR used the Pancake Story as a resource to revise her claim that ε depended on δ. The study focuses on the process that prior knowledge gets incorporated into, refined and/or elaborated to become a new conception (Smith et al., 1993), specifically ways that prior knowledge interacted with the normative resources from the story. DR revised her claim to δ depending on ε after the story. She attended to normative ideas from the story, but they contradicted her prior knowledge. Her explanation of the confusion illuminates how she ultimately aligned her prior knowledge with the normative resources.

DR: I was looking at it like the $\underline{f(x) \text{ depends on the } x}$ and that's how I was saying that epsilon depends on delta because epsilon is related to the $f(x)$ or whatever. But that's just saying the error of the L and the $f(x)$, depends on the a and x. But that's *not* to say that epsilon depends on delta. So, I think that delta depends on epsilon now. Just cus if it's given like this [*the story*] and you're trying to aim at getting within a certain error bound, then you're gonna try to manipulate your entries or whatever to be within a certain error bound.

Interviewer: Ok. Alright. So you changed your mind it seems? Um, so how did that happen? Why did you change your mind?

DR: Because I was given an epsilon and that's kinda the main goal. The main goal is to get the pancake, like that's the main goal. And they gave me a constraint. /…/ And then delta or they didn't give me an error bound for the batter or for the a or x, they didn't give me an error bound. But I know I want to make it small so that it's within the like error bound /…/, or the epsilon. So then I would kinda base my delta on what was epsilon.

Once DR found a way to align her prior knowledge with the normative knowledge resources from the story, DR could prioritize the normative resources. The resource of $f(x)$ depends on x was repurposed to inform the dependence between the errors ($|f(x)\text{-}L|$ and $|x\text{-}a|$), not the error bounds (ε and δ). Earlier DR also differentiated errors from error bounds. The main goal of getting the pancake within the error bound and the fact that epsilon was given but not delta, from the story were then used to conclude that DR had to "base my delta on what was epsilon."

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FRACTIONS SCHEMES AND WHOLE NUMBER UNITS COORDINATION

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Keywords: Number Concepts and Operations, Rational Numbers, Learning Theories

Introduction/Purpose

Across mathematics content domains, the construct of *units coordination* has emerged as an important factor in both learning and teaching. Units coordination is the forming of a structure that allows one to retain, when *assimilating* a *quantity*, the relationships between intermediate *unitizing* activities. For example, a person coordinating three levels of units can immediately and simultaneously think of 15 as both: (a) a unit of five threes, each a unit of three ones, and (b) a unit of 15 ones. A person assimilating 15 with only (b) is coordinating two levels of units; a person assimilating only one level of unit requires *activity* to think of 15 as (b) (see Hackenberg, 2007). Steffe and Olive (2010) describe students' construction of fractions schemes as hierarchical, beginning with the *partitive unit fraction scheme* (PUFS). Partially as the result of *reorganizing* (Steffe, 2001) their ways of thinking about whole numbers to fractions, students construct the *partitive fraction scheme* (PFS), the *reversible partitive fraction scheme* (RPFS), and the *iterative fraction scheme* (IFS). For this study, we tested quantitatively whether there is a positive association between whole number units coordination and fraction scheme construction.

Methods/Results/Implications

We conducted one-on-one clinical interviews with 50 sixth-grade students in the Southeast United States. To assess fractions schemes, we used tasks modified from Wilkins, Norton, & Boyce (2013). For whole number units coordination, we used tasks modified from Hackenberg & Lee (in press). In the poster, we will present excerpts from our prompts and materials used in the assessments as well as cross tabulations of the results. Statistical analysis revealed a strong positive relationship between whole number unit coordination and fraction scheme construction (Somers' $d = .786$, exact $p < 0.001$, two-tailed). An interpretation is that one can expect to reduce the error in predicting students' fractions schemes by 78.6% if their whole number units coordination is known. Five cases did not fit the hypothesized hierarchy, and in each case, a student was assessed with a PUFS, but only one level of whole number unit. Future qualitative research is warranted to investigate this phenomenon.

Acknowledgements

This research was supported by a grant #1118571 from the National Science Foundation.

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WHY DO DIAGRAMS INCREASE LEARNING FROM LESSONS?

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Relevant visual representations, such as diagrams, can increase learning from textbooks (Mayer, 2009). There are two proposed reasons for the benefit of diagrams. One is that diagrams increase learning because they make the corresponding text easier to comprehend. This is because diagrams convey information in a manner that text cannot (Schnotz, 2002). The second is that students make connections between the visual information in the diagram and the verbal information in the text. These connections may deepen their comprehension of the material, thereby increasing learning (Mayer, 2009). Despite these claims, the influence of diagrams on both text difficulty and making connections has not been well examined.

The purpose of our study is to understand why diagrams increase learning from lessons. To address this issue, we randomly assigned undergraduates ($N = 36$) to read a probability lesson either with or without diagrams, while their eye movements were recorded. Students whose lessons included diagrams solved more probability problems correctly at post-test than did students whose lessons did not include diagrams. Students whose lessons included diagrams also had smaller average pupil size and spent less time reading the text than did students whose lessons did not include diagrams. Pupil size and reading times typically increase with task difficulty (Rayner, 1997; van Gog et al., 2009); therefore, this finding indicates that the diagrams lessened the difficulty of reading the lesson. In addition, students whose lessons included diagrams frequently looked to and from the diagram and the text. Their looks to and from the diagram and text may indicate that they were integrating the visual and verbal representations in the lesson (Mason, Tornatora, & Pluchino, 2012).

These findings indicate that both of the previously proposed reasons may explain why students whose lessons included diagrams answered more problems correctly than did students whose lessons did not have diagrams. One is that the diagrams made the lesson text easier to understand; therefore, students could focus their efforts on extracting the content of the lesson, rather than working to comprehend the text. The other is that diagrams encourage students to make connections within the lesson material, which prompts deeper comprehension. These findings enrich our understanding of the benefits of visual representations.

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COLLEGE STUDENTS' APPROACHES TO SOLVING WORD PROBLEMS: A COMPARISON

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This study builds on the work of a previous study (Fox, 2012, 2013). In the previous study, I investigated how pre-service teachers early in their preparation answered word problems involving fractions and proportions. In the new study, I want to expand the population under investigation: instead focusing solely on pre-service teachers, I want to investigate how students at the same institution outside of the education program would answer similar questions. The primary research question guiding this study is: in what ways do college students who are not aspiring teachers provide solutions to word problems that differ from college students who are aspiring teachers? This report focuses on the results of six participants in this study.

This study used the same method as the study for pre-service teachers: I interviewed each participant up to four times. All interviews were recorded using both audio and video recording devices. Across the first three interviews, I asked each participant a total of up to 25 word problems. The word problems were identical to the study with pre-service teachers. In the fourth interview, the participant and I reviewed the questions and responses the participants provided from the previous three interviews. The first three interviews in this sequence satisfied my understanding of Steffe and Thompson's (2000) definition of exploratory teaching, where I can, as a teacher-researcher, "become thoroughly acquainted, at an experimental level, with students' ways and means of operating in whatever domain of mathematical concepts and operations are of interest" (p. 274).

Participants in this study appeared to have different approaches than the early pre-service teachers solving the same word problems (Fox, 2012, 2013). For example, one of the early pre-service teachers used pictures and provided explanations to her solutions (Fox, 2012). Many of the students in the current study, however, tended to provide only a single answer to a question. When probed, what explanations the participants provided typically referenced a recollection of rules they remembered from earlier in their schooling. Many of the participants in the current study mentioned they had not revisited similar concepts in several years, suggesting their own recollection of rules tended to be either faulty or incomplete. Additionally, a variety of explanations exist across the participants in this study. Additional work associated with this study and additional analyses of already collected data could yield confirming results, or provide meaningful insights into additional responses to the same research question.

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HOW STUDENTS BUILD MATHEMATICAL PRACTICES THROUGH ANALYZING OTHER STUDENTS' WORK

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Common Core State Standards for Mathematics, now adopted by 46 states across the country place a sharp focus on the need to assist learners to act as mathematicians; use mathematics to model phenomenon, search for patterns and to describe regularities. Most specifically, the standards demand that the students must learn to construct viable arguments and critique the reasoning of others (practice standard 3). This recommendation for learners' practice is compatible with those proposed for effective teaching: teachers should capitalize on analyzing students' mathematical work for orchestrating learning opportunities for children (Kazemi & Frank, 2004; Cameron et al, 2009). Research has established that attention to "unexpected results and even false solutions offered by students" can enrich student learning (Manouchehri, 2007). This body of work, small in quantity, unanimously suggests that concentrating on analysis of classroom produced work can potentially advance learners' mathematical cognition. Despite this, there is a paucity of research that highlights how learners might be assisted in developing the capacity to learn to analyze and critique the views of others with the goal of learning from the experience.

In this work we examined students' mathematical analysis of peers' mathematical work with the aim to trace the kinds of learning that may have been derived from the experience. The participants included students enrolled in two sections of an undergraduate first semester calculus course, around 20 students in each section. Participants were given three tasks over a period of three months. These tasks were designed to identify both their mathematical approaches to solving the problems and their conceptual understanding of key calculus concepts. The participants worked on each task individually first and submitted their handwritten work to the professor. They were then provided with selective sample of responses on the same tasks. These artifacts were de-identified and also came from students in the other section of the course, not their own. Students analyzed each artifact mathematically and wrote their analyses. Finally, students reworked the original task, often changing their solution.

The data consisted of the handwritten work done throughout the process. Data analysis consisted of three rounds. For the first round, student work was analyzed with respect to mathematical conceptual understanding. Misconceptions, argument structures as well as class trends were identified. The second round focused on the students' mathematical practices on each task as defined in the Common Core Standards (2012). The third round of analysis examined the students' mathematical practices over time, across the three tasks.

Results showed that when examining other students' work and then returning to their own work, students (1) increased the number of mathematical features they were considering, (2) improved their ability to judge the validity of their own mathematical reasoning, and (3) created more coherent and viable arguments. However, some students were convinced by arguments that looked mathematically sophisticated, even if they had substantive conceptual flaws.

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UNCOVERING THE PROCESSES OF YOUNG CHILDREN'S 3D MENTAL ROTATION ABILITIES: IMPLICATIONS FOR LESSON DESIGN

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Spatial reasoning involves thinking about the location of objects in space, their attributes, the paths they take and how they are related to each other (Newcombe, 2010). Of the different types of spatial reasoning, spatial visualization and more specifically, mental rotation, is highly correlated to mathematics performance. In their document on curricular focal points, NCTM (2008) has recommended that at least half of the early years mathematics curriculum be focused on geometry, measurement, and spatial reasoning. As a result, there is a need to develop lesson sequences that focus on spatial reasoning and measures to assess this ability in early childhood. Furthermore, recent studies have shown that spatial reasoning is malleable and can be improved (Uttal et al., 2012). This underscores the importance of laying a foundation for spatial reasoning in the early years to allow for future improvement.

To address the paucity of research on spatial reasoning in the early years and particularly in mental rotation, we have developed a measure to assess mental rotation in young children that is developmentally appropriate. Unlike typical measures of mental rotation, this measure presents children with actual 3D figures, which allows them to physically manipulate the objects. It also lends itself to direct strategies called upon during performance. Analyzing children's task performance showed that even very young children were capable of mental rotation and provided key insight into children's mental rotation strategies. These findings provided the foundation on which subsequent lessons were designed. We conclude by providing an example of a grade 1 lesson designed to develop children's spatial reasoning and 3D mental rotation abilities.

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ABSTRACTING AND GENERALIZING: PROCESSES OF MATHEMATICAL THINKING

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Keywords: Reasoning and Proof

Helping students develop their mathematical reasoning (MR) is a goal of several curricula. Based on this goal, this research project aims to qualify from a theoretical perspective two of the mathematical reasoning processes that are considered to occur in class. To do so, a literature search based on *anasynthesis* (Legendre, 2005) was undertaken. From the analysis of the mathematics education research literature on MR, several processes would appear to be linked to MR. One of these is generalization, but, as has been pointed out by Davis and Hersh (1981) and White (1993), abstraction and generalization are sometimes used as synonyms. To better understand what it means to generalize and to abstract and their relation to MR, a theoretical analysis was carried out.

A commognitive framework (Sfard, 2007) underpinned the anasynthesis of the literature. From a commognitive point of view, MR is a discursive process that derives utterances about objects or mathematical relations by exploring the relations that tie them together. By this, we mean that MR extends an existing discourse about already existing mathematical objects. In contrast, mathematical thinking is a synonym for mathematical discourse. It is composed of particular wordings, mediators, narratives, and routines. It is more than MR. MR is a kind of mathematical thinking process that contributes to its development by broadening it with derived utterances.

The analysis points out that the principal difference between generalization and abstraction is a discursive one. Generalization leads to discourse extension. There is coherence between the old and the new one. No rules are changed; only new information about objects already constructed is built. Abstraction leads to a new discourse, new rules. The old discourse then seems limited, even *passé*, from this new vantage point. At the heart of the matter, one can ask about the role played by reasoning, that is, by inferring, in the process of abstraction. Rather, it could be described as a process that is developed by cycles of individualization of interpersonal discourse and (re)communication of it.

Because generalization and abstraction are linked to mathematical reasoning in the mathematics education literature, a commognitive perspective can have an impact on mathematical reasoning studies in this field. As a matter of fact, this perspective leads to rejecting abstraction as a mathematical reasoning process. While both may be considered processes of mathematical thinking and learning, and can influence each other, one cannot be subsumed within the other.

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COORDINATING REPRESENTATIONS OF COVARYING QUANTITIES: LINKING DYNAMIC GRAPHS & FILLING AREA ANIMATIONS

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Purpose

Researchers have incorporated dynamic environments involving multiple linked representations of covarying quantities when investigating middle and high school students' reasoning about change (e.g., Kaput & Schorr, 2008; Saldanha & Thompson, 1998). While dynamic environments linking animations with graphs might have potential to foster students' interpreting and forming relationships between covarying quantities, students working in such environments seem likely to engage in a wide range of reasoning. In this study (funded by a Faculty Development Grant from the University of Colorado Denver) we investigated how students interpreted and formed relationships between covarying quantities while working in dynamic environments.

Method

This research used design experiment methodology (Cobb et al., 2003) to investigate students' mathematical reasoning in whole class and interview settings involving tasks hypothesized to support students' reasoning about covarying quantities. Johnson led 6 days of whole class instruction with 4 sections of 7th grade students, then conducted follow up interviews with 7 pairs of students, selecting at least one pair of students from each section. During the interview, pairs worked with dynamic environments linking pictorial representations of an adjustable rectangle and a right triangle "filling with area" with a graph relating the changing quantities of area and height.

Results/Discussion

Students reasoning about area as a result of a calculation made comparisons between the shape of an animated object (rectangle or triangle) and the shape of a graph and interpreted variable increase as if it were constant. In contrast, students reasoning about area as a measurable attribute of a rectangle or triangle could envision how area might increase at different magnitudes when interpreting and/or predicting features of a graph relating area and side length. When the use of a procedure to calculate area was problematized for students, distinctions emerged between how students were making sense of area as quantity and how students were considering area as increasing at different magnitudes.

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STUDENTS' USE OF INFORMAL REPRESENTATIONS IN PROOF CONSTRUCTION

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Keywords: Reasoning and Proof, Advanced Mathematical Thinking, Post-Secondary Education

Mathematicians reason from informal representations (IRs), such as graphs, diagrams, and specific examples of more general concepts, when constructing proofs (Thurston 1994). Alcock (2004) found mathematicians to use *algebraic* IRs in their proof construction in order to understand statements, generate arguments, and check arguments. Samkoff et al. (2012) found mathematicians used *visual* IRs for noticing properties and generating conjectures, estimating the truth of an assertion, suggesting a proof approach, instantiating or representing an idea or assertion in a diagram, and validating theorems through diagrams.

As such, mathematics educators suggest that students should base proofs on informal reasoning (Garuti et al. 1998). Case studies, however, show instances of students both successfully (Sandefur et al. 2012) and unsuccessfully (Pedemonte 2007) constructing proofs based on IRs. This study addresses the following research questions: (1) To what extent do IRs help students write proofs? (2) When IRs are unhelpful in students' proof construction, why are they unhelpful? (3) Do students use IRs for the same purposes as mathematicians? (4) Are there common unproductive patterns of behavior in students' usage of IRs in proof construction tasks?

Twelve senior undergraduate mathematics majors participated in task-based interviews completing 7 calculus proof construction tasks. These 84 proof attempts were analyzed for the use of 221 IRs and the purposes for which those representations were employed. Overall, 21 (25%) of the 84 proof attempts were successful. Of the 70 proof attempts using IRs, only 10 (14%) were successful. Of the 14 proof attempts without IRs, 11 (79%) were successful.

Next, we used the frameworks by Alcock (2004) and Samkoff et al. (2012) to analyze the purposes for which students used IRs. Despite 11 out of 12 participants creating 48 algebraic IRs, only 6 IRs (13%) were used for purposes similar to the mathematicians in Alcock's (2004) study. In contrast, at least 7 participants used between 5% and 38% of the 173 visual IRs for the purposes described in Samkoff et al.'s (2012) study.

These data suggest not only that IRs were not necessarily helpful to students in proof construction tasks, but also that a minority of students attempts to use IRs for the same purposes as mathematicians. To investigate why these proof attempts were unsuccessful, we used an open coding scheme and found that students often use inappropriate IRs, make misleading generalizations from special cases, get trapped in time sinks while studying IRs, and use IRs to verify a truth of a statement rather than gain insight on why it is true. These results suggest that simply advising undergraduates to use IRs is not likely to be helpful in their proof construction.

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WHAT CONVINCES STUDENTS BEFORE THEY ARE TAUGHT TO PROVE: EIGHTH GRADERS' PREFERENCE ON MATHEMATICAL ARGUMENTS

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It is well documented that mathematics learners hold different opinions from working mathematicians regarding what kind of mathematical arguments are convincing and explanatory to them (Harel & Sowder, 1998; Tall et al., 2012). However, in school mathematics students are often taught to follow the mechanic procedures to produce proofs without understanding the logic embedded within the arguments (Herbst & Brach, 2006). As a consequence, the type of proofs that students believe would earn them the highest mark can be different from what they would adopt for themselves (Healy & Hoyles, 2000), though little is known about the kinds of mathematical arguments that students would accept for their own use. Among all the age groups, 8^{th} graders are of particular interest since their opinions would help educators better understand students' thinking when they enter the conventional proof-based course in geometry and hence design more appropriate instructional techniques accordingly.

To identify and describe different types of mathematical arguments that children considered as convincing, we augmented Harel and Sowder's proof schemes (1998) and Tall et al.'s (2012) construct of embodiment to explore, from the learners' point of view, the link between evidence and conclusion when considering and evaluating mathematical arguments. Using a survey we collected data on students' judgment of and preference for different types of mathematical arguments. The survey contains four mathematical conjectures from different branches of school mathematics. Each conjecture is then followed by four distinct types of mathematical arguments to justify the conjecture. Students are asked to determine whether each of the presented arguments shows the conjecture is always true and then to choose the argument that they are most likely to adopt when justifying the same conjecture. The survey was designed to be completed online.

Over four hundred participants from five Ohio public schools completed the survey. Initial results indicated that the preference for an argument was highly diverse among individuals in every context. The results also highlighted that each individual's preference for an argument type was highly context based. In order to investigate factors that impact students' preference, the participants were divided into three groups based on their survey responses and a follow-up interview with representatives from each group was conducted so as to further study their thinking. Specifically, the three groups are composed of students who always, sometimes, and never consider an empirical argument as adequate to demonstrate the general validity of a conjecture. Detailed results of participants' responses to the survey as well as findings from the interviews will be shared during the poster session.

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QUANTITATIVE REASONING IN ENVIRONMENTAL SCIENCE: LEARNING PROGRESSION FOR 6TH TO 12TH GRADES

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Keywords: Learning Progressions, Reasoning and Proof, Modeling, Assessment and Evaluation

Purpose

If quantitative reasoning (QR) is to serve as a trigger for interdisciplinary problem-based pedagogies in STEM teaching, then more needs to be known about the progression of students' QR development in STEM. The purpose of this study is to verify a hypothesized learning progression for QR with environmental sciences as a context.

Theoretical Framework

The definition of QR used in this study, derived from the literature as well as prior research conducted by our QR team on a NSF funded project called Pathways (Culturally Relevant Ecology, Learning Progressions, and Environmental Literacy, DUE-0832173) is: Quantitative reasoning is mathematics and statistics applied in real-life, authentic situations that impact an individual's life as a constructive, concerned, and reflective citizen. QR problems are context dependent, interdisciplinary, open-ended tasks that require critical thinking and the capacity to communicate a course of action. Learning progressions are central to learning and curriculum design, a promising model that can advance effective adaptive instruction in QR.

Methodology

Creating learning progressions is an iterative research process that involves grounding the lower anchor in domains that for this study are accessible to 6th graders, then identifying intermediate levels of understanding through which they pass on their way to attainment of the upper anchor. The research subjects include 6th to 12th grade students in partner school districts in 3 different states. Participants will be students in science and mathematics courses selected from middle schools, junior high schools, and high schools within the partner school districts. Data collection will include student closed-form assessments completed in school science or mathematics classes. No new curricular implementation will be undertaken; the project team will only be assessing the student's current understanding of QR.

Results

The result of this research will be a revised learning progression for QR in the context of environmental science. This learning progression provides educators with major indicators of advancement of quantitative reasoning. In addition, the development and revision of the QR assessments themselves provide a potential formative assessment tool for teachers. We will discuss the results of the quantitative and qualitative analysis, as well as exploring the current state of the QR learning progression.

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CHARACTERIZING STUDENT CONCEPTIONS OF SPAN AND LINEAR INDEPENDENCE THROUGH MATHEMATICAL ACTIVITY: THE CASE OF JOE

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In this poster, we share results from a larger research project investigating linear algebra students' understanding of span and linear independence. Situating our work within the existing literature (e.g., Hillel, 2000; Sierpinska, 2000; Stewart & Thomas, 2010), we developed grounded frameworks for analyzing not only students' conceptions of span and linear independence, but also the types of mathematical activity in which students engage as they discuss the mathematical content (Authors, 2013). Briefly, we organize students' conceptions of span and linear independence in four categories: geometric, matrix algebraic, travel, and vector algebraic. Our categorizations of mathematical activities are defining, example generating, problem solving, proving, and relating. The present research comes from a series of interviews designed to utilize these respective frameworks. Specifically, we detail one participant's response to an interview question designed to engage him in problem solving activity while eliciting aspects of his conceptions of span and linear independence.

Our data draws from semi-structured individual interviews (Bernard, 1988) conducted with seven, first-year, honors STEM majors. Each interview lasted between 90 and 120 minutes. Each task in the interview protocol was designed to engage the participants primarily in one type of mathematical activity from our framework. Each task was also designed so that it used a specific type of representation of vectors (i.e., some questions presented specific *n*-tuples in $\mathbb{R}^3$, while others used more abstract notation such as **v** and **w**). For this poster, we focus our interest on one interview question that elicited interesting and diverse responses:

> Given that $\mathbf{v}_1 + 2\mathbf{v}_2 = \mathbf{v}_3$ (where $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in V$, a vector space), develop more than one set of real numbers $\{a_1, a_2, a_3\}$ such that $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = \mathbf{0}$.

We discuss how one participant (Joe) successfully engaged in the task by detailing how he drew from several different aspects of his conceptions of span and independence and coordinated these different conceptions through the activities of example generating and relating to solve the problem. Joe, like most of the participants in our study, used instantiations of *n*-tuples in $\mathbb{R}^3$ in order to find his solution. Unlike his peers, however, Joe additionally relied on a novel geometric representation to develop his response to the task and coordinated this representation with his earlier work, allowing him to provide a more detailed response.

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ON THE EMERGENCE OF MENTAL MATHEMATICS STRATEGIES

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Most studies of mental mathematics concerned with analyzing students' strategies are based on an assumption that is known as the "toolbox metaphor," which implies that in solving a problem, students choose a strategy (a "tool") from a group of predetermined strategies they have acquired in the past (the "toolbox"). Recently, this metaphor has been significantly questioned by a number of researchers (e.g., Threlfall, 2002), who illustrated that there is more to solving mental mathematics problems than the "simple" re-using of already predetermined strategies. The focus becomes more about the adaptation and the creation of ways of solving *in function* of the problem at hand, and not about the use of strategies developed independently of problems. We present an example of a strategy taken from a larger study to support the view that strategies are emergent, thought on the spot, and intertwined with the specific problem at hand.

To solve 741–75, a preservice teacher explained one of her strategies in the following way:

(a) 741 – 75 is like 700 – 75 + 41.

(b) 700 – 75 is like having 7 dollars and subtracting 3 quarters. I am left with $6.25. And, 6.25 is six-twenty-five, so I add 41 to 625.

(c) To do so, I do 5 +1 is 6, 4 +2 is 6, and I have 600, so 666.

When the problem 741–75 was given, the first action undertaken by the student was to find a way to solve it, to "find a way in." This student opted for splitting 741 into the two parts, 700 and 41, and then to subtract the 75 from the 700. This would hardly be a strategy the student would have used if the work had been with paper-and-pencil, where often the standard algorithm would be applied, working through the task in an prescribed manner, mostly independent of the problem given. Each step can be seen as a "new" problem to work on, thereby finding new "ways in" to continue reasoning through the problem. Step (a) is a way to "enter" the problem; the outcome of this step (having to compute 700 – 75) places the student in front of another "problem to solve." Again, the student found a way to continue, and thus opted for finding a money context ($7 minus 3 quarters) to carry out the computation. The outcome of this leads the student to another "obstacle" in the way of the answer, which is to find a way to compute 625 + 41, leading to a strategy of splitting units, tens, and hundreds. That strategy is contingent on the problem given, but is also emergent, engaged with on the spot, when the problem is received.

In this sense, students' ways of entering problems can be seen in the form of "creating" their own problem solving contexts -- that is, of finding a way to (re)organize the problem as one of a "different" sort to solve. These strategies are conceived when students *meet* the task, resulting in a way of solving *in the moment* (and not of stepping back, looking in the "toolbox," and choosing an appropriate strategy). This conceptualization of mental mathematics can offer a new entry into strategy analysis, shifting the focus away from how known strategies are applied toward the creative nature of the mathematical activity in which students are engaged.

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INTRODUCING MANIPULATIVES: TO PLAY OR NOT TO PLAY

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Manipulatives have been repeatedly discussed in the literature and used in classrooms as a solution to bridging the gap between children's grasp of mathematical concepts and meaningful use of written numbers and other symbols (e.g., Fuson & Briars, 1990). For meaningful manipulative use, children must make the connection between the concrete objects and the concepts they represent; hence, manipulatives must be "encoded" as objects representing quantities (Brown et al., 2009). The representational relationships between symbol and concept may be apparent to teachers, but are not always successfully understood by children. In fact, if not used with meaning, manipulatives may hinder learning (McNeil et al., 2009). Because many teachers view and use manipulatives as play objects aimed to make mathematics more enjoyable (Moyer & Jones, 2004), it is unclear what impact such non-quantitative use of manipulatives has on children's ability to encode and use the tools as representations of quantities.

We investigated the impact of three different introductions to manipulatives on children's conceptions of the objects and their ability to use them as representations of quantities. Seventy-three first- and second-grade students ($N = 73$) from three schools in Canada were randomly assigned to three encoding conditions, plus a control group. In the Math condition, children were given a quantitative meaning of red and blue chips (Blue = 1 and Red = 10); the Game Piece group was given a non-quantitative meaning of the chips (i.e., game pieces); and the Free-Play group was encouraged to play with the chips to develop their own representations. After these introductions, we asked the students (a) about their perceived use of the chips, and (b) to show quantities below 50 (e.g., "Can you show me what 24 would look like with these circles?"). All students were then given addition instruction with written numbers in which the chips were used as supportive tools; the same interviews were conducted after the instruction.

We found that the Math group had the largest proportion of children who perceived the manipulatives as having a quantitative use (87%) in comparison to Control (50%), Game Piece (24%), and Free-Play (16%). After instruction, the proportions of perceived quantitative use increased for Control (75%), Game Piece (65%), and Free Play (68%) groups, but the instruction was not enough for them to catch up to the Math group (80%). The results also indicated that allowing children to initially play freely with the manipulatives was significantly associated with using them in a non-quantitative way prior to and after addition instruction (e.g., pre: $X^2(3) = 8.1$, $p < .05$; post: $X^2(3) = 9.6$, $p < .05$). Thus, free play with manipulatives appears to have a negative effect on children's ability to see and use them as quantities, even after instruction.

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EXAMINING MATHEMATICAL THINKING IN ENGINEERNING AND MATHEMATICS STUDENTS

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Keywords: Advanced Mathematical Thinking, Cognition, Post-Secondary Education

Calls for improving the mathematical education of Science, Technology, Engineering, and Mathematics (STEM) students as well as a need for larger numbers of STEM graduates have been widespread in the past decade (Ferrini-Mundy & Güçler, 2009). Recent work examining mathematical literacy at the collegial level has shown that mathematical skills needed by different STEM disciplines are highly specialized (Tague, Czocher, & Baker, 2012). However, there is not yet consensus on how these skills might be supported in coursework students are required to complete (Wankat, 2008). While it might be tempting to suggest that attention be devoted to developing new courses accommodating the particular mathematical needs of different groups, such an approach tends to ignore the structures that constitute human cognitive architecture (Kirshner, Sweller & Clark, 2006). Curriculum and instruction design efforts are less likely to be effective in the absence of an understanding of how the mathematical cognition of different STEM audiences might process information differently. This poster examined the mathematical thinking of one engineering student and one mathematics student from a large Midwestern university as a starting place in addressing this gap. Each student was interviewed and given two open-ended questions and one typical “word” problem. Interviews were each approximately 90 minutes long and students were encouraged to explain their reasoning and thinking processes.

Interviews were analyzed using Valsiner’s (1997) zone theory, a sociocultural lens, for investigating the resources the students drew. Additionally, mathematical modeling cycle (Blum & Leiβ, 2007) allowed for cataloguing the particular phases involved in the participants’ mathematization process. Initial results indicate that the mathematics student was comfortable creating parameters and estimating values whereas the engineering student primary focus was on accuracy. The engineering student also validated not just the numerical estimates he made, but also the methods of solution that he was considering. Differences and similarities between the mathematical thinking of the two students will be discussed as well as possible instructional implications.

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Chapter 6: Statistics and Probability

Research Reports

Brief Research Report Sessions

Poster Presentations

COMPARING THE RELATIVE LIKELIHOOD OF EVENTS: THE FALLACY OF COMPOSITION

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The objective of this article is to contribute to the documented dearth of research on teachers' probabilistic knowledge. Prospective teachers of elementary school mathematics were asked to identify which result from five flips of a fair coin was least likely to occur. Participants, instead of being presented with sequences, were presented with events, that is, sets of outcomes, for five flips of a fair coin. A particular logical fallacy, the fallacy of composition, was used to analyze the juxtaposition of responses and response justifications. As a result, the fallacy of composition was found in the response justifications for both normatively incorrect and correct relative likelihood comparisons. Combing the results of this and prior research utilizing the fallacy of composition demonstrates that logical fallacies are a burgeoning area of research for those investigating relative likelihood comparisons and teachers' probabilistic knowledge.

Keywords: Probability; Teacher Knowledge

Given there is a limited amount of research on "teachers' probabilistic knowledge" (Jones, Langrall & Mooney, 2007, p. 933), one objective of this article is to continue an emergent thread of research investigating prospective mathematics teachers' probabilistic knowledge (Chernoff, in press, 2012a-c, 2011; Chernoff & Russell, 2012a, 2012b, 2011a, 2011b). A second objective of this article is to contribute to a well established domain of research, which accounts for normatively incorrect responses to relative likelihood comparisons (e.g., Abrahamson, 2009; Borovcnik & Bentz, 1991; Chernoff, 2009; Cox & Mouw, 1992; Hirsch & O'Donnell, 2001; Kahneman & Tversky, 1972; Konold, 1989; Konold, Pollatsek, Well, Lohmeier, & Lipson, 1993; LeCoutre, 1992; Rubel, 2007; Shaughnessy, 1977, 1981; Tversky & Kahneman, 1971, 1974). Yet a third objective of this article is to bolster the results of a particular research thread previously published in the conference proceedings of the annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (Chernoff & Russell, 2011a) and the International Group for the Psychology of Mathematics Education (Chernoff & Russell, 2011b)

To meet the objectives stated above, prospective teachers of mathematics, as was the case in prior research, were asked to make a comparison of relative likelihood; however, unlike previous research (for exceptions see, for example, Chernoff, in press; Chernoff & Russell, 2011b), participants were asked not to compare the relative likelihood of sequences, but, rather, events (i.e., sets of outcomes) resulting from five flips of a fair coin. Also breaking from tradition, this research, instead of using the theories, models and frameworks most frequently associated with relative likelihood comparison research (e.g., Kahneman & Tversky, 1972; Konold et al., 1993), uses a particular logical fallacy for the analysis of results. The current set of responses to a task recently introduced to the research literature – the relative likelihood of events task (Chernoff & Russell, 2011b) – once again evidence use of the fallacy of composition (i.e., because parts of a whole have a certain property, it is argued that the whole has that property). Also once again, the fallacy of composition is evident in both incorrect and correct response justifications in the current research. The results from this current research, coupled with the results from the initial study (Chernoff & Russell, 2011b), further validate that certain prospective teachers of mathematics use the fallacy of composition when making relative likelihood comparisons. In line with this point, it will, further, be argued that logical fallacies are a

viable venue for future research investigating relative likelihood comparisons and teachers' probabilistic knowledge.

A Brief Summary Of Prior Research

Research into probabilistic thinking and the teaching and learning of probability has, in the past, seen a focus on normatively incorrect responses. Specific to the field of mathematics education, the theories, models and frameworks associated with heuristic and informal reasoning – rooted in the notions of conceptual analysis (Thompson, 2008; Von Glaserveld, 1995), grounded theory (Strauss & Corbin, 1998) and abduction (Lipton, 1991; Peirce, 1931) – have, traditionally, accounted for normatively incorrect responses to probabilistic tasks. Chernoff (in press, 2012a-c, 2011) and Chernoff and Russell (2012a, 2012b, 2011a, 2011b) have recently provided detailed accounts of (the history of) the theories, models and frameworks associated with heuristic and informal reasoning in the field of mathematics education and are (given the boundaries associated with the present venue) recommended to the reader.

More recently, a burgeoning area of research suggests that fallacious reasoning, more specifically, the use of logical fallacies, can account for certain normatively incorrect responses to relative likelihood comparisons. For example, Chernoff (in press) and Chernoff and Russell (2011b) utilized the fallacy of composition (when an individual infers something to be true about the whole based upon truths associated with parts of the whole) as a framework to anlayse relative likelihood comparison responses. Prospective mathematics teachers were asked to determine which of five possible coin flip sequences were least likely to occur. As was the case in similar research (e.g., Chernoff & Russell, 2011a), the fallacy of composition accounted for normatively incorrect responses to the task. More specifically, the researchers demonstrated that participants referenced the equiprobability of the coin, noted that the sequence is comprised of flips of a fair coin and, as such, fallaciously determined that the sequence of coin flips should also have a heads to tails ratio of one to one. In other words, the properties associated with the fair coin (the parts), which make up the sequence (the whole), are (fallaciously) expected in the sequence. As a result, the fallacy of composition, in addition to the traditional theories, models and frameworks associated with heuristic and informal reasoning, accounted for certain normatively incorrect responses to a probabilistic task.

Chernoff and Russell (2012a, 2011b) further established that certain prospective mathematics teachers, when asked to identify which event (i.e., outcome or subset of the sample space) from five flips of a fair coin was least likely to occur, did not use the representativeness heuristic (Kahneman & Tversky, 1972), the outcome approach (Konold, 1989) or the equiprobability bias (Lecoutre, 1992). Instead, they again demonstrated that participants utilized the fallacy of composition. Worthy of note, the fallacy of composition accounted for both normatively correct and incorrect responses to the new relative likelihood comparison task. As mentioned, this article extends previous research utilizing the fallacy of composition.

The Fallacy Of Composition

The fallacy of composition, as a theoretical framework for the analysis of response justifications for relative likelihood comparisons, has revealed a new approach to account for normatively incorrect responses. Moreover, the fallacy of composition has proven a viable approach for both relative likelihood comparisons between sequences and events. Given that the current research tasks participants with relative likelihood comparisons between events, the fallacy will be used for the impending analysis of results.

Essentially, the *fallacy of composition* occurs when an individual infers something to be true about the *whole* based upon truths associated with *parts of the whole*. For example: Bricks (i.e., the parts) are sturdy (i.e, the truth). Buildings (i.e., the whole) are made of bricks (i.e., the parts). Therefore, buildings (i.e., the whole) are sturdy (which is not necessarily true).

As will be demonstrated throughout the analysis of results, certain participants in this research inferred certain truths associated with individual coin flips to be true for events, that is, sets of outcomes. Prior to this analysis, however, certain methodological components (e.g., Task and Particpants) are discussed.

Task

As seen in Figure 1 below, the task used in this research is the same task recently introduced to the research literature (Chernoff & Russell, 2011b).

Which of the following is the least likely result of five flips of a fair coin?
a) three heads and two tails
b) four heads and one tail
c) both results are equally likely to occur
Justify your response...

Figure 1: The Relative Likelihood Of Events Task

Unlike research involving sequences of coin flips, the relative likelihood of events task possesses a normatively correct response (i.e., "four heads and one tail") and two normatively incorrect responses (i.e., "three heads and two tails" and "both results are equally likely to occur"). See, for example, Chernoff and Russell (2011b) for further details associated with history and design of both relative likelihood comparison tasks and the relative likelihood of events task.

Participants

Participants in our research were (n =) 54 prospective mathematics teachers enrolled in a methods course designed for teaching elementary school mathematics. More specifically, the 54 participants were comprised of two classes, containing 26 and 28 students, taught by the same instructor. Participants were presented with the relative likelihood of events task and were allowed to work on the task until completion. Of note, the participants had not answered any of the other versions of the relative likelihood task prior. Further, the topic of probability had yet to be discussed in class at the time of the research.

Results

Participants' responses fell into three categories. The majority, 38 of the 54 participants (or 70%), incorrectly declared that the events (three heads and two tails and four heads and one tail) were equally likely to occur. Further, fourteen of the 54 participants (or 26%) correctly declared that four heads and one tail was least likely (of the events presented in the task) to occur and two participants (or 4%) declared that three heads and two tails was least likely to occur.

Table 1: The Numerical Breakdown Of Responses

Number of participants	Three heads and two tails	Four heads and one tail	Equally likely
Class A (26)	1	8	17
Class B (28)	1	6	21
Total (54)	2	14	38
Percentage (100)	**4%**	**26%**	**70%**

Given the boundaries associated with the present venue (i.e., the 8 page limitation), the responses from eleven individuals – five from individuals that declared the events equally likely to occur and

six from individuals that declared four heads and one tails as least likely – are now featured in the analysis of results.

Analysis Of Results

The response justifications from certain individuals reveal the fallacy of composition when comparing the relative likelihood of the events they are presented. Given that use of the fallacy of composition is witnessed in the justifications for both incorrect and correct responses to the task and, as such, the analysis of results is organized accordingly.

Both Results Are Equally Likely To Occur (Incorrect Response)

Responses from (more than) 5 of the 38 individuals that declared the events were equally likely to occur evidence (to varying degrees) use of the fallacy of composition. Consider, for example, the responses from Marie Hank and Walter, who, respectively, declare that "either outcome could occur equally," "both outcomes are equally likely" and "it is just as likely for either outcome" because of (for all three of them) the equiprobability of flipping a fair coin.

Hank: Both outcomes are equally likely to occur because the probabilty of getting a head or tail is the same.

Marie: either outcome could occur equally. each time the coin is flipped there is a 50 50 chance that the coin will land with a certain face up.

Walter: It is just as likely for either outcome or any possible outcome (1H+4T, 2H+3T, 3H+2T, 4H+1T) since there is an equal chance of either occuring on each flip.

Discussed in terms of the theoretical framework, although Marie and Frank declare that the "outcome(s)" (i.e., the building) are equally likely (i.e., sturdiness of building) because of the equiprobability of the coin (i.e., sturdiness of the brick), and although Marie makes reference to the equiprobability each time the coin is flipped, their responses do not (explicitly) declare that the "outcome(s)" they are discussing are comprised of five flips of the coin (i.e., the building is made of bricks). Walter extends the argument of Hank and Marie to further declare that not only are the two outcomes they are presented are equally likely, but, further, "any" (of course, Walter is "missing" the all heads and all tails outcomes) of the outcomes are equally likely because of the equiprobability associated with the flip of a fair coin.

The responses of Skyler, Junior and Walter (differently) reference that the events are comprised of five flips of a fair coin. Consider, then, the responses from Sklyer and Junior.

Skyler: Each flip of the coin has the same odds of having heads land face up as having tails land face up. If you flip the coin once, there is a 50% chance you'll get a tail, and 50% chance you get a heads. The next time you flip the coin, the result is not dependent or related to the result of the first flip in any way, so the odds of flipping a tails the second time is still 50%, regardless of what the outcome was the first flip (same goes for head). This continues for all five flips.

Junior: on each flip of the coin there exists a 50:50 chance of heads or tails occurring. therefore, the combinations even when aggregated together have the same chance of occurring.

Like Hank and Marie, Skyler and Junior, in their responses, contend that the events are equiprobable because of the equiprobability associated with the coin flip(s). Discussed in terms of the fallacy of composition, the responses from Sklyer and Junior reference the equiprobability (i.e., sturdiness) associated with each coin flip (i.e., brick); further, their responses reference that the event (i.e., the building) is comprised of coin flips (i.e., bricks); and, as such, the outcome(s) should be equiprobable (i.e., the building made of sturdy bricks should be sturdy) or, in the words of Junior, "even when aggregated together have the same chance of occurring." Skyler and Junior (and, arguably, Hank and Marie) inferred the equiprobability of the events based upon the equiprobability associated with the flip of a fair coin. Put in more general terms, they used the fallacy of composition when determining which event was least likely to result from five flips of a fair coin.

Four Heads And One Tail (Correct Response)

As presented above, the responses from certain individuals that incorrectly declared both results equally likely to occur demonstrate use of the fallacy of composition when comparing the relative likelihood of events. As will be presented below, the responses from certain individuals that correctly determined that four heads and one tail was least likely (of the events presented) to occur also evidence use of the fallacy of composition. However, as will be evidenced in the six responses that follow, the result established through use of the fallacy of composition, that is, that the events are equally likely to occur, is subsequently used to determine that four heads and one tail is least likely to occur. Stated in more general terms, the participants are fallaciously arriving at the "correct" response.

Consider, for example, the similar yet different responses from Gale, Mike and Saul who all make reference to an expected ratio of heads to tails for the outcome.

> *Gale:* Four heads and one tail is least likely because each flip gives you a 50% chance of getting either heads or tails so it is more likely you would get closer to answer a.
>
> *Mike:* a – more likely since 3:2 is closer to 1:1 than 4:1 is. When flipping a coin the outcome should be similar to 1 head to 1 tail although it usually isn't.
>
> *Saul:* Three heads and two tails are more likely b/c there is only 2 options so there is a 50/50 chance of each happening, so it is more likely that the outcomes of heads = tail ratio will be equal or close to equal rather than further apart.

All three of the above responses are similar. For example, all three responses declare that the four heads and one tail outcome is least likely to occur because the ratio of heads to tails, that is, four to one, is farther "away" from the expected ratio of heads to tails, that is, one to one, than the "closer" ratio of heads to tails, that is, three to two. Alternatively stated, as Mike succinctly writes, "[three heads and two tails] – more likely since 3:2 is closer to 1:1 than 4:1 is." Further, also garnered from the responses above, participants (as was the case for participants in the previous section) inferred the equiprobability of the events, that is, the expected one to one ratio of the outcomes, based upon inferred equiprobability associated with the flip of a fair coin.

Discussed in terms of the fallacy of composition, the responses of Gale, Mike and Saul (again to varying degrees) reference: the equiprobability (i.e., the sturdiness) associated with the flip of a fair coin (i.e, the brick); the outcome (i.e., the building) being comprised of five flips of a fair coin (i.e., bricks); and, as such, expected, fallaciously, the outcome (i.e., the building) to have an equal ratio of heads to tails (i.e., the building to be sturdy). Subsequently, Gale, Mike and Saul utilize their fallacious conclusion, that is, both sequences should have a one to one ratio of heads to tails (as is the case for the flip of a fair coin), to analyze which of the outcomes they are presented in "closest" in terms of their expected ratio. In all three instances above, the outcomes with a four to one ratio of heads to tails is declared "further away" and, thus, is "correctly" determined as least likely to occur.

Similar (in conclusion) to the responses of Gale, Mike and Saul, Jesse, Jane and Gus also reference an expected ratio of heads to tails, but, unlike Gale, Mike and Saul, instead of referencing "closeness," reference the unlikeliness or unevenness or inequality associated with obtaining a four to one ratio of heads to tails when expecting a ratio closer to one to one.

> *Jesse:* fair coin would give 50/50 chances so 5 flips should be more equal results than 4:1
>
> *Jane:* You're least likely to flip 4 heads and 1 tail, as opposed to 3 heads and 2 tails. Each flip has a 50% chance of flipping either one. That makes is unlikely that you will flip 4/5 of the same side.
>
> *Gus:* b) because you have a 50/50 chance of getting a head or tail each time you flip. Therefore, getting an even amount of each is more likely than all on one side as in b.

Again, all three of the above responses are similar. The three responses declare that the four heads and one tail outcome is least likely to occur because (with an fallaciously expected ratio of one to one for heads and tails) having "4/5 of the same side" is unlikely or "getting an even amount of each is more likely" or there should "be more equal results than 4:1." Again, the ratio of heads to tail, that is,

one to one, is still farther "away" than three to two, but the terminology used by Jesse, Jane and Gus is different than for Gale, Mike and Saul. Further, also garnered from the responses above, participants (as was the case for participants in the previous section) inferred the equiprobability of the events, that is, the expected one to one ratio of the outcomes, based upon the equiprobability associated with the flip of a fair coin.

Discussed in terms of the fallacy of composition, the responses of Jesse, Jane and Gus (again to varying degrees) reference: the equiprobability (i.e., property of the part) associated with the flip of a fair coin (i.e, the part); the outcome (i.e., the whole) being comprised of five flips of a fair coin (i.e., parts); and, as such, expected, fallaciously, the outcome (i.e., the whole) to have an equal ratio of heads to tails (i.e., the whole to have a property associated with a part). Subsequently, Jesse, Jane and Gus, like Gale, Mike and Saul, utilize their fallacious conclusion, that is, both sequences should have a one to one ratio of heads to tails (as is for the case of the flip of a fair coin), to analyze which of the outcomes they are presented is most "equal" or "even" in terms of their expected ratio. In all three instances above, the outcomes with a four to one ratio of heads to tails is declared too uneven or not equal enough and, thus, is "correctly" determined as least likely to occur.

Concluding Remarks

The fallacy of composition is evident in the response justifications for both normatively incorrect and correct relative likelihood comparisons. For those individuals that incorrectly determined that both events were equally likely to occur, their responses referenced the equiprobability (stated differently by different individuals) of individual coin flips, that the outcomes presented were comprised of five flips of a fair coin and, thus, fallaciously concluded that the events must also be equiprobable. Alternatively, normatively correct relative likelihood comparisons were accompanied with logically fallacious response justifications. The responses from the latter participants also referenced the equiprobability of a single flip of a fair coin, that outcomes were comprised of flips of a fair and, thus, (also) fallaciously concluded that the ratio of heads to tails for the outcomes should as close to one to one as possible. Subsequently, based on this fallacious reasoning, certain participants deemed the outcomes four heads and one tails too "far away" from the expected one to one ratio, whereas other participants deemed the four heads and one tail outcome as too uneven or unequal. As demonstrated, the fallacy of composition is evidenced in both normatively incorrect and correct response justifications.

Discussion

In terms of research investigating relative likelihood comparisons, the use of logical fallacies, particularly the fallacy of composition, is a relatively recent endeavour. However, when the results of the current research are combined with previous research of Chernoff & Russell (2011b), as seen in Table 2, consistencies in the results, numerical and otherwise, are beginning to emerge.

Table 2: Current, Previous And Combined Results

Number of participants	Three heads and two tails	Four heads and one tail	Equally likely
Results of current study			
Class A (26)	1	8	17
Class B (28)	1	6	21
Total (54)	2	14	38
Percentage (100)	**4%**	**26%**	**70%**
Results of previous study			
Total (63)	5	12	46

Percentage (100)	**8%**	**19%**	**73%**
Combined results			
Total (117)	7	26	84
Percentage (100)	**6%**	**22%**	**72%**

Logical fallacies, and, in particular, the fallacy of composition, are proving an effective means when accounting for normatively incorrect, inconsistent and sometime incomprehensible comparisons of relative likelihood (Chernoff, in press, 2012a-c, 2011; Chernoff & Russell, 2012a, 2012b, 2011a, 2011b). As such, and as asserted previously (Chernoff & Russell, 2011b), logical fallacies should be considered, in addition to the traditional theories, frameworks and models, when investigating relative likelihood comparisons (of sequences or events) and teachers' probabilistic knowledge.

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THE EFFECTS OF DIAGRAMS AND QUESTIONING-WHILE-READING ON LEARNING FROM A STATISTICS LESSON

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This study examined the effectiveness of two methods of increasing student learning from posterior probability lessons: diagrams and questioning while reading. Undergraduate students (N = 245) read a lesson in one of three diagram conditions and one of three questioning-while-reading conditions (embedded questions, elaboration interrogations, and control in which they read twice). Diagrams helped students in the control group accurately solve posterior probability problems on the posttest, but only if the information presented in the diagram was also presented in the text. Contrary to expectation, students in the control condition accurately solved more posterior probability problems on the posttest than did students in the questioning conditions. Poor answer quality for the embedded questions and elaborative interrogations is a possible explanation for the unexpected results.

Keywords: Probability, Post-Secondary Education, Instructional activities and practices

Probabilistic reasoning can be difficult for undergraduate students. One type of probabilistic reasoning, calculating posterior probability, is particularly challenging (Kahneman & Tversky, 1972). Calculating posterior probability involves considering both the probability of a particular characteristic or disease in a given population (i.e., the base rate) and the probability that the identification of that characteristic or disease is accurate (i.e., the specificity). Students frequently focus on the specificity information because they consider it more salient than the base rate (Kahneman & Tversky, 1972; Sedlemeier & Gigerenzer, 2001). Therefore, it can be challenging for students to understand how to calculate posterior probability. However, it is important to teach students about probabilistic reasoning, including skills such as accurately calculating posterior probability, so that they can be better consumers and citizens in modern society (Garfield & Ben-Zvi, 2008). The present study examined the effects of two factors that may promote student learning from a lesson on calculating posterior probability: diagrams and questioning while reading.

Diagrams

One technique that may foster students' learning is including relevant visual representations. According to the *multimedia principle*, learning is improved when texts are accompanied by relevant visuals (Mayer, 2009), because students develop a verbal mental model from the text and a visual mental model from the visuals. Students are more likely to make connections among different ideas when they have the information represented in two mental models. These connections prompt deeper processing of the text and increased learning.

At present, it is unclear whether learning is improved when information presented in a diagram is also presented in text. According to the *redundancy principle*, information presented in the diagram should *not* also be presented in the text (Sweller, Ayres, & Kalyuga, 2011), because presenting information in both text and diagram increases the amount of information the student has to process, which may diminish learning (Mayer, 2009). However, the redundancy principle is based on studies using causal diagrams for scientific concepts (e.g., Chandler & Sweller, 1991). It has not been explored with numeric information in math lessons. It is possible

that having the numeric information presented in both diagram and text could facilitate connections between the verbal and visual information. Students may be better able to connect the concepts presented in the text and diagram if the information presented in the diagram is also presented in the text. Therefore, it is uncertain if including the information in the diagram as well as in the text will improve student learning about posterior probability.

Questioning while Reading

A second technique that may improve student learning is to have the students answer questions about the lesson while reading. Two types of questions are commonly used: embedded questions and elaborative interrogations. *Embedded questions* are simple questions about specific ideas stated in the lesson. Embedded questions are useful for directing students' attention to important information. Answering embedded questions has been found to increase learning from text; however, these benefits may be limited to certain student populations (e.g., less-skilled readers, Callendar & McDaniel, 2007). *Elaborative interrogations* are typically "why" questions intended to prompt the student to integrate an idea from the lesson with other ideas from the lesson or with background knowledge (e.g., McDaniel & Donnelly, 1996; Pressley, Symons, McDaniel, Snyder, & Turnure, 1988). Elaborative interrogations are intended to stimulate deeper processing of the lesson, thereby increasing learning (e.g., Ozgungor & Guthrie, 2004). Elaborative interrogation has been found to improve learning across a variety of student populations (cf. Dunlosky, Rawson, Marsh, Nathan, & Willingham, 2013). Given the previous literature on both embedded questioning and elaborative interrogation, it is likely that including these types of questions will improve student learning from a lesson on posterior probability.

The use of questioning while reading may be particularly beneficial when visual representations are included in the lesson. Students do not always attend to the visual representations in text (Schüler, Scheiter, Rummer, & Gerjets, 2012). Embedded questioning may be useful for directing students' attention towards the visual representations, particularly if the questions are about the visual representations. In addition, integrating visual and verbal information typically promotes learning (Mayer, 2009). Elaborative interrogation may be useful for guiding students to integrate the visual and verbal information. This can be accomplished by asking elaborative interrogations that encourage the students to use information in both the diagram and the text.

The present study examined the effects of diagrams and questioning on learning about posterior probability calculations. Given the previous literature on visual representations (e.g., Mayer, 2009), we predict that students will learn more if their assigned lesson includes a diagram. However, it is uncertain whether students will learn more if the information presented in the diagram is also presented in the text. In addition, we predict that students in the two questioning-while-reading conditions will learn more than students in the control condition, who will simply read the lesson twice. Finally, we predict that students who receive both diagrams and questions will learn more than students who receive only one or the other, or neither.

Methods

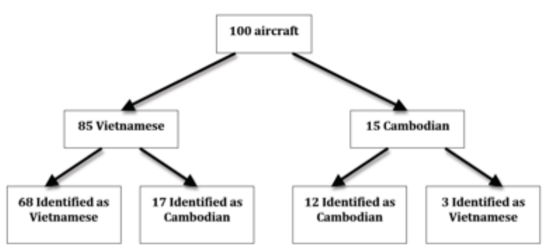
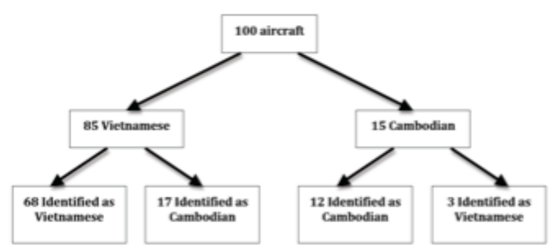
Participants were 248 undergraduates at a large, upper Midwestern university who earned extra credit in their introductory psychology course for participation. Three participants left the study before completing the posttest; their data were not included. Of the remaining 245 participants, there were 158 females and 86 males (one participant did not report gender) with an average reported age of 19.05 years (*SD* = 2.89 years). English was reported as the native

language of 164 students; 80 students reported a native language other than English (one student did not report native language).

Materials

The lesson was adapted from a textbook by Heuer (1999) and included materials from Sedlmeier and Gigerenzer's (2001) experiments. There were two examples of scenarios in which base rate and specificity probabilities were presented, and then the posterior probability was calculated using natural frequencies. For the diagram conditions, there was a diagram for each of the two examples. For the diagram without redundant text condition, the information in the diagram was not also presented in the text. For the diagram with redundant text condition, the information in the diagram was also presented in the text (see Table 1).

Table 1: Lesson Excerpts from Diagram without Redundant Text Condition and Diagram with Redundant Text Condition

Diagram without redundant text	Diagram with redundant text
Imagine 100 cases in which the pilot has a similar encounter. Based on the base rate, we know how many of these encounters will be with Vietnamese fighter jets, and how many with Cambodian. Based on the specific case information, we know that 80 percent Vietnamese fighter jets will be correctly identified as Vietnamese, while 20 percent will be incorrectly identified as Cambodian. Similarly, 80 percent Cambodian fighter jets will be correctly identified as Cambodian, while 20 percent will be incorrectly identified as Vietnamese. This is shown in the diagram below.  This makes a total of 71 Vietnamese and 29 Cambodian sightings, but not all of these sightings were correct. Therefore, when the pilot claims the attack was by a Cambodian fighter jet, the probability that the fighter jet was actually Cambodian is only 12/29ths or 41 percent.	Imagine 100 cases in which the pilot has a similar encounter. Based on the base rate, we know that 85 of these encounters will be with Vietnamese fighter jets, 15 with Cambodian. Based on the specific case information, we know that 80 percent or 68 of the 85 Vietnamese fighter jets will be correctly identified as Vietnamese, while 20 percent or 17 will be incorrectly identified as Cambodian. Similarly, 80 percent or 12 of the 15 Cambodian fighter jets will be correctly identified as Cambodian, while 20 percent or 3 will be incorrectly identified as Vietnamese. This is shown in the diagram below.  This makes a total of 71 Vietnamese and 29 Cambodian sightings, of which only 12 of the 29 Cambodian sightings are correct; the other 17 are incorrect sightings of Vietnamese fighter jets. Therefore, when the pilot claims the attack was by a Cambodian fighter jet, the probability that the fighter jet was actually Cambodian is only 12/29ths or 41 percent.

For the embedded questioning and elaborative interrogation conditions, there was one question after approximately every paragraph. For diagram and questioning conditions, a question was asked after each of the diagrams. For the diagram without redundant text condition, answering the question correctly required using information from the diagram. However, for the diagram with redundant text condition, the question could be answered using information from the text or the diagram (or both). Embedded questions were simple (e.g., "How many fighter jets identified as Vietnamese are actually Cambodian?") and their answers were explicitly stated in the text

and/or clearly presented in the diagram. Elaborative interrogation questions were complex "how" and "why" questions (e.g., "How are the base rate and the specific case information used to calculate the probability the fighter jet is Cambodian?"). The answers to elaborative interrogations were implicit in the text; the participant had to connect different ideas presented in the text and/or diagrams. All lessons began with the reminder that the participant would be asked to answer questions based on the information in the lesson. The lesson in the control condition also began with the reminder that participants needed to read the lesson twice. At the end of the control condition lesson, participants were reminded that if they had only read the lesson once, they needed to go back to the beginning and re-read.

Measures

Pre/posttest. Prior to reading the lesson, participants were asked to solve the following posterior probability problem: "All medical tests have error. For example, the serum test screens pregnant women for fetuses with Down syndrome. The test is a very good one, but not perfect. Roughly 100 fetuses out of 10,000 have Down syndrome. Of these 100 fetuses with Down syndrome, 90 pregnant women will have a positive test result. Of the remaining 9,900 unaffected fetuses, 99 pregnant women will still have a positive test result. What is the probability a pregnant woman who has a positive result on the test actually has a fetus with Down syndrome?" The problem was presented only as text; no visuals were included. The posttest consisted of four posterior probability problems similar to the pretest. The problems were presented as text only; no visuals were included.

Procedure

Participants were randomly assigned to conditions. Participants engaged in the experimental tasks individually in small groups of 2 to 6. All participants in a particular group were in the same condition. All tasks were paper-and-pencil (i.e., they were not administered on a computer). After providing informed consent, participants were given a posterior probability problem to solve as a pretest. Then, they were given the lesson and an answer sheet for their questions, if they were in one of the questioning conditions. The experimenter instructed them to read the lesson carefully because they would be asked to answer questions and solve problems based on its information. Participants in the control condition were instructed to read the lesson twice. Participants in the questioning condition were instructed to answer the numbered questions in the lessons when they came to them, in the appropriate places on the answer sheet. When participants finished reading the lesson, they completed a distractor task of 21 single-digit multiplication and division problems. Then, they completed the posttest. Finally, they self-reported their demographic information. Experimental tasks were completed in a single session, which were approximately 45 minutes in length.

Scoring

Pre- and posttests. Pre- and posttests were scored for accuracy. The number of accurately solved problems was the score for the pretest and posttest. The maximum posttest score was 4.

Answers to embedded questions. Answers to embedded questions were scored dichotomously as correct or incorrect. Correct answers were scored as '1' and incorrect answers as '0.'

Answers to elaborative interrogations. Because the elaborative interrogations were more complex than the embedded questions, their answers were scored using a rubric instead of dichotomously. The rubric had the following categories: ideal, adequate, inadequate, circular, wrong, and missing. Elaborative interrogations were designed to prompt integration of concepts within the lesson. Therefore, an answer was considered ideal if it correctly included three or

more concepts presented in the lesson. Adequate answers correctly included two concepts. Inadequate answers included only one concept. Circular answers repeated the information in the question (e.g., EI: “Why is the base rate important when calculating probability?” Circular Answer: “Because it is important information you need to use to calculate probability.”). Wrong answers provided incorrect or inappropriate information. Missing answers were either responses such as “I have no idea” or the complete absence of answers. An ideal answer was scored as ‘3.’ An adequate answer was scored as ‘2.’ An inadequate answer was scored as ‘1.’ A missing, incorrect, or circular answer was scored as ‘0.’ The scores for all 6 of the answers to the elaborative interrogations were summed for a measure of answer quality.

Results

We first examined the effects of the diagram and questioning conditions on the posttest scores. We separated the participants based on whether they answered the pretest correctly (N = 47) or incorrectly (N = 198). An ANOVA was conducted with diagram condition and questioning condition as fixed factors and posttest scores as the dependent variable. Bonferroni corrections were used for multiple comparisons. Figure 1 presents the data for each of the pretest accuracy groups by condition.

For participants who answered the pretest correctly, diagrams had a positive effect on posttest scores $F(2, 45) = 8.26$, p = .001. There was a benefit of diagrams both without and with redundant text compared to the text-only condition, $t(28) = 3.04$, $p = .01$,Cohen’s $d = 1.01$; $t(28) = 3.98$, $p = .001$, Cohen’s $d = 1.26$, respectively. There was no difference between the two diagram conditions. In addition, there was no effect of questioning.

For participants who answered the pretest incorrectly, there was no overall effect of diagrams on posttest performance. However, there was an interaction between diagram condition and questioning condition, $F(2, 196) = 3.04$, $p = .02$, such that diagram condition affected performance in the read-twice control condition, $t(61) = 2.56$, $p = .04$, Cohen’s $d = .63$, but not the other questioning conditions. In the read-twice control condition, participants in the diagram with redundant text condition had higher scores than did participants in the text-only condition. There were no differences between the diagram without redundant text condition and the other diagram conditions. For participants who answered the pretest incorrectly, there was also a main effect of questioning condition, $F(2, 196) = 13.56$, $p < .001$. Unexpectedly, participants in the control condition had higher posttest scores than did participants in the embedded questioning condition, $t(118) = 4.49$, $p < .001$, Cohen’s $d = .74$, and participants in the elaborative interrogation condition, $t(154) = 4.34$, $p < .001$, Cohen’s $d = .69$.

Pretest Correct **Pretest Incorrect**

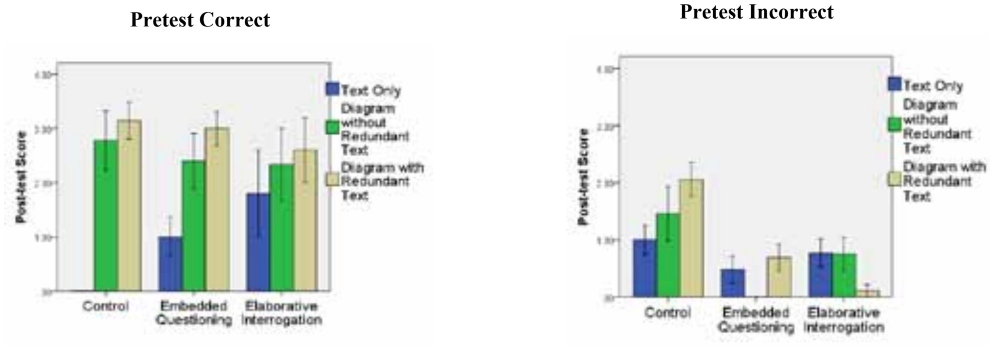


Figure 1: Post-Test Scores by Diagram and Questioning Conditions

The results indicate that diagrams were helpful for participants who initially did well on the posterior probability problem at pretest, aiding them to continue to do well on similar problems at posttest. In contrast, participants who did not understand how to answer the pretest problem benefited from diagrams when they read the lesson twice, but not when they answered questions while reading. Moreover, diagrams appear to be helpful only if they were accompanied by redundant text for participants who answered the pretest problem incorrectly. Contrary to expectations, questioning while reading was not beneficial for learning how to solve posterior probability problems. Indeed, reading twice appeared to yield higher scores on the posttest than either elaborative interrogation or embedded questioning for participants who answered the pretest problem incorrectly.

Given the unexpected finding that embedded questioning and elaborative interrogation conditions were not helpful for learning, we examined the quality of the answers to the questions in the embedded questioning and elaborative interrogation conditions (see Scoring section for scoring of answers). Because of the different patterns of findings based on pretest accuracy, we tested for differences in answer quality based on pretest accuracy. Then, we examined associations between answer quality and learning.

For the embedded questioning condition, participants answered an average of 3.08 questions correctly (SD = 1.70) out of 6 questions total. We conducted a one-way ANOVA with accuracy of answers to the embedded questions as the dependent variable and pretest accuracy as the independent variable. There were no differences in answer quality between participants who answered the pretest correctly (M = 3.00, SD = 1.70) and participants who answered the pretest incorrectly (M = 3.11, SD = 1.72). We used Pearson product-moment correlations to test for associations between accuracy of answers to the embedded questions and posttest scores. There were no associations between answer quality and posttest scores for either participants who answered the pretest correctly or participants who answered the pretest incorrectly.

For the elaborative interrogation condition, participants received an average score of 4.80 (SD = 2.27), out of a maximum possible score of 18. Approximately 37% of the answers were missing, circular, or wrong, 51% of the answers were inadequate, 9% of the answers were adequate, and 4% were ideal. We conducted the same statistical tests with the answers to the elaborative interrogations as we did with the answers to the embedded questions. As with embedded questioning, there were no differences in answer quality between the participants who answered the pretest correctly (M = 4.82, SD = 2.39) and participants who answered the pretest incorrectly (M = 4.69, SD = 1.70). Also similar to embedded questioning, there was no association between the answer quality for the elaborative interrogations and posttest scores for participants who answered the pretest correctly. However, for participants who answered the pretest incorrectly, there was a positive correlation between the quality of the answers to the elaborative interrogations and posttest scores, r = .39, p = .002.

Discussion

The purpose of this study was to examine the effects of diagrams and questioning while reading on learning from a lesson on calculating posterior probability. We had hypothesized that including diagrams in the lesson would improve learning, but were uncertain whether including text redundant with the information in the diagram would affect learning. For students who answered the pretest problem correctly, diagrams improved learning, whether or not they were accompanied with redundant text. However, for students who answered the pretest problem incorrectly, there was a benefit of diagram with redundant text, relative to text only, but only for

the control (i.e., read twice) condition. There was no reliable benefit of diagram without redundant text. These findings are consistent with the multimedia principle, which holds that including relevant visuals in lessons improves learning (Mayer, 2009).

However, our findings are inconsistent with the redundancy principle, which states that information should be presented in the text or diagram, but not both (Sweller et al., 2011). For students who solved the pretest problem incorrectly, there was no effect of redundant text. But, for students who solved the pretest incorrectly in the read-twice condition, there was a reliable benefit of diagrams only if they were accompanied by redundant text. In other words, presenting the information in a diagram *instead of* the text did not appear to be consistently helpful. For some students, the information needed to be in the diagram *as well as* the text for a benefit to be observed. It may be that presenting the information in both the text and diagram helped the students connect the two representations. The integration of the information may have led to more in-depth learning, resulting in better comprehension and application of the lesson.

We had hypothesized that answering questions while reading would have benefited learning. However, there was no effect of questioning for students who answered the pretest problem correctly. Moreover, students who answered the pretest problem incorrectly had higher posttest scores when they read twice as opposed to answer questions while reading. One possible explanation for the unexpected lack of benefit of questioning is that the quality of the answers to both the embedded questions and elaborative interrogations was generally quite poor. According to Jiang and Elen (2011), answering questions while reading only improves learning from text if students answer the questions as they were intended. On average, students answered only about half of the embedded questions correctly. This is surprising given that the answers to the questions were explicitly stated in the text; however, the undergraduate participants may have been confused by the simplicity of the questions presented to them. This confusion could have led to inaccurate answers to the embedded questions and diminished performance on the posttest for students who answered the pretest problem incorrectly. For the elaborative interrogations, students did not provide answers indicative of the deep processing these questions were intended to stimulate. The overwhelming majority of the answers to the elaborative interrogations were missing, circular, wrong, or inadequate.

Given our findings regarding answer quality, it is not surprising that both questioning conditions were detrimental, rather than beneficial, to learning for students who answered the pretest incorrectly. They were likely distracted by the questions and thereby benefited only if they read twice. However, students who answered the pretest problem correctly were not affected by questioning even though their answer quality was similar to those who answered the pretest problem incorrectly. Students who answered the pretest problem correctly may have been better able to focus on lesson content and suppress the distraction of the questions.

The results indicate that better answers to elaborative interrogations were associated with better learning from the lesson for students who lack prior understanding of the content. Therefore, benefits of elaborative interrogation might be observed if answer quality was stronger. We propose two ideas to improve answer quality in future studies. The first is practice training in which students receive examples of what types of answers are expected of them. This practice training would provide a model for the students, which could improve their answer quality (cf. Dornisch, Sperling, & Zeruth, 2011, for discussion). The second is to provide students with feedback on their answers either in the experiment, through their peers while working collaboratively (after training), or through computer-based tutoring. If these ideas could effectively improve answer quality, positive effects of elaborative interrogation might be seen.

Posterior probability, like many probabilistic reasoning concepts, is frequently challenging for students to learn. In this study, we sought to improve student learning on posterior probability problems by incorporating diagrams and questioning while reading into lessons. Our finding that diagrams assisted learning is consistent with the multimedia principle (Mayer, 2009). However, our finding that redundant text has a neutral or positive effect on learning, depending on the population, is inconsistent with the redundancy principle (Sweller et al., 2011). Our findings regarding questioning are inconsistent with our expectations based on previous literature. We believe that poor answer quality is the reason for the unexpected results, and we suggest some methods of improving answer quality. This work provides some guidance about how to improve student learning about probabilistic reasoning, while also highlighting the challenges inherent in this complex domain.

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MIDDLE GRADES TEACHERS' ENACTMENT OF KNOWLEDGE OF SAMPLING

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In this study, we examined the ways in which teachers' statistical knowledge is enacted during instructional practice. Using classroom video recordings from three middle-grades' teachers, we qualitatively analyzed the conversations around a task designed to develop students' knowledge of statistical sampling. Our analyses showed how teachers' selection and use of contexts during instruction impacted students' opportunities for statistical reasoning. Our research extends other literature that has begun to identify the statistical knowledge needed for teaching.

Keywords: Instructional Activities and Practices, Mathematical Knowledge for Teaching, Classroom Discourse

Purpose of the Study

Holding the view that mathematics and statistics are different fields, statistics education researchers (e.g. Groth, 2007) suggest that the knowledge needed for teaching mathematics is different from the knowledge needed for teaching statistics. As such, Groth (2007) has called for researchers to work on mapping the structure of the knowledge needed to teach statistics. To help frame our understanding of what teachers need to know to teach statistics in K-12 settings, we draw on the existing research from the Learning Mathematics for Teaching Project (Ball, Hill & Bass, 2005; Hill, Schilling & Ball, 2004). Specifically, we examine teachers' enactment of specialized knowledge for teaching (SKT), common content knowledge (CCK), knowledge of content and students (KCS), and knowledge of content and teaching (KCT). Our goal is not to provide a framework that encompasses the broad spectrum of knowledge referred to as statistical knowledge for teaching. Rather, in this paper, we focus here on one concept that lies within the "collecting data" component of statistical investigation (as outlined in the GAISE report (Franklin et al., 2007)): sampling. We describe what knowledge is needed to teach this concept and examine how this knowledge is enacted in practice in three middle grades classrooms. We focus on answering the following questions: (i) What knowledge of sampling do teachers need to possess to orchestrate productive statistical conversations? and (ii) In what ways is teachers' knowledge of sampling enacted in practice?

Theoretical Framework

Creating high-quality learning experiences requires more than simply combining practices and curricula; rather, quality teaching encompasses meaningful engagement of students in activities that foster mathematical and statistical thinking (Franke, Kazemi & Battey, 2007). The ability to support this kind of engagement requires teachers to use deep mathematical content and pedagogical knowledge (Ball, Thames & Phelps, 2008).

Mathematical Knowledge of Teaching

Much of the knowledge and skills needed to teach effectively has been encapsulated under the umbrella of mathematical knowledge for teaching (MKT). Teachers with strong MKT know the best ways to represent mathematical ideas to support students' understanding, know how algorithms are developed and why they work, anticipate students' misconceptions and plan

instruction to avoid or resolve them, and understand the typical developmental sequences of students' mathematical thinking (Ball et al., Hill, et al., 2004).

MKT encompasses four main domains of knowledge – common content knowledge (CCK), specialized content knowledge (SCK), knowledge of content and students (KCS), and knowledge of content and teaching (KCT). CCK refers to mathematics knowledge that is common to people across professions. SCK represents subject matter knowledge and skills that are unique to the work of teaching, and therefore is not commonly used in those ways by most other professions (Ball et al., 2008). KCS represents a combination of knowledge of the subject matter with knowledge about how students engage with that content. It comprises, but is not limited to, knowledge of common misconceptions, or what mathematical topics are particularly challenging for students. The fourth knowledge domain, KCT, refers to knowledge of the content combined with knowledge of ways to teach that content (Hill, et al., 2008). This involves knowing which problems are good for developing understanding of concepts, how to sequence problems, and what representations work best to support learning. Although MKT integrates these four domains, in this study we focus on highlighting the ways CCK and KCT are enacted in practice.

Knowledge Needed for Teaching About Sampling

To describe the knowledge needed for teaching sampling, we adopt Groth's (2007) approach by aligning the GAISE framework with the domains of MKT relevant to this study. The *Collecting Data* component of the GAISE framework primarily involves designing a plan to collect data and employing that plan (Franklin et al., 2007). This encompasses both mathematical and non-mathematical activities. Sampling in particular, involves primarily non-mathematical activities (Groth, 2007), such as designing data collection instruments including surveys, establishing quality control measures on data collection procedures, and determining the research design that will best answer the research questions. While much of this knowledge falls under common knowledge, research suggests that students have difficulty grasping key ideas necessary to carry out these activities successfully.

Watson and Moritz (2000) have described a hierarchy to describe the three tiers students typically move through as they develop knowledge about statistical sampling. In the first tier, students draw on their prior personal experiences of samples, such as something you get for free or a small piece, including samples of food or medicine. In a second tier, students are beginning to distinguish the difference between a sample and population and how using random or stratified sampling methods can impact results. Finally, in the third tier students understand the sample as being representative of the population, and students can recognize bias and interpret data based on the sample size and selection method used. Without quality instruction, students struggle to understand the process of sampling as described in tier 3. In a related study, Jacobs (1999) also documented the interesting ways students interpret sampling scenarios. For example, students expressed surprise if survey results contradicted what they expected; others evaluated a sampling method based on its decisiveness, citing that sampling methods that produced results in which one choice resulted in a large preponderance of the data was more valid than those in which the data might be close to a 50-50 split. Other research has shown that students are sometimes reluctant to advocate for a randomized sample. For example, when students were asked to compare three sampling methods (random, convenience, and a method where participants self-selected), 64% of students chose the self-selection method (Shaughnessy, 2007).

Knowledge Enacted in Practice

It has long been thought that a teacher's ability to instruct is strongly connected to what she knows. For several decades, mathematics education researchers have sought to empirically

substantiate the relationship among teachers' mathematical knowledge, instructional practice and student learning. From the results of various studies (see Ball, 1990; Fennema, Franke, Carpenter & Carey, 1993; Lampert, 2001), there is now strong empirical evidence to show that "…there is a powerful relationship between what a teacher knows, how she knows it, and what she can do in the context of instruction" (Hill et. al., 2008, p. 496).

Teachers' MKT is perhaps most visible in practice when they orchestrate instruction in ways that students are engaged in the intended mathematics and the richness of the mathematics is sustained throughout (Sleep, 2012) - what Sleep describes as steering instruction toward the mathematical point. To specifically examine how teachers utilize their knowledge in the act of teaching, researchers have identified characteristics of teaching through which they are able to determine the quality of instruction. Hill et al. (2008) identified six themes classified as *Elements of the Mathematical Quality of Instruction (MQI)*. These include (a) mathematical errors, (b) responding to students inappropriately, (c) connecting classroom practice to mathematics, (d) richness of the mathematics, (e) responding to students appropriately and (f) mathematical language (see Hill et al. (2008) for a full description). Although not an exhaustive list, we found that these elements aligned well with our own observations of the ways teachers' knowledge are enacted in instruction. In the next sections, we highlight how teachers implemented a task that focused on interpreting the limitations of data with bias caused by a non-representative sample.

Methods

The task (Fig 1., adapted from Rossman and Chance, 2008) is designed to have students examine how the sampling method used can impact the data collected. The intent of this task is to develop an understanding that the inferences made from data depend on how that data was collected. Students should understand that to make valid inferences, they should look at data with critical skepticism, which is an important statistical disposition (Wild & Pfannkuch, 1999).

1. On the 12th anniversary of the death of Elvis Presley, a Dallas record company sponsored a national call-in survey. Listeners of over 1,000 radio stations were asked to call a 900 number (at a charge of $2.50) to give their opinion about whether or not Elvis was really dead. It turned out that 56% of the callers felt that Elvis was alive.
   1. Identify the population of interest and the sample actually used to study that population in this example.

      Population:

      Sample:

2. Do you think that 56% is an accurate reflection of beliefs of all Americans on this issue? If not, identify some of the flaws in the sampling method and suggest how it could have been improved.

Figure 1. The Elvis Task

Participants

The participants included three middle grades teachers and approximately 110 students from one elementary and one middle school in a predominantly white, suburban school district in the Midwest. Mr. Jackson was a sixth grade teacher while Mr. Horn and Ms. Ottey taught middle school mathematics to seventh- and eighth-grade students.

Data Sources

We used transcripts of video recordings of classroom instruction and field notes to investigate how teachers' knowledge of sampling was enacted in practice. Three cameras were used; one recorded actions of the teacher, and the second and third focused on individual groups of students for the entirety of a lesson. The videotaping was supplemented with observational

field notes taken by a member of the research team.

Analysis

Members of the research team watched the classroom videos and read the transcripts of lessons where the relevant task was discussed. The transcripts were open coded and examined for statements that captured the teachers' instructional practices. Given our research questions, we developed two categories of codes – discourse codes and statistical knowledge codes. Discourse codes (e.g., revoicing, questioning, telling) aligned most closely with elements (b), (c) and (e) of the MQI, while the statistical knowledge codes (e.g., Knowledge of Sampling) aligned with elements (a), (d) and (f).

Each type of code was counted for each teacher, then the percentage occurrence of each code per category was found to determine dominant discourse types. We also used statistical knowledge codes to do a second round of data analysis. We identified all the segments of the transcripts that had a statistical knowledge code and examined the discourse around each code to describe the ways this knowledge was drawn on during the act of teaching. From the analyses we found different ways in which the teachers' knowledge of sampling were enacted in practice.

Findings

In this section, we contrast how Mr. Horn, Ms. Ottey, and Mr. Jackson introduced the Elvis task. All three teachers made explicit attempts to connect the task to students' prior experiences, but not all teachers were able to engage students in discourse that would allow them to think statistically. Prior to introducing the Elvis task, Mr. Horn asked students to identify the favorite type of music among people in their town. Several students shared, then Mr. Horn followed by asking, "how can we figure this out?" Students suggested advertising in the local newspaper or collecting data online. Calling in to radio stations was also suggested. Students then made a list of different radio stations in their local area. After documenting these stations on the board, he asked students to consider the music genre of each station. The following conversation ensued:

Mr. Horn: You guys had some great ideas about getting a survey out and kind of asking the people of Oakville to respond in a survey. If we did it through the radio, let's say we picked one of these stations.

Student: 100.3

Mr. Horn: 100.3; if we had on the radio, if we had 100.3, that station perform this survey for us and asked the people of Oakville, okay call in and tell us what is your guys' favorite radio station. We had that station list that for us, what do you think would be the most people? The most response to 100.3? What would be the favorite of Oakville? Jeremiah?

Jeremiah: It would, the station that advertised it

Mr. Horn: The station that advertised it so 100.3 plays mostly country right, why would they get more country people?

Students: That's the station.

Mr. Horn: Awww, good say that a little.

Student: That's the station.

Mr. Horn: That's the station that plays country music so people who naturally like country music are going to listen to 100.3. If we only had that radio station put out our survey?

Student: You would only get country people.

Mr. Horn: We'd only get the country people responding right?

Drew: You pick one from each…

Mr. Horn: Awwww, Drew had an excellent idea. In order to make sure that we kind of get everybody within Oakville, why don't we kind of pick one from each little group, if we had a classic rock station, we had a country station…an oldies station.

This excerpt documents Mr. Horn helping his students understand the importance of samples being representative of a population. In particular, he wanted students to understand that when data is collected from a homogeneous group of people, the results may not be generalizable to the entire population. By choosing the context of the radio station, Mr. Horn posed questions to students situated in a context to connect their personal experiences of listening to music on the radio to a larger statistical concept. In addition, Mr. Horn prepared his students to engage in the Elvis task, a problem that also involved responses of radio listeners, which was not representative of the population for other reasons. Thus, Mr. Horn's instructional choices both built on students' lived experiences and prepared students for engagement in a future task.

Ms. Ottey also found ways to connect statistical ideas to her seventh-grade students' prior experiences. During one class session, Ms. Ottey introduced the Elvis problem, and immediately engaged students in a discussion to ensure they understood the two new statistical terms – population and sample. Originally, she asked students, "What is the population of this room?" The students responded by saying twenty-eight until a student mentioned that it should really be thirty, because they had failed to include Ms. Ottey and the researcher in the room. She then asked about whether there were different sub-groups in the room (e.g., girls, adults, brown-haired people, thirteen-year olds), and noted that all of these make up the population. The excerpt below documents the discussion that followed.

Ms. Ottey: …Are you clear on population? [pause] What is a sample? If I wanted to take a sample of this group, who would I sample?

Jamie: Me! All of us!

Ms. Ottey: That would be the whole population. That would be like saying I'm going to give my breakfast survey to every person in the whole school.

Dacia: Three tables.

Ms. Ottey: Oh wait, wait!…What were you going to say?

Dacia: Well, I was just, what was your question?

Ms. Ottey: My question is: What would my sample be? If I were going to take a sample of this room, what would a sample be? [pause] Yes? [pointing to Blake] Oh, yes?

Blake: A person?

Ms. Ottey: I could take one person. That might be a sample. Maybe there's a big survey and I'm, maybe it's like student council. Let's see, who's your student council members? Okay, Jackson, Erica, Kevin, Ariel. So, those four are a sample of our class. And we use those to represent our class, right? Okay, so that would be a sample. Does it include girls? Yeah, but it includes boys, but there's a specific number. There's four. Umm, so we wouldn't have an eighth grader and have them represent our team or our class, right? Cause there are no eighth graders here. So, a sample is a small part of the population.

As Ms. Ottey helped her students to distinguish population and sample, she drew on her knowledge of her students and statistics to make several important teaching moves. For example, she recognized that these terms might be difficult for her students to distinguish, so she asked directed questions to ensure the students had a collective understanding of the terms.

By situating the discussion in terms of the student council members, the students could see how a sample is contained within a larger population. Furthermore, the members of the student council represent the members of the class, much like the data from a statistical sample are used

to represent a population. She extended this conversation by helping students to recognize the importance of considering the membership of the sample by emphasizing the need to include particular subgroups (girls) and exclude those that would not be representative of the class (eighth graders). Although the metaphor of the student council was helpful for understanding the purpose of a sample, it did have limitations. Typically, the students in a class elect their representatives on the student council. However, in statistical studies the population does not typically decide which of their members will constitute a sample. One might argue that Ms. Ottey should have disclosed this difference to her students.

As the examples of Mr. Horn and Ms. Ottey have demonstrated, using contexts that students understand can be a useful tool in helping students to think statistically and learn statistical concepts. Contextualizing a problem can also be used as a motivational tool. Mr. Jackson attempted to help his sixth-grade students recall who Elvis Presley was by playing his music.

Mr. Jackson:, …raise your hand if you know who Elvis is? This guy here. Right? [plays Elvis Presley's *Hound Dog*] … Right? [Some students begin moving arms as if they are dancing in their seats.] And Chad and Karl like to dance to it. And Bradley too. Okay? So, if we read this problem, if we read this problem: the twelfth anniversary of Elvis Presley, Bradley. On the twelfth anniversary, a Dallas record company…

Mr. Jackson introduced the Elvis problem in a way that would help students recognize Elvis Presley. For some students, the music might provide motivation to further understand the problem. However, playing music did not provide a means to engage students in statistical discourse. As evident in the excerpt, while the music played, students were dancing to the music. After giving the students several minutes to talk about the Elvis task in small groups, Mr. Jackson debriefed the discussions as a whole class.

Mr. Jackson: Do you think that 56% is an accurate reflection of beliefs of all Americans on this issue?

Gabe: Yes!

Mr. Jackson: And if not, you need to tell us some of the flaws in this method and suggest how could they improve it, how could the radio station actually improve it, okay? But before we answer that question, let's go and answer the first one. What is the population of interest? What's the population of interest? Bradley?

Bradley: People around the state.

Mr. Jackson: So you think people just around the state? Okay, what do you think?

Lanie: Everyone around the United States of America.

Mr. Jackson: Okay, Lanie thinks the population of interest is everyone around the United States of America.

Lanie: Unless it's a local radio stat—

Mr. Jackson: Unless it's just a local radio station, right, Nick?

Nick: Yes.

Mr. Jackson: But what was the sample that they actually took? Jordan?

Jordan: People who listen to the radio station.

The students' misunderstandings about sample and population are prevalent in this discussion. First, Gabe's comment suggesting that such a high percentage of Americans would believe Elvis was still alive suggests he doesn't have a clear understanding of the context. Bradley incorrectly identified the population as "people around the state." Finally, Jordan misunderstood what the sample was, identifying all people who listened to the radio station, rather than those listeners who paid $2.50 and called in.

Mr. Jackson's students had been given ample time to discuss and critique the task in small groups, so why did these misunderstandings persist? One possible explanation is that Mr. Jackson, unlike Mr. Horn and Ms. Ottey, had not framed the context and launched the task in a way that really allowed students to make sense of the important statistical ideas in the task and connect it to their personal experience in a meaningful way,. Furthermore, Mr. Jackson's primary discourse move in this excerpt was to repeat students' responses. He did not react to either correct or incorrect thinking in constructive ways.

Discussion and Conclusions

From our data, we observed that Ms. Ottey and Mr. Horn had clear understandings of population (set of entities to which inferences will be made) and sample (representative subset of the population), evident in their discussion with the students. However, to help students grasp the idea of "representativeness," having CCK of population and sample was insufficient. To build this understanding, the teachers framed the classroom discussions within meaningful contexts that connected to the students' experiences, demonstrating aspects of their KCT. In particular, by referring to the composition of the Student Council and foregrounding the need to collect data from fans of different music genres, the teachers made it clear that an appropriate sample should include all the types of people from the population. These contexts were conceptually-rich, allowing students to distinguish between population and sample and identify methods to reduce sampling bias. Mr. Jackson's choice of context was quite different; he chose to play and discuss an Elvis song prior to the discussion. While this prelude may have captured the students' attention, in contrast to the other classrooms the context did not help build meaning of the statistical concepts involved in the task. In fact, based on his interaction with the students, one could question whether Mr. Jackson's had CCK related to these concepts. Engaging students' statistical thinking requires that teachers understand the concepts (Hill et al., 2008), select appropriate contexts (Boaler, 1993; Sullivan, Zevenberger & Mousley, 2003), and know how to frame statistical questions in ways that allow students to build connections between their prior experiences and a task's embedded statistical concepts.

Teaching quality is related to how well teachers can draw on and integrate these domains of knowledge in the act of teaching to engage students in the intended mathematical (or statistical) work (Sleep, 2012). In this regard, maintaining the richness of the statistics (content), resolving misconceptions and responding (in)appropriately to students' statements were elements of practice that were enacted differently across classrooms. Although Mr. Jackson did continuously question the students, he did not deeply probe students' responses or use follow-up questions in ways that encouraged students to think critically about the ideas being discussed (Franke, et al., 2009). His interactions with students were primarily repeating students' statements, rather than fostering a discussion that could lead to the intended statistical idea. His missed opportunities to counteract students' incorrect statements served to perpetuate misconceptions held by the students. Conversely, Mr. Horn and Ms. Ottey were better able to guide the discussion towards the statistical point. They wanted students to develop a healthy skepticism of data and to acknowledge potential bias caused by non-representative samples. To do so, they drew on the experiences of the students, identified their current understanding of population, and used this information to craft questions and responses that guide the conversation towards to the intended statistical point.

As the field moves towards building a knowledge base for teaching statistics, important questions to frame this discussion include: what contexts best support students' understanding of

sampling, and statistical concepts more broadly? What are the affordances and constraints of different contexts? What contexts are effective for particular groups of students (different grade levels, rural vs. urban etc.)? What instructional practices are most effective to foreground the intended statistical ideas? The results of this paper add to this emerging body of knowledge by providing usable knowledge about ways to build meaningful contexts for statistical thinking around students' experiences. Also, we provide insight into what teacher moves work and how to deploy these moves to foreground the intended statistical ideas.

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COLLABORATIVE DESIGN WORK OF TEACHER EDUCATORS: A CASE FROM STATISTICS

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Our study, situated within graduate-level courses for teachers focused on statistics, illustrates work across institutions for making sense of teachers' statistical thinking. Using a common assessment instrument, we identify and discuss four items that indicate strong statistical thinking and two that highlight concepts with which teachers struggle. We discuss potential course elements that may be contributing to areas of success. Implications for collaborative course (re)design and shared assessment items are discussed.

Keywords: Data Analysis and Statistics, Teacher Education-Inservice/Professional Development

Introduction

Over the past 30 years, a considerable body of research has addressed students' statistical thinking from primary through tertiary schooling (cf., Shaughnessy, 2007). Over the same time frame, the role of statistics and probability has received attention in the secondary curriculum in the United States (National Council for Teachers of Mathematics, 2000; Common Core State Standards Initiative, 2010). However, as Shaughnessy and others have noted, any effort to improve the teaching and learning of statistics in secondary schools depends on the statistical knowledge of secondary teachers. To this end, many graduate programs in mathematics education and professional development efforts have begun to include more opportunities for teachers to develop their own statistical reasoning abilities, and to learn about pedagogical issues in teaching statistics. But the research base on secondary teachers' statistical reasoning is sparse (c.f., Batanero, Burrill, & Reading, 2011), and thus limits the capacity for teacher educators to design effective courses and other professional development experiences for teachers. In this paper, we report on our emerging work on collaborative design of a course for secondary teachers that focuses on teaching and learning statistical reasoning.

Working at two different institutions, the authors identified several important commonalities in the statistics education courses that we had developed independently. These commonalities included shared foci on key elements of statistical reasoning, such as the central role of variation, distribution, and the sampling distribution in developing students' inferential reasoning. Our common goals for our courses also included a focus on the central role that statistical software designed for the learning of statistics (such as *Fathom* (Finzer, 2001) and *TinkerPlots* (Konold, 2005) should play in supporting teachers in their own learning of statistics and in developing their approaches to teaching statistics. We shared a common commitment to taking the time needed to develop teachers' understandings and, hence, in both courses "coverage" of topics was on occasion sacrificed for depth of understanding. Finally, as teacher educators working in statistics, we had a common goal of gathering and analyzing data that would provide evidence about teachers' statistical reasoning and provide an empirical basis for a collaborative course design on the teaching and learning of statistics. Thus, we are not conducting a comparative course study, but rather we seek to investigate how evidence on teachers' statistical reasoning from both courses can be the basis for an on-going collaborative design of an effective statistics course for secondary teachers. In this paper, we focus on the following question: *How can assessment items on statistical reasoning provide a basis for the (re)design of learning experiences for teaching statistics at the secondary level?*

Theoretical Grounding

Recently, researchers have discussed and investigated teachers' statistical knowledge for teaching, using various frameworks and perspectives in their work (e.g., Groth, 2007, Burgess, 2011; Lee & Hollebrands 2011; Noll, 2011). Each of the emerging frameworks has identified teachers' own statistical literacy and thinking as a foundational or cornerstone aspect of their ability to teach statistics. Not many would argue this point. Though our courses aim to more fully develop secondary teachers' understanding for teaching statistics, the focus in this paper is on several key concepts in statistics and teachers' reasoning abilities with these concepts, without regard to their understanding of how to teach these concepts. If statistical literacy and thinking are essential for teachers, then teachers should have a strong foundation that is above that of students who have taken a collegiate level introductory statistics course, most of which are at the level of an AP statistics course taught in high schools. Thus, we turned to literature on assessing statistical literacy, thinking and reasoning (e.g., Garfield & Chance, 2000) for assistance in how to best assess the literacy and thinking of our teachers. Several researchers have engaged in developing, validating, and administering test items that aim to assess conceptual understanding and statistical literacy and thinking, rather than skills. One coordinated effort (delMas, Garfield, Ooms, & Chance, 2007) resulted in the Comprehensive Assessment of Outcomes in a First Statistics course (CAOS, https://apps3.cehd.umn.edu/artist/caos.html) and the collection of items at the ARTIST website (Assessment Resource Tools for Improving Statistical Thinking).

Methods

Course Contexts and Participants

Course1. Authors 2 and 3 designed and taught a one-semester graduate-level course in mathematics education to engage teachers with a range of tasks involving the investigation and exploration of statistical concepts using the software package *Fathom* (Finzer, 2001). The statistical content of the course consisted of investigations into variation and distribution, sampling distributions, confidence intervals, and inferential statistics. In addition, the course included various readings and discussions about (a) the nature of statistical reasoning and how it compares to other forms of mathematical reasoning and about (b) secondary students' learning and statistical reasoning. *Fathom* was used to support teachers' learning by providing an interface that would allow them to flexibly explore multiple graphical representations (e.g. shifting between box plots, dot plots and histograms) while being able to easily compare data sets, and to make changes to the data so as to explore conjectures. *Fathom* also provided the simulation tools necessary to create sampling distributions and representations of a population, a sample, and the sampling distribution. We saw this as critical to developing the teachers' knowledge of sampling, in order to build an understanding of formal inference.

There were 13 teachers who participated from Course1. Four participants were pre-service teachers (in a graduate licensure program), five were in-service teachers, two were in a masters program full-time, and two were doctoral students in mathematics education. Eight of the participants were female and five were male. All participants had completed the equivalent of an undergraduate major in mathematics, with all but one having had at least one course in statistics.

Course2. At a different university, another graduate-level, semester-long course in mathematics education was designed and taught by Authors 1 and 4. The course was intended for secondary and tertiary teachers of introductory statistics courses. The course had similar content focus as Course1, except that the instructors deemed it was necessary to go deeper in several explorations with data early on, and thus confidence intervals and formal inference were not discussed. Teachers in Course2 also engaged in similar readings and discussions as those in Course1. *TinkerPlots* was the main tool used in the course, though teachers also had some experiences with *Fathom*. The choice of

TinkerPlots was purposeful to engage teachers in more exploratory data analysis and to emphasize reasoning about distributions and statistical measures in multiple visual ways.

The 16 teachers in Course2 consisted of two preservice teachers enrolled in an M.A.T. program; eight teachers in a masters program with six currently teaching (3 of whom taught AP Statistics) and two in graduate school full-time; six PhD students in Mathematics Education, all of whom had recent secondary teaching experience; and one Statistics PhD student currently teaching a college-level introductory statistics course. Twelve teachers were female and four were male, with three international students within their first year of graduate work in the US. All teachers had completed an equivalent of an undergraduate degree in mathematics or statistics, and all had taken at least a first level graduate course in statistics for social science majors (content about equivalent to Advanced Placement Statistics).

Instrument

In the final week of both courses, all participants completed a 20 item multiple choice test with items in six categories: graphical representations, sampling variation, inference, data collection and design, bivariate data and probability. This selection of categories represents key concepts in introductory statistical literacy and thinking, and potentially provides us with a range of topics for which we can use as a basis for the continuing (re)design of our courses. Most items were drawn from validated instruments such as the CAOS test, with a few selected from the larger ARTIST database (https://apps3.cehd.umn.edu/artist), and one item drawn from the work of Zieffler, et al. (2007) because of its focus on informal inference (emphasized in both courses). All items have been used previously with college students, and we have known results for 19 of the 20 items that would allow for a comparison to the results from our teachers.

Analysis

We were interested in identifying the common successes and struggles with statistical reasoning displayed by our teachers, across both courses and institutions, as evidenced in their responses to this 20-item assessment. The common successes and struggles would provide us with some evidence about the areas where teachers had strong understandings of the measured concept and areas where their struggles (known from the research on college students) persisted through to the end of both courses. This evidence in turn becomes the basis for the (re)design of the courses in ways that will build on the strengths of teachers' statistical reasoning and will address areas of persistent difficulty. Thus, we first focused our attention on the items on which at least 75% of the teachers chose the correct response in both classes. We then attended to items that both sets of instructors felt assessed concepts they had spent considerable time developing in their courses, yet the response rates did not show evidence of strong statistical reasoning for either group of teachers. Due to space limitations, we discuss four questions from the strong statistical thinking category, and two items from the struggling statistical thinking category.

Results

The overall performance of our teachers on six selected items from the post-test is shown in Table 1. For comparison purposes, the last column indicates performance results of students from introductory statistics courses from the work of others. In all six items, our teachers demonstrated much higher abilities in statistical thinking.

Table 1. Comparative Results for Percent Correct on Selected Items on Post-test.

Item	Measured Learning Outcome	Course1 N=13	Course2 N=16	Comparison to Others' Results (N varies)

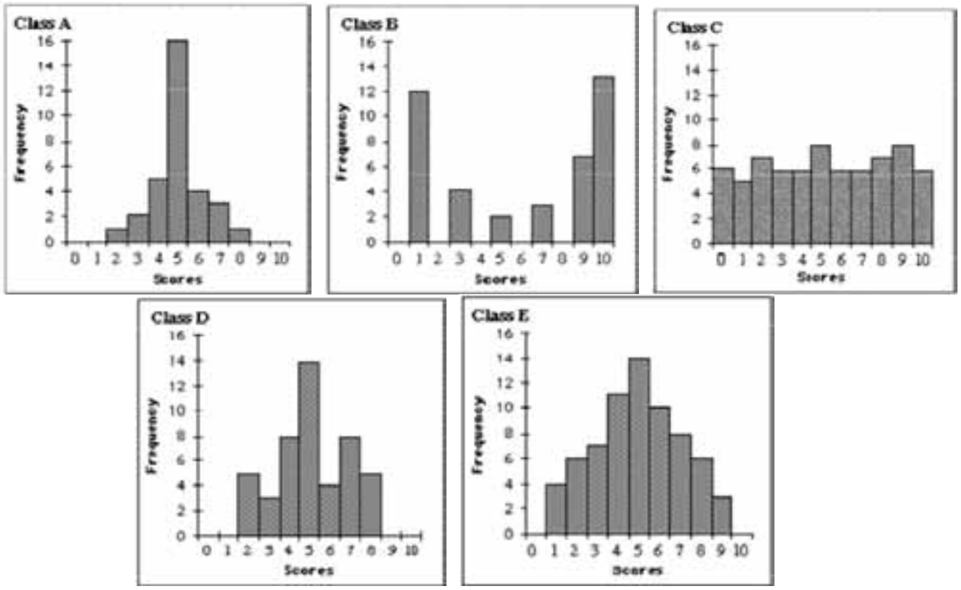
Category: Strong Statistical Thinking				
4	Ability to correctly estimate standard deviations for different histograms. Understands highest standard deviation would be for graph with the most spread (typically) away from the center.	92%	75%	46.9%[a]
7	Ability to use a frequentist approach to estimate probability of events for tossing an irregular shaped object, rather than apply an equiprobable approach	100%	75%	59%[b]
14I	Understanding of the law of large numbers for a large sample by selecting an appropriate sample from a population given the sample size.	100%	100%	65.2%[a]
14II	Ability to select an appropriate sampling distribution for a population and sample size.	77%	75%	44%[a]
Category: Struggling Statistical Thinking				
3	Understanding that a distribution with a median larger than the mean is most likely skewed left.	46%	63%	39.7%[a]
5	Understanding that statistics from small samples vary more than statistics from large samples.	54%	63%	31.9%[a]
[a]Results from CAOS administration reported by delMas et al. (2007) with participants (N) varying per item from 724-749. [b]Results reported from ARTIST database (delMas, personal communication, January 31, 2013).				

Evidence of Strong Statistical Thinking Across Groups

Most teachers in both groups (92% and 75%) were able to identify Class B (Figure 1) as the distribution that would likely have the largest standard deviation, because more of its scores are far from the mean. The concept of standard deviation was discussed explicitly in each course and teachers had an opportunity to examine research items similar to this in their reading and discussion of the work of delMas and Liu (2005). It is promising that these teachers tend to reason much better than students who have completed an introductory statistics class, of which less than 50% correctly responded to this item. Thus, explicit attention to reasoning about standard deviation of a distribution and reading about students' reasoning with this concept may help support teachers' reasoning.

Our teachers have no problem identifying a distribution for a single large sample chosen from a given population (100% correct in both courses), but did slightly less well on their ability to choose a correct graph representing a distribution of 500 sample means (Figure 2). On both parts of the item, our teachers did much better than the typical student as reported from delMas et al. (2007). This is especially encouraging since teaching the difference between a sample and a distribution of sample statistics is a particularly important, and hard to teach, concept in introductory level statistics at the collegiate and high school level (Saldanha & Thompson, 2002). Both courses spent considerable time engaging with these concepts using technological tools and reading literature about typical students' difficulties with distinguishing between a sample and a distribution of sample statistics.

Five histograms are presented below. Each histogram displays test scores on a scale of 0 to 10 for one of five different statistics classes.



Which class would you expect to have the highest standard deviation, and why?

A. Class A, because it has the largest difference between the heights of the bars.

B. Class B, because more of its scores are far from the mean.

C. Class C, because it has the largest number of different scores.

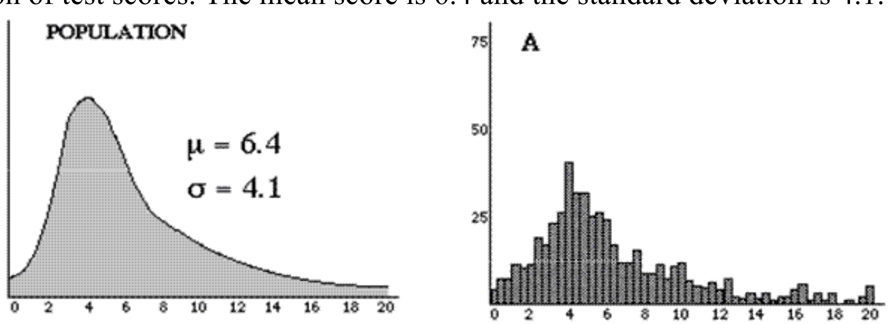
D. Class D, because the distribution is very bumpy and irregular.

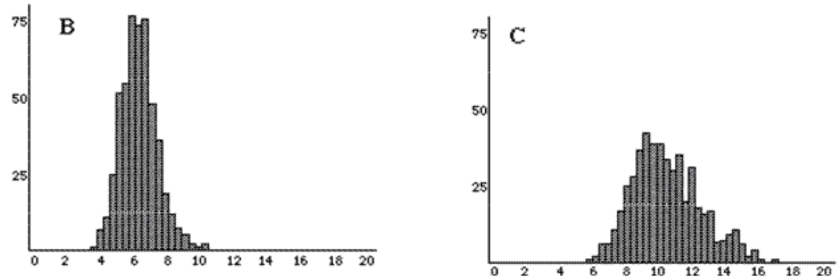
E. Class E, because it has a large range and looks normal.

Figure 1. Item 4 – Choosing Distribution with Highest Standard Deviation

Teachers in both groups also demonstrated a strong understanding of how to estimate probability of events for tossing an irregular shaped object using a frequentist approach (Figure 3). None of the teachers chose the assignment of probabilities based on a classical equiprobable approach (choice A). Three students from Course2 chose none of the above for their response; thus their reasoning for how to approach this task is unclear. If we want teachers to teach their students that an equiprobable distribution is not always the best estimate for the probability of events, then the high success rate on this item is particularly promising. Only 59% of students completing the same item in the ARTIST database chose the correct response, and 22% thought a classical approach (choice A) was the best (delMas, personal communication).

Four graphs are presented below. The graph at the top [left] is a distribution for a population of test scores. The mean score is 6.4 and the standard deviation is 4.1.





I. Which graph (A, B, or C) do you think represents a single random sample of 500 values from this population?
A. Graph A B. Graph B C. Graph C

II. Which graph (A, B, or C) do you think represents a distribution of 500 sample means from random samples each of size 9?
A. Graph A **B. Graph B** C. Graph C

Figure 2. Item 14 -- Understanding Sample Distribution and Distribution of Sample Means

A game company created a little plastic dog that can be tossed in the air. It can land either with all four feet on the ground, lying on its back, lying on its right side, or lying on its left side. However, the company does not know the probability of each of these outcomes. They want to estimate the probabilities. Which of the following methods is most appropriate?

A. Since there are four possible outcomes, assign a probability of 1/4 to each outcome.
B. Toss the plastic dog many times and see what percent of the time each outcome occurs.
C. Simulate the data using a model that has four equally likely outcomes.
D. None of the above.

Figure 3. Item 7 – Probability Measurement from a Frequentist Approach.

Teachers' Consistent Struggles Across Both Groups

In both courses, significant attention was given to understanding the relationship between a distribution and its measures of center. The teachers in both courses engaged in exploratory data analysis with *Fathom* (Course1) or *TinkerPlots* (Course2) and experienced how dynamically moving a data value in a graph impacted the mean and deviations from a mean. In addition to the focus on standard deviation discussed earlier, both courses also included readings and discussions in which common conceptions of the mean were discussed (e.g., Shaughnessy, 2006; Zawojewski & Shaughnessy, 2000). Despite this focus, many teachers did not correctly choose the appropriate distribution (Histogram b) for the given statistical measures (see Figure 4).

A study examined the length of a certain species of fish from one lake. The plan was to take a random sample of 100 fish and examine the results. Numerical summaries on lengths of the fish measured in this study are given.

Mean	26.8 mm
Median	29.4 mm

Standard Deviation 5.0 mm
Minimum 12.0 mm
Maximum 33.4 mm

Which of the following histograms is most likely to be the one for these data?

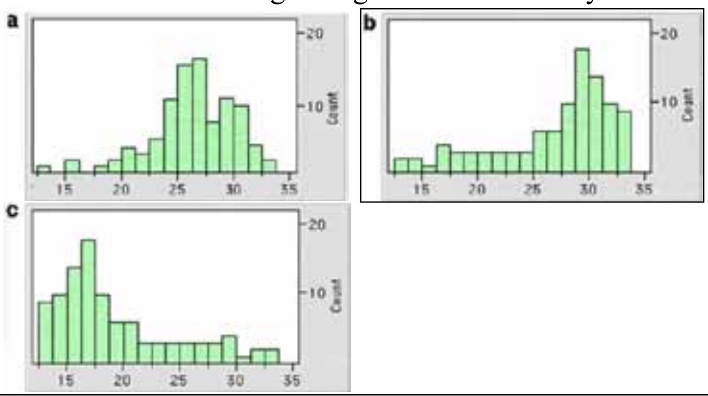


Figure 4. Item 3 – Choosing Distribution Given Statistical Measures

Given our attention to the role of sample size in variation from expected proportions, we were surprised that many of our teachers did not respond correctly to item 5 (Figure 5). While our teachers did better as a group than introductory statistics students, the most common incorrect answer (E) chosen by introductory statistics students, was also the incorrect response chosen most often by our teachers. This choice demonstrates applying equiprobable reasoning to the situation rather than considering effect of sample size.

A certain manufacturer claims that they produce 50% brown candies. Sam plans to buy a large family size bag of these candies and Kerry plans to buy a small fun size bag. Which bag is more likely to have more than 70% brown candies?

A. Sam, because there are more candies, so his bag can have more brown candies.
B. Sam, because there is more variability in the proportion of browns among large samples.
C. Kerry, because there is more variability in the proportion of browns among smaller samples.
D. Kerry, because most small bags will have more than 50% brown candies.
E. Both have the same chance because they are both random samples.

Figure 5. Item 5 – Understanding Role of Sample Size in Variability from Expected.

Discussion and Conclusions

Looking across the strengths and struggles of our teachers, two particular trends surfaced. First, teachers seemed to exhibit strong distributional reasoning when reasoning *from* graphs (items 4 and 14), but had difficulty predicting a graph of a distribution when given only statistical measures (item 3). Similarly, teachers were able to apply non-equiprobable approaches to a probability estimation task (item 7), but were prone to resort to applying equiprobable reasoning in a context about comparing likelihood of results from different sample sizes (item 5).

These results point to implications for us as course designers and for item and task development to measure our teachers' reasoning more systematically. First, we want to be sure to include a stronger focus in our courses for teachers to reason from graphical representations of distributions, as

well as towards predicting graphs of data with certain statistical characteristics. We also need to draw teachers' attention to students' use of equiprobable reasoning and how it can interfere with probability judgments. Such course improvements can include more targeted readings and discussions, as well as purposeful task design, particularly with technology tools. Our results also point to a need for multiple choice items and open response tasks that may better assess teachers' (a) reasoning from and to graphs of data, and (b) ability to apply equiprobable reasoning appropriately. With our focus on using dynamic statistics software, we aim to develop new assessment tasks and items that take advantage of these tools.

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HIGH SCHOOL MATHEMATICS TEACHERS' STATISTICAL QUESTION POSING

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The purpose of this study was to examine high school mathematics teachers' statistical question posing when engaged in statistical investigations in a graduate level mathematics education course. Sixteen teachers completed tasks at the beginning and end of a course on teaching and learning statistics in which they were asked to propose several questions that could be answered with multivariate data sets. In the course, teachers were provided opportunities to receive and provide feedback on statistical question posing. Analysis of the questions posed indicated that teachers asked many deterministic, computational questions, primarily univariate in nature, at the beginning of the course. On the final task, they asked no computation-only questions and many more broad and contextual questions, usually bivariate in nature.

Keywords: Data Analysis and Statistics, Teacher Education-Inservice, Teacher Knowledge

Introduction

New school mathematics standards have increased the focus on statistics and data analysis (Common Core State Standards Initiative, 2010), with research calling for an emphasis at the secondary level on statistical investigations (Franklin et al., 2007; Wild & Pfannkuch, 1999). The beginning of a statistical investigation includes choosing a focus and creating questions you will attempt to answer by collecting and analyzing data. Thus, statistical question posing is an important skill for both students and teachers engaging in statistical investigations (Arnold, 2008). In addition, as Arnold's (2008) review of other studies indicates, it is a skill with which both groups have difficulties. Lee et al. (under review) propose that teachers need to be asked to investigate and learn how to ask broader statistical questions. Statistical questions have been described in literature as distinct from mathematical questions in that they are contextual in nature and nondeterministic (Franklin et al., 2007). Makar and Fielding-Wells (2011) add that statistical questions should be ill-structured and ambiguous as opposed to the typical well-defined problems typically given in school mathematics. The purpose of this study is to examine the types of statistical questions that teachers posed about multivariate data sets at the beginning and end of a graduate level course on teaching and learning statistics.

Theoretical Background

It is important to consider statistical questions not just as the initial phase of an investigation, but as Konold and Higgins (2003) describe, as part of a back-and-forth process between data and researcher in which new questions may emerge. Several researchers have created classification schemes to differentiate types of statistical questions. Arnold (2008) defined statistical questions as summary, comparison, or relationship questions. Summary questions are usually about a single data set and serve to describe the data in some way. Comparison questions are used to compare a single variable across two or more subsets of data. Relationship questions are used to explore possible connections between two data sets. Lee et al. (under review) proposed distinguishing between broad and precise questions, where precise questions "focused on a specific goal or hypothesis, which could involve simple or complex analysis" and broad

questions required “a more open problem-solving process and perhaps more exploratory data analysis” (Table 3, p. 7).

Methods

This paper reports on sixteen teachers’ responses to an initial and a final task assigned in a graduate level class focused on teaching and learning statistics. For several weeks in the course, teachers had experiences aimed at assisting them in learning to pose statistical questions with given data sets. The participants consisted of two preservice teachers enrolled in an M.A.T. program; eight teachers in a masters program with six currently teaching (3 of whom taught AP Statistics) and two in graduate school full-time; six PhD students in Mathematics Education, all of whom had recent secondary teaching experience; and one Statistics PhD student currently teaching a college-level introductory statistics course. Twelve teachers were female and four were male. All teachers had completed an equivalent of an undergraduate degree in mathematics or statistics and all had taken at least the first level graduate course in statistics for social science majors (content about equivalent to Advanced Placement Statistics).

Tasks for Assessment

Two parallel tasks were used for the purpose of assessing teachers’ ability to pose statistical questions from a given multivariate data set and to use technology tools to explore and answer their question of interest. The initial task, given in the first week of the semester, was modified from Connected Mathematics: Data about Us (p. 40). Teachers were given data about sixth grade students’ jump roping that included three variables, two of which were categorical (class and gender) and one quantitative (number of consecutive jump ropes). Teachers were asked to “Generate several possible questions that could be answered with this data.” After generating questions, teachers were instructed to explore one of the questions (either by hand or using a technological tool), describe their exploration and findings, and note any new questions that arose as they explored the data. They were provided with data files for Excel, TinkerPlots, and Fathom. Most teachers had been exposed to these technologies in earlier courses.

The final task, given in the last week of the semester, asked teachers to generate possible questions that could be answered with a new data set. The task was parallel in structure to the initial task and provided a quantitative variable (students’ scores on the PSAT math section), as well as two categorical variables (grade level, and class). We note that there were three non-native English speakers in the class for whom the context of the PSAT was unfamiliar and information was provided on the possible range of scores for the PSAT math section.

Tasks for Learning

Upon examining the results of the initial task, the researchers (who were the course instructors) determined that more time needed to be spent on distinguishing statistical from deterministic questions. In the second week, teachers discussed differences between statistical and mathematical thinking and analyzed tasks for their potential to promote statistical reasoning. In the third week, teachers used TinkerPlots and discussed how its use could enhance their exploration of the initial Jump Rope task. Then, teachers examined the Census at School project’s questions and measurement guide (http://www.amstat.org/censusatschool/). Teachers downloaded random samples of 100 cases from the project’s respondent database and during the fourth week formed groups and proposed statistical questions to explore with the data. Each group provided feedback on another group’s questions and then revised their own questions.

Teachers struggled with posing statistical questions during these activities; therefore, teachers and instructors jointly developed a list of guidelines to consider. It was suggested that

questions should: invite multiple graphs and/or measures; provide room to explore other questions of interest; directly relate to context; consider why someone would care about the answer to the question and/or find the question interesting. In addition, teachers cautioned against posing questions with one answer or interpretation; pointing to specific measures or procedures in their questions; and forgetting about context. These guidelines match nicely with the recommendations of Makar and Fielding-Wells (2011) that statistical questions be interesting, challenging, and relevant; statistical in nature; and ill-structured and ambiguous. In other words, the questions should be answered through gathering and interpreting data and the wording of the question should require interpretation and negotiation among students.

Codes

All questions posed on the initial task and the final task were blinded and compiled into two new documents. Though we began by initially coding questions as broad or precise according to Lee et al. (under review), our analysis quickly emerged into a constant comparative method (Strauss & Corbin, 1990). In categorizing the questions, clear differences emerged among the types of questions coded as precise. Thus, new categories of codes emerged as we examined similarities and differences among the questions posed. For example, some questions only required looking at the data to find an answer. More complex precise questions involved computing a measure and comparing across groups or computing different measures for a single group and comparing the differences between the measures. Descriptions and examples of the codes generated are provided in Table 1.

Table 1: Question Codes

Code	Definition	Examples from Teachers' Work
Look at Data	Looking for special cases in the data.	Did a boy or girl have the most jumps?
Compute	Compute a specific measure on one subset of the data.	What is the mean of consecutive jump ropes in Mrs. A's class?
Compute and Compare	Compute a specific measure or measures and compare between two subsets of data.	What is the difference between the mean number of jumps for the girls in Mrs. A's class and the mean number of jumps for the girls in Mrs. B's class?
Display and Describe	Visually display the data and describe it.	Construct a box and whisker plot. Are there any outliers?
Broad	Invite exploration of data; do not point to a specific measure or method.	Which class seems to be better at jumping rope?
Contextual	Refer back to context; may not be answerable by data.	What might these values imply about the genders?
Inappropriate	Inappropriate/nonsensical statistical question	What is the mean and median of the number of girls in both classes?

Findings

The most dramatic difference between the types of questions posed by teachers on the initial and final task related to the Compute category. On the initial task, this was the most frequently posed type of question. In contrast, on the final task, no questions of this type were posed by any of the teachers. The number of Look at Data questions also decreased from twelve to two. Other differences were the increase in questions coded Broad or Contextual on the final task compared

to the initial task. It is worth noting that 14 out of the 16 teachers were able to compose broad questions on the initial task, but the tendency was to ask more Compute questions.

Table 2: Results

	Initial Task		Final Task	
Code	**# of questions posed**	**# of teachers posing**	**# of questions posed**	**# of teachers posing**
Look at Data	12	5	2	2
Compute	69	11	0	0
Compute and Compare	32	11	11	6
Display and Describe	5	1	0	0
Broad	32	14	58	16
Contextual	8	2	16	9
Inappropriate	7	2	5	3

Conclusion & Summary

This study examined the nature of questions posed by teachers at the beginning and end of a graduate level course on teaching and learning statistics. It is apparent there was a change in the types of statistical questions posed by these teachers between the initial and final tasks. While both of these tasks involved posing questions from given data, we acknowledge that it is also important for teachers to learn to design instruments and collect their own data. Given the time consuming nature of engaging in the entire statistical cycle, it seems advantageous to improve teachers' skill in posing questions related to existing data. It is possible that learning to pose statistical questions from data might also create better understanding of the connections between research questions and data collection measures. Further research is warranted to determine which learning experiences in the course contributed to the change in questions posed and to consider implications for changes in the teachers' instructional practices.

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LEVELS OF REASONING FOR UNDERSTANDING THE CONCEPT OF PROBABILITY

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The purpose of this work is to describe the evolution of the reasoning of a high-school student on the concept of probability from the inferences he is able to make with this concept. Based on Jones' framework which describes probabilistic reasoning, excerpts from interviews are analyzed in which important characteristics of the student's reasoning are revealed. It is shown that the difficulties the student faced in different moments are: giving meaning to random quantification, overcoming the law of small numbers, and managing variation in a convenient way. It is observed that two important elements to overcome these difficulties are: an informal knowledge of the law of large numbers and reasoning with relative quantities. The work concludes with a hierarchy that describes the probabilistic reasoning of this student.

Keywords: High School Education, Probability, Technology.

Introduction

The level of understanding a subject reaches on a concept is associated with the meaningful and plausible inferences he is capable of formulating with it. Concerning the concept of probability, the most important inferences are "the *unpredictability* of random phenomenon in the short-run [and] *predictability* in the long-run trends in data (i.e., the law of large numbers)" (Sthol & Tarr, 2002). In order to formulate such inferences it is necessary to make both the frequency and classic approaches of probability intervene and articulate. To elucidate this process of (inferential) articulating, the framework proposed by Jones, Thornton, Langrall and Tarr (1999) is suggestive since it shows a wide description of probabilistic reasoning and its interconnections. However, there are two important modifications made in the present work regarding this framework: 1) only one subject is studied and 2) the subject is older than those examined by Jones. As a consequence, the following question arises: How is the evolution of a high-school student's reasoning on the concept of probability according to what Jones' framework establishes?

Conceptual Framework

Jones *et al.* (1999) propose a framework that describe and predict students' probabilistic reasoning (8-14 years old) based on four levels: *Subjective*, associated with a idiosyncratic or non-quantitative reasoning; *Transitional*, it begins to recognize the significance of quantitative measures, but frequently there are regressions; *Informal Quantitative*, it involve quantitative reasoning of the form "more of", "less of", "3 out of 5"; *Numerical*, incorporate reasoning with relative numbers and its operations. The authors analyze probabilistic reasoning through six constructs: *sample space*, *experimental probability*, *theoretical probability*, *probability comparisons*, *conditional probability*, and *independence*. Notwithstanding, in this work we regard the fourth ones, because of they are enough to account for the reasoning about the concept of probability of student studied here.

Methodology

This work is a case study that follows up the evolution of the reasoning on the concept of probability in one single student. He studied the third year of high-school (17 years old) and had a good arithmetic and algebraic reasoning, though he had never studied any probability topics. Only one student was considered in order to explore the underlying ideas between classic probability and relative frequency in a deeper way, and his age was determined because such ideas have a complexity which is not easy to handle by younger students (Ireland & Watson, 2009; Konold, Madden, Pollatsek, Pfannkuch, Wild, Ziedins, Finzer, Horton & Kazak, 2011). Fifteen one-hour long interviews were conducted. In two of them the problem in Figure 1 was developed including its simulation using the Fathom software. Both sessions were videotaped, transcript, and analyzed.

In box A there are 3 black chips and 1 white chip. In box B there are 6 black chips and 2 white ones (fig.1).

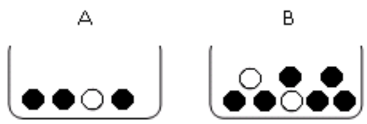

fig. 1

A box is chosen and a chip is drawn randomly from it. A prize is won if it is a black one. Juan considers that choosing box B is the most convenient action because no matter both boxes have a proportional quantity of black and white chips there is a larger number of black chips in B.

a) What do you think about Juan's explanation?
b) How would you measure the possibility that the event "drawing a black chip from box A" occurs? Which value would you assign to it?
c) How would you measure the possibility that the event "drawing a black chip from box B" occurs? Which value would you assign to it?
d) How would you prove the answer given in a)?

Figure 1: Probability Comparison Problem

Results

First episode. Miguel wrongly believes that Juan's statement in the problem is right and answers: "I think it is better [to choose box B] because of the [greater] number [of black chips when compared with box A]". Later, he is asked: Which value would you assign to the possibility of drawing a black chip from box A? In his answer, he questions the meaning itself of the assignation: "What do you mean a value?... that's what I don't understand, which value. They are values of what or what…". After a brief intervention of the interviewer, the student shows he perceives the equality in the proportions of black chips: he says there is a "75% of drawing a black [chip from box A]" and a "75% [of drawing a black chip from box B]". However, despite he has apparently calculated the probability of drawing a black chip from each box and observed they are equal, he maintains his appraisal on the situation: "I think Juan is right". As a consequence, this reasoning is placed in a *Subjective* level (Jones *et al.*, 1999).

Second episode. Now, Miguel is asked to draw randomly samples of size 10 with replacement from box A (with one white chip and three black ones) and another one from box B (with two white chips and six black ones), and to register the results in Table 1.

Table 1: Results From the 10 Extractions in Each Box (B = white chip and N = black chip)

Number of drawings	1	2	3	4	5	6	7	8	9	10
Box A	N	N	N	N	N	N	B	B	N	N
Box B	B	N	N	N	B	N	N	N	N	N

Based on these results, Miguel was asked: Which box would you choose if you wanted to get a black chip? He answers he would pick box A and explains: "the first chips drawn turned out to be black". Then, he is asked to draw other samples of size 10, one from box A and another from B. He obtains ten and seven black chips respectively, which strengthens his belief. He is asked now to predict the frequency of white and black chips in 100 and 1000 possible repetitions of the experiment in each of the boxes (Tables 2 and 3).

Table 2: Prediction of the distribution of 100 extractions

Error! Objects cannot be created from editing field codes.

Table 3: Prediction of the distribution of 1000 extractions

Error! Objects cannot be created from editing field codes.

Miguel's predictions as well as his assignation of values 80% and 70% respectively to the possibility of drawing a black chip from boxes A and B reflect his wrong belief that there are more possibilities of getting a black chip from box A. It is convenient to note that this reasoning is based on a bias called the *law of small numbers* (Tversky & Kahneman, 1982). However, Miguel has set in play the necessary elements that constitute the basic frame of probability: the proportions of chips in the boxes and the relative frequencies; but he has not articulated them yet in a convenient way. Therefore, his reasoning is at a *Transitional* level (Jones *et al.*, 1999).

Third episode. Afterwards, Miguel is asked to represent the situation in the Fathom software and to make the corresponding simulations in which he obtains the following results:

Table 4: Results of the computer simulation of 100 extractions

Error! Objects cannot be created from editing field codes.

Table 5: Results of the computer simulation of 1000 extractions

Error! Objects cannot be created from editing field codes.

When asked what he observes, Miguel focuses his attention in circumstantial characteristics (Table 4): "The more black chips [are in the boxes] […] the less [black chips] are [in the simulation]". Later on, as a result of the interviewer's questions on the cases of 100 (Table 4), 1000 (Table 5) and 10,000 (represented in the software) drawings, Miguel focuses his attention in the similarity that exist between the relative frequency of black and white chips obtained in the simulation and the proportion of black and white chips in the box. Besides, he observes that this similarity grows closer as the number of trials increases. Nevertheless, he considers box A offers a greater possibility of drawing a black chip since he reasons with absolute numbers caused he values the differences between results as meaningful: "Black chips come out more often from box A than box B. Here [Table 5] […] there are two [chips] less, but here [Table 4] […] there are six times more". It's evident that the Miguel´s observation that box A offers a greater possibility of drawing black chips turned out a relatively strong belief because, in absolute terms, this situation was favored by the results in the physical and computer simulations. Thus, his reasoning remains in the *Transitional* level (Jones *et al.*, 1999).

Fourth episode. Miguel assigns the values 77% and 71% to the possibility of drawing a black chip from boxes A and B respectively, reasserting what happened in the *third episode*. This assignation is only supported by the results of 100 drawings (Table 4), so the interviewer questions the student so that he assesses how appropriate these values are to the other cases of 1,000 (Table 5) and 10,000 drawings. As a consequence, Miguel integrates two observations he had made during the *third episode* into his reasoning; on one hand, that the relative frequency of black and white chips obtained through simulation is similar to the proportion of black and white chips in the box; on the other hand, that this similarity grows closer as the number of trials increases: "it is almost a quarter of white chips and three quarters of black chips in 10,000… They are the same, no matter the box [which one choosing]". He assigns the appropriate values to these possibilities at the same time: "75%". So, his reasoning has now reached the *Informal Quantitative* level (Jones *et al.*, 1999).

Conclusions

Through the interviews, Miguel´s reasoning regarding probability goes from the *Subjective* to the *Informal Quantitative* level, preparing him to the *Numerical* level. Based on these results, we then proposed as a conclusion a hierarchy that describes the reasoning of this student on the concept of probability. This hierarchy integrates the levels proposed by Jones *et al.*, (1999): *Subjective*, in which no meaning is attributed to the assignment of a number to an event, or a number is assigned in a naïve way without the possibility to make inferences; *Transitional,* in which a number is assigned to an event in an intuitive way based on the analysis of the random device (v.gr. the contents of color chips in the boxes) or on the relative frequencies, but without adequately relating them, which prevents the formulation of pertinent inferences; *Informal Quantitative,* in which the frequency and classic approach are used to assign a number and some relationship among subjective assignment, analysis of the random device and frequencies are perceived; in this way inferences start to be created, but there is no reasoning with relative numbers, so variability is not assimilated; *Numerical*, in which both approaches are set in play to assign a number to an event and relationships among subjective assignment, analysis of the random device and frequencies are perceived. Particularly, there is reasoning with relative numbers to adequately consider the variability and, in consequence to make better inferences.

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STATISTICAL REASONING DEVELOPED BY UNIVERSITY STUDENTS EXPLORING DATA IN A COMPUTER ENVIRONMENT

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The Problem

Lajoie & Romberg (1998) identify as an important task to the development of reasoning and statistical thinking in high school students, formulating research questions and following through with data creation, collection and interpretation. In this sense, we established the following research questions: what types of representations and software resources the students use to the answering of the questions about a data collection? Have they employed all the potential of the representations that the software provides for the data analysis? How is characterized the statistical reasoning developed by the students in the exploratory data approach through the use of the technology?

The theoretical perspective adopted firstly considers, from a point o view of an expert in statistics who requires formulate questions about the collection of data, by implicating a set of knowledge and abilities to make sense to the complex relationships and information expressed in some graphics and descriptive tables with statistical summaries. Secondly we considered that the use of computer technology in the data analysis not only represents an amplifier resource of the computing capacity and construction of the graphical representations but constitutes at the same time a resource with potential of reorganization of the mental activity of the users.

The research involved 34 first grade university students of the computers science program (between 18-19 years old) while taking the probability and statistics course. The research was carry out while addressing topics of data analysis that are contemplated in the curriculum focusing to the use of an explorative approach with the dynamic software Fathom (Finzer 2000). The final activity in the study consisted of a collection of data of automobiles on sale in Mexico.

Conclusions

In general the students utilized the previewed representations and it was observed correct analysis, but not enough from the point of view of an expert, since in the interpretations there not always were utilized all the information given by the representations that they built. It is conjecture that their statistical knowledge was not enough to make a more complete interpretation. Even students utilized with frequency more than one representation (graphics and summary tables) their interpretations were based most of the times on the identification of descriptive measures provided by the tables. The results shown that the global comprehension of the data is a difficult aspect to many of the students, however the plenty information of the representations that they utilized, it is located over the frequency showed with a local approach of the data and its interpretations. In this manner their statistical reasoning was correct but incomplete in most of the cases, for it is required a high level of knowledge of the concepts of data analysis to give sense to the information provided by the diversity of the representations provided by software as the one utilized in this research.

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INDIVIDUAL DIFFERENCES IN CALCULATING POSTERIOR PROBABILITY: DO STATISTICS EDUCATION AND MATH PROFICIENCY MATTER?

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Educators may be interested in the factors that are associated with probabilistic reasoning so that they can better understand the needs of their students. The purpose of this study is to examine individual differences that may predict success in solving a posterior probability problem, specifically prior statistics education and math proficiency. Previous research findings on these relations have been mixed, and they differ depending on the type of reasoning involved. As one might expect, statistics education is often associated with better performance on probabilistic reasoning tasks (Fischbein & Schnarch, 1997). However, statistics education is associated with an increase in the misconception that all events are equally probable (i.e., equiprobability bias), perhaps due to the emphasis on randomness in statistics education (Morsanyi et al., 2009). Focusing on the randomness of an event occurring, without considering other factors that affect probability, may increase the equiprobability bias. There are also mixed findings regarding relations between math proficiency and performance on probabilistic reasoning tasks (e.g., Johnson & Kuennen, 2006; Stanovich & West, 1998). For example, Stanovich and West (1998) found that SAT math scores were positively associated with being able to identify important information in a posterior probability problem, although this did not result in improved accuracy in posterior probability problem solving.

Undergraduate students ($N = 210$) were given a posterior probability problem to solve and provided self-reports of their statistics education and standardized math test scores (ACT or SAT). Approximately 20% of the students solved the problem correctly. Students who answered the problem correctly ($M = .59$, $SD = .55$) tended to have taken more statistics courses than those who solved the problem incorrectly ($M = .37$, $SD = .50$), Exp(B) = 2.25, $p = .02$. In addition, students who solved the problem correctly reported ACT or SAT scores in higher percentiles ($M = 94.76$, $SD = 4.80$) than those who solved the problem incorrectly ($M = 89.54$, $SD = 11.16$), Exp(B) = 1.09, $p = .01$. These findings suggest that for student populations with weak mathematics skills or with little statistics background (e.g., introductory statistics students), posterior probability problems may require special attention.

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MISTAKES IN DRAWING THE SHAPE OF A DISTRIBUTION BY MIDDLE SCHOOL STUDENTS

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The Common Core State Standards for Mathematics (National Governors Association Center for Best Practices Council of Chief State School Officers, 2010) includes studying the shape of a distribution as a statistical topic for middle school students. Shape is a fundamental idea for understanding the concept of distribution and provides an organizing structure to conceive data as an aggregate entity instead of individual values (Bakker & Gravemeijer, 2004; Reading & Canada, 2012).

Drawing graphs to predict the shape of a distribution in hypothetical situations, namely without having data, is a useful instructional design to foster students' understanding of shape and distribution (Bakker et al., 2004). Zieffler, Garfield, Delmas, and Reading (2008) also suggest having students draw a graph of a population based on a sample to promote their reasoning about the shape of the population. This implies that experience drawing graphs might have an influence on students' reasoning about the shape of a distribution. Thus, there is a need to understand more about the graphs students draw to predict the shape of a distribution. To better understand students' drawing of distributions, this study investigates the following research question: *What mistakes do 6th graders make when asked to draw a shape for an unknown population based on a sample?*

This work is part of a larger study investigating middle school students' informal inferential reasoning. Four 6th graders of varying mathematics achievement levels participated in the study. A paper-pencil test was administered, followed by an interview to gain a deeper understanding of students' thinking in answering the questions. The test includes two problems, along with two sub-questions each. Each problem required participating students to draw a shape for the population based on a given sample. The results illustrate the misconceptions that surfaced in their graphs of unknown populations based off of small samples. Implications for analyzing and learning from students' thinking in drawing shapes will be presented.

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THE LINEAR LOGISTIC TEST MODEL AND MODELING CONCEPTUAL UNDERSTANDING IN MATHEMATICS ASSESSMENT

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In the report, Adding it Up: Helping Children Learn Mathematics, the Mathematics Learning Committee from the National Research Council (Kilpatrick, Swafford, & Findell, 2001) suggested that student learning and performance could be enhanced if conceptual understanding was taught at the same time as procedural fluency. This major new direction placed a great emphasis on the idea of conceptual understanding as a main strand within mathematics education. Cognitive diagnostic assessment (CDA) is an important thrust in measurement designed to assess students' cognitive knowledge structures and processing skills in relation to item difficulty (Leighton & Gierl, 2007). There has been an increasing demand in education to develop assessments that map and measure the psychological processes involved in conceptual understanding. Fischer (1973) introduced a model, called the linear logistic test model (LLTM) that is capable of bridging cognitive processing models and psychometric models.

The purpose of this study is to provide some evidence to help determine if the LLTM functions well and is robust when the Q-matrix is properly specified, underspecified, balanced misspecified, and overspecified. A simulation study was conducted to explore the performance of the linear logistic test model (LLTM) when the relationships between items and cognitive components were misspecified. Marginal maximum likelihood and empirical Bayes were used in SAS to estimate parameters for cognitive components and item difficulty, as well as person ability, respectively. Factors manipulated in the simulation study included overall type of misspecification (under-, balanced-, and over-specification), percent misspecified (1%, 5%, 10%, and 15%), sample size (20, 40, 80, 160, 320, 640, and 1280), Q-matrix density (64% and 46%), skewness of ability distribution (-0.5, 0, and 0.5), and test length (20, 40, and 60 items). Statistical bias, RMSE, CI coverage, and CI width were computed for estimates across the set of replications.

As the Q-Matrix moved away from the truth, cognitive components and item difficulty estimates became progressively more biased (positive bias with under-specification, and negative bias for balanced- and over-specification). Results were interpreted for the design factors and recommendations for the application of LLTM in assessment studies were provided.

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HOW CAN MATHEMATICS EDUCATORS FOSTER HIGH SCHOOL STUDENTS' RISK LITERACY?

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The purpose of the study is to investigate high school students' understanding of risk with respect to the coordination between likelihood and impact. The objectives are: 1) to investigate students' informal knowledge of risk and their informal coordination between likelihood and impact; 2) to explore ways in which the formal instruction can foster the coordination between likelihood and impact. In this study, risk of an event is defined as the measure of both likelihood and impact (Pratt, et al., 2011). The study is a case study of teaching risk literacy in Ontario in a grade eleven enriched classroom during the probability and statistics unit (23 participants, 19 girls and 4 boys).

Based on the initial assessment, students were divided in five groups and given an activity in which they had to assess the safety of nuclear power plants using data including likelihood and impact. In the second part of the activity, students were asked to estimate an impact of an accident based on empirical data provided. The students were then asked to assess the risk of both nuclear and coal accident by combining the data on probability and impact. The coordination was initially presented to the students as the product of probability and impact. After this, the students were provided the set of coordinates with the probability on the vertical axis and the impact on the horizontal axis. The students were then asked to plot the values for the probability and impact for both nuclear power plants and coal power plants.

Three themes emerge from the research: informal knowledge about impact, the role of content knowledge, and the role of critical elements. There is evidence that students possess pre-existing informal (intuitive) knowledge of impact. From pedagogical viewpoint, this fact is encouraging since in many other domains (such as assessment of probability) there is strong evidence that individuals' intuitions are often erroneous. As it was seen from the intervention, this informal knowledge has a potential to be used in the instruction.

The initial assessment in both classrooms shows that students use different language to talk about impact (e.g., "massive" and "dangerous"). Through the introduction of quantitative data, the informal qualitative descriptions can be transformed into quantitative in order to make descriptions more formal and ready to use in the mathematical analysis. For example, the word "massive" can be transformed into a numerical value consistent with the statistics presented in the activity. Students' responses suggest that at least some of them have used their own personal knowledge about energy sources to answer the question. This is consistent with Pratt et al.'s (2011) view that the coordination between likelihood and impact has to be viewed in conjunction with students' personal beliefs and values. Data suggest that the coordination between likelihood and impact has to be taken into account together with critical elements. Students would benefit from being aware of the "worry questions" (Gal, 2005) such as the nature of the terms used in statistical inference as well as being aware of where data came from. For example, some students did question the meaning of "significant" in the statement that there were only three "significant" accidents. However, many students ignored the term and concentrated on the numerical information alone.

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CAPTURING EARLY ELEMENTARY SCHOOL STUDENTS' INFORMAL INFERENTIAL REASONING THROUGH STORYTELLING

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The importance of developing statistical literacy in schools is advocated by numerous researchers (e.g. Shaughnessy, 2007; English, 2012; Pfannkuch, 2011, Makar & Rubin, 2009). One growing area of research on statistical learning centers on Informal Inferential Reasoning (IIR). Makar and Rubin (2009) describe IIR as having three components: expressing uncertainty, making and evaluating claims, and drawing explicitly from the data when making or evaluating claims. There is limited research about IIR at the early levels of elementary school but this type of informal reasoning can be a powerful precursor to formal statistical learning in the middle grades (English, 2012). This poster details the results of a study that focused on providing opportunities for young children to make informal inferences about data through an approach called Storytelling-Questioning. The goal of Storytelling-Questioning was to provide young students with fruitful opportunities to engage in IIR within the context of the normal classroom practice of listening to stories. Storytelling-Questioning consists of the researcher reading students a story, and asking strategic questions both planned and unplanned, that are grounded in the story's context and content, and built on the students' responses. This work represents a new use for stories (given its focus on IIR) and builds off recent interest in using storytelling in mathematics (e.g. van den Heuvel-Panhuizen, van den Boogaard, & Doig, 2009).

Four groups of three students (half from kindergarten and half from 2nd grade), participated in the Storytelling-Questioning episodes. The goal of this research was to answer the question, *What information about early elementary school students' Informal Inferential Reasoning does Storytelling-Questioning elicit?* The results of this research speak to the affordances of eliciting IIR through questioning during storytelling. The results demonstrate that Storytelling-Questioning approach provided ample opportunities for students to engage in IIR. Stories proved a natural avenue for students to make and evaluate predictions based on data in the story. Expressing uncertainty about a claim given that the sample data presented is on aspect of IIR. Stories provided a context in which students were able to acknowledge some elementary ideas of variability by expressing uncertainty. The story and the expressions of uncertainty tied closely together. The story seemed to facilitate students' recognition of the uncertain nature of predictions based on a sample. In addition to these results, this poster will also reflect the ways in which Storytelling-Questioning might be improved to become a more efficient method of engaging young students in this crucial type of statistical reasoning.

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A FRAMEWORK FOR STUDYING SECONDARY MATHEMATICS TEACHERS' AFFECT TOWARDS STATISTICS

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Secondary mathematics teachers' affect (emotions, attitudes, and beliefs) towards statistics is an important construct in light of the increasing importance of statistics in the school curriculum (McLeod, 1992; Shaughnessy, 2007). A characterization of this population's affect regarding statistics requires careful consideration of relevant factors such as teachers' knowledge, experience, identity, views of statistics and mathematics, and conceptions related to the teaching and learning of mathematics and statistics. The proposed framework for studying secondary mathematics teachers' affect towards statistics is part of a study whose fundamental purpose is to analyze and describe secondary mathematics teachers' emotions, attitudes, and beliefs towards statistics. The framework for the study establishes the research domain as the *affect system*, which is primarily comprised of emotions, attitudes, and beliefs towards statistics. The designation *affect system* has been adopted for the study since this population of teachers' emotions, attitudes, and beliefs are heavily influenced by knowledge, experience, identity, conceptions, and curriculum. Therefore, *affect system* refers to the complete organization of interworking parts that define secondary mathematics teachers' emotions, attitudes, and beliefs towards statistics. The larger framework of the proposed *affect system* establishes three general categories: emotions, attitudes, and beliefs. The three general categories are then divided into smaller categories. For instance, emotions are set apart from attitudes and beliefs by a strong relationship with context (mathematical and statistical); in light of this, local affect versus global affect will be explored. Teachers' feelings associated with mathematical intimacy and integrity within a statistical context will also be studied as an aspect of emotion. Attitudes will be distinguished from emotions by their categorization as automatized emotions that have become more stable over time (Phillip, 2007). Attitudes are conceptualized through teacher identity, including identities pertaining to self, mathematics, teaching mathematics, statistics, learning statistics, teaching statistics, and social contexts. While attitudes and beliefs are separate constructs (McLeod, 1992; Phillip, 2007), they are necessarily connected. The conception of teacher identity described as part of attitude impacts teacher knowledge and experience, which are elements of beliefs in the proposed framework. Within the construct of beliefs, knowledge and experience impact conceptions, which potentially influence teacher practice. Through their practice, teachers develop and respond to beliefs about the curriculum related to mathematics and statistics. This strengthens the proposed connection between teaching practice and conceptions.

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Research Reports

Brief Research Reports

Poster Presentations

STUDENTS' CONCEPT IMAGES OF AVERAGE RATE OF CHANGE

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In this paper, we describe students' concept images of average and average rate of change and the similarities and differences between those concept images. We do so by describing the students' ways of thinking and ways of understanding average and average rate of change, and how the students' meanings for average influenced their conceptions of average and instantaneous rates of change. We describe the importance of everyday meanings for average in students' conceptions of rate, and propose how instruction might be tailored to address this link. We conclude by discussing implications of this work for teaching average and instantaneous rates of change in single and multivariable calculus, and suggest important directions for future research.

Keywords: Advanced Mathematical Thinking, Cognition, Cognition, Learning Trajectories (or Progressions)

Introduction

The purpose of this paper is to explain how students leverage their understanding of average in their conception of average rate of change in differential and multivariable calculus and the implications of that leveraging for ideas that rely on a concept of average. Rate of change is foundational to calculus because it allows a student to represent how fast a quantity changes with respect to one or more other quantities. While average rate of change has a specific mathematical meaning in calculus, the word *average* may have lexical ambiguity because of its use in statistics and everyday language (Barwell, 2005). We hypothesized that students' understanding of average created confusion as they learned about average rate of change in calculus, and that they developed meanings for average rate of change that relied on an everyday understanding of the word average. We sought to characterize what students' concept images for average rate of change by focusing on their ways of thinking about average and their subsequent ways of thinking about average and instantaneous rate of change. We use *ways of thinking* as Harel and Koichu (2010) do to mean "a cognitive characteristic of a person's ways of understanding associated with a particular mental act" (Harel & Koichu, 2010, p. 117). In this paper, we describe the theoretical underpinnings of the study, identify how our assumptions about student thinking drove the study's design, illustrate our methodology and coding, and present a framework that characterizes students' concept images of average and average rate of change. We argue that specifically addressing the different uses of average in mathematics, statistics, and everyday language is crucial to students developing a coherent understanding of average and instantaneous rate of change in calculus and propose ideas for helping students develop the conceptions of average and instantaneous rate of change that we intend.

Theoretical Framework

In this study, we focused on representing students' understanding and thinking, which by their nature are models that are shaped by our inferences based on students' actions and words. This study relied on the assumption that ways of thinking and ways of understanding (Harel &

Koichu, 2010) reside and develop at the level of the individual and reveal themselves in the decisions and actions students undertake.

Given our assumptions about learning and thinking, we used Vinner's (1983) definition of *concept images* as an orienting framework because students' understandings of average were largely imagistic in nature. We drew on Vinner's (1983) definition of *concept image* as the set of properties associated with a concept together with the all the mental pictures the student has ever associated with the concept. Vinner (1983) differentiates between the concept image and the concept definition (the verbal definition typically used to introduce a concept) and proposes that while handling a concept requires both image and definition, "in thinking, almost always the concept image will be evoked" (p.293). That is, while engaging in mathematical thinking, students tend to use their mental pictures of a concept rather than a symbolic or verbal definition.

We analyzed students' concept images for average and average rate of change. We looked for and asked specifically about similarities in how *average* was used in find-the-mean computational problems and in questions about two- and three-variable functions' average rates of change. We looked for students' *ways of understanding* both average and average rate of change and their *ways of thinking* about these two topics. We make the same distinction between *ways of understanding* and *ways of thinking* that Harel and Koichu (2010) do. That is, a *way of understanding* is the product of a mental act – a single moment as a student grapples with a mathematical situation, while a *way of thinking* "a cognitive characteristic of a person's ways of understanding associated with a particular mental act" (Harel & Koichu, 2010, p. 117). By 'characteristic,' Harel and Koichu mean a distinguishing trait. A *way of thinking* is a pattern of ways of understanding.

The result of analyzing students' concept images under the framework of *ways of understanding* and *ways of thinking* are descriptions of how students think about average and average rate of change, expressed imagistically as they are in students' minds.

Literature Review

Understanding rate of change is foundational to ways of thinking about ideas in calculus, yet many students possess difficulties reasoning about rate (Carlson et al., 2001; Rasmussen, 2000; Thompson & Silverman, 2008). Students' difficulties understanding rate of change include problems interpreting the derivative on a graph (Asiala et al., 1997) and focusing on cosmetic features of a graph (Ellis, 2009). Thompson (1994) found that the difficulties students displayed in understanding the fundamental theorem arose from impoverished concepts of rate of change and incoherent images of functional covariation. Thompson described a coherent way of thinking about average rate of change of a quantity as, "if a quantity were to grow in measure at a constant rate of change with respect to a uniformly changing quantity, then we would end up with the same amount of change in the dependent quantity as actually occurred"PAGE # ?. However, we observed that this way of understanding was difficult for students to achieve.

We hypothesized this difficulty with average rate of change could be attributed to meanings students associate with the word *average*. Students' meanings for words used in technical domains are connected to past experiences with the word (Lemke, 1990). *Average* is used both in everyday language and in mathematics and thus may have lexical ambiguity, or multiple meanings (Barwell, 2005). Meanings statistics students hold for *average* include "definitions that were not indicative of the center and…responses that were not obviously connected to the idea of average as a measure of center or what is typical" (Kaplan et al., 2009, p.11). Given these meanings for average and our hypotheses about students' use of average, we investigated

calculus students' meanings for average in the context of compute-the-mean tasks. We then had students complete average rate of change tasks to see if their meanings for average affected their understanding of average rate of change.

Method

Subjects and Setting

We recruited sixteen multivariable calculus students from a pool of volunteers from six sections of multivariable calculus. We chose this course because it was the students' first exposure to functions of more than one variable in mathematics. This allowed us to observe the students' initial fits and starts with systems with more than one quantity, and to adjust our subsequent questions to more clearly understand their thinking. Each student participated in a pre and post interview. The pre and post interviews questions were designed to gain insight into the students' ways of thinking about function and rate of change. The pre-interview questions were open-ended and focused on single-variable functions and rates. The post-interview questions were also open-ended and consisted of questions about both single and multivariable rates of change.

Analytical Method

Data analysis was multi-phased. We used the pre-interviews to characterize ways of thinking about and understanding function and rate of change. We identified common behaviors and responses across interviews using grounded theory (Corbin & Strauss, 2008) with a particular focus on students' concept images for average rate of change and instantaneous rate of change. Our analyses from the pre-interviews suggested that students relied on colloquial definitions of average in their representations of average rate of change, and that those definitions were prevalent in both two and three dimensions. We designed the post-interviews to gain insight into students' images for average, average rate of change, and their thoughts about how those uses of average were related. We identified and transcribed important passages that gave insight into students' concept images for average in its use in statistics and calculus. We identified a set of concept images for both average and average rate of change using open and axial coding (Figures 1 and 2).

Category	**Criteria**
Normal, typical, mediocre, common	Student uses the word 'normal,' 'mediocre,' 'typical,' or 'common' to describe 'average.'
Mean	Student uses the word 'mean' to describe 'average' or as a synonym for 'average'
Median, middle, center, balance point	Student uses the words 'median,' 'middle,' 'center,' or 'balance point' or talks about the average as being the middle or center of the data
Overall summary, representative value, value used to compare, estimate, expected value	Students talk about the average as a number that presents an overall summary of the data; a number that is representative of all the data; the average as an estimate/approximation or expected value for a new data point; or talk about using the average to compare data
Mode, most common number	Student uses the word 'mode' as synonymous for 'average' or talks about average as the most

	common number

Figure 1: Students' Concept Images for Average

Category	Criteria
Arithmetic mean of slopes	Student talks about summing slopes and dividing by number of slopes summed whether it is a finite or infinite number of slopes is irrelevant to the student.
Expected or most common slope	Student uses the word 'most common', 'expected', 'typical' to describe average rate of change. The student expects the average rate of change provides information about 'all' of the slopes.
Constant rate of change	Student describes average rate of change as the constant rate of change required to produce the same change in the function over the original interval of input.
Smoothing out of all the slopes	Student describes the average rate of change as the all of the slopes smoothed out. Student describes decreasing the 'choppiness' of the slopes.

Figure 2: Students' Concept Images for Average Rate of Change

Results

The following results are representative of our findings for the sixteen students. We highlight three students' responses that represent the major categories of thinking and understanding we identified. We found that students carry their meanings for average into their thinking about average rate of change. We will demonstrate this with excerpts from student responses to the tasks shown in Figure 3. The interviews included two additional compute-the-mean tasks similar to question 1 and two additional conceptual rate of change problems similar to questions 2 and 3.

Questions
(1) The data given below represent the masses of six fishing lures. What would the average mass of the lures mean?
(2) Suppose we define a function *f*, so that $f(x)=e^{-\cos(2^x)}$. Discuss the *process* you would use to determine the average rate of change of the function with respect to *x* over the interval [2.0, 2.2]. (3) Suppose we define a function f, so that $f(x,y)=e^{-\cos(xy)}$. Discuss the *process* you would use to determine the average rate of change of the function. What information do you need to know to complete this process?

Figure 3: Representative Interview Tasks

Responses across these tasks demonstrate the use of concept images for average in the concept images for average rate of change. For instance, Brian's concept image for average is one of 'smoothing out,' a phrase he used explicitly when talking about average rates of change (Response to 2). We also infer that Brian's use of average rate of change being a constant rate as related to the idea of smoothing out; that is, the constant rate is a smoothed-out number made from finding the mean of instantaneous rates (Response to 3). Jordan's concept image for average included the property that an average is a 'typical value' (Response to 1) and she correspondingly thought about average rate of change as a typical slope (Response to 2). Jane's concept image for average included the property of an average being 'common' (Response to 1) and she thought average rate of change as the most common value (Responses to 2 and 3).

Brian: [Response to 1] I see the average as kind of like adding everything up into a big ball, and then smoothing it out into equivalent pieces.

[Response to 2] I see the average rate of change like a constant rate of change. Like, how fast the function would need to change to produce the same change in y over the same change in x, but at a constant rate. You take the change in y over the change in x, that kind of smooths it out to determine it for you.

[Response to 3] Now, well, this is harder but I still know I am finding a constant rate of change. However, to pick a constant rate, you have to specify a direction in space, or there would be infinite average rates of change. So, you still have a change in the function on top, but divided by a change one variable or the other. It tells you a constant rate of change.

Jordan: [Response to 1] Well, I sum the masses, then divide by how many there are, which tells me what their mass was mostly, or typically.

[Response to 2] Well, I am finding the slope between two points here Right, so I find the change in y over the change in x. That just tells me a typical slope.

[Response to 3] Again, I probably am finding a slope, an average slope, so I need a change in something over a change in something else. Probably a combination of z, x and y? Again, it would just tell me a typical slope.

Jane:[Response to 1] Well, it would be the most common mass of all of them, kind of giving me information about the seventh lure.

[Response to 2] The average rate of change tells me the most commonly occurring rate of change of all the rates of change, infinite of them in the interval. I find the change in y over change in x, and it tells me that, the most common value.

[Response to 3] Sure, I'm still finding an average rate, so I need something to divide into something else, probably change in z over a change in a combo of x and y. Gets me to the same point, the most commonly occurring rate of change. Like summing up all of the rates of change, and dividing by how many there are in the interval.

Discussion

Students' Concept Images

A concept image includes the properties of the concept, any mental pictures a student has ever associated with the concept, and a concept definition. Vinner (1983) writes that a concept

definition is “a description of our concept image [and is] either taught to us or made up by us when we are asked to explain the concepts to somebody” (p.294). By ‘made up,’ Vinner means that students construct definitions as a result of their experiences with the concept.

We asked students to define ‘average’ and ‘average rate of change’ and the compare the meaning of the two, in addition to the questions in Figure 3. Many students expressed difficulty defining average (“everyone knows what average means”), and as a result we think their attempted definitions make good snapshots of their concept images. The majority of students’ concept images contained properties from everyday language and mathematics. For example, Jordan talked about an average mass as a typical mass and an average rate of change as a typical rate of change.

Mathematically, average may refer to mean, median, or mode (Triola, 2006) and students’ explanations reflected this ambiguity. Jane’s use of ‘most common’ and Jordan’s use of ‘what it is mostly’ are reflective of mode-as-average. Other students (excerpts not included) talked about average as a middle value or a balance point, reflecting average-as-median. While students described median and mode in their concept images for average, all students’ calculations for average mass were arithmetic means.

Students’ concept images for average rate of change reflected their concept images for average as a most common, typical, middle, or smoothed out value. For instance, Jordan talked about average as a most common value and talked about average rate of change as “the most commonly occurring rate of change of all the rates of change, infinite of them in the interval.” Brian talked about average as a smoothed-out value and described average rate of change as “You take the change in y over the change in x, that kind of smooths it out to determine it for you. I find the change in y over change in x, and it tells me that, the most common value.” While students frequently referred to computing average rate of change as taking the change in one variable divided by the change in another (e.g., Brian, Jordan, and Jane’s responses to 3), this seemed to be a rote procedure. That is, students did not seem to have an image of x,y and z as quantities or an image of a quantity changing with respect to another quantity at a constant rate.

We also identified a disconnect between the way students thought about average rate of change and their procedures: that is, they tended to think about it as a ‘smoothing out,’ a median, or a mode, but the procedure of change in one variable over change in another is more reflective of average-as-mean. We concluded that students were unsure when the different meanings for average were appropriate for a situation.

Ways of Thinking and Understanding

Harel and Koichu (2010) differentiate *way of understanding* as an in-the-moment process a student uses to construct meaning and *way of thinking* as a characteristic way of understanding. We looked at students’ ways of understanding average and average rate of change in computational problems, then looked for similarities in ways of understanding that would constitute ways of thinking. We found that students’ ways of thinking about average rate of change mirror their ways of thinking about average.

Students’ ways of understanding average included properties like normal, typical, mediocre, common, arithmetic mean, median, middle, center, balance point, mode, most common number, smoothed-out value, representative value and expected value (Figure 1). Their ways of understanding average rate of change included arithmetic mean, expected value, most common rate of change, typical rate of change, rate of change representative of all rates of change, and a smoothed-out rate of change (Figure 2). The overlap in ways of understanding across context

indicate that students bring their way of understanding average to their way of understanding average rate of change. Moreover, these ways of understanding constitute ways of thinking for many students. We base this conclusion on the observation of patterns in students' ways of understanding across three compute-the-mean problems and four conceptual average rate of change problems.

We conclude that students' ways of thinking included a number of elements from ways of understanding an idea in different contexts. In the case of average rate of change, these ways of thinking seem to prevent students from thinking about average rate of change as a quantity changing at a constant rate with respect to another quantity.

Implications

Instantaneous Rate and Extensions

Our characterization of students' concept images for average and average rate of change has implications for how students think about instantaneous rate of change and for how they extend their way of thinking about rates of change to three dimensions. Many of the students who thought about average as an arithmetic mean determined that average rate of change was an arithmetic mean of a finite number of instantaneous rates of change. They could hold this conception in mind because they thought about an instantaneous rate as slope (a picture) without a measuring process attached to it. Thus, for these students an instantaneous rate was similar to the weight of a lure, and the average rate of change was an arithmetic average. This way of thinking ignores the limiting process in measuring a rate of change, and does not focus on measurement of quantities to determine how fast one is changing with respect to another. When the students attempted to determine how to interpret and measure rate of change in space, they relied on their image of instantaneous rate as a degree of slant of a line (without problematizing direction), and the average rate of change as an arithmetic mean of the degrees of slant. This way of thinking allowed the students to ignore the issue of direction in space and the limiting process that makes possible the computation of a derivative. These ways of thinking support a non-quantitative conception of rate of change for two reasons. First, the measurement of quantities is unimportant. Second, and partially because of the first, the limiting process is not necessary for the student. These issues confound the understandings we intend that students have, and likely constrain students from seeing rate as a quantification of how fast quantities are changing.

Recommendations for Instruction

We propose that students must understand that average rate of change is a comparison of *change* in *quantities,* and that statistical average is a quantity that characterizes a number of quantities. We propose that the distinction between the uses of average can be supported using quantitative and covariational reasoning. (Thompson, 1994, 2011) In both statistic and calculus, students' understanding of average relies on quantities and their measures and students often associate the two because each uses division. However, average rate of change requires that students measure changes in quantities (quantitative reasoning), and compare those quantities to determine how fast one is changing with respect to another (a constant rate, requiring covariational reasoning). We believe that students must understand that while average rate of change and average in statistics use similar calculations, the result of those calculations represents different quantities. While space permits us from detailing a quantitative and covariational approach to teaching rate of change here, we have ideas for particular learning trajectory-type tasks that teach rate of change from these two perspectives.

Future Directions

Our results suggest that students use their concept images for average to make sense of average rate of change. These concept images often lack the element that rate of change is a measurement of the change in one quantity with respect to the change in another quantity. This leaves students to impart different meanings to [f(b)-f(a)]/(b-a) and these meanings are based on their meanings for the word average. In other words, they apply concept images that are not appropriate for the situation. Moreover, not thinking about a rate of change as quantities changing results in students not seeing a need to make the change in one of those quantities approach zero: that is, there is no need in students' minds for a limiting process. We believe that students' development of the meanings instructors intend requires not just taking into account, but rather using productively, their concept images for topics related to new material. Having documented these concept images, the next step is to determine how instructors can use these to their productively in teaching calculus.

We hypothesize that a focus on quantitative and covariational reasoning may help students develop the intended meanings for average rate of change, the limiting process, and instantaneous rate of change. While the importance of quantitative and covariational reasoning has been highlighted in algebra and differential calculus, limited work has studied the role of these ways of thinking in multivariable calculus. Our work suggests that these ways of thinking are critical for upper-level calculus, and future work is needed to determine how to best foster these ways of thinking.

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ACCOMMODATING THE ENGLISH LANGUAGE LEANER IN MATHEMATICAL DISCUSSIONS

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This study explores the communicative accommodation strategies employed by a bilingual tutor and an ELL student during the discussion of word problems involving secondary algebra and geometry concepts. Mathematics discussions are conceived as communicative events wherein ELL student comprehension is promoted through provision of simplified linguistic input. The reported findings reveal that ELL student accommodation entails multiple dimensions, taking multiple (non)verbal forms with varied degrees of orientation toward language versus ideas and relative focus on literalness versus metaphoricity.

Keywords: Equity and Diversity, Classroom Discourse, High School Education

There is growing interest among mathematics educators in helping English Language Learners (ELLs) overcome the complex linguistic demands of mathematics instruction and promote their participation in mathematical discussions as legitimate members of the classroom community. To help mathematics teachers attain this goal, educational researchers have identified a variety of strategies. Mostly anecdotal in nature, this literature falls short of providing mathematics educators with more sophisticated, theory-based understandings of how to effectively accommodate the communicative needs of ELL students. Teachers in the lower-grade levels have also incorporated a variety of means to communicate mathematics and science content to ELL students. Commonly found in practitioner journals, this literature identifies a wide variety of communicative strategies that elementary and middle-school teachers can adopt in their own classrooms. Few strategies are identified specifically for teacher adoption at the high-school level.

In the present inquiry, we set out to attend to the above limitations in the mathematics education literature by using accommodation theory, drawn from the field of communication, to explore communicative strategies employed by bilingual speakers to construct meaning during the discussion of word problems involving secondary algebra and geometry concepts. More specifically, we tackle the following research question: What accommodation strategies do a bilingual tutor and an ELL student adopt to overcome comprehension problems and communicate effectively during discussion of mathematics word problems? In the following section, we discuss our theoretical perspective.

Accommodation Theory

Drawing on theoretical and empirical work from the field of communication, we conceive of mathematics as a school subject whose communication requires mastery over a specific *register*, that is, a specialized subset or context-specific variety of language (Saville-Troike, 2003). Furthermore, mathematics word-problem discussions are viewed as dyadic communicative events, wherein a bilingual tutor (treated as a "native speaker" of the language of mathematics) facilitates the interpretation of complex written input by an ELL who is also a "non-native mathematics speaker" with limited proficiency in the mathematical register. To make communication possible and

overcome potential comprehension challenges, the tutor supports the novice learner by contextualizing input, making it more easily understandable or interpretable (Gibbons, 2003). Expert communicators interpersonally adapt (Burgoon, Stern, & Dillman, 1995) to their recipients' communicative abilities, leading toward a more accessible learning experience for ELL's (both in message understanding and reciprocation) (Hatch, 1983).

Scholarly efforts to analytically account for speakers' tendency to make speech adjustments such as accentual modifications in interpersonal encounters have been heavily influenced by *speech accommodation theory* which, according to Street & Giles (1982), posits that "people will attempt to converge linguistically toward the speech patterns believed to be characteristic of their recipients when they… desire a high level of communicational efficiency" (p. 213). When successful, such accommodation and mutually influential behavior results in the reduction or elimination of language barriers as well as proclivity toward *communicative convergence* or similarity in speakers' ways of talking (Ireland et al., 2011), typically being well received (i.e., positively evaluated) by listeners.

In the present study, we conceive of mathematics discussions as communicative events wherein a mathematics tutor strategically incorporates verbal and nonverbal accommodation. As Allen (2000) points out, "teachers who use the FL [Foreign Language] as the language of instruction typically make both verbal and nonverbal accommodations to facilitate learners' comprehension of the language" (p. 155). By combining accommodation talk and gestures, the tutor draws on the student's current understandings and experiences as resources (Moschkovich, 2002) for comprehension of the textual contents of mathematics word problems.

A variety of strategies have been identified to promote ELL students' mathematical literacy, knowledge, application, and skills. This literature points to multimodal instruction (the combined use of multiple modes of communication) as an effective means to accommodate ELL students. However, how such multimodality can be systematically designed and implemented by mathematics teachers is not completely clear. Further research is necessary in the area of multimodal mathematics instruction to clarify exactly how it can benefit ELL students and foster mathematics learning. As argued by Chval & Chavez (2012), teachers' use of multimodal strategies must be "purposeful… [and] target the development of language and mathematics for the ELLs in their classroom" which is not a task to be taken lightly; it is a task that "requires thoughtful conversations and planning" (p. 265).

Methodology

In order to explore the potential of multimodal dynamics in support of ELL mathematics learning, we conducted a microethnographic case study of tutor-student interactions. This particular inquiry adopted a qualitative research approach, a phenomenological underpinning, and aligned with socio-constructivist perspectives on human interaction (Robson, 2002). Patton (1990) states that case studies are used when "one needs to understand some special people, particular problems, or unique situation in great depth…" (p. 54). Our phenomenological orientation reflects our concern with participants' first-hand experience of a communicative phenomenon, namely accommodation. This case was explored through descriptive data systematically collected through open-ended research methods (video-recorded observations) and analyzed inductively to build a naturalistic account of a bilingual tutor's accommodation strategies when discussing mathematics word problems with an ELL student.

Our dataset was composed of ten hours of video-recordings of bilingual exchanges (Portuguese/English) between one of the authors (Alan Oliveira) and a female ELL student (Cassie) during one-on-one mathematics tutoring sessions. The curricular material was written exclusively in English, while the discussions were bilingual with Portuguese as the dominant language, the predominantly used linguistic code (Saville-Troike, 2003) in the oral mode of communication. As such, our video-recordings offer a unique opportunity to examine discourse strategies deployed by a

bilingual tutor to accommodate an ELL student's communicative needs and hence enable her to participate, by drawing on her bilingual capabilities, in mathematical word problem discussions involving secondary algebra and geometry concepts.

To examine accommodation strategies adopted by the mathematics tutor and ELL student to promote comprehension, we first transcribed the video-recordings of the tutoring sessions and then conducted a *microethnography,* a study of language-mediated social interaction in minute detail through an up-close and exhaustive examination of how people use language and other forms of communication to realize the social work of their daily lives (Erickson, 1996). Gee and Green (1998) define it as "a research language that describes the ways members of a social group construct the structures of daily life" (p. 132).

Our microethnographic analysis was conducted at the level of *episodes* or *key cultural scenes* (Erickson, 1996); short stretches of naturally occurring discursive interactions (tutor-student dialogues) with variable number of utterances. This microethnographic analysis initially revealed a recurrent, three-stage pattern in the bilingual problem-solving approach: aloud translation, mathematical transcription, and mathematical calculation. More specifically, the tutor and ELL student first translated each word problem aloud from English to Portuguese, then transcribed the word problem into an equation or diagram (drawn on a large whiteboard in front of the classroom), and lastly made their mathematical calculations both orally and in writing. Based on this initial finding, a written report was then produced describing and illustrating the tutor's main accommodation strategies in each of the three different stages of their bilingual problem-solving approach.

Multiple strategies were adopted during data collection and analysis to ensure validity and reliability. First, our approach to data collection was aligned with a tradition of educational research known as *action research* (Lewin, 1946). As the mathematics tutor facilitating the discussion, Alan was actively involved with the phenomenon being researched. This ensured familiarity and an insider perspective on the bilingual context of instruction. Second, *peer debriefing sessions* with external researchers (not directly involved with the tutoring sessions) were frequently held during data analysis, wherein key scenes were examined collectively, individual analyses shared, and interpretations discussed extensively. The emergent account was gradually adjusted to include any variation that surfaced from this reflective group interpretation. These sessions were frequently held to triangulate emerging interpretations of the data and guard against individual researcher biases (Robson, 2002).

Findings

We now describe and illustrate general patterns which occurred during the bilingual problem-solving. Attention is given sequentially to accommodation strategies adopted during three different phases: aloud translation, mathematical transcription, and mathematical calculation.

Aloud Translation

Accommodation in this initial phase of the bilingual problem-solving process took various forms, verbal and nonverbal. Its main aim was to fill gaps in the ELL student's vocabulary that became evident when she was unable to orally translate a particular word problem from the source language (English) to the target language (Portuguese). The two main accommodation strategies deployed to ensure student comprehension of the written content of linguistically complex word problems were *literal translation* (i.e., explicit word-for-word translation of unknown terms) and *borrowing* (taking words from a source language when no equivalent exists in a target) (Fawcett, 2003). One instance of borrowing occurred when Cassie encountered difficulties translating the words "nickels" and "dimes," two words for which there are no one-to-one equivalent translations in Portuguese which compromised her ability to demonstrate her knowledge of mathematical meaning.

Upon noticing Cassie's inability to demonstrate her knowledge of the mathematical meanings associated with the two English words "nickels" and "dimes," Alan interjected by posing a *confirmation check* (Long & Sato, 1983), a type of question whose main function is to check whether students have understood previous discourse. His query "You do know that each coin in the United States has a name, right?" is multifunctionally designed, in the sense that, it serves not only as a request for the student to explicitly confirm her comprehension, but also as a source of critical cultural knowledge for the student to comprehend the book writer's intended meanings (i.e., it is both a prompt and a hint). This cultural hint is then elaborated upon by Alan who proceeds to disclose the mathematical meanings behind each term used for informal refence to particular coins in English. Despite the unavailability of equivalent terms for "nickels" and "dimes" in Portugese, Cassie is then able to successfully perform her aloud translation of the problem through borrowing.

Nonverbal accommodation during aloud translation took the form of *deictic gestures* (McNeill, 1992). More specifically, Alan used his index finger to point to objects nearby and designate the contextual location of the referent of unknown words or expressions. This particular accommodation strategy was deployed when Cassie experienced difficulty translating the expression "minute and hour hands of a clock;" he reacted by pointing with his index finger to the clock in the back of the room. Cassie's difficulty stems from the fact that Portuguese speakers use the word "ponteiros" which literally translates into English as "pointers." However, rather than simply providing Cassie with a direct tranlation of the non-literal meaning of the word "hands," Alan encourages her to independently deduce its meaning from direct examination of a real clock in the immediate classroom context. His pointing serves as an *illustrator*, that is, a type of gesture designed to physically illustrate the meaning of accompaning speech (Burgoon, Buller, & Woodall, 1989). By witholding and delaying his provision of a direct translation ("A pointer is what he is talking about"), Alan encourages Cassie to actively and independently translate words based on the available information, context, and her own experience rather than being passively told its meaning in the target language.

Mathematical Transcription

Verbal accommodation in this second problem-solving stage took the form of *simplified rewording*. Because issues with translation had already been handled at this point, verbal accommodation typically took the form of restatement within the target language, as opposed to word transposition across languages. A good example of this form of accommodation took place during mathematical transcription of a problem which involved comparing line segments in a given ratio on the same line. Although Cassie had already been able to translate the word "respectively" into its Portuguese equivalent "respectivamente" without any problems, she had difficulties demonstrating that she had a clear understanding of the meaning of this term during mathematical transcription of the word problem into a geometrical diagram. This was evident during an exchange in which Cassie described how the diagram should be drawn.

By succesfully translating the word problem aloud,Cassie initially gives the impression that student comprehension has occurred and teacher accommodation is not needed. However, when Alan probes further by prompting Cassie to describe how to mathematically transcribe the word problem, it becomes apparent that she does not comprehend the meaning of the word "respectively" as used in the mathematical register. This lack of comprehension becomes particularly evident when Cassie fails to succesfully answer Alan's query "so, M and N respectively, so M is whose midpoint?" By re-articulating the word problem in everyday Portuguese terms, Alan provided Cassie with simplified input, thus accommodating the student's communicative needs not only in English but also in the mathematical register in her own native language.

Nonverbal accommodation during mathematical transcription took the form of *iconic gesticulation* (McNeill, 1992), that is, hand gestures that shared some sort of spatiotemporal correspondence (e.g., trajectory, direction) with abstract ideas or processes being representationally

communicated. This visual form of accommodation was deployed by Alan when he sought to help Cassie comprehend why the mathematical statement “the length of each side of a square (represented by the letter L) is 3 more than the length of each side of a regular pentagon (represented by the letter P)” was correctly transcribed as “L = P +3.”

Alan uses his arms and hands to iconically express mathematical equality in terms of vertical spatial relations (up/down). Through his arm movements equality is spatialized as a state wherein both his hands (each representing one side of a mathematical equation) are at equal heights. Achieving this state requires either lowering the higher hand (i.e., subtracting three units from one side of the square) or raising the lower hand (i.e., adding three units to one side of the pentagon). Alan’s iconic gesturing provides an image of mathematical equality, a conception with which Cassie seems to be struggling.

Mathematical Calculation

Verbal accommodation in this final phase also included deployment of simplified rewording within the target language. For example, while calculating the angles in a geometrical diagram Alan noticed that Cassie attempted to do her mathematical calculations through visual inspection despite the note “figure not drawn to scale.” Despite being able to translate the mathematical expression “not drawn to scale” aloud, Cassie’s subsequent actions (in this case mathematical calculation) reveals that she has not in fact comprehended the message, leading Alan to restate the mathematical register in simpler layterms she can more easily comprehend: “it’s just a little scheme, it’s not measured” and “it’s not measured, they did not measure it carefully.” A very similar instance was observed in a different problem with the mathematical word “adjacent,” which was first translated aloud by Cassie as “adjacente” (its Portuguese cognate) and later simplied by Alan (also in Portuguese) as “near, side-by-side” and “right by one’s side.” In both instances, student comprehension entailed literal translation as well as simplified rewording whithin the target language. Put differently, literal translation did not always suffice as a means to accommodate the ELL student’s communicative needs, especially when dealing with academic or mathematical words. Because this lexicon is not commonly found in everyday parlance, ELL students may occasionally hear it and yet not fully understand its specialized meaning within the mathematical register.

During mathematical calculation, there was also noticeable instances of combined verbal and nonverbal accommodation, usually in the form of *conceptual asides* – brief interruptions in the normal course of mathematical calculation to fill gaps in the student conceptual knowledge. One particularly interesting conceptual aside took place during discussion of a problem where the length of two of the sides of a triangle was provided. Upon noticing that Cassie lacked familiarity with the triangle inequality theorem, Alan interrupted the mathematical calculation to provide a conceptual aside. Rather than simply telling Cassie what this mathematical theorem formally states (i.e., that the sum of the lengths of two sides of a triangle must always be greater than the length of its third side) or the correct inequality for the given triangle, Alan engaged Cassie in a short discussion about a generic triangle drawn separately from their ongoing mathematical calculations. Not only is the triangle inequality theorem communicated in simple lay terms, but it is also made visually accessible through metaphorical spatialization, thus accommodating Cassie’s communicative needs and ensuring her comprehension.

Discussion

The reported findings reveal that teacher accommodation of ELL students entails multiple dimensions. In addition to taking varied (non)verbal forms (the previously known dimension Verbal–Visual), accommodation can also be analytically placed along two additional continua (Linguistic–Epistemic and Literal–Metaphorical), depending upon their degree of orientation toward language versus ideas and the relative focus on literalness versus metaphoricity, respectively. These two additional dimensions are discussed below.

Linguistic and Epistemic Orientation

Our findings revealed that tutor accommodation during bilingual problem solving had varied degrees of linguistic and epistemic (conceptual) orientation. These varied degrees were evident in the fact that the tutor's accommodation strategies were sometimes more focused on promoting ELL student comprehension of linguistic meanings (the textual contents of word problems) than on mathematical meanings (mathematical concepts or ideas ecompassed in the word problem or required for its resolution). At other moments, the primary focus was on mathematical meanings while linguistic meanings remained secondary. Linguistically-oriented accommodation was characterized by an English-Portuguese focal alignment, wherein the tutor took on the role of a foreign language educator who helped the ELL student comprehend a word problem by explicitly establishing semantic equivalence (House, 1997), that is, identifying equivalent words in the target language. This type of accommodation invariably entailed *code switching* (Saville-Troike, 2003), strategic interlinguistic changes from English to Portuguese, typically taking the form of "label quests" (Martin, 1999).

By contrast, epistemically-oriented accommodation strategies had a Portuguese-Mathematics focal alignment, typically entailing *register switching*, intralinguistic changes from an academic to a conversational register (Cummins, 2000) within the same language (Portuguese). By strategically rewording academic and nonliteral mathematical words, the tutor assumed the role of a mathematics teacher who sought to help a mathematics learner comprehend specialized mathematical meanings needed for accurate mathematical transcription and calculation. Accommodation strategies with such epistemic orientation were aimed at enhancing the ELL student's conceptual knowledge (as opposed to improving her English proficiency). It is clear from this literature and our findings that effective ELL accomodation needs to be more focused on language meaning than language form, and that attention to conceptual understanding, in either the ELL student's first or second language, is necessary for academic achievement.

Literal and Metaphorical Accommodation

Previous educational efforts aimed at accommodating ELL's communicative needs have been limited to student comprehension of literal input, for the most part overlooking the need for more metaphorical forms of accommodation. This tendency to focus accommodation of ELL students on the concrete is particularly evident in mathematics educators' frequent use of manipulatives (Fernandes, 2012), science educators' deployment of pictorial and replica models (Buck, Mast, Ehlers, & Franklin, 2005), and language educators' use of realia (real artifacts or cultural objects) (Ash, Tellez & Crain, 2009). However, as it pertains to mathematics instruction, such a trend is problematic in the sense that it is inconsistent with current scholarly work in the field of cognitive science which has highlighted the centrality of metaphoricity to mathematical thought and revealed the underlying metaphorical structure of basic mathematical ideas. The present shows how metaphorical accommodation in mathematics instruction can be effectively accomplished through strategic deployment of iconic gesticulation. More specifically, we found that iconic gestures served an important accommodation function during mathematical transcription and calculation, metaphorically structuring abstract mathematical concepts of algebraic equality and representing the triangle inequality thereom in nonliteral space. Such a finding is consistent with existing research showing that professional mathematicians use iconic gesturing to make metaphorical mathematical meanings more accesssible and comprehensible to others (McNeill 1992) and that non-native language learners can effectively employ gesticulation to overcome communicative difficulties by compensating and eliciting help in the course of multilingual exchanges (Gullberg, 2011).

Significance and Future Research

The main significance of the present study is that it introduces a theoretical framework for systematically examining ELL student's comprehension in mathematical discussion. The present

study increases the linguistic diversity of this research base by focusing on Portuguese. Our findings also have practical significance for classroom instructors of mathematics. In essence, the reported tutoring practices underscore how effective accommodation for ELL students during mathematical discussion requires continuous assessment of student comprehension. Such finding is consistent with the available literature which clearly shows that tutors, teachers, and peers successful in assisting ELLs are consistently attentive, noticing and adjusting to (mis)understanding as they go (Gerena, 2012). Rather than recurrently resorting to a single communicative strategy, the tutor deployed a variety of techniques as he continuosly assessed and adapted to the ELL student's ability to comprehend the contents of mathematical discussion.

The reported findings point to potentially productive direction for future research. Evidence exists that many mathematics teachers lack awareness of the central role of language in mathematics learning as well as the importance of making linguistic accommodation for ELL students (Fernandes, 2012). In a similar vein, Echevarria, Vogt and Short (2008) point out that few content teachers receive in-service training on how to effectively teach ELLs and very few pre-service teacher education programs include instruction on teaching this rapidly growing student population. Such a state of affairs suggests a need for teacher educators to facilitate professional development that can make mathematics teachers more aware of the multiple dimensions of linguistic accommodation and use these dimensions as guiding principles for systematically designing and implementing mathematics instruction that can effectively promote ELL student comprehension of mathematical principles and concepts in their classrooms.

The present study takes a first step in this direction by identifying several strategies that mathematics teachers can potentially combine for the purpose of accommodating students in larger classroom settings. Nonetheless, despite this valuable initial insight, further analytical consideration will need to be given to how the identified strategies can be effectively integrated by teachers to promote multilingual accommodation in regular mathematics classrooms.

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CONSTRUCTING IDENTIFICATIONS WITH MATHEMATICS PEDAGOGIES, DIS/ABILITIES AND "OUR DOMINICAN SELVES"

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How do children develop identifications with mathematics over time, seeing themselves as agents in their math classrooms (or not)? This ethnographic and interview study followed nine Latino/a children with and without learning dis/abilities through two years of mathematics in a high-poverty urban school. The children participated in two distinct mathematical pedagogies (discussion-based and procedural-based) that differently constructed ability and dis/ability in mathematics. Individual children constructed unique and dynamic self-understandings as math learners over time, using the cultural resources of the multiple contexts in which they participated, including positioning through language use, ethnicity, gender, and dis/ability. Children made connections between equity and discussion-based mathematics, and between procedural mathematics and fixed conceptions of mathematical ability.

Keywords: Equity and Diversity, Curriculum, Affect, Emotion, Beliefs, and Attitudes, Gender

Purpose of the Study

As a classroom teacher and a teacher educator, I have met many children and adults who readily identify themselves as particular kinds of math learners, such as "I'm just bad at math" or "I'm not a math person." My work asks how such self-understandings around mathematics develop. How do they change? How do these self-understandings matter for learning over time? I am particularly concerned about these questions for children who have historically had less access to high level mathematics: Black and Latino children, children from low-income families, girls, and those with disabilities. This study is situated within a high-poverty urban school with Latino/a children in order to better understand the ways in which social positioning intersect with mathematics, understanding that Latino/as continue to be overrepresented in special education particularly in the categories of learning disabilities and behavioral disorders (Losen & Orfield, 2002; Artiles et al., 2010).

Theoretical Frameworks

This study assumes that self-understandings are created through participation in social worlds. Within "figured worlds" (Holland et al., 1998), participants create and recreate particular cultural practices and discourses (Nasir & Hand, 2006), with particular habits of making sense both of others and oneself. A single mathematics classroom has multiple figured worlds intersecting within it (Esmonde & Langer-Osuna, 2011), providing multiple sets of resources for understanding oneself as a math learner. This study focuses on processes of identification, providing analysis of the process through which children appropriated various discourses and practices around mathematics (Holland et al., 1998; Bakhtin; 1981).

All figured worlds position participants, particularly in terms of gender, race, class, ethnicity and dis/ability. While all of the children in this study were officially listed as Hispanic, they described themselves in terms that moved beyond limited categories: "I speak Spanish but also English"(Ana, Second Interview) and "I am from the Dominican Republic but I was born here"

(Ruby, Second Interview). Following feminist scholars such as Gloria Anzaldúa (1987), the purpose of my work is to understand people as complex and dynamic rather than static. I also approach disability with the same attention to complexity rather than categorization. Disability studies in education (DSE) recognizes that although individuals may have natural biological variation in how they learn (*neurodiversity)*, it is the social effects of difference that disable (Gabel, 2002; Reid & Valle, 2004). Disabilities depend on context for their form and meaning: an individual may be seen as learning-disabled in one mathematics classroom and not in others (McDermott et al., 2006).

Methodology and Data

As a white, non-disabled, middle class woman researching the experience of Latino/a children from low-income homes, I understand social positioning as not only a theoretical, but also a methodological issue. In order to understand the multiple figured worlds interacting in the classroom as an outsider, I needed a methodology that was sensitive to the complexity of experience. I also knew that I needed multiple interviews with children in order to understand processes of identification with mathematics over time.

Research Questions

1. In one middle school mathematics classroom, how do cultural practices construct and display particular kinds of mathematical ability and disability?
2. How do children, both labeled as learning disabled and not, construct and enact understandings of themselves as math learners?

Participants

Located in a large city, Central Academy Middle School predominantly enrolled Latinos (92%) with a poverty rate of 85%. Although this was a two-year study, this paper will focus on the seventh grade mathematics class. This inclusion class included 24 total students, 12 of which had an Individual Education Plan for learning disabilities. Students with IEPs in special education were placed into a single classroom at each grade level with two teachers, in this case a math and a special educator. Ms. Marquez, the math teacher, was Latina. She had nine years of experience teaching math at the secondary level. Ms. Alton, the special educator who joined the class halfway through, was African American. She was in her first year of teaching. Out of nine focus children, six were girls and three were boys. Six out of nine had IEPs while three did not. Two children with IEPs were also classified as English language learners. All of the children were identified as Latino/a.

Data Collection and Analysis

I made twenty-six visits to the ninety-minute seventh grade mathematics class. For each visit I wrote field notes and twelve classes were also video-recorded and transcribed. Using grounded theory (Glaser, Strauss & Strutzel, 1968), I developed claims about the cultural practices and positioning of the math classroom, and then tested those claims concurrently. To understand the mathematical practices, I separated the ninety-minute class periods into different activities, coding each segment for type of mathematical pedagogy. I carefully tracked children's learning of integers over time, using data from participant observation, analysis of video recordings, student work and a final assessment of computation.

I interviewed the nine focus children twice in their seventh grade year. The first interview was semi-structured, in pairs to increase discussion. The final interview was individual. I analyzed these interviews through narrative analysis (Riessman, 2007).

Results

Ms. Marquez's classroom followed a fairly consistent schedule. Children came in, sat in assigned seats, and quietly began to solve a Warm Up math problem. Ms. Marquez would then lead a discussion based on the problem. Ms. Marquez would then split the children into three groups. On most days in the fall, children rotated between these groups, sometimes working independently and at other times with a teacher.

Early in the fall, Ms. Marquez asked me if I had noticed the "two different sides" of her math class, what she called "critical-thinkingish" and "state-examish"(field notes, 9/28/10). I had noticed that most of the ninety-minute classes began with 30-45 minutes of "critical thinkingish" math, or discussion-based math, before shifting to "state examish" or procedural math. Each way of doing mathematics had its own set of discourses, practices and positioning. I understood each as a separate figured world.

The figured world of discussion-based mathematics emphasized the practices of problem-solving and discussion, particularly sharing multiple strategies and asking questions. During discussion of Warm Up problems in the fall, Ms. Marquez typically asked for multiple strategies, listening to the children's answers and representing their thinking. When children disagreed, she facilitated discussion, using strategies such as revoicing (O'Connor & Michaels, 1996). In the fall, children also occasionally worked in groups on open-ended problem solving which also emphasized multiple strategies. Ms. Marquez spoke with a questioning voice during these activities, placing accountability in the hands of the students (Engle & Conant, 2002).

The work in the second half of the class tended to be procedural mathematics. Children were given packets of work sheets in order to practice a new procedure or review an old one. In the figured world of procedural mathematics, valued practices were being able to solve a range of computational problems independently and quickly, ideally using only memory. The discourses of procedural mathematics were concerned with efficiency and speed. During procedural mathematics, Ms. Marquez emphasized following already determined procedures, placing accountability in the teacher.

Ms. Marquez always began a mathematical topic using discussion-based practices and gradually transitioned to procedural work. Work on integers began with extended discussion of multiple strategies to solve integer addition and subtraction problems, often modeled on the number line. One week later, these discussions were replaced with memorization of set rules for addition and subtraction of integers. Once the rule was established, children tended to use the rule because it had the most status. By the middle of the year, children told me the number line was a "crutch" and "for beginners"(Federico, First Interview).

Some children used the figured world of discussion-based mathematics to understand themselves as math learners. One boy, Luis, explained the difference between discussion-based math and procedural math: "problems that give you problems" versus "worksheets which are nothing"(Luis, First Interview). Luis preferred discussion-based math, enjoying this kind of work so much that he would sometimes hide open-ended problems under his worksheets, working on them secretly. His narratives about mathematics emphasized his persistence in problem solving and his creative solutions. More commonly, children used the figured world of procedural mathematics to understand themselves as math learners. A girl named Ana told me in an interview that she was the kind of learner who had to "practice and practice" until she "got it"(Ana, First Interview). Many children emphasized two kinds of math learners, those who "get it right quick" and those who "struggle"(Arturo, First Interview).

In the fall, children engaged in both kinds of mathematical practices. Out of thirty-four different class activities documented in the fall, twenty were procedural and fourteen were

discussion-based. In the second half of the year, the balance shifted. From January until May, Ms. Marquez eliminated almost all discussion-based math, focusing instead on preparing for the test. Out of thirty coded activities from January to June, twenty-eight were procedural and two were discussion-based.

This shift in pedagogies caused some children to appear more disabled, and others less so. In the fall, Ms. Marquez saw Luis as an innovative problem-solver, while she questioned Ana's ability to understand conceptual work. Luis became a concern for Ms. Marquez in the spring because he was not able to memorize rules. Ana, who identified with procedural mathematics, had the opposite trajectory in the class. In the spring, Ana was on the honor roll, no longer a concern as she was able to memorize procedures called for on the state exam. During discussion-based activity, Luis was deeply engaged. When activity was procedural, Luis was frequently disengaged. During the spring, as the class was focused on preparing for the state exam, Luis's special education teacher, Ms. Alton, repeatedly referred to Luis as "a behavior"(field note, 3/23/11), alluding to a special education designation that Luis did not have. Both Luis and Ana had a label of learning disability. Their differences as learners remind us that there is no one mathematical profile of a child with a learning disability. Almost all the children had some kind of shift in status as the math pedagogy changed, except for a small set of children who seemed able to do both kinds of mathematics.

These two figured worlds of mathematics, that of discussion-based and procedural mathematics, were not the only figured worlds circulating in the classroom. There was also the figured world of schooling, comprised of practices such as lining up, sitting still, and talking in particular ways in the classroom. There was also the figured world of friendships, which was characterized by practices of care work, such as working together with a friend on a math problem. A girl named Esa told me about three kinds of kids in the classroom: "*chistosos*, serious people, and outsiders"(Esa, First Interview). *Chistosos* made jokes in the class, but also tended to engage enthusiastically in procedural mathematics. Serious people were those in the class who took both the practices of schooling and the practices of friendship seriously. Outsiders were those who placed less importance on social relationships. Like the groupings of the high school students that Paul Willis (1977) found in his study of working class youth in English high school, these groups were engaged not only in participating in the practices of the school, but in creatively reworking those practices. The *chistosos*, for example, used hip-hop to express their quickness at procedural mathematics, remixing the figured worlds of the classroom.

Outside of the creative remixing of the *chistosos*, mathematics was separated from language, culture and children' experiences. Luis incorporated issues of social identities into his mathematical understandings, telling me for example that he imagined zero on a number line as the border between Mexico and the United States when he was adding and subtracting integers. In interviews, children repeatedly discussed being taught mathematics at home and in the Dominican Republic in the summers. Such experiences were not part of their experience in math class. Although most children and Ms. Marquez were bilingual, math was discussed only in English. Children told me that they felt this separation. When I asked a girl named Desi if she ever spoke Spanish in mathematics class, she told me no, that Spanish was for when they were "our Dominican selves" and not for "being serious in some subject"(Desi, Second Interview). The figured world of friendship was the only one in the classroom that was bilingual.

The final figured world I found in this classroom was that of dis/ability. Throughout the fall, grouping within this class was heterogeneous. In January, I noticed a new adult in the class, a soft-spoken teacher named Ms. Alton. After class Ms. Marquez told me that the new groups were

tracked, designed to keep all the children with IEPs together so that they could receive services from Ms. Alton (field notes, 1/4/11). One group was all children with IEPs, another group included no children with IEPs, and another group was a combination. The teachers never explained to the class the reasons behind these groupings. Some children took up the discourses of schooling to explain the grouping, telling me that the highest group was “independent” because they are “most trusted” by Ms. Marquez (Clementine, First Interview). Other children took up discourses of disability that I also heard from Ms. Alton; some children “learn differently” and “need different methods”(Federico, Second Interview; Ms. Alton, Teacher Interview).

Most of the focus children took up the discourses of ability and disability in procedural mathematics to understand how the groups were formed. A boy named Arturo told me about two types of learners: those who “get it right quick” and those who “struggle”(First Interview). If you did not know immediately how to solve a problem, you struggled. If you struggle, you “barely understand”(Albert, First Interview). As procedural math dominated, children increasingly used this binary of *get it fast* versus *struggle slow* to understand themselves and others. Comparing the first set of interviews with the second set of interviews, these binaries were increasingly used to explain both oneself as a learner, and to classify, organize and sort others.

Not all the children constructed their understanding of themselves as math learners from the figured world of procedural math. Two of my focus children consistently used the discourses and practices of discussion-based math to understand themselves as math learners. In his first interview Luis did not believe that there was a difference between different learners in his math class, even though “if I am thinking about it in one way, and another person thinking about it in another way, he might be smarter than me at that, but no body is better than nobody else” (Luis, First Interview). Luis kept a similar narrative across the year, insisting on equity in the math classroom. Competence is open for each new problem, a very different conception of competence than the increasingly fixed ideas of most of the children. Luis’s words are an example of a script I heard from multiple kids: “many ways, none is better”(Carmen, Second Interview). This quote comes from a girl named Carmen, describing to me how gender operated in the classroom. Carmen understood herself using the figured world of procedural mathematics, telling me one day she was “smart” at math (5/25/10) and the next “I hate math because I suck at it”(field notes, 10/12/10), each time because of how comfortable she felt with the procedures being used. But Carmen moves out of this binary when she discusses gender and ability and mathematics, using the discourses of discussion-based mathematics. While most children used the binaries of get it fast or struggle slow to understand themselves and others as math learners, when they reflected on situations involved gender or racial equity, they used a different discourse about mathematical competence.

I looked not only at the way in which children constructed self-understandings, but how those self-understandings evolved over time. Over the course of the year, I saw that some children were identifying more with mathematics, while others were distancing themselves. For those who continued to predominately author themselves as agents in mathematics, some used discourses from the figured world of procedural math. Federico used discourses of disability to author himself as simultaneously fast and slow. Others also authored themselves as agents in mathematics using discourses from discussion-based math. Luis, for example, even though he was positioned as unsuccessful in the spring, still authored himself as an agent in math, citing his own persistence and creativity.

About half the focus kids did not author themselves as agents in mathematics. Two of the girls, Rita and Ana, told narratives of de-identification with mathematics, telling me repeatedly and in multiple ways—I used to like math, now I don't. Both of these girls were on the honor roll in the spring, but rejected what they described as the stressful process of memorizing rules. A final trajectory was not of de-identification, but of never identifying with mathematics. These children, also both girls, told stories of perpetual struggle and resistance around mathematics, particularly focused on troubling relationships with teachers and family members around mathematics. My focus children were fairly evenly distributed between these four trajectories. While these patterns emerged, all the children constructed unique and dynamic mathematical selves, built from materials of multiple figured worlds.

Discussion

Several strands of educational research have recently focused on the ways in which children conceptualize their own ability in school subjects. Complex Instruction, developed by Elizabeth Cohen (1999) explicitly addresses these self-understandings, asking teachers to stress multiple abilities. The work of Carol Dweck (2000; 2007) on mindsets focuses on the critical importance of these self-understandings, affecting achievement. Developing flexible understandings of one's ability that hinge on effort rather than innate ability is a critical goal of educators.

First, my work shows that pedagogy mattered for these kinds of self-understandings. Second, the connection between equity and competence demonstrated that the children had narratives of equity that they brought into their narratives when gender or race was in play. They also brought forward powerful narratives from disability studies in education about being "labeled"(Desi, Second Interview) by the way teachers perceive their disabilities. Yet that same child, Desi, would shift back into static conceptions of ability and disability when discussing mathematics. I heard narratives of equity from Desi in all areas of her educational life except for in mathematics, suggesting both that mathematics has a particularly long road ahead of us in order to change the way children think of themselves as math learners, but also that we can build on powerful narratives of equity outside of mathematics.

Finally, and perhaps most important I worked on describing the process through which self-understandings develop. The kinds of self-understandings that I saw as fixed when I began this study, such as "a math person," were certainly not fixed. Attending closely to self-understandings over time, I saw kids define themselves in different ways at different times, based on the complex context of the classroom and their lives. I learned that children engage in multiple practices in math class, are positioned within these practices, and position themselves. Through their engagement in intersecting figured worlds: procedural and discussion-based mathematics, friendship, schooling and disability, the children worked and reworked their relationships, identities and positioning. The authoring of mathematical selves was always in process, making sense of these multiplicities. But possibilities were narrowed when the curriculum narrowed, when more children used the binary of get it fast or struggle slow to understand themselves.

I found that these self-understandings mattered both in their engagement, and in the choices they make about their mathematics education. Children used self-understandings to orient themselves towards mathematics work every day. There was a dynamic, dialectical relationship between self-understandings and mathematical activity for all the children.

Mathematical ability and disability were differently defined within different pedagogies in these classrooms, causing young people with disabilities to be seen as more or less disabled as

teaching practices changed. The cultural practices of schooling and sorting children, here the intersection of high-stakes testing, special education and tracking, not only reflected, but constructed certain ideas of ability and disability. When I began this study, I expected to see ability and disability change based on pedagogies, but I thought it might be a subtle difference, not the pronounced pedagogical shift that I found. This shift, coming as it does from the conflict between mathematical reform pedagogies and current high stakes testing reforms, is not limited in scope to this classroom, but widespread. Neoliberal policies which value multiple-choice test scores above all else will continue to create classrooms in which children that cannot memorize procedures and facts are disabled.

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THE GROUPS OF STUDENTS PROBLEM: INSIGHTS ABOUT MULTIPLICATION AND IMPLIED ORDER IN COMBINATORIAL ENUMERATION

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Counting problems have applications in probability and computer science, and they provide rich contexts for problem solving. Such problems are accessible to students, but subtleties can arise that make them surprisingly difficult to solve. In this paper, students' work on the Groups of Students problem is presented, and an important issue related to multiplication (a fundamental aspect of counting) is discussed. Examples from two categories of students are presented – those who were able to make sense of a correct solution to the counting problem, and those who were not. Evidence is provided that many students failed to attend to the implied order in the multiplicative process, and a productive way of thinking that emerged for some students is also shared. Pedagogical implications and suggested avenues for further work are provided.

Keywords: Post-secondary Education, Advanced Mathematical Thinking, Probability

Introduction and Motivation

As one aspect of discrete mathematics, combinatorial enumeration (the study of counting problems) has received attention in both K-12 and undergraduate curricula, emerging as an important facet of problem solving and probability (e.g., CCSS, 2010; English, 2005). While counting problems are often simple to state and explore, they contain sophisticated mathematical ideas. In spite of their accessibility, there is evidence that students are often unsuccessful when solving counting problems (e.g., Eizenberg & Zaslavsky, 2004; Hadar & Hadass, 1981; Author, Date). Given that discrete and combinatorial topics are something with which mathematics and computer science students should have facility, there is a need for mathematics education researchers to look harder at students' work on counting problems and to attend to potential ways in which students may succeed on such tasks. In this paper, results from student work on one particular *Groups of Students* counting problem are reported. Through the exploration of student work on this problem, the research goal addressed in this paper is to present and discuss students' difficulty in recognizing a key issue related to multiplication – that multiplication implies a process with ordered stages. Some students never resolved this issue, but those who did used outcomes as a means of making sense of the issue. By focusing on one counting problem, we can go into significant mathematical detail, pulling apart some especially salient mathematical issues that might arise more broadly for students. The goal is to use the case of this particular counting problem to identify issues that might be more generalizable for students as they solve other counting problems. This work suggests that students' conceptual understanding of multiplication in counting may be limited, and that students' correct and incorrect assumptions about multiplication should be examined in greater detail.

Literature Review and Theoretical Perspective

There is ample evidence in the research literature that students struggle with correctly solving counting problems. For instance, Hadar & Hadass (1981) identify pitfalls that students face in solving counting problems, and Eizenberg and Zaslavsky (2004) report low success rates for undergraduates. Authors of combinatorics textbooks also underscore the difficulty of counting problems, as Martin (2001) calls the first chapter of his college-level textbook "Counting is Hard," and Tucker (2002) stresses that his counting chapter "is the most important and difficult chapter" in

his textbook on applied combinatorics. Several mathematics education researchers have tried to identify reasons for such struggles, examining strategies that might be productive for students (e.g., English, 1991), studying the effects of implicit combinatorial models (Batanero, Navarro-Pelayo, & Godino, 1997), and investigating students' verification of counting problems (Eizenberg & Zaslavsky, 2004). Halani (2012) looked beyond student difficulties in an attempt to identify ways of thinking that students employ in solving counting problems, and Author (Date) suggested that focusing on student-generated connections might be a productive way to understand how students conceptualize counting problems. In spite of such research, on the whole there is still much to learn about students' thinking related to counting, specific causes for student difficulties, and ways in which such difficulties might be allayed. In this paper, one potential factor of confusion related to multiplication in counting is highlighted.

Theoretically, this work is framed within a focus on mathematical issues of multiplication and order, and within Author's (Date) notion of a set-oriented perspective. Multiplication is undeniably foundational in counting problems. Informally, the multiplication principle is the notion that the total outcomes of a counting process is the product of the outcomes of independent stages of that process, and many refer to this as the "fundamental counting principle" (e.g., English, 2005; Sengadir, 2009). Tillema (2007) has focused on middle school students' multiplicative reasoning in the context of counting tasks, and his work suggests that multiplicative reasoning on such tasks is far from trivial. Students at a variety of levels often view multiplication as a simple task, and they tend to gloss over the details of why multiplication works on counting problems. However, while multiplication is often straightforward, situations can arise in which students encounter subtleties about multiplication in counting problems. In addition, issues of order are perennial in counting problems, and the question of whether or not outcomes should be ordered can determine which counting processes or formulas are appropriate. Several researchers have identified order as being a key element of students' counting (Batanero, et al., 1997; Mellinger, 2004), and the determination of whether or not order "matters" in a counting problem is not an insignificant task. Additionally, Author (Date) has argued for the importance of sets of outcomes as related to order. Author advocated a perspective that counting involves enumerating sets of outcomes, and that counting activity can be considered in light of structuring those outcomes. The findings below, and particularly the work of students who were successful at understanding the correct solution, are situated within this set-oriented perspective.

As the results and discussions show, the work described herein highlights a subtle aspect of multiplication – that multiplication can imply ordered stages in a counting process. This work emphasizes that this implied order in multiplication is perhaps not something to which students naturally attend. This paper presents student work on a task for which failure to attend to this important property of implied order in multiplication leads to problematic issues. The case is made that this aspect of students' multiplicative reasoning may deserve more explicit pedagogical attention and should be investigated further.

Methods

Participants. Twenty-two post-secondary students participated in individual, videotaped, 60-90 minute-long interviews. The students were junior, senior, and graduate mathematics students who had experience with fundamental combinatorial ideas such as binomial coefficients. The structure of these interviews was first to give students five counting problems to solve on their own. Then, students subsequently returned to these problems, during which time they were presented with alternative answers to evaluate (often these alternatives were reasonable but incorrect answers, developed in pilot work.) Typically, this resulted in students comparing two answers, both of which could seem reasonable, but one of which contained an error. This facilitated situations in which students evaluated incorrect but seemingly reasonable answers.

Tasks. The students each solved five problems, but in this paper the *Groups of Students* problem is emphasized. For this problem, both a correct answer and an incorrect answer are provided. While many correct and incorrect answers could be presented, two answers are given that are most relevant to subsequent discussion.

The Groups of Students Problem. The Groups of Students problem states, “In how many ways can you split a class of 20 into 4 groups of 5?” A correct answer to this problem is $\left[\binom{20}{4}\cdot\binom{15}{5}\cdot\binom{10}{5}\cdot\binom{5}{5}\right]/4!$. To arrive at this solution, five students are first chosen to be in a group, and there are $\binom{20}{5}$ ways to do this. Then, for each choice of the first five, five of the remaining students are chosen to be in another group, and there are $\binom{15}{5}$ ways to do this. Similarly, five more are chosen to be in a group, $\binom{10}{5}$, and then finally the last five to be in a group, $\binom{5}{5}$. These four terms are multiplied together because for each group chosen in a respective stage, there are a certain number of options available at each subsequent stage. However, it is noteworthy that this multiplication process assumes a first, second, third, and fourth group – the groups were developed in a particular order. Because of this, the product must divided by 4 factorial since the groups are not meant to be labeled or distinguished in any way. A typical incorrect answer is $\binom{20}{4}\cdot\binom{15}{5}\cdot\binom{10}{5}\cdot\binom{5}{5}$, which neglects the division by 4 factorial. Reasoning behind each of these answers are further detailed in the results section.

Data analysis. Initial data analysis involved transcription of the interviews. Then, the Groups of Students problem was targeted, and the author carefully analyzed and categorized students’ solutions to this problem. In an initial pass different solutions and strategies were characterized, and the implied order in multiplication was identified as mathematical theme of interest. Videos excerpt that related to multiplication were re-examined, and a narrative (Auerbach & Silverstein, 2003) about the students’ performances on the Groups of Students problem was constructed.

Results

The results are organized into two sections. First, overall numerical results are given, emphasizing that there was a low success rate on the problem. Second, to highlight particular mathematical issues that arose, work from two different students is presented, one who was ultimately *unable* to understand the correct answer, and one who was *able* to understand it.

Overall results. On the whole, students struggled with the Groups of Students problem. Only three of 22 students answered the question correctly on their first attempt (Table 1 provides a breakdown of students’ initial answers). Discussions during the interviews confirmed that the students had interpreted the problem as intended (as involving splitting the groups into four indistinguishable (not distinguishable) groups of five students).

Table 1: Frequencies of Initial Answers

Expression	Initial Answer	Correct/Incorrect	Frequency (%)
Expression (1)	$\left[\binom{20}{4}\cdot\binom{15}{5}\cdot\binom{10}{5}\cdot\binom{5}{5}\right]/4!$	Correct	2 (9%)

Expression (2)	$\frac{20!}{5!\cdot 5!\cdot 5!\cdot 5!\cdot 4!}$	Correct	1 (5%)
Expression (3)	$\binom{20}{5}\cdot\binom{15}{5}\cdot\binom{10}{5}\cdot\binom{5}{5}$	Incorrect	8 (36%)
Expression (4)	$\binom{20}{5}\cdot\binom{15}{5}\cdot\binom{10}{5}$	Incorrect	3 (14%)
Expression (5)	$\binom{20}{5}+\binom{15}{5}+\binom{10}{5}+\binom{5}{5}$	Incorrect	2 (9%)
	Other	Incorrect	6 (27%)

Student data. In this subsection, work of one representative student is presented from each of two categories of students (see Table 2; student work from categories in bold are presented in detail). First, work is presented of one of seven students who initially answered the problem incorrectly and was *unable* to make sense of why division by 4 factorial was necessary. The student's inability to recognize the implicit order in multiplication is emphasized, and reasons for this phenomenon are suggested. Second, work is presented of one of nine students who initially answered the problem incorrectly but was *able* to make sense of division by 4 factorial. The student's utilization of a specific outcome allowed him to recognize the need for the division by 4 factorial. The two students' work is contrasted. In the Discussion section, major themes from these findings are synthesized and implications and avenues for further study are proposed.

Table 2: Categorization of Students

Category description	Frequency (%)
Students answered the problem correctly initially	3 (14%)
Students answered the problem incorrectly initially, were presented with a correct solution, and were UNABLE to understand the correct solution	**7 (31%)**
Students answered the problem incorrectly initially, were presented with a correct solution, and were ABLE to understand the correct solution	**9 (41%)**
Students answered the problem incorrectly initially; did not revisit the problem	3 (14%)

Nic – Initially incorrect, unable to make sense of division by 4 factorial. Nic originally arrived at the incorrect Expression 3. In his work he had written down Gp 1, Gp 2, Gp 3, Gp 4 (see his work in Figure 1**),** although he did not reference them in his answer. In fact, his discussion revealed that while he wrote different group numbers, he interpreted the problem correctly (that the groups were to be indistinguishable), saying that the question was asking about a situation, "where you're just splitting them into the groups and then that's it." Given his proper interpretation of the problem, the fact that he wrote down group numbers is noteworthy. He stated that he did so simply to "keep an order" in his head, but he did not seem to realize that this work suggested that he was unintentionally treating the groups as distinguishable.

> *Interviewer:* Okay. Cool, the Group 1, Group 2, Group 3, Group 4, did that help you in that counting process, or why did you distinguish between Group 1, Group 2, Group 3, Group 4?
> *Nic:* All it really did was to, basically I have 20 choose 5, so that gives me a group, is basically what I was thinking of, so I need to make sure I subtract those kids off for the next possible

group… So it was basically just helping me keep an order on where I was at in the process, so, it was more keeping an order for me in my head.

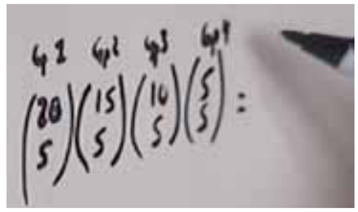

Figure 1: Nick's Group Labels Above Each Binomial Coefficient

When Nic revisited the problem and was given the correct Expression 1 to evaluate, he was unable to explain the division by 4 factorial. There was evidence that the unintended distinguishability of the groups seemed to be a key factor in his inability to reconcile the division by 4 factorial. In Nic's thinking, his process of writing down the labels for the groups was simply organizational, and he did not realize that by performing the multiplication, he was actually employing a counting process with implicitly ordered stages. The excerpt below provides evidence for why Nick might not have recognized that multiplication has implied ordered stages. He intimates that the binomial coefficient, the "choosing," already indicated that, "you're not permutating [sic]." This suggests that Nic felt the issue of order (and not making the groups distinguishable) was already handled in his answer to the problem.

Nic: In how many ways can you split a class of 20 into 4 groups of 5?...I think that [his initial answer, Expression 3] is still right. Because when you do this choosing, when you do this already takes into account that you're not permutating. This one [the alternative solution, Expression 1] you're doing it twice. So this [Expression 1] is probably undercounting I think.

There is further evidence that Nic intended for his labeling of the groups as helping him keep track of things, focus on groups and not students. However, he did not recognize that there was an implicit distinguishability in his answer through the multiplication. Additionally, he did not notice that by saying there was a Group 1, Group 2, Group 3, and Group 4, he was implying that there was a first, second, third, and fourth ordered stage to his process.

Interviewer: Okay, now in your solution you had written a Group 1, Group 2, Group 3, Group 4 above it. What was that?

Nic: Uh, that was just show that I was doing it by groups rather than students, that we have 4 different possibility – 4 different groups that the students could go into that wasn't anything for them to be – the groups weren't labeled or anything, that's just knowing that you have to split 20 things into 4 spots, rather than actual names.

Even after spending significant time on the problem, Nic was ultimately not able to understand or explain the correct Expression 3. Nic's work on the Groups of Students problem shows an instance of a student who initially answered the problem incorrectly, and who was unable to make sense of the division by 4 factorial, even when the correct answer was presented to him (he was one of seven such students). In seeing this struggle, what comes to light is the fact that there was confusion for Nic about whether his multiplication somehow already caused the groups to be distinguishable and ordered in some way. His inability to see this is addressed further in the Discussion section.

Owen – Initially incorrect, able to make sense of division by 4 factorial. Owen had initially generated Expression 4, and his explanation highlights the sequential order in which he came up with the answer. When asked why he did not write the final term of choosing 5 of 5 students, Owen said, "There's just one way to do that, there's only one. They're just sitting there in their group already."

Owen: Ah, interesting. 4 groups of 5. Well, this one I mean, it seems pretty easy to pick the first group, you know, I have 20 people, choose 5…Great. Well, this one I feel kind of friendly with,

'cause then whoever I chose, I have 15 people left, I just choose 5…I only have 10 people left, and I choose 5. There's 5 people left. That's how I'd count it. I don't know if that's right or not.

When Owen returned to the problem later in the interview, he was given Expression 1 to evaluate. Upon reading it, he said, "I don't understand why they would divide by 4 factorial I guess…I'm immediately confused by that." After taking some time to think about what might have been going on, Owen was eventually able to explain what had happened in the problem, and why he would indeed need to divide by 4 factorial. To do so, Owen appealed to a particular outcome, showing how the incorrect Expression 4 could lead to an overcount. Owen started to write an example, but not wanting to write out 20 objects, he instead considered splitting up 8 students into four groups of two. He wrote down ABCDEFGH, and he wrote the corresponding solution that he would have done with 8 students, $\binom{8}{2}\cdot\binom{6}{2}\cdot\binom{4}{2}\cdot\binom{2}{2}$. He then took different colored pens, and, as he explained the solution below, he underlined each binomial coefficient with a different color (pink, red, orange, and green, respectively; see Figure 2).

Owen: I'm going to order my selections, because I think of them as happening in some order. So it's like this first [draws a pink line under $\binom{8}{2}$], and I'm going to do this second [draws a red line under $\binom{6}{2}$], this third [draws an orange line under $\binom{4}{2}$], and this fourth [draws a green line under $\binom{2}{2}$], right

Figure 2: Owen's Color Coding of the Outcome

Then, as seen in the excerpt below, Owen recognized that when he chose the groups of students to be AB, CD, EF, and GH, he could have chosen them in some order; the colors represented a particular order in which he chose that division of the 8 people. As he talked about choosing a pair of students, he drew a color under that pair. He drew lines under AB, then CD, then EF, then GH in that first example, and he drew lines under AB, then EF, then GH, then CD in the second.

Owen: What if, when I first chose 2 students, I picked, I'm going to make it really obvious, A and B [draws a pink line under AB]… Cool, then I have 6 remaining. Then I pick C and D [draws a red line under CD]. Oh, what if I have 4 remaining? Oh I'm going to pick these 2 [draws an orange line under EF, then draws a green line under GH].

He then noted that he could pick the same division of 8 people (AB, CD, EF, GH) in another way, and he drew colors above the letters to represent picking the pairs in a different order. This time, he drew AB first, then EF, then GH, then CD.

Owen: Well, that's one way to count, pick 'em, right? Color, change the colors drawn above again, so…you can pick the pairs in different orders… To make it obvious I'll draw the colors. Look, I'm just drawing the same exact pairs, different colors up here…To symbolize that if I did

this, I could have picked him, this same pair twice, right, but second I could have picked CD or I could have picked EF…And then third I could have picked EF or GH…And last, CD or GH, the remaining pairs…Same exact pairs, counted twice here [Expression 4], counted only once here [Expression 1] because you're dividing out the ordering of the pairs.

Owen is an example of a student who initially got the problem incorrect, but who was ultimately able to make sense of the correct Expression 1 when it was presented to him. Owen's work on this problem is noteworthy because it shows how he identified a particular outcome (or an outcome in a smaller case) in order to address the overcounting issue and ultimately decide which expression was correct. His work suggests that he recognized that the multiplication was imposing order on his process, producing distinguishable groups. He cleverly used a color-coding scheme to indicate exactly how the same division of 8 students was actually counted more than once by the incorrect expression. His work here suggested that focusing an example in the set of outcomes of the smaller problem allowed him to complete the problem correctly.

Discussion and Conclusion

The above findings show examples from two groups of students who faced varying levels of success on the Groups of Students problem. As they struggled with making sense of the division by 4 factorial in Expression 1, evidence emerged about an important aspect of students' thinking about multiplication. In particular, the Groups of Students problem revealed that students did not always recognize that multiplication necessarily implies ordered stages. The ordered stages are difficult to detect in this particular problem because the problem counts fairly abstract objects, and because the problem solution includes binomial coefficients. For instance, Nic assumed that his initial solution took into account the fact that he wanted the groups to be indistinguishable, and he assumed that the multiplication of the binomial coefficients did not imply ordered stages. This was presumably partly because of the presence of binomial coefficients (which tend to signify order *not* mattering), causing him not to realize that, whether he intended to or not, the act of multiplying was bringing order into the problem. This contributed to his failure to realize that he was really incorporating order, and, thus, to answer the problem correctly, he needed to account for the groups being distinguishable by dividing by 4 factorial.

The Groups of Students problem brings to light significant issues regarding multiplication and order, both of which are absolutely fundamental aspects of counting. Students' poor success on this problem (both initially, and even after being given the correct answer) suggests that multiplication is not as well understood as might be expected. Thus this problem highlights an important issue involving multiplication and suggests that this implied order in multiplication is something that should receive more explicit attention among students who are learning to count. The findings also suggest a way to fix this issue, however. Owen's work, for instance, suggested that while he initially could not understand the division by 4 factorial, it was an appeal to a particular outcome that enabled him to make sense of why the division was necessary. Indeed, there were 9 students who initially got the problem wrong but were ultimately able to justify the division by 4 factorial. For all 9 of them, they used particular outcomes (in a manner similar to Owen) to come to understand the correct solution. The role of outcomes in identifying an overcount contributes to a growing body of evidence (Author, Date) that outcomes provide a productive perspective for counters to emphasize. Thus, more can be done in helping students articulate how the multiplication counting process actually generates outcomes, and by tying multiplication to outcomes productive and correct conceptions of multiplication can be strengthened, particularly in the context of counting. In examining students who were unsuccessful, problematic ways of thinking emerged that could be suggestive of broader issues as students count, especially pertaining to a lack of understanding of implied ordered stages in multiplication. In examining students who were successful, a particularly productive way of thinking arose, as students drew upon the set of outcomes to effectively reconcile and explain a key

mathematical issue. These students were able to articulate how the multiplication and the order were interacting in the incorrect Expression 3 to generate some outcomes more than once.

Pedagogically, students may benefit from engaging with problems like the Groups of Students problem, which raise subtle aspects of important counting concepts like multiplication and order. Multiplication cannot merely being seen as a trivial, taken-for-granted process, but rather teachers must strive to develop deep multiplicative reasoning in students, explicitly tying properties of multiplication to counting processes.

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A LEARNING TRAJECTORY FOR EARLY EQUATIONS AND EXPRESSIONS FOR THE COMMON CORE STANDARDS

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This paper describes the learning trajectory for Early Equations and Expressions (EELT), one of the 18 learning trajectories presented at www.TurnOnCCMath.net, developed to interpret the Common Core Standards for Mathematics based on research on student learning. The EELT is foundational for introducing concepts of early algebra and setting the foundations for the most advanced mathematical ideas in the later grades of schooling. The theoretical framework and research literature that the unpacking was based on are presented by giving examples from the descriptors of the standards.

Keywords: Expressions, Equations, Standards, Learning Trajectories

Learning Trajectories and the Common Core Standards for Mathematics: TurnOnCCMath.net

The Common Core State Standards for Mathematics (CCSS-M) (CCSSO, 2010) have been adopted by 45 states and the District of Columbia. Major goals include strengthening students' mathematical conceptual understanding, weaving eight key mathematical practices throughout mathematical instruction and learning, and bringing U.S. educational standards to par or better with education in countries whose preparation of students in mathematics is stronger than that of the U.S., based on international assessments (CCSSO, 2010). The CCSS-M represents a major effort to improve the coherence in learning expectations across states. The new standards include significant changes in depth, emphasis and timing of instruction in numerous topics. Some topics are introduced earlier, some later, than in previous standards from various states, and several with more intensive or expanded treatment (e.g. early number and operations; ratio and proportional reasoning; statistics and probability).

How then to translate the intent of the new standards into instructional practice, to interpret the CCSS-M in ways that strengthen students' conceptual development? Research on student learning by mathematics educators during the past 20 years or more has generated *learning trajectories* (LTs) as an organizing framework for student conceptual growth (Clements & Sarama, 2004; Confrey et al., 2009; Simon, 1995). Depicting student learning of major mathematical ideas over time, LTs describe student prior knowledge of particular ideas, and networks of instructional experiences that support students in transiting likely intermediate states of understanding on their way to robust domain goal understanding (Confrey et al., in press).

Learning trajectories can provide a framework for organizing the instructional core (Confrey & Maloney, 2011)—that spectrum of instruction, instructional practices, enacted curriculum, and classroom discourse and assessment that comprise the learning environment that children experience—to focus on the consistent development of student conceptual understanding over time.

[1] On October 22 of 2013, PME-NA committee was informed of a mistake in authorship. Therefore, the paper authorship herein differs from the version of this paper included in the PMENA 2013 Conference Proceedings that were posted on the conference website on Monday, October 21, 2013. The authors of this paper listed here are correct as of Thursday, October 24, 2013.

The CCSS-M calls for incorporation of "research-based learning progressions detailing what is known today about how students' mathematical knowledge, skill, and understanding develop over time" (CCSSO 2010, p. 4). Our research priorities are to bring to bear the research on student learning to support educators in interpreting the CCSS-M and support practical instructional implementation of the CCSS-M.

We have constructed 18 learning trajectories that cover all of the K-8 CCSS-M standards (http://www.turnonccmath.net) (Confrey et al., 2011; Confrey, 2012). A hexagon map of the CCSS-M was created as a visual model for navigating the connections among major topics and standards and illustrating learning trajectories within the standards. Each learning trajectory (LT) comprises descriptors that incorporate the Common Core Standards into text- and graphics-based descriptions of student movement from naïve to more sophisticated mathematical understanding. The descriptors identify (1) conceptual principles, (2) student strategies and representations or *inscriptions* along with misconceptions; (3) meaningful mathematical distinctions and multiple models; (4) coherent structure or schemes of reasoning as topics become more complex; finally, (5) bridging standards that are added to provide conceptual continuity for more fully articulated learning trajectories than could be provided in the compact CCSS-M. The iterative development of these learning trajectory descriptors, and their continual improvement and strengthening based on feedback and review, have been described previously (Confrey, 2012). In this paper, we describe in more detail the learning trajectory for Early Equations and Expressions (EELT).

Early Equations and Expressions Learning Trajectory

Algebra "encompasses the relationships among quantities, the use of symbols, the modeling of phenomena, and the mathematical study of change" (NCTM 2000, p. 37). By linking these topics to, and developing them simultaneously with, topics of early mathematics, the aim of early algebra in elementary schools is for students to establish the foundations for development of algebraic reasoning in middle and high school (Carraher et al. 2007). The main goal of algebra is for students to learn how to form generalizations by seeing the "general through the particular" and "the particular in the general" (Mason et al., 2005), to shift students' focus "from thinking about relations among particular numbers and measures toward thinking about relations among sets of numbers and measures, from computing numerical answers to describing and representing relations among variables." (Carraher et al., 2007, p. 266). For competency in algebraic reasoning, student learning must encompass and coordinate several different "big ideas" such as generalized arithmetic, functional thinking, the concept of equality, and the concept of a variable (Blanton & Kaput, 2011; Carraher et al., 2008; Carraher et al., 2007).

The goal of the EELT was to describe the development of students' reasoning about patterns and the early foundations of functional relationships (Section 1, "Exploring Patterns and Their Relationships"), equality and equations (Section 2, "Exploring and Solving Equations"), and expressions, including variables as quantities that vary (Section 3, "Working with Expressions"). These interrelated concepts span grades K through 7, concurrently during many of the grades, with standards from the Operations and Algebraic Thinking and the Expressions and Equations domains of the CCSS-M. Other topics important in the foundation of algebraic reasoning are covered in other TurnOnCommonCoreMath LTs: Inequalities and more advanced notions of functional thinking are emphasized in the Linear and Simultaneous Functions LT; operations including inverse operations, and properties are explored in detail in the Addition & Subtraction, and Division and Multiplication LTs, and ratio, rates, and proportional reasoning are developed in the Ratio and Proportion and Percents LT. Explicit connections among these and the EELT are made, where appropriate in each of these LTs. The EELT contains a total of 17 CCSS-M and 6 Bridging Standards, with their corresponding descriptors. The following sections describe the unpacking of the EELT as it was

developed and presented in www.TurnOnCCMath.net.

Section 1: Exploring Patterns and Their Relationships

Section 1 describes the development of student reasoning in identifying repeating and growing patterns, and the characterization of relationships between two patterns. The CCSS-M introduces arithmetic patterns in Grade 3 (Standard 3.OA.9), but does not explicitly treat geometric patterns, which are also a type of growing pattern. Repeating patterns can act as the basis for introducing growing patterns, because they are foundational for expressing initial generalizations of rules, functional relationships, proportional thinking, and number theory (Threlfall, 1999; Warren, 2005; Zazkis & Liljedahl, 2002). The CCSS-M introduces shape patterns in Grade 4 (Standard 4.OA.5), however, research has demonstrated that non-numeric patterns such as pictorial, verbal and symbolic should occur in earlier grades to help students recognize generalization in relationships that are "independent of the numbers or objects being operated on" (Warren, 2005, p. 759). Building on such insights, the learning trajectory therefore incorporates introduction of students first to non-numeric patterns (both repeating and growing) and then to numeric patterns (both repeating and arithmetic/geometric); we added four bridging standards to support these (Bridging Standards K.EQX.A, 1.EQX.A, 1.EQX.B and 2.EQX.A).

We also discerned a systematic scheme for pattern exploration (Figure 1), an example of *coherent structure* (element 4, above). For each type of pattern, student understanding is strengthened through first (a) identifying the terms and the core unit, (b) extending the pattern to subsequent terms, (c) identifying missing terms, (d) debugging mistakes, (e) recognizing common structure among patterns (i.e. the pattern 2, 4, 6, 8, … has the same structure as the pattern 86, 88, 90,…) and finally (f) creating new patterns from a given core unit and initial element.

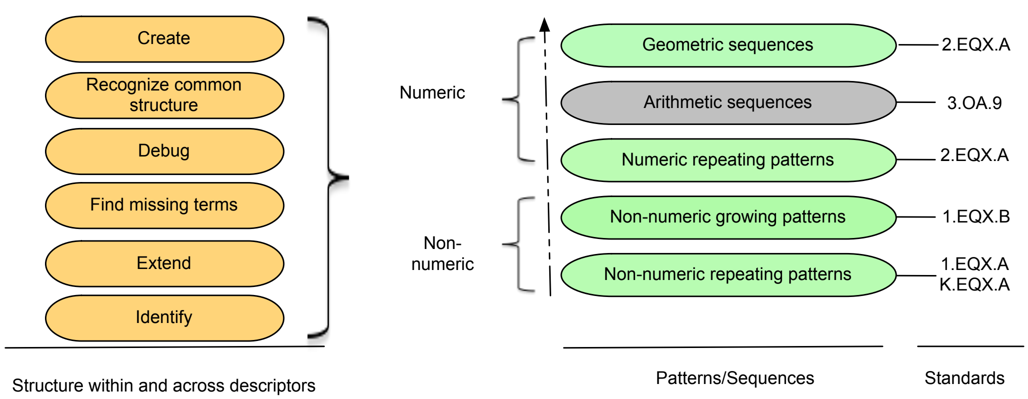

Figure 1: CCSS-M Standards in Grey and Bridging Standards in Green

A more sophisticated description of patterns, identifying a rule for a given pattern in standard 4.OA.5 (i.e. given the pattern 2, 7, 12, 17, … students identify the rule as "Add 5"), supports the second goal of this section, namely the identification and characterization of relationships between two patterns (Standard 5.OA.3). A correspondence description for two patterns expresses a rule that relates corresponding value pairs of two patterns. A foundation of the development of correspondence rules is a covariation relationship, which explains how the values of two patterns change simultaneously (Blanton & Kaput, 2011; Confrey & Smith, 1991, 1994, 1995). The distinction between covariation and correspondence relationships can be introduced through contextual problems, for example the following: "A roadrunner, being pursued by Wile E. Coyote, runs 3 meters every two seconds. Make a graph that describes the distance the roadrunner travels in relation to time. Predict the roadrunner's location after 5 seconds." The learning trajectories highlight

multiple representations: e.g. for characterizing functional relationships, the use of diagrams, tables, graphs, and dynagraphs (Figure 2).

Time (s, seconds)	**Distance m (meters)**
0	0
2	3
4	6
6	9
8	12
10	15

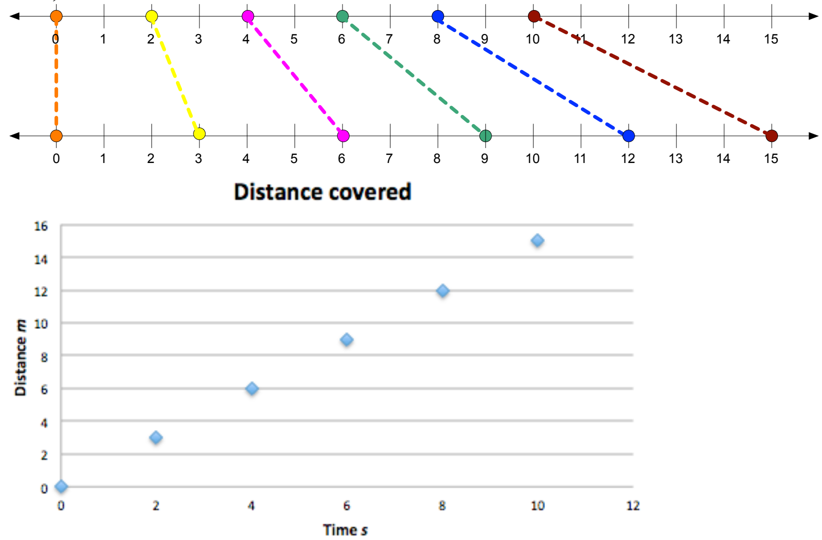

Figure 2: Table, Dynagraph and Graph as Representations

The two patterns in this example covary: values in the first column increase by 2 while values of the second column increase by 3. A correspondence description here delineates a rule that describes one of the sequences in terms of (dependent on) the values of the other: any value to distance covered by the roadrunner (m) is $^3/_2$ times the corresponding value of the time (s) in seconds. Students in 5th grade are not expected to write a symbolic equation for this type of relationship. However, they are expected to extend the patterns and identify the rule (correspondence). Studies have shown that students are capable of reasoning about covariation and correspondence relationships as early as first grade (Blanton, & Kaput, 2004; Martinez & Brizuela, 2006; Stephens et al., 2012). By 5th grade in the EELT, students have experience exploring both covariation and correspondence relationships and a variety of arithmetic and geometric sequences, all of which lay a strong foundation for the development of ratio and functions in middle school (Carraher & Schliemann, 2007; Zazkis & Liljedahl, 2002).

Section 2: Exploring and Solving Equations

The goals of Section 2 focus on a) understanding the concept of equality and solving equations and b) the concept of a variable as representing a single unknown. Equality is a pivotal concept with multiple meanings. It is key to development of a progressively sophisticated understanding of and flexibility with equations. It is developed from a way to make the equations "balance," to determining which equations are true or false, leading to the more sophisticated solving of equations (Carpenter, & Levi 2000; Carpenter et al., 2003; Van de Walle et al., 2013). The aim is for students to recognize that the equal sign signifies a claim that the quantities or expressions on either side of the equal sign represent the same value; this serves as a foundation for creating equivalent algebraic expressions at the higher levels of the learning trajectory.

The CCSS-M introduces the meaning of the equal sign in equations in Grade 1 (Standard 1.OA.7). Before formal introduction to the equal sign and writing equations, students learn to model the equality of (a) two additive parts (which can be joined) and their sum, and (b) equipartitioned parts (which can be reassembled) and their original collection or whole. To link students' prior understanding of joining and separating in counting (Samara & Clements, 2009; Nunes & Bryant, 1996; Fuson et al., 1983; Piaget, 1965) and equipartitioning and reassembly (Confrey et al., 2009) to

the concept of equality, we added a preceding bridging standard (1.EQX.C) that contains multiple well-known ways to concretely model equations, such as unifix cubes, counters, and drawings. Balance scales also serve as visual models for equations: children use different combinations of quantities to represent and achieve balance (equality) and to show equivalence between an original collection and all of the corresponding fair shares reassembled (Baroody & Ginsburg, 1983; Van de Walle et al., 2013). Balance diagrams (Figure 3) anticipate the balancing of equations in algebra (Herscovics & Linchevski, 1994). Another visual model for equations is the "bar model" (Figures 4a/b), in which children use different combinations of horizontal bars to establish claims of equal lengths or to show equivalence between a collection of items and the reassembly of equipartitioned fair shares (Fong Ng & Lee, 2009; Hoven, 2007). For example, students may recognize that a 12-inch paper strip shared among three people can be represented with 4-inch long strips per person, which can then be reassembled lengthwise to form the original 12-inch paper strip (Figure 4a). They may also demonstrate that the three fair shares of 4-inch strips all have the same length (Figure 4b). We conjecture that as students coordinate such models with the writing of equations, they reveal misconceptions about the role of the equal sign, such as using it as an operator symbol (Carpenter & Levi, 2000; Carpenter et al., 2003; Kieran, 1981). Teachers can then, in a timely way, instructionally address such a misconception.

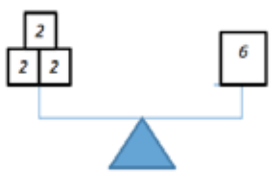

Figure 3: Balance Scale

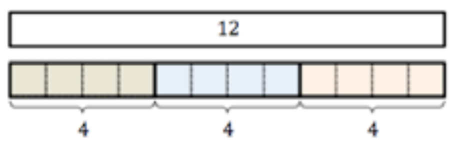

Figure 4a: Bar Model 1

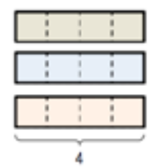

Figure 4b: Bar Model 2

Following the writing of equations, students begin to solve equations with single unknowns. They progress from addition and subtraction equations to division and multiplication equations relating three whole numbers (Standards 1.OA.8 and 3.OA.4 respectively), move to equations that involve four whole numbers (3.EQX.A, bridging standard added) and then to the more sophisticated solving of two-step word problems using all four operations in equations (Standard 3.OA.8). For the solving of all the types of equations above, the EELT includes four different strategies suggested by research (Carpenter, & Levi 2000; Carpenter et al., 1996): a) *substitution* of different values, including checking to see if the two sides of an equation have equal value; b) using *models* to illustrate equivalence, such as rearranging arrays (i.e. 6 x 2 is equal to 3 x □); c) applying inverse operation (as in "fact families"), such as rewriting 35 ÷ 7 = □ as 7 x □ = 35 to solve; and d) applying known properties (commutative property of addition and identity) to solve problems, such as recognizing that the equation 1 x □ = 35 ÷ 7 can be simply □ = 35 ÷ 7 through application of the identity property (1 x □ = □).

Section 3: Working with Expressions

Section 3 of the EELT has goal understandings of a) distinguishing between variable as representation of a single unknown value and variable as a quantity that varies, according to context, b) using variables in different contexts to represent quantities, and c) identifying and generating equivalent expressions at a more advanced level.

Students first encounter the *variable* as representing a single unknown value in equations (Section 2) but now broaden the concept to represent a quantity that varies, in exploring expressions (Standard 6.EE.2.a). Studies have shown that students of an early age can represent quantities that vary (Carraher et al., 2007; Marum et al., 2011). Again, multiple representations support

strengthening student understanding: variables should be represented in many forms including multiple letters, shapes, and symbols, in part to avoid the development of misconceptions and prototypes such as believing that letters are general referents (i.e. *h* stands for height of multiple people) (MacGregor & Stacey, 1997) and that different symbols can have the same value (Carpenter et al., 2003). Contextual problems such as the following, from Brizuela & Earnest (2008), support distinctions: "Raymond has some money. His grandmother offers him two deals: *Deal 1*: She will double his money. *Deal 2*: She will triple his money and then take away 7. Raymond wants to choose the best deal. What should he do? How would you figure out and *show him* what is the best to do? Is one deal *always* better? Show this on a piece of paper." Generating expressions for these problems, students recognize that a) a variable can represent a parameter whose value determines the characteristics or behavior of other quantities [i.e. if m is the value money above, then Deal 1 is $2\bullet m$ and Deal 2 is $(3\bullet m)$-7], b) while the value of a parameter can vary, its value is always fixed in the context of a specific problem, and c) the same variable used in expressions related by context must represent the identical value, but different variables may represent the same value or different values (Blanton & Knuth, 2009-2013). By exploring which deal is better to take, students distinguish between an equation or inequality being true for *all* values of x, and for *some* (or only one) values of x.

While exploring equivalent expressions, students typically identify two expressions as equivalent if, for any value substituted for the variable, the values of the two expressions are equal (Standard 6.EE.4). This implies that a single counterexample (a value for the variable that yields unequal values of the expressions) demonstrates the expressions' non-equivalence (Carpenter & Levi, 2000). Students express numerous common misconceptions and errors when simplifying expressions, but by substituting values for the variables before and after simplifying, and looking for counterexamples, they are able to strengthen their understanding and improve their accuracy. This experience is critical for generating equivalent expressions, bringing many of students' skills with operations, in a variety of contexts, together with their work in algebra, to explore and demonstrate the power of algebra as a generalized solution method (Standards 7.EE.1 and 7.EE.2), thereby advancing their algebraic reasoning and problem-solving abilities.

Implications and Discussion

The EELT builds on previous research on how students' thinking progresses within patterns and sequences, equations, and expressions, and is foundational for the development of their algebraic thinking. Aiming to show the progression of "big ideas" within the CCSS-M, the LT maintains the grade level of each mathematical idea as it is presented in the CCSS-M (for instance, not introducing the variable as a quantity that varies until 6th grade, despite research showing that this can be done earlier) while providing bridging standards to fill gaps in the CCSS-M for instructional continuity and a coherent structure (e.g. introducing repeating patterns and geometric sequences). Blanton and Knuth (2009-2013) have suggested five curricular learning progressions for algebra education: a) equality, expressions, equations and inequalities, b) functional thinking, c) variables, d) generalized arithmetic, and e) proportional reasoning. Ongoing research such as theirs will provide additional fundamental insight for revising the EELT and other learning trajectories of TurnOnCCMath.net. Interpretation and implementation of the CCSS-M using learning trajectories can spur strategic critical areas of further research into student learning of mathematics, and contribute to constructive critique of the CCSS-M, both of which will lay the groundwork for systematic revision of the CCSS-M in the coming years.

Acknowledgements

Supported by National Science Foundation, Qualcomm, and Oak Foundation. We gratefully

acknowledge K. Lee, T. Avineri, K. H. Nguyen, A. K. Corley, and J. Nickell (members of the EELT writing and reviewing team) and Pedro Larios, APS, Inc., (website engineering).

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THE CONSTRUCTION OF TRIGONOMETRIC AND INVERSE TRIGONOMETRIC FUNCTIONS

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This study uses APOS Theory to study students' construction of the sine and cosine functions and their inverses. The conjecture of the constructions students would need to do in order to attain a process conception of these functions is tested by using semi-structured interviews with a group of 11 undergraduate students. Analysis of the interviews suggests a small number of basic mental constructions that students would need to do, that account for observed difficulties, and that hence would need to be stressed during instruction.

Keywords: High School Education, Instructional Activities and Practices, Algebra and Algebraic Thinking

The trigonometric functions and their inverses are important enough in mathematics and its applications to warrant more attention than they have received in the mathematics education research literature. In this paper we use APOS Theory to discuss how students may construct the sine and cosine and their inverse functions. Our research questions are:

1. What basic mental constructions can be conjectured that students need to do in order to have a process conception of the sine and cosine functions and their inverses?
2. Which of the conjectured basic mental constructions students can be observed in students' work when they use these functions in problem solving activities?
3. Which of the conjectured basic mental constructions seem to be lacking when students use these functions in problem solving activities?

Background

While there is an increasing amount of publications in the mathematics education research literature that deal with trigonometry, due to space restrictions we will not include a review of that literature but rather only mention and refer to three articles.

Bagni (1997) observed in an example that a sizable number of students have trouble finding all solutions of $\sin x = 1/3$. Also, for some students the value of the trigonometric functions have no clear sense at real numbers other than the integer multiples of $\pi/6$, $\pi/4$, and $\pi/3$. He relates these observations to the effect of the didactical contract on students. Weber (2005) argued that an instructional path where trigonometric operations are understood as geometric processes to be both physically performed and reflected upon may result in deeper conceptual understanding of these operations as functions. In her doctoral thesis, Brown (2005) examined the transition from right triangle trigonometry to trigonometry in the coordinate plane, and from there to the graphical representation of sine and cosine. She found that many students had a fragmented or incomplete understanding of sine and cosine, with many not connecting the right triangle, unit circle, and graphical representations. While in her exploration and discussion she seems to favor starting with a ratio definition of trigonometric function and moving from there to the unit circle, we adopt a different approach, as the emphasis of our study falls heavily on the unit circle representation. This will enable us, among other things, to obtain information on students' notion of inverse trigonometric functions, a topic omitted by Brown. While we agree

with many of her observations and recommendations, our approach leads us to consider different issues and constructions in students' learning of trigonometry.

Theoretical Perspective

Action-Process-Object-Schema (APOS) theory (see for example, Dubinsky and McDonald, 2001), is a well known theory and hence we only give a brief description. An action is a transformation of a mathematical object by individuals, following explicit step by step instructions or using a memorized fact or procedure. An action is perceived as external by the individual. As an action is repeated and the individual reflects on the action it can be *interiorized* into a process. As a process, the transformation may be done mentally; it may be reversed and may be coordinated with other processes. The outcome of the process may be imagined without actually having to perform it. In contrast to an action, a process is perceived as internal. When a person reflects on actions applied to a particular process, the person may become aware of the process as a totality. In this case it is said that the process is *encapsulated* into an object. A mathematical schema is a coherent collection of actions, processes and objects, and other previously constructed schemas, which are synthesized to form mathematical structures utilized in problem situations (Baker, Cooley, & Trigueros, 2000). It is important to note that although it might be thought that in APOS theory there is a linear progression from action to process to object and then to having different actions, processes, and objects organized in schemas, this often appears more like a dialectical progression where there can be partial developments, passages and returns from one to another conception (Czarnocha, Dubinsky, Prabhu, & Vidakovic, 1999). What the theory states is that the way a student works with diverse mathematical tasks related to the concept is different depending on his or her conception.

The application of APOS theory to describe particular constructions by students requires researchers to develop a genetic decomposition – this is a conjecture of specific mental constructions students may make in understanding mathematical concepts and their relationships. Some clarifications are pertinent. The initial genetic decomposition is based on the researchers' knowledge of mathematics, teaching experience, history of the mathematical development of the concept, and any available data. A genetic decomposition for a concept is not unique, it is a model of how a concept may be constructed; different researchers can develop diverse genetic decompositions for a particular concept, but, once one is proposed, it needs to be supported by research data from students. Frequently, data may reveal mental constructions which students make that are different from the ones conjectured, as well as specific conjectured constructions that cause special difficulty for students. This may lead to the development of learning activities for students to help them make the necessary constructions and/or to a revision of the genetic decomposition to better reflect the constructions that students actually make. Eventually the cycle of data gathering, classroom activity implementation, and genetic decomposition revision, stabilizes in a genetic decomposition that serves both to describe the constructions that students actually make as well as to guide instruction.

While we will not include our complete initial genetic decomposition in this paper, we will outline some of the constructions, particularly some specific processes that form part of it.

The Process $t \uparrow P(t)$

One of the processes needed in the construction of sine, cosine and their inverse functions is the process of taking a given real number *t* (preferably not an integer multiple of $\pi/4$ or $\pi/6$, at the beginning) and locating the terminal point $P(t)$ of the unit circle that corresponds to *t*. Building this process requires repeating and reflecting upon actions such as those suggested by

Weber (2005), a version of which appears in Figure 1. A student able to imagine doing this without having to explicitly do it can be said to have the process denoted $t \to P(t)$.

Symmetries of the Circle as a Process

The student will also need to interiorize into a process the action of given a point $P(t)$, represented geometrically or as an ordered pair, finding the other three corresponding points $P(-t)$, $P(t+\pi)$, and $P(\pi - t)$ on the unit circle, in the respective representation, geometrically or as ordered pairs. Further actions that students need to interiorize include: given *t*, locate any point of the form $P(t+n2\pi)$, $P(-t+2n\pi)$, $P(t+\pi+2n\pi)$, $P(\pi-t+2n\pi)$ as well as $P(t+n\pi/2)$, for integer *n*, and find their coordinates, given those of $P(t)$. This process plays an important role in the reversal process used to define inverse trigonometric functions. We will refer to it as the "symmetry of the circle" process. Explicit use of these symmetries while solving trigonometric equations may be regarded as evidence that a student has interiorized these actions into a process.

Process of Projection

Now the sine and cosine functions may be defined by following the $t \to P(t)$ process with a process of projection onto the *y* axis (to define the sine function), or onto the *x* axis (to define the cosine function). We will refer to the concatenation of these two processes as the definitions of the basic trigonometric functions. A process conception requires being able to imagine this concatenation. What we are stressing is that the definitions of the sine and cosine functions require two separate steps: given a real number, assigning to it the coordinates of a point on the unit circle, and then following that process with a projection onto the appropriate axis.

While it is true that formally, anyone defining the trigonometric functions starting with a unit circle approach would at least implicitly use the above decomposition, we are underscoring here that there are three main ideas that would merit being stressed during instruction: the process of given a real number, assigning to it a point on the unit circle; using the symmetries of the circle; and that the definition of the sine and cosine functions follow a two step process (locating a point on the circle followed by a projection). In keeping with Bagni (1997) and Weber (2005), the above constructions need be done for real numbers, not only for integer multiples of $\pi/6$ and $\pi/4$. After having interiorized the process of definition of the sine and cosine functions, the process may be coordinated with the use of technology, allowing students to, among other things, compute values of the sine and cosine functions with the precision afforded by the technological artifact used.

Reversal of the Process of Projection

Now, the process of definition of the basic trigonometric functions will need to be reversed in order to define the inverse trigonometric functions. Note a process conception of the definition is needed in order to reverse it. Hence one needs a process of reversal of the appropriate projection. The process starts with a number which is either an *x* or *y* coordinate of a point on the unit circle, produces a geometric representation of the number as a point on the *x* or *y* axis, and then produces the geometric representation on the unit circle of the corresponding point or points having that *x* or *y* coordinate. The symbolic representation of those points as an ordered pair of real numbers could be obtained coordinating the points obtained from the projection reversal process with the equation of the unit circle, or, equivalently, with the Theorem of Pythagoras.

After reversing the projection, to obtain one or two points (as geometric entities) on the unit circle, the process $t \to P(t)$ needs to be reversed in order to obtain a number *t* determining one of the points. This reversal needs to be an interiorized mental process, and may be constructed

with activities analogous to the ones in Weber (2005), or equivalently, using the unit circle and pipe cleaner as suggested by Figure 1. This process reversal may be done with geometric arguments in the case of special values. Observe that memorizing tables of special values will only result in an action conception, not in the needed mental process. The action of obtaining one real number t corresponding to a given point $P(t)$ of the unit circle, will now need to be coordinated with the symmetry process to produce all real numbers t determining the corresponding points.

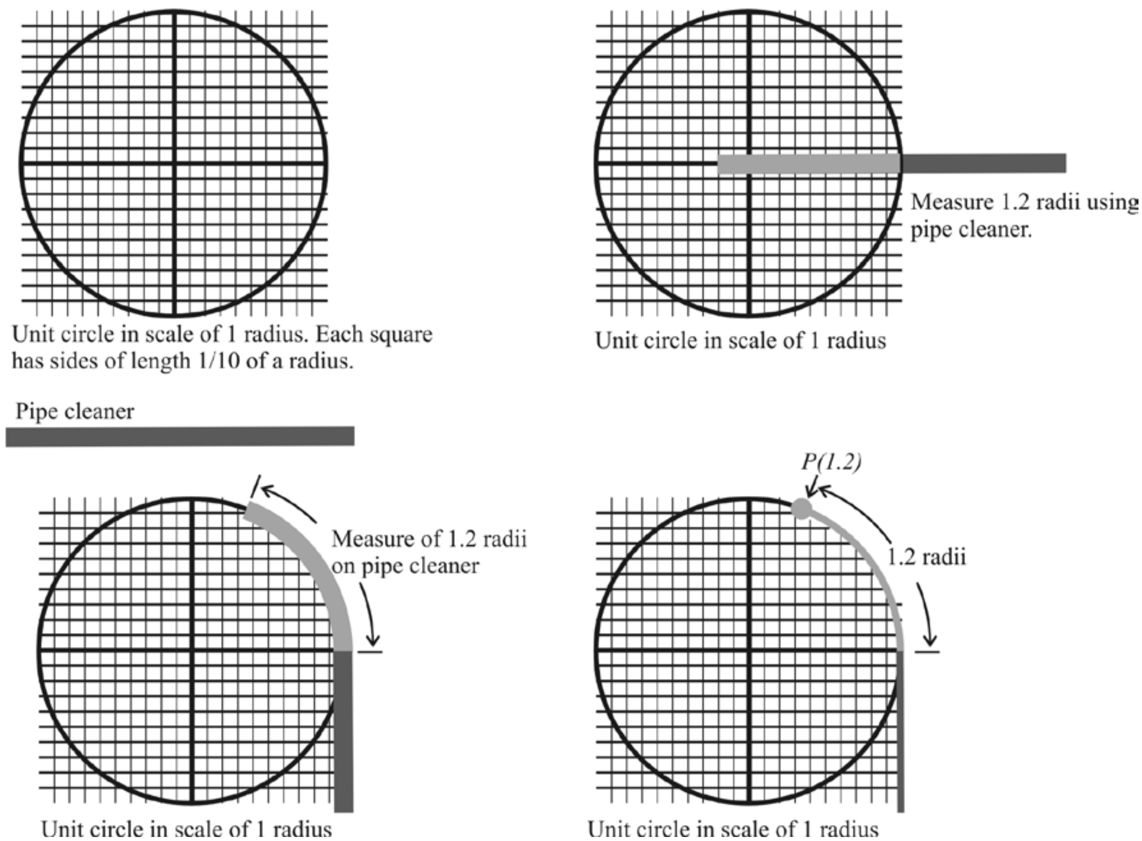


Figure 1: Using a Pipe Cleaner to Locate Point $P(1.2)$ on the Unit Circle.

The chain of actions that starts with a number (represented as an x or y coordinate), goes on to identify the point or points on the unit circle having that number as an x, or respectively y coordinate, and then identifies all the real numbers corresponding to the point or points on the unit circle, may be interiorized into a process that we call reversal of the definition. This process starts with a coordinate and produces the collection of all real numbers corresponding to the points on the unit circle (one, two, or none) having that coordinate. By its nature, this process does not define a function.

The Process of Range

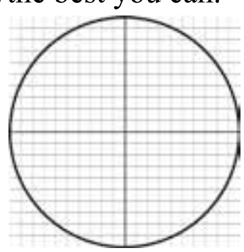
The action of coordinating the above definition reversal process with the function schema is needed in order to have the inverse trigonometric functions. In particular, the coordination of the process of reversing the definition with, what we will call, the process of Range (of the corresponding inverse trigonometric function), is interiorized into a process of inverse trigonometric function. As will be seen, an action conception of Range (having memorized the

range of the inverse sine and cosine) will not enable students to build a process conception of inverse trigonometric functions. We conjecture that building a process conception of Range requires explicit attention and will not follow from the mere statement of a formal definition. Students will need to interiorize actions that explore ways of restricting the domain of the sine and cosine functions so that the resulting function is one to one and the restricted domain is connected and as large as possible. These actions should include both, unit circle representation and the graphs of these functions. The process resulting from the interiorization of these actions would allow students to argue for the need and also the convenience of restricting the domains of sine and cosine as they normally are. Otherwise, students will be constrained to having an action conception of Range as a memorized fact. The coordination of the process of exploration of possible restrictions of the domains of sine and cosine with the concept of range of an inverse function is what we call the Range process (of an inverse trigonometric function). Once students have a process of inverse trigonometric functions (coordination of definition reversal and Range), it may be reinforced coordinating it with the use of technology through actions that explore and compare the mental constructions that have been made.

Method

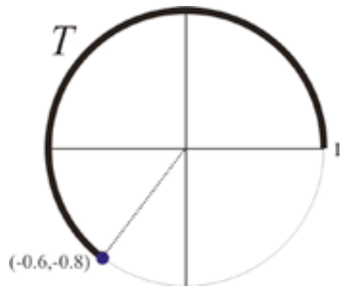
The study took place at a mid-size public university in Puerto Rico (approximately 14,000 students) with a large science and engineering component (at least 60% of the students). Semi-structured interviews were conducted with 11 students in order to test a preliminary genetic decomposition. Of the students interviewed, 3 were above average, 5 average, and 3 below average according to their professor. Students were chosen this way in order to be able to observe a wide range of different behaviors when engaged in mathematical tasks. All students were enrolled in the same section of a precalculus course, where they used a traditional textbook (Stewart, Redlin, and Watson, 2011) and received traditional instruction. The interviews were recorded and transcribed, and records of students' written responses were kept. The interviews were analyzed independently by the researchers, by comparing student constructions to those predicted by a preliminary genetic decomposition. They were then discussed, and differences were negotiated. Student performance on the interview questionnaire was graded and scores were kept. Seven of the 12 interview questions were pertinent to the present article:

1. Use the unit circle in the figure and the pipe cleaner [a string, as in Figure 1] to approximate the value of $\sin(1.2)$ the best you can.



2. Use the unit circle in the figure and the pipe cleaner to approximate the value of $\sin^{-1}(0.8)$ as best you can.
3. Given that $\sin\left(\frac{10\pi}{7}\right) = -0.974927912$. Find $\sin^{-1}(-0.974927912)$.
4. Draw on a unit circle *all* terminal points $P(t)$ corresponding to numbers *t* of the form $4t = \frac{\pi}{3} + 2n\pi$, *n* an integer, and that are between 0 and 2π.

5. Find all solutions of $\cos(t) = -\frac{1}{2}$ in $[0, 2\pi)$.
6. Find all solutions of $\sin(t) = -\frac{3}{5}$.
7. We start on point $(1, 0)$ on the unit circle and travel counterclockwise on the circle a distance T, as shown in the figure below, ending at point $(-0.6, -0.8)$. What is the value of T?



Results

The Process *t* ↑*P(t)*

Recall this is the process of assigning a point on the unit circle to any given real number. Of the interviewed students, 8 of 11 had difficulty assigning a point to a number.

When originally asked to estimate the value of sin(1.2) using the pipe cleaner and the unit circle, and after having shown Ricky how to use the pipe cleaner to measure arc-length on the unit circle (as in Figure 1):

Interviewer: Is there some way you can use the pipe cleaner to estimate $\sin(1.2)$?

Ricky: I don't think so, I don't know… It is finding 1 and 1.2 [he is referring to finding $P(1)$ and $P(1.2)$] and I'm finding it difficult … well, sine is y, … drawing a line towards y [referring to the y axis]

Later on, after a long intervention where the interviewer helped Ricky locate $P(1.2)$:

Ricky: So, sine of 1.2, what comes to mind is … when one says that sine is y … is drawing a line to the y axis and that would give me a value.

Note that Ricky has difficulty with the $t \to P(t)$ process (seems to have an action conception) but seems to be aware of the need to project.

Symmetries of the circle as a process

Six of the 11 interviewed students had difficulty using the symmetries of the circle. Maria, the student with the second best performance in the interview questionnaire (71%) is an example of a student not having interiorized the use of circle symmetries as a process. In Problem 7, after using a calculator to compute $\cos^{-1}(-0.6)$ she needs to find the corresponding point on the third quadrant:

Maria: Use this angle … to do something, the answer then is… add pi.

The Process of Projection

We saw the case of Ricky, who had not interiorized the $t \to P(t)$ process but seemed to recognize the need to follow this with a projection onto the y axis in the definition of the sine function. In the case of Diana, it is the other way around. Diana was the only student not showing any difficulty with either the $t \to P(t)$ process or the process of symmetries of the circle, and by far the student having the best all around performance on the interview

questionnaire (92%). Three instances where she failed to immediately obtain a correct response had to do with her failure to do the corresponding projection:

Diana: [In problem 1, after correctly using the pipe cleaner to locate $P(1.2)$ without any help] In this case, sine of 1.2 is equivalent to, the *y* value of a terminal point.

Interviewer: How would you estimate $\sin(1.2)$? You may use the pipe cleaner as you see fit.

Diana: I don't know if to apply $a^2 + b^2 = c^2$ … I don't know, nothing comes to mind …

Latter when doing Problem 7, Diana again fails to project onto an appropriate axis (in order to use either $\cos^{-1}(-0.6)$ or $\sin^{-1}(-0.8)$):

Diana: … What is the value of *T*? … one would need to find … the measure of the angle between pi and the terminal point and add to it … ay! It seems to be so easy!

Interviewer: What are you thinking?

Diana: I'm thinking in … it doesn't occur to me …. [She was eventually able to use right triangle trigonometry and the inverse tangent function to successfully complete the problem]

Reversal of the Process of Definition and Inverse Trigonometric Functions

All 11 students interviewed failed at one point or another, to apply an inverse trigonometric function. Indeed, of the 12 questions in the interview questionnaire, the four problems in which students achieved the lowest overall score were problems directly dealing with the inverse sine or inverse cosine functions (36% in Problem 2, 13.6% in Problem 3, 31.8% in Problem 6, and 30% in Problem 7). In the interviews we observed students who showed difficulty using an inverse trigonometric function due to different reasons: not having interiorized the $t \to P(t)$ process, or the projection process, or the circle symmetries, or the Range process.

For example, Gina seems to not have interiorized the projection into a process (in problem 2), hence she shows difficulties with its reversion:

Gina: Well I have that sine represents the *y*... [She goes on to measure 0.8 correctly on the pipe cleaner but locates $P(0.8)$ rather than a point with *y* coordinate equal to 0.8].

Gina is then directed to compute $\sin^{-1}(0.8)$ on the calculator but she still fails to relate this to the 0.8 on the *y* axis:

Gina: … the angle of 53 degrees would give the point $P(t)$.

Interviewer: I'd like to see that angle, where is it? [Referring to the unit circle in the figure used for problems 1 and 2.]

Gina: Well, it would be around here, I would draw an approximate radius, since I have nothing to measure it with [Even though a horizontal line crossing the *y* axis at 0.8 is part of the figure, she does not use it or refer to it].

Another common observation is that students may be able to do an action or process of reversing the definition when working in the context of special values, like Problem 5, and yet fail to use the process of reversing a definition in the more general context, as in Problem 6. Indeed, 6 students had problem 5, either partially or completely, correct and problem 6 completely incorrect. This suggests that the actions being interiorized by students should be carefully chosen to include the general context as well as the context of special values.

It is important and interesting to notice that none of the students used a drawing in trying to make sense of problems 5 and 6 (solving $\cos(t) = -1/2$ and $\sin(t) = -3/5$). This stresses the need and importance of the actions that give explicit attention to the geometric representation of the unit circle when reversing the process of definition of sine and cosine.

Observed results sustain the need to help students develop the conjectured constructions.

Summary

In order to construct a sine or cosine function, students need a process ($t \to P(t)$), where they start with a real number and assign to it a point on the unit circle, and they need to follow that process with a projection onto the appropriate axis. Hence, we argue that instruction would need to emphasize that the definitions of these functions follow a two-step process and it would also need to emphasize that the definition applies to any real number, not only special values. It was found that the two steps in this construction, the process ($t \to P(t)$) and the projection, are two independent processes that account for many student mistakes in problem solving situations.

We also observe that the construction of an inverse trigonometric function requires essentially, reversing a definition. Reversal of the definition process, consists of reversing the projection by starting with an x or y coordinate of a point on the unit circle and ending with one or two points on the unit circle, followed by reversing the process $t \to P(t)$ and coordinating with the symmetries of the unit circle to obtain all real numbers determining a point on the unit circle having the given x or y coordinate. In particular, this suggests that instruction would need to emphasize activities that lead students to interiorize actions exploring the symmetries of the circle. It was also conjectured that an inverse trigonometric function results from the coordination of the definition reversal process with the appropriate Range process. It was found that student' difficulties, particularly in problems dealing with inverse trigonometric functions resulted from lacking some of the aforementioned processes. It could also be deduced that instruction would need to aim at having students build a process of Range for the inverse trigonometric functions, relating unit circle and graphical arguments of why the sine and cosine functions are restricted they way they are when defining inverse trigonometric functions. It was observed that not having this process, students are left with a memorized fact, hence an action conception of Range, making it difficult to recall and use the range of an inverse trigonometric function in problem situations. It was clear that the notion of an inverse trigonometric function is a complex construction that accounts for many student mistakes in problem solving situations.

Much work still needs to be done: developing activities to help students do the conjectured constructions, class testing, performing a second round of interviews including students' use of graphical and numerical methods in problem solving, and refining the genetic decomposition accordingly. Nevertheless, the conjectured process constructions seem useful in describing student behavior in problem solving, in organizing research, and in helping guide instruction.

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DEVELOPING PROCESSES FOR LEARNING HIGH SCHOOL MATHEMATICS

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High school mathematics students often complete homework and study for unit tests without support to consider how these actions could contribute to their mathematical learning. Learning to learn mathematics invites students to bring into view how they learn mathematics. The constructivist grounded theory study reported in this presentation describes how grade 12 students inquired into the systemically defined and externally imposed learning strategies that they perceived as static and superficial. The Framework for Developing Processes for Learning Mathematics *illuminates the complexity of becoming aware, incorporating suggestions, verbalizing possibilities, and (re)forming intentions to shape personal processes for learning. Viewed as dynamic and authentic, the processes for learning mathematics students developed shaped them as learners who made sense of mathematical ideas.*

Keywords: High School Education, Learning Theory, Metacognition

Mathematics education reforms have emphasized students' personal development of mathematical ideas (National Council of Teachers of Mathematics, 2000; Western and Northern Canadian Protocol, 2008). Absent from these reforms and from experiences in many high school mathematics classrooms is explicit discourse about the processes of learning – both identifying the strategies students use to learn (e.g., homework, taking notes, test preparation, study groups) and then how to individually adapt those strategies. While study how-to books abound (e.g., Coles, White, & Brown, 2003; Ooten & Moore, 2010; Peltz, 2007), the publications are didactic in their approach and divorced from the particular contexts of learning mathematics, rather than working from individual students' current intentions and processes for learning mathematics.

Within mathematics education, research using the psychological construct of metacognition (Brown, 1978; Flavell, 1976) has supported improvement in students' mathematical thinking (e.g., Hamilton, Lesh, Lester, & Yoon, 207; Schoenfeld, 1987). This research report extends the work of these cognitively-based studies to address the complexity of learning which view mathematics students as persons in the process of becoming. Research in mathematics education has identified successful students as those who understood their personal learning processes (Dahl, 2004; Smith, 1999), and this study extends this recognition to explore students' *development toward* becoming capable mathematical learners.

Purpose of the Study

The empirical research being presented in this research report is part of a larger study that addressed the question: *What is the nature of students' learning when they engage in conversations to shape their personal processes of learning high school mathematics?* The purpose of the study was to understand how high school students learn to learn mathematics. Results of the larger study demonstrate that students were able to see themselves as capable learners of mathematics because of a learning-based orientation in the world that simultaneously grew as they developed approaches to independently learning academic mathematics. This research report will focus on one facet of my theorizing about students' experiences of learning to learn mathematics in order to closely examine how students shifted from the use of

systemically-imposed learning strategies to personally-developed processes of learning. Novak and Gowin (1984) help tease out the distinction between “learning and knowing. They are not the same. Learning is personal and idiosyncratic; knowing is public and shared.” (p. 5) In taking up this notion of learning, I am proposing an alternate perspective on students’ engagement in high school mathematics classes, namely the possibility for inviting students to shape processes for learning mathematics in ways that are personally dynamic and authentic.

Mode of Inquiry

Constructivist grounded theory [CGT] (Bryant & Charmaz, 2007; Charmaz, 2006, 2009) returns to the symbolic interactionist root of grounded theory while looking through a constructivist lens as an interpretive process for inquiring into dynamic phenomena. Within this postmodern orientation, theory is constructed by a researcher on a provisional basis and contingent to the context. There is an “emphasis on processes, making the study of action central” (Charmaz, 2006, p. 9), recognizing that shifts in people’s actions and experiences signify growth and changes within the people and their interactions. The researcher, seen as a subjective knower, is immersed in the research setting while co-constructing qualitative data with participants. As data is analyzed abductively, the researcher moves from rich empirical data through levels of abstraction toward developing a mid-range interpretive theory. Processes like coding, memoing, categorizing, theoretical sampling, saturation, and sorting are offered as “systematic, yet flexible guidelines for collecting and analyzing qualitative data … rather than formulaic rules” (Charmaz, 2006, p. 2). The reflexivity of the researcher results in explicating the theorizing as both process and product enabling other researchers to apply and extend the work.

The focus of this research, students’ experiences of learning to learn mathematics, is supported by CGT’s framing to notice and interpret the growth of individuals. Grounding interpretation in students’ experiences, rather than applying extant theoretical frameworks, supports the uniqueness of the study in attending to the development of mathematical learners. Theoretically, constructivism is the predominant epistemological orientation to the teaching and learning of mathematics (Bishop, 1985; Davis, Maher & Noddings, 1990), often used in conjunction with symbolic interactionism for mathematics education research (Cobb & Bauersfeld, 1995; Sierpinska, 1998; Voigt, 1994). The use of CGT responds to the growing importance in theorizing to make progress within the field of mathematics education (Hiebert, 1998; Proulx, 2010).

Research Context and Participants

The study was situated in an academically-focused suburban school in a city in Western Canada. Thirteen grade 12 students who were taking a pure mathematics course volunteered to participate in the study. Their pure mathematics courses were offered in a didactic format where the teacher lectured, students copied out worked solutions to examples, and then worked through similar questions independently as homework. The students were enrolled concurrently in a course, *Mathematics Learning Skills*, that provided support for their mathematical learning. In the class, students worked on homework and requested help from the teacher. Within the *Learning Skills* course, I assisted the teacher in coaching students to improve their approaches to learning mathematics while simultaneously collecting data. The teacher also participated in the study to provide contextual information and offer her perspective on emerging analysis.

Data Collection

Data collection occurred over four months. After observing each class, I wrote detailed field notes of students’ (inter)actions in the class and descriptions of daily informal conversations with the teacher. Students took part in bi-weekly interactive journal writings (Mason & McFeetors,

2002). They responded to prompts about the progress of their learning strategies and I replied in order to interact with their ideas, modeling thinking about learning and fostering a relationship with each student. Students were placed into one of three small groups with a focus on developing a learning strategy as a group (transitioning from notes to homework, developing big ideas from completed homework, and studying for unit tests by creating summary sheets). Each small group met for three to five sessions of approximately 30 minutes each. The students also participated individually in two informal interviews as a retrospective look at their progress in shaping their learning strategies. Each interview was approximately 30 minutes and occurred halfway through the study and at the end. While the interactions were intended as multiple sources of data, they also afforded students the opportunities to develop processes of learning to support their mathematical learning and to notice improvements in learning. Providing these opportunities was framed by Dewey's (1938/1997) notion of experience which is characterized by continuity and interaction and where activity is transformed into experience through the reflective act.

Data Analysis

Using line-by-line coding and the constant comparative method (Glaser & Strauss, 1967), data analysis involved the development of codes for students use of learning strategies across all forms of data. The codes, such as "do questions" and "see the process", remained close to the students' words and were refined through several passes through the data. The codes were stored in a spreadsheet which enabled parsing codes to create initial categories. Names of categories, such as "becoming aware" and "incorporating suggestions", are descriptive of the students' actions and abstracted from the data to highlight the process-based nature of learning to learn mathematics. I constructed the *Framework for Developing Processes for Learning Mathematics* through the interpretive act of theorizing by exploring the relationships of the categories. The categories and framework will be described in the results section below.

Perspective

Rather than using an interpretive framework, I adopted Blumer's (1954) notion of sensitizing concepts to "merely suggest direction along which to look … providing clues and suggestions" (pp. 7-8). Drawing on Blumer's work as a symbolic interactionist responds to the misconception that grounded theory studies begin atheoretically. Rather, the sensitivities of the researcher – what the researcher is drawn to attend to because of her/his experiences of conducting research, scholarship in the field, and interests – are acknowledged, explored, and employed as a starting place in the collection and analysis of data. The purpose of a CGT research project is not to refine sensitizing concepts but to interpret the participants' experiences.

My sensitizing concepts developed out of two of my related research projects. In one project, high school students' success in non-academic mathematics was interpreted as an emerging of voice, where students came to say things about themselves as mathematical thinkers and learners with the intention of (re)forming their identity (McFeetors, 2003, 2006). In another project, students' trajectories of learning, through choices of high school mathematics and science course, indicated that students selected among mathematics courses in relation to their identity as learners, and their ways learning within courses was connected to who or what they perceived as sources of mathematical knowledge (Mason & McFeetors, 2007). The four sensitizing concepts are: intentions, voice, identity, and relationships with sources of knowledge. After each explanation, I provide examples of literature for those who desire a more thorough discussion.

Intentions are internal constructs which give meaning to actions. These thoughts and desires arise from attention to previous experiences and to the consequences of actions, often through

reflection. When students are intentional, they are acting with the intentions they have formed and hold, to move toward a particular aim. This aim, as an end-in-view, is fluid and the method of moving toward it contains ambiguity. Intentions point to what students want to do or achieve and a notion of how they might go about doing. So, intentions both mark an aim and a process. (See Bereiter & Scardamalia, 1989; Searle, 1983.) *Voice* points toward having space and confidence to say things and to do so, a reflective stance to make sense of experience through conversation, and being deeply implicated in actively shaping oneself. Voice is dynamic concept, one in which a student's voice is continually being refined through experience and through the voicing of the experience and growth of self. (See Baxter Magolda, 1992; Confrey, 1998.)

Identity is an understanding or sense of self. It is a dynamic processes, where the (re)forming of identity is continually undertaken through experiences and relating with others. While occasionally marked by large shifts, (re)forming identity is more often seen as shaping a way of being in the world and understanding that way of being. Shaping an identity is the ongoing negotiation of a student's relationship with mathematics, learning, schooling, others – identity is malleable and complex. (See Britzman, 1994; Sfard & Prusak, 2005.) *Relationships with sources of knowledge* – such as teachers, peers, and textbooks – point to students' beliefs about knowing and coming to know (epistemological stances) which are inextricably connected to the experiences of learning mathematics. The relationships could include dependence, independence, and interdependence and are often illustrated through examples of where authority in mathematical knowledge lay and through the choice of approaches to learning. (See Belenky, Clinchy, Goldberger, & Tarule, 1997; Chickering & Reisser, 1993.)

Results

In students' first journal, they listed the prescribed ways to learn mathematics, such as study, review, copy notes, work with others, and do homework. Learning strategies were ways students were told by their teachers to learning mathematics and were labels that did not probe the steps students would need to take to enact the learning strategies. The uniformity in naming learning strategies arose from the systemically defined characteristic of learning strategies, that school as a normative structure has systematized these procedures without consideration of these particular students. Teachers, as actors in the system, compelled students to use the learning strategies through their authoritative stance, resulting in externally imposed ways to learn content. Although the students desired to succeed in mathematics class and acknowledged that these learning strategies should support that success, they struggled to implement them to any effect. Vanessa illustrates this in her first journal when she writes "I tried reading my math notes when I get home. But I find that when I try to do the homework and understand the notes, I've already forgotten how and what to do by the end of the day."

The unquestioning implementation of learning strategies demonstrated the students' lack of intentions and personal investment in procedures espoused by the school system. Students perceived the learning strategies as being static and superficial. Teacher demands, especially for notes and homework, led students to use the strategy as prescribed even if they did not see it as supporting their learning – the strategy was static both in its implementation and lack of contribution to learning mathematics for understanding. The learning strategies contributed in a superficial form of learning (memorization of mathematical procedures), glossed over the challenges of learning, and viewed one way of learning as equally effective for all students. However, not only were the learning strategies static and superficial, but these characteristics positioned the students as static persons whose learning was also superficial. Through the use of simplistic learning strategies, students did not have opportunities to be changed by deep

engagement in a learning process and mathematics. They continually saw themselves as ineffective learners and prioritized the surface goal of high marks. Against this backdrop, the students and I began to explore the ways they were struggling to learn academic mathematics.

This exploration is better characterized as an inquiry into learning strategies. My interpretation of the students' engagement resulted in the *Framework for Developing Processes for Learning Mathematics*. The four facets of the framework provide an emerging picture of how students engaged in inquiring into learning strategies, moving toward the development of personal processes of learning mathematics. These forms of engagement in shaping ways of learning mathematics represent the categories constructed from and grounded in the students' data. Being enacted simultaneously and in an interrelated fashion, *becoming aware*, *incorporating suggestions*, *verbalizing possibilities*, and *(re)forming intentions* were the ways in which students shaped how they were learning mathematics through their *Learning Skills* class and the procedures for data collection for the study.

The students were *becoming aware* of the learning strategies, the limitations of those strategies, and the personal nature of their mathematical learning through their inquiry. Kylee, who had previously created a system of cue cards for learning terms in biology, noticed their limitation for mathematics through her second journal, "I realize how much of my time I waste making Q-cards (sic) before my test when I could instead be studying them," and found an opening to refine an existing process of learning. Shane's insight in our first interview, "I would focus on learning how these numbers work and now I guess how the numbers work is a concept in itself, but I never thought of it that way" signals a new awareness that his mathematical learning could fit his identity as a conceptual learner. The growth in students' awareness was situated in a space where their voices could be heard and valued, even in its tentative state.

As the students shaped ways of learning in the small group sessions, they were *incorporating suggestions* from their peers and from me. Kylee described my interactions with her as "you weren't telling me to do something or getting mad because I did that on a math test. You were just encouraging", as I offered alternatives and worked from students' current capabilities in learning mathematics. Drawing on both peers' and my support, Elise exemplifies the notion of incorporating suggestions by actively modifying suggestions (as opposed to compliantly implementing teachers' learning strategies). In her recording "big ideas" at the end of a homework assignment, Elise asserted that "what we wrote down here broke down what it actually meant … So, I understand what to do when I have a question" showing that she had adapted my suggestion of identifying one or two main concepts from a lesson which I had suggested. Elise also modified Danielle's approach of using sticky notes on summary sheets as "I kind of like the way she does it, but I think it works better for me the way I do it." While the students listened to the ideas of others, they recognized that they themselves were expert sources of knowledge about how they learned.

When students deliberated on how to modify ways of learning and put their ideas into words, they were *verbalizing possibilities* for ways to learn mathematics. Danielle demonstrated that sometimes these were internal conversations, where "I was just sitting on the bus, and I was thinking … how would I be able to separate my ideas and stuff." In my field notes I also recorded that "as Danielle handed in her fifth journal, she explained aloud how she would use different colors of sticky notes to represent different kinds of content such as definitions, formulas, and examples." After noticing that Danielle wrote "With [the] method I developed, my ideas are organized and laid out in a way that really helps me understand" in the journal itself, her oral utterance is an example of her first attempt to articulate for herself the details of a new

possibility for her process of learning at the end of a unit. As we worked together in a small group on this studying process, Danielle not only refined her explanations for the use of sticky notes as an organizational technique, but used them to demonstrate how she connected mathematical ideas across a unit of content. The verbalizing of possibilities occurred both orally and in writing through my conversations with students and demonstrated that students, like Danielle, could be sources of knowledge for how they learn and perceived their voices as being valued in the learning context.

Students were *(re)forming intentions* for particular ways of learning mathematics as they moved away from the unquestioning use of learning strategies. Grace realized a memorization-based approach to academic mathematics was not sufficient, and explained in our first interview how she refined two processes of learning with the intention of developing mathematical understanding. First, the students' talk in her study group shifted from "how you got the answers" to "we discuss why you're doing it" when completing homework questions. While this shift supported Grace's growing belief about mathematics as a process, the intention she had for her collaboration with peers evolved from comparing answers to mathematical discussion. Second, in authoring various forms of notes, Grace explained she added on side notes – that were "in my words … [s]o it's easier for me to understand" – to the teacher's notes she copied down. She formed the intention of putting mathematical ideas in her own words which impacted all her processes of learning mathematics. The students' intentions developed out of shaping the learning process and increased the complexity of the process of learning. As students became aware that learning processes could be adapted to fit who they were as learners, the intention was not only how the process would support their mathematical learning, but they also became intentional about learning to learn mathematics.

As students inquired into the learning strategies they were told to use to learn mathematics, they were developing processes for learning. Examples of these processes of learning (juxtaposed with strategies in parentheses) included creating summary sheets (study), making and using cue cards (review), authoring various forms of notes (copy notes), collaborating with peers (work with others), and learning from homework (do homework). Processes for learning mathematics are ways in which students make sense of mathematical content and are developed by students in response the particularities of each one. In contrast to learning strategies, processes for learning were perceived by the students as being dynamic and authentic. A process of learning was dynamic because the students continued to shape it and noticed their peers doing the same. Additionally, process for learning supported students' construction of mathematical understanding, were the discipline of mathematics was seen as malleable and the content was of their own making. There was authenticity in the processes for learning, not only because the students were aware of the effectiveness of the processes but because they authored the processes and could describe the development of the processes. The students experienced growth as mathematical learners, a dynamic process, and came to have a nascent authorial stance as they saw themselves as personally developing their processes for learning.

Discussion

The *Framework for Developing Processes for Learning Mathematics* represents the complexity of students' learning when they engage in an approach that opens up space for them to develop processes of learning. This extends beyond the limits of how-to literature on studying to contribute to a way of being with students that supports their learning to learn mathematics. By explicating four ways of engaging in learning to learn mathematics, I hope to invigorate research into metalearning in mathematics education in order to refine these constructs.

This study is an example of an alternative research design which could serve to broaden approaches to inquiring into students' mathematical learning. Researchers may find the methods of data collection, which students found to be legitimate processes in learning to learn, useful in their research with high school students. Additionally, researchers may find CGT to be a methodological approach to research in mathematics education which holds much promise for theorizing informed by the experiences of students and teachers in schools.

Acknowledgement

This research was funded by the Social Sciences and Humanities Research Council of Canada and the University of Alberta.

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PRE-SERVICE TEACHERS' MEANINGS AND NON-CANONICAL GRAPHS

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Research continues to illustrate the important role of covariational and quantitative reasoning in the context of function and graphing. The same body of literature has emphasized that students and teachers often construct meanings for function and graphing that do not foreground these reasoning processes. In order to gain deeper insights into such meanings, we conducted clinical interviews with ten pre-service secondary teachers. In the present work, we illustrate the construct of shape thinking in relation to their graphing activity during the clinical interviews. We draw particular attention to the implications of shape thinking, including constraints generated by meanings rooted in such thinking, when confronted with non-canonical situations.

Keywords: Teacher Education-Preservice, High School Education, Algebraic Thinking

Students' function meanings remains a critical area of research in mathematics education (Oehrtman, Carlson, & Thompson, 2008), with a growing body of literature characterizing the role of covariational and quantitative reasoning in supporting students' function concept (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Castillo-Garsow, 2012; Ellis, 2007; Moore, 2012; Thompson, 1994, 2011; Weber, 2012). For instance, Castillo-Garsow (2012) identified that the manner in which students' conceive quantities and how they change in tandem has serious implications for their images of exponential growth. Likewise, Moore (2012) and Weber (2012) illustrated the importance of students' attention to dynamic relationships between quantities in the context of trigonometric functions and two-variable functions, respectively.

Although it is apparent that quantitative and covariational reasoning are important for students' function concept, students' mathematical experiences typically lack a fundamental focus on such reasoning (Oehrtman et al., 2008; Smith III & Thompson, 2008; Thompson, in press). In turn, these students construct function meanings devoid of imagery that involves varying quantities' values. As cases in point, Thompson (1994), Goldenberg and colleagues (1992), and Weber (2012) documented that students have a tendency to conceive graphs as pictorial objects with various global properties that are not grounded in quantitative reasoning. Weber and Thompson termed such ways of thinking as *shape thinking* (Weber, 2012).

We extend the shape thinking construct by exploring pre-service secondary mathematics teachers' (PSTs') activity during clinical interviews (Goldin, 2000) designed to offer insights into their meanings, particularly in the context of graphing. Based on several classroom events that suggested PSTs engage in shape thinking, we designed interview tasks such that shape thinkers might face perturbations in their meanings. By generating such situations, we characterize shape thinkers' meanings, including constraints shape thinkers face when confronted with situations that are not supportive of shape thinking. Against the backdrop of our results, we discuss several implications of shape thinking and provide conjectures about the nature of students' mathematical experiences that contribute to shape thinking.

Theoretical Framing

When one speaks of meanings, he or she is speaking of a pervasive, yet complex and often ill-defined term (Thompson, in press). Thus, it is necessary that we provide a brief description of our use of the term *meaning*. Drawing on the works of several individuals who concerned themselves with epistemology

(e.g., John Dewey, Ernst von Glasersfeld, and Jean Piaget), we consider meanings to be constructions unique to an individual that organize her experiences. As Thompson (in press) described, and drawing on Piaget's description of understanding and meaning, the act of constructing meaning entails assimilation; to construct a meaning is to construct a scheme through repeated reasoning that enables one to organize her experiences.

Due to the assimilatory nature of meanings, meanings influence how an individual makes sense of her future experiences. But, we caution that claiming meanings are assimilatory does not imply that meanings are static cognitive structures to be applied as is. Rather, meanings become more and more stable through an individual repeatedly reconstructing these meanings to make sense of her experiences (Thompson, in press). Meanings remain viable as long as they continue to enable an individual to organize her experiences in a way that is internally consistent. In the case that a particular meaning and experience leads to a perturbation, an accommodation or reorganization is then necessary to reconcile this perturbation.

Graphing, Shape Thinking, and Quantitative Reasoning

Stemming from a study exploring calculus students' meanings for two-variable functions, Weber (2012) characterized shape thinking as, "an association the student makes with a function's graph. For example, a student might associate a function's graph with a particular shape with physical properties while another student might associate a function's graph with a representation of quantities' values" (p. 17). In the present work, we use the phrase *shape thinking* to refer to this first aspect in which an individual's meaning for a graph is inferred directly from the pictorial image and perceptual properties of the physical shape. In contrast to this form of reasoning, Weber deemed activity rooted in reasoning about covarying quantities as *expert shape thinking*. For clarity reasons and to avoid implying that shape think is necessarily developmental, we do not use the phrase shape thinking to refer to the act of conceiving a graph as an emergent representation of how two quantities vary in tandem.

To provide contrasting examples of shape thinking and reasoning about covarying quantities, consider a student tasked with determining the formula for the graph in Figure 1.

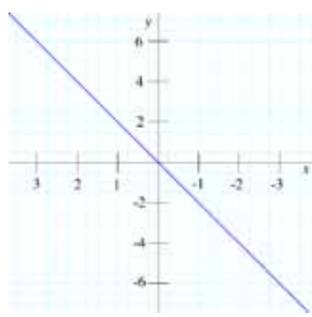

Figure 1: A Graph of $y = 2x$

When providing a formula for the above graph, a shape thinker may first associate the graph with a formula of the form $y = mx+b$ because she associates $y = mx+b$ as defining a line. From there, she may conclude that $m = -1$ by reasoning that m represents the tilt of the line and the line is downward sloping at a 45 degree angle with the horizontal axis. Lastly, she concludes that $b = 0$ because the graph passes through the origin. In such a solution (e.g., $y = -x$), the students' activity foregrounds previously defined properties that are tied to perceptual attributes of the shape (e.g., straightness and tilt).

To provide a contrast to the above solution, another student may approach the problem by identifying that for each point captured by the curve, the y-value is two times as large as the x-value, leading the student to conclude that $y = 2x$. As another example, the student may determine that the curve captures

paired values such that for any change in x, the change in y is two times as large, and since the initial value (e.g., when $x = 0$) for y is 0, then $y = 2x$. Whereas a shape thinker focuses on more global and pre-defined properties of the graph, essentially conceiving the graph "all at once" to determine the associated formula, these latter two solutions are generative in that the graph and its associated formula are conceived in terms of a dynamic relationship between quantities' values; the latter students' activities foreground a dynamic projection of points onto two axes to represent the quantities' values.

We agree with Weber's (2012) stance that shape thinking is not entirely bad. Shape thinking can support an individual in quickly inferring different representations and properties of a relationship conveyed by a graph (e.g., a line in the Cartesian coordinate system conveys a relationship that is also represented by $y = mx+b$). But, a problem arises when shape thinking is in the absence of underlying meanings tied to inferences about quantities that vary in tandem (Goldenberg et al., 1992; Oehrtman et al., 2008; Thompson, 1994; Weber, 2012).

Methodology

In order to better understand the nature and implications of shape thinking, we conducted semi-structured clinical interviews (Goldin, 2000) with ten undergraduate PSTs at a large university in the southeastern United States. We chose the PSTs on a voluntary basis. Each interview lasted approximately 90 minutes and all participants were given the same set of interview tasks. The PSTs were in their third undergraduate year, had already completed a minimum of two semesters of calculus plus two courses past the calculus sequence, and were currently enrolled in a functions and modeling course for PSTs. The course formed their first content course in the pre-service secondary mathematics teacher education program. We videotaped, transcribed, and then analyzed the interviews using conceptual analysis techniques (Steffe & Thompson, 2000) with a goal of characterizing the ways of thinking and meanings that supported their activity on the tasks.

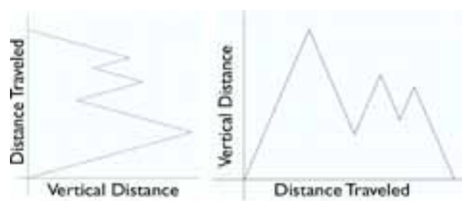

Figure 2: Debating Graphs

The choice of PSTs for this investigation was influenced by a variety of factors. First, our focus on shape thinking arose out of a classroom event during a previous year in the functions and modeling course. This event consisted of a debate that ensued as a group of PSTs argued whether two graphs with different orientations represented the same relationship (Figure 2). For some, the orientation of the axes and visual features of the graphs were significant components of their thinking leading them to maintain that the two graphs did not represent the same relationship; to these PSTs, the two graphs were visually different and thus had to represent different relationships. Based on this event, we anticipated that a subset of the following semester's PST cohort would rely upon shape thinking. Further, we chose to work with PSTs because their meanings matter in that they influence their teaching practices and their future students' learning (Simon, 2006; Thompson, in press). We hoped to better understand their meanings and subsequently improve teacher preparation based upon our understandings of their meanings. For instance, better understanding shape thinkers' meanings may provide insights into how to perturb their meanings and generate learning opportunities.

Task Design Example

When characterizing meanings, it's important that the researcher identify constraints in an individual's ways of operating, as these constraints aid the researcher in building more viable models of students' mathematics (Steffe & Thompson, 2000). In order to gain deeper insights into PSTs' meanings, we designed numerous tasks such that a shape thinker might encounter such constraints. As shape thinking is rooted in reasoning about pictorial objects in and of themselves, meanings rooted in shape thinking require that the situation conform to conventions upon which such meanings were abstracted. Thus, in our design of the tasks, we designed several non-canonical situations in order to better understand shape thinking by identifying how deep-rooted shape thinking may be, characterizing constraints engendered by shape thinking, and uncovering other meanings that that the PSTs used to reconcile problematic situations.

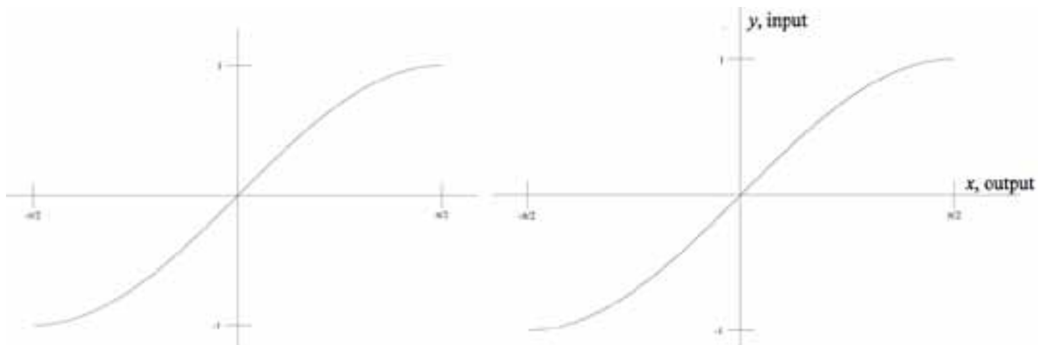

Figure 3: A Task to Investigate Shape Thinking

To illustrate one such task, consider the two graphs in Figure 3. The task proceeds: (i) first a PST determines a graph of the inverse sine function when given a graph of the sine function; (ii) the PST is then asked to comment on a graph (Figure 3, left) and prompt claiming that a student produced such a graph as an answer to part (i); (iii) lastly, the PST is presented with and asked to comment on a modified version (Figure 3, right) of the previous graph and a student explanation. The narrative provided with (iii) is, "*Well, because we are graphing the inverse of the sine function, we just think about x as the output and y as the input*. When giving this explanation, the student added the following labels to their graph. (*Student quote in italics*)."

We designed parts (ii)-(iii) to provide information about the PSTs' meanings in the context of a situation that was non-canonical (e.g., considering axes as simultaneously representing input and output quantities of two functions). To a shape thinker, a function's name and a particular pictorial object are inherently connected. For this reason, we conjectured that the solutions in parts (ii)-(iii) might be problematic to a shape thinker because each presents one curve as conveying *two* functions, $y = \sin(x)$ and $x = \arcsin(y)$ for $-\pi/2 \leq x \leq \pi/2$.

Results

As conjectured, the task described in the prior section revealed a number of instances that were suggestive of shape thinking. As a first example, consider Beth's (names are pseudonyms) response when determining a graph of the inverse sine function for part (i) of the task.

Beth: I should know this. Umm…I mean…(*pauses for 30 seconds and writes* $y = \sin(x)$).

Int: So how are you thinking about finding the inverse of that? Or the graph of the inverse?

Beth: (*Writes* $\sin^{-1} y = x$). So I mean, obviously it's just (*writes* $\sin^{-1} x = y$) which doesn't help at all. (*Laughs*). Umm, but, I'm thinking that it's either going to be something like…actually I'm kind of embarrassed because I really should know this.

In this exchange, Beth first attempted to recall a shape from memory (e.g., "I should know this"), an approach that was not successful. Beth then symbolically manipulated $y = \sin(x)$ into $\sin^{-1} x = y$, but obtaining this formula was not satisfying to Beth because her difficulty still lied within attempting to remember the shape of the graph (e.g., "I really should know this"). Following the above exchange, Beth made a few conjectures regarding the shape of the inverse sine graph, but as the discussion continued it became apparent that these conjectures were attempts to recall the graph from memory. Beth explained that she could not confidently determine an answer. We then progressed to parts (ii) and (iii) of the task to see how Beth would make sense of a posed solution to the problem.

Int: So a student came to you with that graph (*Figure 3, left*) claiming that it was a graph of the inverse sine function. What would you say to that? Could that be true?

Beth: Could that be true? Umm…(*thinks silently for 15 seconds*)…mmm, no. No.

Int: Why are you thinking no?

Beth: …I'm thinking this just kind of looks like…sine graph – like the plain sine graph. (*Laughs*). Which is going to be different. So, no…

(*Interviewer later gives Beth part (iii), Figure 3, right*).

Beth: I mean I guess what I'm like thinking about, like struggling with thinking is that like, like I don't know if, or if...an inverse function...like the graph of an inverse function, like, can't be the same as the original graph. Or can it? Like, I wouldn't think that it could. But maybe there's something I don't know. (*Sighs*).

Beth's responses and uncertainty in parts (ii) and (iii) are viable when viewed through the lens of shape thinking. Given this perspective, two different functions cannot be represented by the same graph because otherwise they would be the same function; a shape or curve has only one associated function name as a label. Thus, even though Beth was willing to admit there may have been something she did not know, she was unable to determine a way in which a student might conceive the graph as both the sine function and sine inverse function.

While Beth's perturbations appear to stem from her belief that different functions must have different shapes, another PST experienced conflict for apparently different reasons when responding to part (iii).

Megan: Oh, Um, Um. We just can't do that (*in response to Figure 3, right*).

Int: OK, say a little bit more.

Megan: (*Laughs*). It's not, that's not, his definition is not necessarily wrong. But you can't just label it like that. Um, why? Why can't you do that? I don't know, I feel like he's missing the whole concept of a graph…Like, I know you can call whatever axis you know if you are doing time and weight or volume whatever…But not necessarily with the sine graph. Like the sine graph's like a…it's a graph that everyone knows about…They are just missing the concept of graphing.

Whereas we did not identify an instance in which Beth focused on input and output quantities, Megan identified that the posed graph involved denoting the axes in various ways relative to input and output quantities. Megan seemed willing to accept reversing the roles of the input and output for some quantities such as time and weight, but not in the case of the given graph. We believe that her comment, "it's a graph everyone knows about" indicates that for the given graph, the input and output axes are fixed because that particular curve is designated as the sine function. Hence, changing the roles of the axes in the provided graph produces a conflict.

We see that both Megan and Beth experienced unique perturbations when faced with a non-standard approach to graphing the inverse sine function. Yet, underlying both perturbations is a similar phenomenon that rests in their uniquely associating the given curve with the sine function. Due to the nature of this association, the given curve could not also be considered as the inverse sine function, even in the case that a PST (e.g., Megan) acknowledged the possibility of considering different axes for the input and output quantities. In each case, because the curve was "known" by a particular label, it was not possible to be given another label.

Another form of shape thinking we observed was instances in which the PSTs' reasoning was focused on the global properties of a curve in ways that were not attentive to emergent quantitative

relationships. For example, Beth was presented with the following hypothetical student's graph of $y = 3x$ (note the non-standard orientation of the x and y axes in Figure 4).

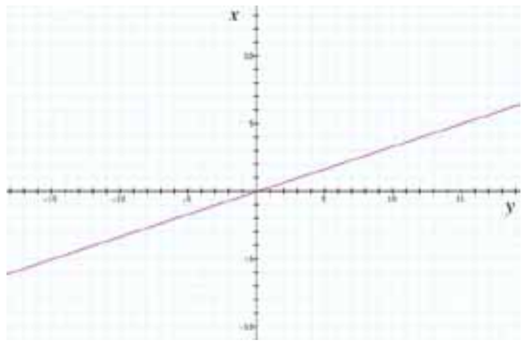

Figure 4: A Hypothetical Student's Non-standard Graph of $y = 3x$

> *Int:* How would you respond to the student or how might the student be thinking about this?
> *Beth:* Umm…like this (*spinning paper 90 degrees counterclockwise*). (*Laughs*). Like, because if you turn it this way then this (*traces left to right along the x-axis which is now in the horizontal position*) and this (*traces top to bottom along the y-axis*) and it would be still not right though (*spinning paper back to original orientation*).
> *Int:* And how would you respond to this student if they said, "Well here's how I'm thinking?"
> *Beth:* I guess…I mean the only way I can think of it is like this (*spinning paper 90 degrees counterclockwise*) and it's still wrong because this (*a line which is now sloping downward left to right*) is negative slope. So I would just, I would just explain to them, like the difference between the x- and y-axes and umm …show them like the difference between positive and negative slopes also. Because that's something that, like, when I was in middle school we, like, learned kind of like a trick to remember positive, negative, no slope, and zero (*making hand motions to indicate each*). Like where the slopes were. And it's stuck with me until now so it's important to know which direction they're going…

Beth's actions convey several instances of shape thinking. First, Beth's insistence that the line has a negative slope after rotating the paper 90 degrees counterclockwise to orient the x-axis horizontally and her reference to remembering slope-curve orientation pairs provides evidence that her meanings for slope are connected to a particular shape (e.g., a line sloping downward left to right implies negative slope). Beth's repeated moves to reorient the graph and her later rejection of the hypothetical student's solution is also evidence of her reliance on shape thinking. Beth's rotating of the graph was an attempt to achieve a conventional axes orientation, which would enable judging the solution against her image of the proper graph. Because Beth was unable to achieve an orientation compatible with her image of the appropriate graph, she maintained that the graph was "wrong" and described that she was not sure what the student did.

Although shape thinking was a common way of reasoning for many of the PSTs throughout the interviews, a few of the participants utilized other ways of reasoning to make sense of the given situations. For instance, in contrast to Beth's interpretations of the non-standard graph of $y = 3x$, another PST, Jacob, did not rely upon shape thinking. He instead made sense of the hypothetical student's work by conceiving the graph in terms of a quantitative relationship. After taking a few moments to inspect the graph (Figure 4, axes unlabeled), Jacob decided to consider a few of the paired values. This led him to conclude that the student was "plugging in" values on the y-axis (e.g., consider x from the formula as along the typical y axis). After seeing the student's work with the vertical axis being labeled as x, Jacob accepted the student's work as correctly conveying the relationship $y = 3x$. Although this type of

quantitative reasoning was rarely observed during the interviews, we include it as a contrast to the other results in order to highlight the implications of relying upon shape thinking as one makes sense of graphs.

Discussion and Concluding Remarks

Over the course of the interviews, the PSTs predominantly engaged in shape thinking, which is unsurprising given the body of literature documenting meanings of this nature (e.g., Thompson, 1994, in press; Weber, 2012). What was unforeseen was how the PSTs' meanings influenced their activity on problems that problematized shape thinking. The conflictions the PSTs faced during the interview tasks provide insights into the possible genesis of shape thinking. Instances of shape thinking became apparent when the given interview tasks broke common practices or conventions in school mathematics (e.g., axes orientation). As such, the PSTs' struggles can be partly framed as a consequence of constructing meanings inherently tied to these conventions and encountering interview tasks that did not conform to the conventions from which shape thinking was abstracted. The PSTs' propensity to engage in shape thinking highlights that such thinking had become an integral part of these PSTs mathematical meanings (e.g., meanings for the sine function, for inverse, and for slope). It follows that they likely had repeated experiences in which to (re)construct these meanings as viable. In other words, by repeatedly encountering situations that conformed to particular conventions, they abstracted meanings that were inherently tied to these conventions. For example, Beth's slope meanings were tied to shapes reliant on a particular axes orientation that is pervasive in mathematics.

Obviously mathematical conventions are critical supports to mathematical activity. But, a problem arises when meanings are tied to these conventions in ways that restrict one's ability to reconcile situations that are internally consistent but do not follow such conventions. For instance, the graph in Figure 4 is a quantitatively correct representation. Yet, several PSTs were not able to provide a viable explanation for the student's proposed graph because their meanings required that the graph be in a more conventional orientation. Such an outcome speaks to a possible drawback of mathematical conventions in the context of student learning; by repeatedly experiencing situations that conform to particular conventions, these conventions can become constraining aspects of student meaning. Then, as these students later become teachers, the conventions constrain how they may operate with their students and interpret student work.

Glasersfeld (1995) stated, "Actions, concepts, and conceptual operations are viable if they fit the purposive or descriptive contexts in which we use them" (pg. 14). The prevalence and deep-rooted nature of shape thinking among the PSTs in the study implies that their meanings had consistently been "purposive and descriptive" during their previous mathematical experiences. By breaking from common graphing conventions, the tasks in this study created situations in which shape thinking did not provide a viable solution. This problematized the PSTs' meanings, which would have generated teaching and learning moments in the classroom or another setting (e.g., a teaching experiment). Further research should explore how such tasks could be used to support students' development of meanings that entail reasoning with quantitative relationships. Additionally, further research is necessary to characterize possible implications of particular conventions in K-12 mathematics in the context of student learning.

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MATHEMATICS STUDENTS TRANSITIONING INTO THE WORKFORCE: INFLUENCES ON CAREER CHOICE

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Currently, there is a shortage of Americans entering STEM careers. Furthermore, women remain underrepresented in these fields. In this paper I discuss the career decisions of three undergraduate women mathematics majors who, during their undergraduate years, expressed an interest in obtaining a career in mathematics but lacked substantial knowledge of potential careers. I explore how these women transitioned from college into the workforce and how their career decisions were based, in part, on aspects of their identities and on environmental opportunities. The implications of this work provide knowledge to the field of how to retain certain individuals in the field of mathematics.

Keywords: Gender, Post-Secondary Education, Affect, Emotion, Beliefs, and Attitudes

Currently, there is a shortage of Americans entering STEM (Science, Technology, Engineering, and Mathematics) careers in the United States (National Academy of Sciences, 2010; Huang & Cohen, 2012). Although there does not appear to be a consistent definition amongst national organizations of what is classified as a STEM career, an unspoken definition appears to be any career that requires the knowledge of a Bachelor's degree in a STEM field (excluding careers in K-12 teaching). The shortage of those entering STEM careers is not necessarily due to a lack of talent, but rather is based on the reality that Americans, for various reasons, simply are not choosing to enter these fields. Furthermore, national statistics demonstrate that women remain underrepresented in mathematics and science fields (BLS, 2011). Given the national need for more individuals to enter these disciplines, the underrepresentation of women in these fields should be of widespread concern.

There are many critical junctions within individuals' lives when they make decisions that influence their future career opportunities. Prior research demonstrates that women choose to leave the STEM career pipeline at disproportional rates at every junction along this trajectory. Girls as early as eighth grade demonstrate less interest in obtaining a future career in mathematics or science than their male peers (Riegle-Crumb, Moore, & Ramos-Wada, 2011). Women are less likely than men to declare a college major in a STEM field and are more likely than men to switch their major away from a STEM field prior to graduation (Seymour & Hewitt, 1997). What is unique to mathematics, however, is that women choose to major in mathematics at the same rate as men (NCES, 2010), but are less likely to enter graduate school in mathematics and are more likely to choose to leave graduate programs in mathematics prior to completing the degree than men (Cleary, Maxwell, and Rose, 2010; Herzig, 2004).

There are a multitude of factors that influence women's decisions to leave the mathematics and science fields. Some of the most common mentioned in the literature are the absence of role models, STEM curricula and pedagogy, women's attitudes and beliefs about the fields, cultural pressure, and a 'chilly climate' (Blickenstaff, 2005). Furthermore, it has been argued that not necessarily the same factors affect all areas of science equally and the factors may differ for science and mathematics careers (Riegle-Crumb, Moore, & Ramos-Wada, 2011). Therefore, it is important to study women in mathematics separately from those in other STEM fields.

In my work, I analyze what factors influence undergraduate women mathematics majors' academic and career decisions. In this paper, my analysis surrounds the career decisions made by women who, during their undergraduate careers, expressed an interest in obtaining a career in mathematics but lacked substantial knowledge of potential careers. This paper discusses how these women transitioned from college into the workforce and highlights the role that aspects of their identities and environmental opportunities played within their career decisions.

These women's experiences and career decisions are worthy of a focused analysis because these individuals desired a career in mathematics and had the proper education to obtain one, but lacked knowledge of potential careers. Learning about what led to their career decisions provides knowledge to the field of how to retain certain individuals in the field of mathematics.

Theoretical Framework

Many scholars have explored social and cultural reasons for the shortage of women in STEM careers (Blickenstaff, 2005; Hanna, 2003; Spelke, 2005; Wallon, 2005). One perspective is that of the "glass ceiling", suggesting that women are not given the same opportunities as men to succeed in the STEM fields through means of discrimination. That is to say that external barriers prevent women from being successful in these fields. Another perspective is that the shortage of women in these fields is a result of choices made by women. In other words, women are making the decision themselves not to pursue STEM careers. This perspective gives agency to women and views them in control of their own life decisions (Damarin, 2000).

In my work, I follow this latter perspective, focusing on the educational and career choices made by women. This perspective does not ignore external barriers or cultural norms. Rather, it views women as having the choice of how to allow these factors to influence their beliefs and decisions.

When considering what influences an individual's choice of career, research demonstrates that the situation is complex. Eccles and her colleagues (2011, 1994) have developed a framework on achievement-related choices that takes into account the role gender plays within these decisions. In this framework, educational and vocational choices are most directly related to the individual's expectations for success in a field and the subjective task value (STV) that they attach to a field. STV is described as having four major components: 1) utility in facilitating one's goals or external rewards, 2) enjoyment, 3) ability to manifest one's identities and core values, and 4) the cost of engaging in the activity (Eccles, 2011).

In this framework, an individual's beliefs are based on a complex interplay of multiple factors, such as cultural norms, personal experiences, input by significant people in their lives, gender-role beliefs, and perceived ability, all of which are affected by one's gender. It is this framework that has informed my analysis of the choices made by the women in my study.

Methods

The study described in this paper is a part of a larger longitudinal study analyzing the experiences and career selection of women mathematics majors. As part of the larger study, I conducted a series of three 90-minute interviews with 12 undergraduate women mathematics majors at two different universities in the Midwestern United States to learn what motivated these women to major in mathematics and to learn about their future career goals. I employed the interview technique known as in-depth, phenomenologically-based interviewing and followed the Three-Interview Series protocol as described by Seidman (1998). The interviews were conducted during the student's junior or senior year of college and generally took place over the span of one month. Then, three to five years after the initial interviews, I conducted a

one-hour follow-up telephone interview with eight of the original 12 participants, to learn what influenced their career decisions since graduating from college. For this paper, I will focus on a subset of three of these participants as described below.

All 12 of the original participants were U.S. citizens, had Caucasian heritage, grew up in the Midwestern United States, and attended college immediately following high school. Of the original 12 participants, half were planning to become middle or high school mathematics teachers. The remaining six participants had consciously made the decision not to have a career in teaching. Of these six, three had specific career goals that they were working towards. The other three participants, however, did not have any specific career plans, in part because they did not know what careers were available to them in the field of mathematics. It is these three participants who are the focus of this paper.

All interviews were audio-recorded and transcribed. The four interviews for each of the participants (three while they were students and one 3-5 years later) were critically analyzed and coded inductively with regards to these women's desired career qualities and what influenced their actual career paths. For each participant, common themes were noted between the participant's first set of interviews and the follow-up interview three to five years later, emphasizing validity within the data. Profiles were then created for each of the participants in order to accurately represent each woman's account (Seidman, 1998). The profiles were then analyzed for common themes arising between the women's experiences.

Findings

The three participants I focus on in this paper have earned bachelor's degrees in mathematics. They all chose this major because they had strong mathematical identities and because they enjoy the field of mathematics (Piatek-Jimenez, 2013). Despite their choice of major, however, while they were undergraduates they were unaware of what careers they could obtain with a degree in mathematics. In fact, all three of them specifically stated that the biggest obstacle to entering a career in mathematics is not knowing what their options are. In this section I provide details of how each participant transitioned into the workforce. In the following section, I analyze the commonalities found amongst these participants' experiences and decisions with regards to Eccles' (2011, 1994) framework.

Kelly

While an undergraduate, Kelly was unaware of potential careers available to her upon graduation with a major in mathematics. People frequently told her that mathematics is a good field to enter because of the many career options available, but when pressed, no one seemed to know what these options were. In an attempt to learn more about potential careers, Kelly spent a day job-shadowing a banker and attended a talk by a mathematician at the National Security Agency, but determined that neither career was a good fit for her.

When considering what qualities she wanted in a career, Kelly mentioned that she did not want a desk job. She considers herself to be a social and active person and wanted a job with human interaction and physical activity. She claimed, "Working at a desk is 'ugh'; I like to get up and move around."

Because of her Spanish minor, Kelly initially planned to study abroad during her senior year of college, however her plans changed when she learned how expensive studying abroad would be. Instead, Kelly opted to graduate a semester early and travel abroad once she graduated:

> Part of the reason why I decided to go abroad was because I wanted to study abroad, but I didn't want to pay all the money to study abroad. So, I thought a good goal was for me to graduate early and then use that extra semester to go abroad for four months or for six

> months... The first country I went was Russia... My purpose was to pick a Spanish speaking country, but I just, Russia sounded like a little bit more fun. So, but then I did end up going to a Spanish speaking country after that... [In] Russia and South Korea I taught English, and then, in the Dominican Republic I worked as an event coordinator at a resort.

Even though Kelly initially planned to live abroad for only four to six months, these short-term plans lasted two and a half years. For each position, Kelly was hired for a fixed amount of time and after each contract expired she would return to the U.S. for approximately one month before leaving again for the next country. “I could have stayed in each of those places, but I just, I wanted a new experience, something different.” When asked if she had considered math-related positions, too, Kelly responded:

> The type of position that I wanted was the short-term position, and the math positions were all relocating there for four or five years, and that was something that I wasn’t willing to do. So, I was just going for the short-term positions.

Most recently, Kelly obtained a job in her home state as a middle school Spanish teacher. Although she had been looking for another job abroad, she had not yet found one that was a good fit for her. She then learned through her father that there was a position opening mid-year for a Spanish teacher in her hometown. Given the need of the school district, they hired her without a teaching credential. Since Kelly did not have another position abroad lined up yet, she decided to take advantage of this opportunity that arose.

Although Kelly had initially determined that she did not want to become a teacher, now that she is in the classroom, she is happy with this decision for the short-term:

> I’m enjoying my job for the most part and everything like that, but I don’t know if that’s something that I could really do for long-term... I’m going to be focusing on teaching at least for the next five, ten, years or so... I’ve just kind of fallen into [teaching] and it seems to be working out well, so yeah, I might as well pursue it a little bit and see what happens.

Although Kelly is currently a middle school Spanish teacher, she has begun an online program to earn a teaching credential in mathematics, with the hope of eventually finding a position teaching mathematics. She explains this decision by saying, “Even though Spanish seems to be where things are taking me now, math has always had a soft spot for me in my heart. I just, I think it’s fantastic!”

Mandy

Despite her decision to major in mathematics, Mandy did not know what she wanted to do once she graduated. Mandy even felt embarrassed when people would ask her what she was going to do with her degree because she would have to tell them that she did not know. She simply knew that she did not want to have a career doing pure mathematics. “I just like applying math to things. I don’t want to do pure math… I want to apply my math skills to something bigger than math. So, to another field [such as] biology maybe.”

Upon graduation, because Mandy did not know what career path to pursue, she decided to go to graduate school. “I think part of it was I didn’t know what my options were for finding a job.” She ended up applying to two very different graduate programs. One of the programs was in applied mathematics while the other was in information communication sciences. Mandy learned about the latter program through a colloquium talk given at her university. In making her decision of which program to attend, Mandy claimed, “It almost came down to a coin flip between [the two programs]. I got assistantships at both of them, but I just felt like I needed a change of pace from math and I wanted something with more communication tied in.” In the end, Mandy decided to do the Master’s program in information communication sciences. Even

though she was not certain whether this program would be a good fit for her, knowing it was only an 11-month program, she figured, “I can do anything for a year.”

Throughout graduate school, the director of the program took it upon himself to mentor Mandy. The same professor also assisted Mandy in obtaining multiple job interviews, one of which lead to her current position. Mandy has found a job as an IT consultant at a consulting company in a large city. The majority of this company’s clients are in the healthcare sector. Mandy described her current work as follows:

> I’m helping with a wireless appointment, and now we are doing one in a new hospital here. So the access points are installed and we walk around with a survey tool tweaking the coverage areas and making sure that the radio frequencies are at the power levels to make sure that it’s at the right level without being too high. So, I really like that cause it’s like a puzzle.

Mandy claims that while she may not be using specific content that she learned as an undergraduate, she is using many of the skills she obtained as a mathematics major, “the problem-solving, the thinking, the analyzing, all those skills I gained learning to write proofs.”

Mandy really enjoys her job because she likes doing “technical” work. She also enjoys that she is out in the field each day. “I *love* that I don’t sit at one desk all day. I like that it changes it up.” The only concern that Mandy has about her current job is that she travels approximately half the time, and if in the future she gets married or has children, Mandy does not want to continue traveling so extensively. She does state, however, that she would still want to have a job in a “technical field.”

Nicole

Like the other participants, Nicole did not know what careers were available to her upon graduation. Her father was a high school mathematics teacher, but Nicole did not want to enter the field of teaching because she desired “a more glamorous job.” She knew, however, that she wanted to stay within the field of mathematics.

During her senior year, Nicole saw an advertisement on Facebook for “Math for America.” Through this program, she would be able to earn a Master’s degree from Columbia University Teacher’s College with all expenses paid and then teach in New York City’s public schools for four years after completing her degree. Despite the fact that Nicole had originally decided not to enter the profession of teaching, given that she was close to graduation and did not know what other options she had, she chose to apply to this program.

At the time, Nicole clearly stated that she did not envision herself remaining as a teacher for a long period of time. She simply saw this as a short-term opportunity and not as something that would lead to a future career. The fact that it was a short-term program specifically appealed to her because she felt it would give her time to discern a long-term career goal.

Upon completing her Masters degree, Nicole learned that the “Math for America” program would continue to pay for her to take a college class each semester for the next four years while she taught in the New York public schools. She decided to take advantage of this opportunity and began a PhD program in mathematics education:

> Because I was getting, like I could get those classes for *free* I kind of thought, “Well, I could do this PhD and that would be cool” I guess. But I mean that’s really kind of; if it hadn’t of been for that, I probably wouldn’t have started it.

Since that time, Nicole has been taking two courses a semester and is currently on track to complete her PhD over the span of four years.

Upon completing her PhD in mathematics education, Nicole is still unsure of what career path she will take. "I don't really know what I can, what I'm going to do with it." One option she is considering is a teaching job at a university, but she clearly states that she would only be interested in a position teaching mathematics, rather than teaching education courses, demonstrating that she still wants a career with an emphasis on mathematics.

Reflecting on her decision to join "Math for America" after college, Nicole says that she does not know if she would make the same decision again:

> I don't know... I wouldn't say that I regret it… I'm not sure if I would be able to do anything differently if I could go back… I don't know that I'm that happy with what I'm doing right now… I guess I'm as unsure as I was three years ago… I kind of think I just really didn't know what I wanted to do. (sighs) That kind of started to get me to change my mind [about teaching].

Nicole admits that while she is not necessarily happy teaching, four years out of college she is no more knowledgeable about careers in mathematics than she was when she graduated. She also clearly states that the reason she chose to go into teaching was because she felt like it was her only option. She does admit though that one thing that was appealing to her about teaching was "I didn't really want to have an office kind of job."

Analysis

Although these three women transitioned into the workforce in very different ways, there are some common themes that permeate their stories. At the time of graduation, all three of these women remained interested in wanting careers in the field of mathematics. Because of their lack of awareness of careers that also fit other desirable job qualities (for example, not wanting a desk job), they chose short-term options instead of focusing on long-term plans. For Kelly and Nicole, their short-term jobs were in fields they knew they did not want to remain in for a long period of time, but that satisfied other short-term goals, such as traveling abroad or earning a Master's degree. These decisions appear to be the results of trying to postpone making long-term career plans. In Mandy's case, initially unsure whether her short-term plans (her Master's program) would lead to a long-term career, she chose an option that could potentially prepare her to have a career in a STEM field.

According to Eccles et al. Model of Achievement Related Choices (Eccles, 2011), the two sets of beliefs that most directly influence an individual's career choice are expectations for success and subjective task values. The first of these, expectations for success, influenced these women's choice of major (Piatek-Jimenez, 2013), but did not appear to influence their career decisions. This might be, in part, because these women were entering short-term careers. Since their jobs were not intended to be long-term, their expectations for success was less influential on their decisions.

On the other hand, the role of subjective task values appeared to be instrumental in their short-term career decisions. In particular, their decisions were based on a synthesis of the components of enjoyment and the ability to manifest one's identities and core values. These three women all identify as "math people", yet they also identify as social and active individuals. Their limited knowledge of mathematical careers, however, prevented them from being able to obtain a career in mathematics that satisfied all three of these aspects of their identity. As a result, they placed their identity of being social and active as a priority, at least for the short term.

Both Kelly and Nicole found jobs that allow them to be both social and active at work, yet they are not content with their current positions. They had both chosen short-term jobs hoping to later learn about mathematical careers that fit their desired career qualities. Despite being three

to four years beyond graduation, they are no closer to being knowledgeable about such careers. Therefore, they are still searching for their desired long-term career.

Mandy, on the other hand, entered a graduate program in information communication sciences, which has allowed her to obtain a career that is active, social, and "technical" in nature. Although she is no longer in a mathematical field, it appears that being in a different STEM field is satisfying this portion of her identity. Mandy is aware, however, that later in life if she gets married, her identity will change and that her current position may not fit the needs of those new aspects of her identity. Therefore, it appears that Mandy has successfully transitioned into the workforce by obtaining a career that satisfies her needs, at least until these needs change.

It is also interesting to note the role that environmental opportunities played in these women's present career paths. Each of these women are currently following a career trajectory that was proposed to them at a time when they were discerning their next educational or vocational decision. Given their limited knowledge of mathematical careers, any opportunity that arose became a consideration, even when this opportunity did not entirely satisfy their desired career qualities. It is also important to note that when these women needed to make concessions, they chose to follow a career path that more closely aligned with gender-role affirming aspects of their identities (Morgan, Isaac, & Sansone, 2001).

Furthermore, all three of these women expressed receiving support from their families in their career decisions; however, they also believed that their parents would have been supportive regardless of their actual career choice. This unwavering support, as well as having the financial means to explore short-term options, may have also influenced these women's career decisions.

Discussion

Previous studies have focused on specific reasons individuals have consciously chosen to leave the field of mathematics or other STEM disciplines (Herzig, 2004; Seymour & Hewitt, 1997). The results of my study suggest that some students leave the field of mathematics not specifically due to the desire to leave, but rather because of a lack of awareness of careers in the field. In order to retain more people in the field of mathematics, this population is important to study. These individuals should be among the easiest to maintain in the field, given their desire for a career in mathematics.

This lack of knowledge of mathematical careers exists amongst both male and female mathematics students (Adhikari et al., 1997; Piatek-Jimenez, 2008). Therefore, this is likely to be a reason that both men and women do not remain in the field of mathematics. Given the need to increase the mathematics workforce in the U.S., a national agenda to inform talented students about mathematics careers may be successful in retaining more students within these fields.

The dispersion of career information may assume a number of different forms. A handful of books have been published to assist students in learning about careers in mathematics (Burnett, 2002; Lambert & DeCotis, 2005; Sterrett, 2002). More one-on-one mentoring of mathematics students by university faculty may be useful. Companies and graduate programs may consider visiting universities to actively recruit potential candidates. Furthermore, mathematics departments might consider synthesizing these ideas by creating a capstone course, whose primary responsibility would be to assist students in the transition into the workforce by utilizing the books listed above and guest speakers to inform students of career options in mathematics.

Simply providing information of such careers may not be sufficient, however, if these careers do not encompass other qualities these individuals desire from a career. Given the role that gender plays on one's subjective task values (Eccles, 2011), men and women generally value different qualities within a career. For example, in general, women tend to have a stronger

preference for careers that involve social interaction, helping others, and flexible schedules (Eccles, 2011; Morgan, Isaac, & Sansone, 2001). Therefore, a variety of career opportunities that encompass these qualities need also be proposed.

Finally, when needing to choose between aspects of their identities, intentionally or not, the women in this study placed gender-role affirming qualities as a higher priority than qualities that challenge their gender-roles. More research needs to be done to learn if this is a wide-spread trend, and whether or not these decisions are also influenced by culture or social class.

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STUDENTS WHO SWITCH OUT OF CALCULUS AND THE REASONS THEY LEAVE

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A substantial percentage of college students who enrol in Calculus I intending to take more calculus decide at the end of the semester not to continue with calculus. This represents a huge loss in terms of the need for more students to pursue a major in one of the science, technology, engineering and mathematics (STEM) disciplines. In this report we examine the characteristics of STEM intending students who begin their post secondary studies with Calculus I and either persist or switch out of the calculus sequence, and hence either remain or leave the STEM pipeline. The data used for this analysis comes from a unique, in depth national survey aimed at identifying characteristics of successful programs in college calculus.

Keywords: Post-Secondary Education, Instructional Activities and Practices, Assessment and Evaluation

Introduction

As detailed in the recent report in the USA from the President's Council of Advisors on Science and Technology (PCAST, 2012), there is tremendous need for more students with degrees in science, technology, engineering, and mathematics (STEM). For example, the PCAST report predicts that, over the next decade, approximately 1 million more STEM graduates above and beyond the current level of STEM graduate production will be needed in order to meet the demands of the national workplace. One strategy for meeting this need is to increase the retention of STEM majors. In fact, the PCAST report predicts that simply increasing the retention of STEM majors from 40% to 50% would go a long way to meeting this need.

As reported by Seymour (2006), students leave STEM majors primarily because of poor instruction in their mathematics and science courses, with calculus often cited as a primary reason. Therefore, in order to develop more successful retention strategies, the field is in need of a deeper understanding of what distinguishes those who continue with calculus from those who do not.

The purpose of this report is to examine the characteristics of STEM intending students who begin their post secondary studies with Calculus I and either persist or switch out of the calculus sequence, and hence either remain or leave the STEM pipeline. The data used for this analysis comes from a unique, in depth national survey aimed at identifying characteristics of successful programs in college calculus. In this report we answer the following three research questions: (1) What is the profile of students who choose not to continue with calculus? (2) What are the reasons that students give for switching out of calculus? (3) What characterizes the behavior of switchers and the behavior of their instructors?

Background

Researchers in Higher Education have extensively studied factors related to student retention at the post-secondary level, often focusing on the effects of student engagement and integration on persistence (e.g., Kuh et al., 2008; Tinto, 1975, 2004). According to Tinto's integration framework (1975), persistence occurs when students are socially and academically integrated in the institution. This integration occurs through a negotiation between the students' incoming

social and academic norms and the norms of the department and broader institution. From this perspective, student persistence is viewed as a function of the dynamic relationship between the student and other actors within the institutional environment, including the classroom environment.

Guided in part by this theoretical and empirical work in higher education, this paper reports on a five-year study of Calculus I instruction at colleges and universities in the United States. The first phase was a large-scale national survey of Calculus I instruction at two- and four-year colleges and universities. The survey was restricted to what is known as "mainstream" calculus, the calculus course that is designed to prepare students for the study of engineering or the physical sciences.

The second phase of the study consists of case studies examining Calculus I instruction at seventeen colleges and universities identified as having a notable measure of success with their Calculus I program. Success was defined in terms of both the percentage of students who had successfully completed the course and the percentage of students who maintained or increased their interest in continuing the study of mathematics beyond Calculus I, controlling for the varying academic strengths and interests of the entering students at different institutions.

Methods

The large-scale national survey of mainstream Calculus I instruction was conducted across a stratified random sample of two- and four-year undergraduate colleges and universities during the fall term of 2010. Preparation for the surveys included a literature review leading to a taxonomy of potential dependent and independent variables followed by constructing, pilot testing and refining the survey instruments (Lodico, Spaulding, & Voegtle, 2010; Szafran, 2012).

A total of six on-line surveys were constructed: one for the calculus coordinator; two for the calculus instructors of which one was administered immediately before the start of the course and the other immediately after it ended; and three for the students of which one was administered at the end of the second week of the course, one just before the end of the course, and the last one year later to those students who had volunteered their email addresses. In addition, instructors reported on the distribution of final grades and submitted a copy of the final exam. All surveys were completed online, and no incentives were given for completing the surveys. For the analysis reported here, only the three student surveys were used.

The survey was sent to a stratified random sample of mathematics departments following the selection criteria used by Conference Board of the Mathematical Sciences (CBMS) in their 2005 Study (Lutzer et al, 2007). Following the strategy of CBMS, we separated colleges and universities into four types, characterized by the highest mathematics degree that is offered: Associate's degree, Bachelor's degree, Master's degree, and Doctorate. Within each type of institution, we further divided the strata by the number of enrolled full time equivalent undergraduate students, creating from four to eight substrata. We sampled most heavily at the institutions with the largest enrolments. In all, we selected 521 colleges and universities: 18% of the Associate degree colleges, 13% of the Bachelor's degree colleges, 33% of the Master's degree universities, and 61% of the Doctoral universities. Of these, 222 participated: 64 Associate degree colleges (31% of those asked to participate), 59 Bachelor degree colleges (44%), 26 Master's degree universities (43%), and 73 Doctoral universities (61%).

There were 660 instructors and over 14,000 students who responded to at least one of the surveys. There is complete data (the first five surveys completed and linked with each other) for 3103 students enrolled with 309 instructors at 125 colleges or universities. However, in order to

answer our research questions we did not need to restrict ourselves to the completely linked data set. Instead, we needed either a student end of term survey or follow up survey.

Results

Depending on a student's initial intention to continue with calculus and whether they switched or persisted with their intention, we used multiple questions across surveys to classify students into four categories: Culminaters, Persisters, Switchers, and Converters. Culminaters are those students who began and ended the course not intending to take Calculus II. These students typically only need Calculus I for their major. Persisters were those students who initially intended to take more calculus and did not change this intention. Switchers, on the other hand, were those students that started Calculus I intending to take more calculus, but then by the end of the term (or one year later) changed their plans and opted not to continue with more calculus. Finally, Converters were those students who initially did not intend to take more calculus but by the end of term changed their mind and wanted to continue taking more calculus. Out of a total of 7260 students for which we could code in terms of one of the four categories, there were 1,789 Culminators, 4,710 Persisters, 671 Switchers, and 90 Converters.

Persisters and Switchers constitute the two main categories of STEM intending students. For STEM intending students in our sample, we found that 12.5% of them were classified as Switchers. In order to improve retention of STEM majors, we need to understand how Switchers and Persisters are similar and different. The following analysis focuses on comparing Persisters and Switchers.

To address our first research question, we compared Switchers and Persisters across a number of variables. In this report we provide results from gender, ethnicity, career path, and academic preparation. Data for each of these variables was collected on the start of the term survey.

Of the students who reported gender information, 41.5% (1317) of STEM intending students were female and 48.5% (1856) were male. In comparison to males, the percentage of female Switchers is significantly higher, indicating that women are more likely to leave a STEM major. Specifically, only 11% of 1856 males were identified as Switchers whereas 20% of the 1317 females were Switchers (χ^2 (df = 1, n = 3173) = 49.14, p < .001). Contrary to the observed differences among gender, there were no statistically significant differences by ethnicity (χ^2 (df = 7, n = 3169) = 3.210, p = 0.865).

We also analyzed differences by career path. Switcher rates differed significantly by career choice, with Engineers switching at very low rates (5.9%) and the biological sciences switching at much higher rates than average (24.8%). These results made us interested to know how this was related to differences in gender. As shown in Table 2, the switching behaviors within career choices varied significantly based on gender (χ^2 (df = 15, n = 3141) = 102.9, p < .001). For example, in the biological sciences almost 30% of females switched while only 17% of males did so. Similarly, in the fields of math, physical sciences and computer science women switched out at two to three times the rate of males in the same fields.

Table 1: Relation Between Career Choice and Gender

	Male		Female	
Career Choice	Persister	Switcher	Persister	Switcher
Biological Sciences	82.55%	17.45%	70.35%	29.65%
Life, Earth, and Environ. Sciences	81.33%	18.67%	82.40%	17.60%
Math and Physical Sciences	95.04%	4.96%	85.88%	14.12%
Engineering	94.45%	5.55%	93.11%	6.89%
Computer Science	88.39%	11.61%	77.27%	22.73%
Math or Science Teacher	86.84%	13.16%	82.98%	17.02%

We addressed academic preparation by examining coursework taken in secondary school, Advance Placement (AP) pass rates, SAT scores, self reported algebra skills, and end of term self assessment of preparation. We conjectured that Switchers were less well prepared than Persisters when they began their postsecondary study of Calculus I. In broad terms, this conjecture turned out not to be the case. There was no statistically significant difference between the percentages of Switchers compared to Persisters who took Calculus in high school (χ^2 (df = 1, n = 2676) = 2.12, p = .15). Similarly, the mean SAT score for Switchers (M = 642, SD = 86.97) was not significantly different than that for Persisters (M = 651, SD = 75.823), t(2710)= 2.233, p = .076. It was the case, however, that Persisters had significantly higher mean AP Calculus BC scores (but not AB scores) than Switchers. However, the number of students who took the BC exam and enrolled in Calculus I was relatively small.

To address our second research question (reasons why Switchers are choosing to not continue on in calculus), we looked at Switchers' and Persisters' responses to an end of the term survey question in which students could check off multiple reasons for not continuing with calculus. The most frequently given reason for not taking Calculus II was a changed major, with 38.9% of Switchers selecting this option. Because students were allowed to select multiple responses, we were interested to know the overlap between reasons. Specifically, we were interested in the other reasons Switchers who changed majors gave for not continuing on in calculus. As shown in Table 2, this analysis shows that of the Switchers who replied that they are not taking Calculus II because they changed their major, 31.4% also replied that their experience in Calculus I made them decide not to take Calculus II.

Table 2: Reasons Switchers Give for Not Taking Calculus II

Reason for not taking Calculus II	
My experience in Calculus I made me decide not to take Calculus II	31.4%
To do well in Calculus II, I would need to spend more time and effort than I can afford	28.7%
I have too many other courses I need to complete	27.6%

I do not believe I understand the ideas of Calculus I well enough to take Calculus II	18.8%
My grade in Calculus I was not good enough for me to continue to Calculus II	11.5%
I never Intended to take Calculus II	6.1%

This result is consistent with Seymour's (2006) finding that students frequently leave STEM majors because of their experience in their introductory courses, including Calculus I. This finding necessitates a better understanding about the nature of the Calculus I experience, which leads to our third research question.

For the third research question, we investigated several different variables to understand students' experiences in Calculus I, including student behavior in and out of class and student description of their Calculus I instruction. For example, students were asked to report how frequently they did each of the following activities during class, from never (1) to every class session (5): contributed to class discussions, were lost and unable to follow the lecture or discussion, asked questions, and simply copied whatever was written on the board. For each of these questions, we conducted an independent-samples t-test to compare responses for Switchers and Persisters. As can be seen in Table 3, there was a significant difference in the responses for the amount of time spent contributing to class discussions between Switchers and Persisters, time spent lost and unable to follow the lecture or discussion, and time spent simply copying whatever was written on the board, but there were not significant differences between Switchers and Persisters on time spent asking questions. These results indicate that Switchers report spending less time in class contributing to class discussion, more time lost and copying down what is written on the board, and the same amount of time asking questions as reported by the Persisters. Taking these together, Switchers report being *less engaged* than Persisters during class. This is despite the fact that their mathematical preparation was not significantly different from that of Persisters.

Table 3: Student Reports of in-class Behavior.

During Class:	**Persister**	**Switcher**
I contributed to class discussions.**+	2.69 (1.25)	2.47 (1.17)
I was lost and unable to follow the lecture or discussion.**	1.89 (0.99)	2.18 (1.02)
I simply copied whatever was written on the board.**	2.86 (1.36)	3.26 (1.32)
I asked questions.+	2.38 (1.12)	2.34 (1.07)

Note. * = $p \leq .05$, ** = $p \leq .001$, + indicates that Persister mean greater. Standard Deviations appear in parentheses below means.

In terms of out of class behavior, we found that similar to Persisters, about 70% of Switchers worked at a job for at most five hours per week. Moreover, the vast majority of Switchers spent about the same amount of time studying calculus as did Persisters. Moreover, a statistically significant greater percentage of Switchers reported visiting their instructor's office hours either weekly or monthly (56.2% versus 48.1%) and going to tutoring on a weekly basis (25.6% versus

15.2%) than Persisters. Thus, this data suggests that Switchers are making the effort to be successful. Compared to Persisters, they do not work more on an outside job, they are studying as much or more, and they are seeking academic help more so than Persisters. All of this, together with the reasons that Switchers give for not continuing on with Calculus, suggests that a closer look at what happens in the classroom is warranted.

To examine student reported classroom instruction, we conducted a factor analysis on the questions on the end of the term survey pertaining to instructor pedagogy. The 27 items of the student end of term survey were subjected to principal components analysis (PCA). Prior to performing PCA, the suitability of data for factor analysis was assessed and supported the factorability of the correlation matrix. Principal component analysis revealed the presence of four components with eigenvalues exceeding 1, though inspection of the scree plot revealed a clear break after the second component. The two-component solution explained a total of 46% of the variance, with Component 1 contributing 36% and Component 2 contributing 10%. There was a weak positive correlation between the two factors ($r = 0.250$). The results of the analysis are used to create new variables representing these components, hereafter referred to as "Good Teaching" and "Progressive Teaching."

"Good Teaching" included questions where students rated their instructor on the extent to which he or she listened carefully to their questions and comments, allowed time for them to understand difficult ideas, presented more than one method for solving problems, asked questions to determine if they understood what was being discussed, discussed applications of calculus, encouraged students to seek help during office hours, frequently prepared extra material, gave assignments that were challenging but doable, graded exams fairly, and gave exams that were a good assessment of what was learned. "Progressive Teaching" included questions where students rated their instructor on the extent to which he or she required them to explain their thinking on homework and exams, required students to work together, had students give presentations, held class discussions, put word problems in the homework and on the exams, put questions on the exams unlike those done in class, and returned assignments with helpful feedback and comments.

Based on the frequency distributions, we grouped these two variables into thirds, representing low, medium, and high levels of Good and Progressive Teaching. Table 4 shows how low and high levels of Good Teaching and low and high levels of Progressive Teaching relate to the percentage of students who were Switchers. The percentages in Table 4 should be compared to 14.8%, which is the switching percentage for the sample of students who responded to the instructor pedagogy questions on the end of term survey.

Table 4: Switching Rate for Low and High Levels of Good and Progressive Teaching

	Low Good Teaching	High Good Teaching
Low Progressive Teaching	18.8%	13.6%
High Progressive Teaching	14.0%	8.9%

As shown in Table 4, the type of instruction seems to make a significant in student retention (χ^2 (df = 2, n = 3294) = 10.124, p = .006). "Progressive" teaching, which includes instructional approaches that more actively engages students, is associated with lower switching rates. Indeed, a high level of progressive teaching coupled with high levels of good teaching reduces the switching rate from 18.8% to 8.9%. This finding is consistent with Tinto's integration framework

(1975), which connects persistence to students' social and academic integration; engaging students in class are at once integrating them into the academic community, both socially and academically. These findings indicate that Switchers reported having different classroom experiences than Persisters. Their instructors were less likely to actively engage them (working by themselves or with a classmate on problems, having a whole class discussion, asking students to explain their thinking, etc.), they were less likely to contribute to class discussion, and more frequently found themselves lost in class.

Ellis, Kelton and Rasmussen (2013) conducted a follow up analysis to determine if the above results are due to Switchers and Persisters being in different classes, or instead if these students are in the same class but having different instructional experiences. They found that student reports within a class are on average in agreement with instructor reports. In addition, they found that Switchers reported being less engaged than Persisters within the same classes, even when controlling for gender, major, socioeconomic status, and preparation.

Additional studies that include classroom observations are needed to further study the effect of instructional approach on student retention in a STEM major. Nonetheless, these findings are consistent with prior research summarized in the PCAST report and with the seminal work of Seymour and Hewitt (1997).

Conclusion

Up until now there has been little large-scale data collected on who elects to study Calculus I at a university. Additionally, little is known about the effect of Calculus I on student intention to pursue a career in mathematics, science, or engineering. Even information as basic as the US national success rate and the percentage of students in university Calculus I who successfully complete the course has not been reported. This large-scale national study is making a significant contribution to what we know about Calculus I (for example, see Bressoud, Carlson, Mesa, & Rasmussen, 2013).

Findings from this report illuminate the types of students who are switching out of STEM majors, as well as their experiences in Calculus I. It is clear that many of the students who intended to take Calculus II but did not were hard working and well prepared. When asked why they no longer intended to take Calculus II, Switchers reported not continuing with calculus because they changed their major, citing a negative experience in Calculus I and spending too much time and effort in Calculus I as the second and third most responses. When we look more deeply at their experience in Calculus I, Switchers and Persisters report different experiences. This suggests that instructional variables such as actively engaging students, having students explain their reasoning, etc. may make a difference in retaining STEM majors. While many may have conjectured that such a finding is the case, this is the first large scale, national study in calculus to provide data for this position.

Acknowledgements

Support for this work was funded by the National Science Foundation under grant No. 0910240. The opinions expressed do not necessarily reflect the views of the foundation.

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RELATIONAL PROBLEM-BASED LEARNING: ADOLESCENT GIRLS' EXPERIENCES WITH AN INCLUSIVE PEDAGOGY FOR MATHEMATICS

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Although the Gender Achievement Gap is closing in mathematics, the "interest gap" in pursuing STEM fields is not. One recommendation for encouraging young women to continue on in mathematical studies is to find instructional methods that allow them to feel more included in the learning process – fostering attitudes such as self-confidence, empowerment and agency. In this study, the journeys of five adolescent girls who were currently studying secondary mathematics with a pedagogical approach that sought to encourage such attitudes named Relational Problem-Based Learning (RPBL) were explored and analyzed to describe the relationship between the pedagogical approach and the girls' attitudes.

Keywords: Gender, Affect, Emotion, Beliefs, and Attitudes, Instructional Activities and Practices

In recent decades, mathematics educators have been called upon to not only change mathematics education to ensure accessibility for the diverse range of students in our culture but to allow for equitable learning environments, including changes specifically focused on underrepresented and underperforming students in our society due to gender, race/ethnicity, class or socio-economic status (SES) (Leder, 2003;McGraw, Lubienski, & Strutchens, 2006). Correspondingly, others have recognized the importance of alternative instructional methods for marginalized groups in the discipline of mathematics and ways in which they might best be served in the classroom (Gutstein, 2007; Ladson-Billings, 1995; Lubienski, 2007; Spielman, 2008). The overall goal here being to increase the number of total underrepresented students interested in continuing with future mathematical studies and increasing the diversity, both in people and ideas, in the field of mathematics, and STEM (Science, Technology, Engineering and Mathematics) fields in general – creating a better field for America and the future.

It is true that for the past decade, the achievement gap in mathematics education has been closing remarkably. Females have higher grades in math and science courses in secondary schools than males, scores between male and female students on the NAEP exams have grown closer than ever (although males are still slightly higher) and the number of women choosing certain mathematics and science undergraduate majors has increased annually (Hill, Corbett, & St. Rose, 2010). However, it has also become evident that men still outnumber women in STEM fields in graduate degrees and professional work in STEM fields. In their 2010 report, *Why So Few?,* one of the American Association of University Women researchers' conclusions is that girls' achievements and interest in mathematics are "shaped by the environments around them" (Hill, Corbett, & St. Rose, 2010). For girls especially, it seems the mathematics classroom environment has a great influence on their attitudes towards learning and it is greatly affected by the relationships and beliefs that are forged in those classrooms. Ideally, the instructional methods that are used would allow all students, regardless of gender, race/ethnicity or SES, the safe, secure space to build those relationships and beliefs that would make their learning experience optimal. Therefore, it should be a goal of mathematics educators to find instructional approaches that satisfy the relational needs of a diverse group of learners. However, given the

inequities that persist in STEM fields and the problems that exist in retaining women in STEM careers, it remains of crucial importance to examine girls' learning and paths to STEM fields of work and study.

In this qualitative study, I investigated the attitudes of adolescent girls who experienced one such instructional approach designed to satisfy these relational needs of many types of learners. Based on a fundamental theory of interhuman connection of learning, a problem-based learning classroom at the secondary level was found in which one sub-population of underrepresented students (adolescent girls) experienced mathematics in an alternate learning environment. Specifically, I desired to explore the five attitudes of enjoyment, self-confidence, value, empowerment and agency towards learning mathematics and the mathematics classroom. I addressed the following research question:

> What is the nature of the relationship between girls' attitudes towards mathematics and their learning of mathematics during and after experiencing it in an Relational Problem-Based Learning (RPBL) environment? How do they describe their experiences?

Theoretical Framework

To situate this study, and hence my own framework for mathematics education, I must put forth the following two premises as stated by Burton (2002):

- Learning in the mathematics classroom is social, not individual
- Coming to know mathematics depends upon active participation in the enterprises so valued in that community of mathematics practice that they are accepted within that community

Within this view, mathematics knowledge is understood to be constructed within the classroom community in which it exists and a learner "knows" mathematics based on the values that are prescribed within that community. Further, and in contrast to a more traditional view of learning mathematics, I situate mathematical learning, and learning in general, within the context of the greater relational approach to knowing – where "knowers are social beings-in-relation-to-others", and these relationships must be built on respect and care, not oppression and power (Thayer-Bacon, 2004). According to this view, education has a relational character and it is just that relationship between the teacher and the student, and even possibly the student and her classmates, that affords the community the opportunity for the interaction in education (Biesta, 2004). The task then is to craft a pedagogical framework for mathematics instruction that facilitates relational learning and construction of knowledge and it should incorporate the ideologies that enable as many students as possible the freedom to create those connections and relationships.

A Pedagogy of Feminist Relation

My theoretical framework , which includes relational trust (Bryk & Schneider, 2003; Raider-Roth, 2005), relational authority (Bingham, 2004), relational equity (Boaler,2008), voice and agency (Taylor & Robinson, 2009), has at its roots what was historically known as Feminist Mathematics Pedagogy, stemming from the gender difference movement of the '90's (Becker, 1995; Boaler, 1997; Burton, 1995; Solar, 1995; Willis, 1996). The intersections and overlaps of these constructs are not coincidental and therefore, I cannot ignore the feminist influence which initiated the concept of valuing learning from a humanistic standpoint and appreciating

individual differences and voices within the mathematics classroom. Solar (1995) posited an inclusive pedagogy based on a postmodern epistemology and identified concrete attributes that characterized the "four dialectical aspects" of feminist pedagogy : (a) passivity and active participation, (b) silence and speech, (c) omission and inclusion, and (d) powerlessness and empowerment. The framework is also corroborated by another model of a feminist mathematics classroom (Anderson, 2005) in which empowerment, agency, development of authority, valuing of intuition, and honoring of voices were key components. The characteristics described in both of these models are consistent with the main tenets of postmodern feminist epistemology that resists dichotomous thinking and focuses on subjective thought and multiple perspectives (Hesse-Biber & Leavy, 2007). Further, it is well known that most traditional pedagogies in mathematics include process-driven and rather objective perspectives of mathematics which create "environments in which most students surrender agency in order to follow predetermined routines" (Boaler & Greeno, 2000).

However, it is important to point out the focus on the humanistic and inclusive aspect of the feminist pedagogy and not the dichotomous "feminine" view on learning. At one point in educational theory this concept of "connectedness" was specifically formalized to support women's and girls' ways of knowing and learning, and specifically in mathematics education (Becker, 1995; Belenky, Clinchy, Goldberger, & Tarule, 1986). However, more recently opponents of gender difference theory in mathematics education promote an "unfixing" of the differences "to see mathematics as an opportunity to develop relations with others and re-make themselves" (Mendick, 2005b). Mendick goes on to say that

> By aligning separate-ness with masculinity and connected-ness with femininity, these approaches feed the oppositional binary patterning of our thinking and in the final analysis reiterate it (p 163).

Supporters of this more humanizing approach to the multiplicities of student relationships with mathematics agree that rethinking gender differences in a larger framework would benefit both boys and girls. It may be possible to do this if mathematical learning were viewed in less of an oppositional way (male vs. female, objective vs. subjective, etc.) and in more of a interhuman relational way – appreciating all of the various needs of connection including being "authentic" and "feeling seen" by the other (Raider-Roth, 2005). So although this framework has its roots in and draws support from Feminist Mathematics Pedagogy based on "women's ways of knowing" it is in not fully dependent on or continually presumptive of it.

Recent interest in fostering mathematical practice standards of perseverance and reasoning through problems has motivated many teachers to investigate the instructional method often referred to as Problem- or Project-Based Learning (PBL). Originating almost thirty years ago in medical schools, PBL has made its way into secondary school classrooms because of its ability to engage students in the multi-disciplinary skills that are needed; including communication, collaboration, reasoning & sense-making (Savery, 2006). In secondary mathematics, little research has been done to date regarding the development of the use of this pedagogy, however implementing PBL in the high school classroom can have many mathematical advantages for learners (Schettino, 2011/2012). The discourse and opportunities for open-ended questioning are just some of the many characteristics of this pedagogy that has many intersections with the feminist and relational pedagogies described above.

Methods

For the purpose of this study, I have defined RPBL as an approach to curriculum and pedagogy where student learning and content material are (co)-constructed by students and teachers through mostly contextually-based problems in a discussion-based classroom where student voice, experience, and prior knowledge are valued in a non-hierarchical environment utilizing a relational pedagogy. A narrative inquiry methodology was chosen for consistency with the theoretical framework and for a relational way of viewing the student's learning experience in both a personal and social way. Because of my intense and intimate relationship with the school, students that were potential participants and the teachers in the study, it was important for me to be as transparent as possible in my research methods and to triangulate the data. I had been involved with the curriculum writing of the PBL materials, training of the teachers and instruction at the school in which the study was being conducted. To maximize validity, I constructed methods of data collection that covered many different perspectives of the student experience. The five young female participants in the study diversely represented the population of girls that were registered in the course that year ranging in age from 14-17. Figure 1 below shows the overall plan of data collection over a six month period for all participants who were adolescent girls in the second term of a year-long integrated algebra and geometry course at Greystone Academy – an independent, single-sex boarding and day school in the northeast United States.

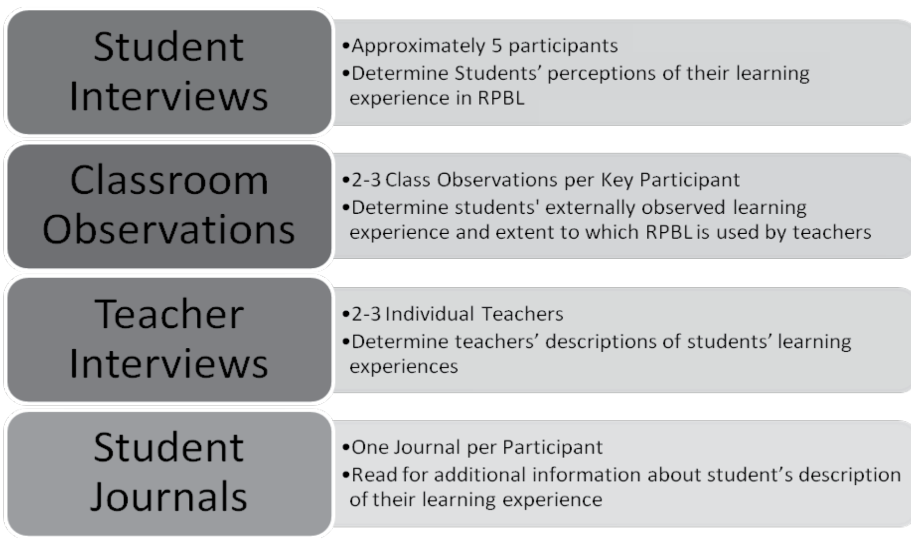

Figure 1: Summary of Purposes of Collected Data

Data Analysis: The Listening Guide

In keeping with the theoretical framework of education as a relational phenomena, I used the Listening Guide (Brown & Gilligan, 1991, 1992; Gilligan, Spencer, Weinberg, & Bertsch, 2003), a voice-centered, relational approach to narrative data analysis. With this method, a researcher employs multiple readings – or "listenings" - of interview transcripts, in each reading a different perspective of the participant's voice is identified and "listened for" (Doucet & Mauthner, 2008) because one's discourse has multiple layers. "I-Poem"s are "written" from first and second-person pronoun phrases from interview transcripts. One researcher stated that the "I-Poem" attends to an associative stream of consciousness carried by the first person voice running through a narrative, rather than being contained by the full structure of sentences" (Kiegelmann, 2009, p.77). In other words, the I-Poem allows you to see a secondary structure to the story that

is brought out by focusing on only the pronouns and how the participants is describing her experience through those voices alone. After creating and coding the I-Poems, they would start showing patterns of alternating voices, but in the third or fourth listenings, you could actually *hear* the voices in tension with each other. The themes of these poems and the poems themselves were member-checked with the participants for accuracy and meaning.

Results and Discussion

Listening to the stories of these five girls, speaking with their teachers (of whom I was one) and reading through their journals allowed me to get an intimate and full portrait of their complex experience not only within the classroom, but of their relationship with mathematics presently but from their past as well. Each girl had different experiences from their past, but it was clear, and probably not surprising, that those past experiences had ramifications on their attitudes at the present. Table 1 shows a summary of the students' varied characteristics and interests in mathematics, as well as their abilities.

Table 1: Student Participant Information

Name	Leona	Isabelle	Kacey	Sarah	Alanna
Grade	10	9	10	9	9
Teacher	(Author)	Johnson	(Author)	Brown	(Author)
Race	White	Mixed	White	White	African-American
SES	Upper	Middle	Middle	Upper Middle	Lower
Ability	Low	Middle	Low	Middle	High
Interest	Low	Medium	High	Low	Low
Boarder/Day	Boarder	Boarder	Boarder	Day	Day

The Listening Guide allowed me to code each of the pieces of data for different aspects of their stories. Their I-Poems gave descriptions with their voices both from the first and second person standpoint giving perspectives on the five attitudes that helped complete the pictures that the plot listening outlined. Although in different ways, each girl attributed some positive change in their attitudes towards learning mathematics to some characteristic of the pedagogy of the RPBL environment.

For example, Sarah, Leona and Alanna had all had prior experiences where they had not been asked to inquire in mathematics. However, when exposed to that type of thinking, they did enjoy it, especially in a relational way. Leona's innate talents of communication and debate made her a natural for discussion and when she realized she could apply this to mathematics class, it became much more fun and comfortable. Once Sarah was encouraged to follow her natural sense of inquiry and extension questions, she enjoyed being valued for those skills. Alanna needed to know that her high intelligence was of use in a problem-based classroom by allowing her independence to work for the class instead of being a distraction in a direct-instruction classroom.

With respect to confidence, the only girl in the study who was of high mathematical ability was Alanna and she claimed to have strong self-confidence ("I was never not confident"), but

would also claim inconsistently her fear of some types of mathematics. So even in a girl with high ability, there was some residual feeling of uncertainty or cultural norm that she wasn't supposed to feel secure in her skills in mathematics. The other girls grew in their concept of self-confidence throughout the year to different places, but their sense of strength seemed to stem from their experiences relating to their peers in the classroom. The more the internal process of learning was made external by discussion, problem presentations, verbal questioning and answering by students, the more they were able to "see" that process unfolding and being constructed and feel more confidence that everyone was on the same page, or least that it was out in the open. Kacey's confidence, which was such an integral part of who she is, allowed her to make the most of the relational aspect of this course. She questioned freely, conjectured regularly and took risks openly in almost every class. It was extremely natural for her and she showed others how to "do mathematics" in many ways.

The girls' sense of empowerment seemed to change by viewing mathematics through a lens of multiple perspectives, valued inquiry and connection to prior knowledge. All girls expressed appreciation for the opportunities to look at problems from multiple solution methods that originated with student presentations. One of Alanna's I-Poems moved from the second person disassociative "you" voice towards the first person, showing more ownership for her learning in the class.

more opportunity for you to affect your own learning
you get another idea
you just do it
you see something
you notice things for yourself

I like to see it
I notice things for myself
I have a better grasp
I'm just learning
I'm just that person
I think
I think I'm that independent
I want to learn it for myself

Agency was viewed differently by many of the girls depending on how much agency they had come to this RPBL class with initially. All of the girls' views of the five attitudes that were triangulated with their teachers' comments, my classroom observations and their journal writings are summarized in table 2.

The RPBL Framework

From the stories these girls told I could draw conclusions about certain aspects of the RPBL that connected to each of their attitudes, as they mentioned them over and over. The tenets of RPBL pedagogy that coincided most often with the codings from the girls' stories were: 1) Connected Curriculum 2) Justification not Prescription 3) Ownership of Knowledge Production and 4) Shared Authority. It is imperative that the curriculum used be based on scaffolded problems that are decompartmentalized such that students can appreciate the connected nature of the mathematics they are learning. The instructor also needs to set a culture of a natural focus on "why" in solutions and foster inquiry with interesting questions that value curiosity and assess creativity. The whole classroom community and structure needs to encourage individual and

group ownership by the use of journals, student presentations, extension questions, teacher wait time, and other discourse moves. The purposeful dissolution of any authoritarian hierarchy with deliberate discourse moves to improve equity and send the message of valuing risk-taking and all ideas will created a sense of shared authority.

Table 2: Cross-Student Comparison of Five Attitudes in Learning

Student	*Relationship with Enjoyment*	*Concept of Self-Confidence*	*Vision of Value*	*Realization of Empowerment*	*Journey towards Agency*
Sarah	With support and praise, surprisingly enjoyable when allowed to follow her own inquiry	Still developing, mending the damage of comparison, competition and silence	Math is more of a tool for science, but problem solving is a life skill that is valuable	Realized strong voice of inquiry for extension questions, leadership skills and ability to take risks	Still dependent on teacher for validation, but strong increase in independence and interdependence in directing learning
Leona	Enjoyment comes from deeper understanding, reasoning – knowing the "why" and communicating	Improvement because the class played on her strengths of communication and relational skills	Math is 'something you must do', but the problem solving skills help in other disciplines	Strong newfound sense of her own voice in the classroom, but grappling with her power in her learning	Relational nature helped bring out her own agency in learning, appreciated multiple perspectives immensely
Isabelle	Views problem solving as fun, like doing a puzzle, enjoys relating others by helping and being helped	Sense of confidence not necessarily related to mathematics, but gave her an advantage in discussion and risk-taking	Sees the relevance of big picture problem solving or calculation skills, but does not see its necessity	Realized the possibility of the subjective nature of mathematics and how voicing her own ideas and/or listening to others added to the learning experience	Students can be more agentic because their understanding is "out in the open" - no one "keeping up appearances of understanding since they are constantly questioning
Alanna	Pride, satisfaction and fulfillment from accomplishing learning independently, in her way and above her expectations	Confidence in her ability remained constant, but concept of self as a student changed drastically (see empowerment and agency)	Mathematics is a worthwhile subject to study, but the class was always boring until RPBL	Being an independent learner doesn't mean working alone, it means collaborating with your peers and building knowledge together.	Seeing things with "her own eyes" is more meaningful than "being told" by a teacher, found appreciation for agency in her learning
Kacey	Enjoyment flourished even though her achievement did not	Initial strength did not wane and allowed for risk-taking and sustained inquiry	Mathematics value is in its "beauty" not necessarily in its everyday usefulness or applications	Grows from a individual and group sense of ownership of knowledge that is co-constructed, stems from connections made through the problems	Juggled balance between independence and interdependence in her learning – but the agency and direction in learning came from her and her peers.

Implications for Future Research

Because of the positive nature of the experiences of these girls in relation to mathematics and the mathematics classroom, it would be wise to do a larger scale study on these attitudes with respect to RPBL pre- and post- classroom intervention. Clearly, no intent of generalizability is implied from the narrative inquiry, only obtaining a rich description of their experiences in the hope of understanding it and their perspective of it better. However, should further research find that RPBL is an effective means by which girls' attitudes towards learning mathematics can be improved a great deal of professional development will be needed for teachers as well as curriculum work and support. All of which will need to be assessed for effectiveness and delivery within the classroom potentially for underrepresented groups of all students.

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PARENT PERCEPTIONS OF MATHEMATICS LEARNING AT HOME AND IN SCHOOL

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Parents of forty-nine elementary school students who attend an alternative urban after school mathematics program speak out about impediments of and supports to the productive teaching of mathematics and their suggestions for what is needed to help promote such teaching at home and school. This qualitative research study acknowledges parent voices in finding a common ground to negotiate potential conflicts and areas of dissonance and agreement in order to further the dialogue about creating meaningful mathematics teaching and learning for urban children.

Keywords: Affect, Emotion, Beliefs, and Attitudes, Elementary School Education, Equity and Diversity, Instructional Activities and Practices

Purposes of the Study

Parents can make valuable contributions to the discussion of improving mathematics teaching and learning. In 2011-2013 this stance led us to document our participant parent's beliefs, attitudes and suggestions about meaningful math learning at home and in school. Those data, then, form the basis of the present paper.

Theoretical Framework

The literature stresses that parents' involvement and their beliefs about mathematics relate directly to how children learn mathematics in school [Eccles, 1993]. Examining home and school contexts and parent perspectives is seen as crucial for improving mathematics learning of students whose access and engagement is limited [Allexsaht-Snider & Hart, 2001]. Schools with high levels of parental and community involvement report increased student learning, attendance and self-directed behavior [National Institute on Out-of-School Time, 2000]. Family involvement in schools is therefore central to providing high quality education for all students. However, most parent involvement initiatives assert school values over those of parents [Civil, 2001; Tate, 2005, 2007]. The results of this study support that stance.

The In Addition After School Mathematics Project is situated in a public elementary school of a major metropolitan community. In 2011/13, we served forty-nine students in grades 3, 4, and 5 and their parents. The families are of low socio-economic status. Our definition of "parent" includes any prime caregiver such as grandmother or uncle. The In Addition students are always selected by random lottery. The sole criterion for acceptance rests on a commitment for children to attend two hours a day, three days a week, from September to May. Parents and their children attend two weekend retreats, one in October and one in May. We meet with parents once a month for math workshops at the children's school. The multiethnic group of children and their parents are of African American, Latino, Chinese, and European heritage. The first language of many parents is other than English. We provide translation services where needed.

Modes of Inquiry

This narrative research study analyzed written, interview, videotaped, and focus group data provided by parents and surrogate parents of the urban after school mathematics program. In this paper we maintain anonymity of people and places as part of trustworthiness in keeping our promises to them. Further, we use the term "parents" to subsume all primary caregiver adults who provided data. We sought to amalgamate what parents reported are impediments of and supports to productive teaching of mathematics, as well as suggestions for what is needed to promote such teaching at home and school.

Data for this study were collected via open-ended questionnaires, focus group sessions, field logs, and video-taped segments of In Addition classroom events. Audio taped focus group sessions and questionnaires centered about two questions: "What helps the learning of math at home? in School?" and "What hinders the learning of math at home? in School?"

Audio and video-taped sessions were transcribed. Questionnaire results were categorized. Transcripts were coded in a qualitative, analytical process [Wolcott, 1990] by studying the text, chunking it into meaningful categories and re-categorizing into over-arching themes.

Each of three members of the research team analyzed independently. The group met to compare, contrast and discuss emergent categories and insights as well as discrepancies. This resulted in overall agreement.

Results

Following is a selection of themes resulting from our analysis that seem particularly pertinent to this presentation. They strongly emphasize that:

1. At home it is best to teach children in the same way as teachers do in school.
2. We as parents must act as substitute teachers because there isn't enough time in school.
3. We feel pressure to know the math being taught and how to teach it.
4. What helps the learning of math at home is to apply the math to real life situations.
5. We try to use similar math learning strategies that worked for us in the In Addition parent workshops.
6. To us, teachers and teaching are seen as making the difference in how children learn math.
7. What harms math learning at home is homework and lack of supportive time, space, and materials.

Discussion

Parents voiced that they accepted the tasks of parents-as teachers as their responsibilities; tasks that most probably were defined by school personnel to them and to their children. It seems reasonable to assume that school personnel saw the "parents–as-teachers" function perfectly acceptable and necessary. Because of our experience in other schools and school districts as well as the literature [Cobb, 2002; Herrold & O'Donnell, 2008] we are of the mind that this transfer of responsibility may happen with great force in many schools and with many families in many places.

If this is a trend, then the implications are obvious: math in school is seen by many as a matter of making the way through the math textbook. If there is not sufficient production and or time in school, then math is a matter to be taken care of at home by the children and their parents. The message to parents would be that they need no special math skills or pedagogical talents to complete the work other than those they possess already. This conclusion is bolstered by our data wherein parents described that they receive no or limited direction and materials such as textbooks for use at home.

The pathos inherent in these findings is that our parents buy into them. And why shouldn't they? Most probably this pattern has been true as well for their education. In addition, they are not often invited to interact with school personnel to learn what math to teach and how to do that.

We believe parents can bridge the educational experiences of school and home and see it as imperative to incorporate parents' unique understandings into the ongoing dialogues about reform. Research also indicates that in order to enhance the educational experience of "disadvantaged" students and increase access, equity, and engagement for all students inclusion of parents in the education of their children is essential [Moles, 1993; US Department of Education, 1997]. However, our parent data results, 1, 2, and 3 (see above) are in agreement with the findings of [Eccles 1993]. In order for math reform efforts to move forward, much more attention must be paid to helping parents shift to a more problem solving vision of teaching and learning mathematics.

Our After School Math Project is based on a philosophy of how children best learn, honed over decades of professional involvement by the project founder, as original developer, and continuously shaped and reshaped by all the team members. Our philosophy echoes the recommendations of the National Research Council [National Research Council, 1989; National Research Council 2001], the National Institute on Out-of-School Time [National Institute on Out-of-School Time, 2000] and the National Council of Teachers of Mathematics [National Council of Teachers of Mathematics, 2000]. These leading organizations emphasize that it is essential to provide children with opportunities to be active participants in the development of their own mathematical understandings and to be instructed by teachers using strategies that begin with children rather than with themselves.

Our work with parents is guided by this philosophy. Our parents enter into In Addition activities with enthusiasm, interest, and deepened insights. Yet there is an irony in what we learned. With all of our efforts to work closely with parents about meaningful teaching and learning of mathematics, with all of their voiced understandings and joy, our study results; 1, 2, and 3, attest that the "old school" vision prevails when parents talk of how best to teach children at home. This seems quite reasonable. These parents deeply want their children to succeed, to "make it." And they have read their schools correctly. If teachers assign particular homework, these parents believe that it is their responsibility to take on the role of teacher no matter what is asked.

We are heartened by study results 4, 5, 6, and 7 which indicate that parents see the value of some broader, more child-centered views of mathematics teaching and learning. This illustrates the conundrum.

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ONE HIGH SCHOOL STUDENT'S DEVELOPMENT OF MATHEMATICAL DISCOURSE ON TRANSLATION

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The aim of this study is to explore one Turkish high school student's learning of the geometric concept of translation using a communicational approach. Our approach formulates learning as occurring through the changes in students' discourses as they interact with mathematical concepts. Our results confirm that the student's learning about translation was closely related to the changes in particular elements of his discourse. We conclude that the framework we used provides rich tools with which to examine stages of learning as well as the elements of discourse that play important roles in mathematical learning.

Keywords: High School Education, Learning Trajectories (or Progressions), Geometry and Geometrical and Spatial Thinking

Introduction

Transformation is one of the significant concepts in learning of geometry that allows students to develop broad concepts like congruence and similarity (Jones, 2002). There are many studies that focus on students' learning on transformational geometry from cognitive perspectives (e.g., Clements, Battista, & Sarama, 2001; Hollebrands, 2003). However, few studies have explored the issue through a socio-cultural approach. This study explores one Turkish high school student's learning of the geometric concept of translation using a communicational approach to cognition, which considers learning as the process of change in students' discourse (Sfard, 2001). This study is part of a larger research that explores students' development of discourse on transformational geometry. For space reasons, we only consider the concept of translation here and we address the following questions: How does a (10^{th} grade) student's discourse on the concept of translation develop? More specifically, how do the elements in his discourse change as he continues to participate in the discourse on translations?

Theoretical Background

In this paper, we use Sfard's (2008) communicational approach as a theoretical framework. According to this lens, thinking is a special form of communication and the individual who is thinking is considered to be self-communicating. Sfard (2008) explains communication as a dialogical endeavor; it can be intra-personal or interpersonal. Learning is about changing the ways we think and how we communicate what we think (Sfard, 2001). Learning here is conceptualized as change in one's discourse and students' thinking can be examined through their discourses (Sfard, 2001). According to Sfard (2008), discourse is a "special type of communication made distinct by its repertoire of admissible actions and the way these actions are paired with re-actions" (p.297). If the content of a discourse is mathematical, Sfard (2008) describes it as a mathematical discourse.

There are four elements of mathematical discourse: word use, routines, visual mediators and narratives (Sfard, 2008). *Word use* is one of the distinctive characteristics of discourse, the vocabulary that learners use. There are four phases in the development of word use: first one is *passive use*, where the learner is "not able use a [mathematical] word in her own speech" (Sfard,

2008, p.181). The other is *routine-driven use*. In this phase, a mathematical word is uttered to signify the learner's actions on the mathematical object. In *phrase-driven use*, "words become linked to constant phrases rather than with the whole routines" (Sfard, 2008, p.181). Last phase is *object-driven use*, where words signify mathematical objects. *Routines* are repetitive patterns in the actions of learners that are characteristic of the discourse. *Visual mediators* are visible objects that are operated through the communication process, like symbols, graphs, drawings, tables. *Narrative* is any sequence of any utterances indicating the relationships among mathematical objects.

The purpose of this study is to investigate one high school student's development of the concept of translation by using the four elements in his discourse.

Methodology

The study was conducted with one 10th grade student named Okan (a pseudonym) in a medium-sized, urban high school in Turkey. The data for this work was collected through three task-based interviews across six weeks to explore Okan's development of mathematical discourse particular to the geometric concept of translation. Okan was selected as a participant because he was observed in the classroom as an uninterested student but occasionally gave interesting answers to the teacher's questions. Although his grades were low in geometry, the informal interviews with the teacher suggested that his geometrical learning was enhanced throughout the discussions in the classroom. The researchers prepared three different task-based interviews about translations. Each interview lasted about 20 minutes and was video-taped. First interview was conducted before the teacher introduced the topic of translation in the class. Second interview was right after the translation class. The third one was conducted 25 days after the second interview.

Each interview consisted of four types of tasks on the concept of translation. Across the three interviews, each type of task was equivalent to those of the same type in terms of the nature of the translation problem presented and the skills the student needed to use when solving the problem. To ensure equivalence and the validity (Patton, 2001) of the tasks, five experts reviewed the tasks in terms of their parallelism and content. The interviews were conducted in Okan's native language and then translated from Turkish into English by the authors. Data were analyzed in terms of word use, visual mediators, routines and narratives (Sfard, 2008).

Results

We analyzed word use considering the context of the discourse in a qualitative way rather than a quantitative way. Throughout the interviews, passive word use about translation was not observed in Okan's discourse.

Interview I

During this interview, Okan's word use was mainly routine-driven. For example, when we asked him what it means to translate an object, he said that it "is the process of positioning the object from one location to another." In terms of visual mediators, Okan's focus was on the vectors and how he can split them into their corresponding *x* and *y*-components. However, his use of these mediators was not consistent. In fact, during the first interview, Okan had no consistent routine with which he translated mathematical objects. He used some strategies for the problems in a haphazard rather than repetitive way. Therefore we could not identify any routines in his discourse at this stage. Okan was also confused about which narrative to endorse regarding translation. Results indicate that, before the teacher introduced translation, Okan considered

translation as the process of repositioning the location of an object but did not have robust understanding of the concept.

Interview II

The second interview was conducted after the teacher introduced the topic of translation. In this interview Okan's word use was predominantly phrase-driven. For example, when we asked him what translation is, he said "translation is when the objects are moved towards particular directions within the coordinate system". There were also some object-driven utterances in his discourse. For example, he said that "Translation is a transformation that enables the placement of objects within the coordinate system".

There were four kinds of routines (three geometric and one algebraic) determined in Okan's second interview. Every time Okan translated a geometric shape and gave examples for translation, he used these two geometric routines (GR1 and GR2): GR1. Translating one vertice of a given geometric object along the components of the vectors visually and then using the property of translation (it preserves size and orientation) to draw the rest of the translated object. GR2. Translating each vertice of a given geometric object along the direction of the vector and carrying the vertices to its translated location and then connecting the vertices of the translated shape using the property of translation.

After Okan translated objects using GR1and GR2, the researchers asked Okan to find the coordinates of each translated vertice. Okan's responses to these questions elicited one geometric (GR3) and one algebraic (AR1) routine: GR3. Finding the coordinates of the vertices visually using the grids provided in the questions. AR1. After using GR3, Okan regularly wanted to double-check his answer algebraically by adding the coordinates of each vertice of the original shape with the corresponding coordinates of the vector.

Vectors were the primary visual mediators in Okan's discourse during the second interview. He considered vectors as objects that move the geometric shapes as indicated by one of his geometric routines (GR2). He also used vectors algebraically, treating them as ordered pairs as indicated by his algebraic routine (AR1). One of the endorsed narratives that Okan used in his words, routines and visual mediators was: "Translation is the placement of objects in the coordinate system".

When we compare Okan's discourse in this interview with his first interview, we see the following changes in the elements of his discourse: (a) His mainly routine-driven word use changed to being mainly phrase-driven, and (b) his haphazard solution strategies changed to consistent routines in his discourse. He became close to objectifying translation as a form of transformation that preserves the size and orientation of any given geometric object.

Interview III

The third interview was conducted 25 days after the second interview. In this interview, Okan's word use was mainly object-driven. When talking about translation, Okan frequently referred to the translated geometric objects as the end states of the translation process. There were also some phrase-driven utterances in his discourse. For example, while Okan was translating a geometric object, he said "When an object is translated, area, circumference, and the distance between the points [of the original shape] do not change".

In this interview, we identified one geometric routine in Okan's discourse. He used this routine in three of the tasks on which he worked. Throughout the tasks, his approach was generally based on geometric rather than algebraic routines. His geometric routine was GR1. In terms of visual mediators, Okan mainly operated with vectors geometrically rather than algebraically as indicated by his routine GR1. One of the most frequently endorsed narratives in

Okan's discourse was: "If the object is translated...the distances between the points [vertices of the original geometric object] do not change."

When we compare Okan's discourse in this interview with his second interview, we explored the following changes in the elements of his discourse: (a) His mainly phrase-driven word use changed to being mainly object-driven, (b) his four different types of routines (GR1, GR2, GR3 and AR1) were reduced and stabilized to one geometric routine (GR1), and (c) his phrase and object-driven utterances consistently treated translation as an object long after Okan's second interview. Accordingly, his endorsed narratives started resembling actual mathematical facts rather than idiosyncratic utterances he endorsed at the very beginning of his learning process.

Conclusion and Discussion

In this study, we explored Okan's learning of the geometric concept of translation, which was conceptualized as the change in his discourse (Sfard, 2008). Note that this framework does not only focus on students' use of language but also takes into account other elements in their discourse such as their actions and visual mediators. Our analysis of Okan's discourse revealed that these elements of discourse are in close relationship with each other as he shifted his routine-based discourse to an object-based one in the context of translations. The learning trajectory Okan went through in the course of the interviews resulted in changes in each element of discourse, indicating that students need to command all the elements of mathematical discourse in order to enhance their thinking (Güçler, 2013).

One critical aspect of mathematical learning is to change the talk about processes of mathematics to the objects of mathematics (Sfard, 2008). Okan's discourse reflected such development as his routine-driven word use gave way to phrase- and object-driven word use. For the concept of translation, this means a shift from considering translation as actions formed on geometrical objects to considering translation as a distinct mathematical object that belongs to a class of other mathematical concepts (i.e., transformations). A detailed analysis of student discourse helps identify the characteristics of students' trajectories and discourse regarding these important conceptual shifts in transformational geometry.

Acknowledgments

This study is supported by the Scientific and Technological Research Council of Turkey (TUBITAK).

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ADDRESSING THE REVERSAL ERROR WITH A VISUALIZATION TOOL

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In this study, we examine engineering students' engagement with the common Student-Professor type problems using a web-based tool designed to develop a conceptual understanding of formulating valid mathematical models. We analyze one student's thinking on five tasks with this tool. Findings suggest that interacting dynamically with verbal, visual, symbolic, and numeric representational spaces of such problems allowed the student to better understand the relationship between coefficients and variables.

Keywords: Algebra and Algebraic Thinking, Modeling, Technology

Background

Consider the well-known problem: Write an equation to represent the following statement: 'There are six times as many students as professors at this university.' Many studies have examined the "student-professor problem" and issues surrounding it (see e.g., Clement, 1982; Cohen & Kanim, 2005). In spite of its seeming simplicity, this problem and its variations (henceforth called SP-type problems) have caused difficulty for students at all levels. The most common error is the *reversal error*; writing $P = 6S$ instead of $S = 6P$. This error has been studied repeatedly, yet its complexity continues to intrigue researchers (Cohen & Kanim, 2005). We have encountered this error while working with college engineering students. In this paper, we focus on the research question: in what ways can an interactive visualization tool help college students analyze and makes sense an SP-type problem? Because of the long history of student difficulties with SP-type problems, we believe this is a useful place to focus our attention.

Many researchers have attributed difficulty with SP-type problems to a "direct-translation" approach (Clement, 1982; Fisher, Borchert, & Bassok, 2010). With this strategy, one makes the sequence of algebraic symbols match the sequence of objects in a word problem. For example, given "There are six times as many students as professors," the reversal error is made by translating directly to 6 times S equals P. MacGregor and Stacey (1993), however, found that when phrasing questions so that a direct translation produces a correct response, students still tended to make the error. They attribute difficulties to an underlying cognitive model of mathematical relationships in which models are based on comparison rather than equality. Others have identified this phenomenon as using a "static comparison" strategy (Clement, 1982; Cohen & Kanim, 2005), where "$6S$ " and "P " are treated as six students and one professor respectively (instead of as the number of students and professors). The equal sign is seen here as representing correspondence rather than equality (Cohen & Kanim, 2005; Palm, 2008).

Two distinct behaviors have been seen in students who successfully solve simple word problems: An *operative approach* involves the use of a hypothetical operation that produces an equivalence relation (Palm, 2008). Fisher et al. (2010) found that writing a non-standard relationship by developing such equivalence relationships decreased the appearance of the reversal error significantly. The second pattern is what Clement (1982) called the *substitution pattern*, where students substitute numbers into the equation's variables and then engage in the operative approach.. In this study, we encourage students to engage in the patterns suggested above through the use of visualizations of the problem space that we create though our Room-Metaphor and Test-Case approaches (described below) in a computer environment. Yazdani (2008) also found that asking students to draw a picture, figure, table or graph

greatly improved students ability to grasp problems on which reversal errors had previously been made and found that visualizations helped students detect errors and see limitations in previously used strategies. In this study, we focus on a visualization of the mathematical expressions themselves in an effort to further enhance students' abilities to solve word problems without the reversal error.

Framework

Zazkis, Dubinsky, and Dautermann's (1996) Visualization/Analysis (V/A) model suggests that an integration and interplay of visual and analytic thinking is often needed for understanding mathematical concepts. The V/A model begins with an act of visualization, which can be the actual drawing of a picture or the expression of a mental image. This is followed by an act of analysis in which the person reasons about what was visualized. Then follows a second visualization step, enriched by the analysis, which then leads to a second act of analysis, etc. (Stylianou, 2002). As the learner develops a clearer understanding of a mathematical problem, the iterations increase in sophistication and enhance conceptual understanding (Zazkis et al., 1996; Stylianou, 2002). Stylianou (2002) expanded the framework to identify particular "analysis steps" which we have elected to use as codes (Table 1) in our data analysis to address ways in which visualization helps a learner understand SP-type problems.

Table 1: Data Codes

Code	Definition
IAC	Collect new info from a visualization to understand implications of prior actions;
MEI	Use new info gained to further explore the problem; Use patterns to refine visualization
SNG	Create a visualization to better understand the problem or next steps in problem solving
MSP	Pose questions (to one's self) to check understanding and verify correct procedures.

Methods

We interviewed engineering students as they solved SP-problems to identify possible designs for our tool, resulting in our development of two interventions to use in tandem. First, we developed an interface to prompt the use of test cases and provide instant feedback when they were not compatible with constructed equations. This strategy was included with the intention of encouraging students to adopt a substitution pattern of thinking (Clement, 1982). We also developed a second, pictorial approach called the "room metaphor" approach, which involves showing a variable as a room that holds some number of boxes. This room metaphor addresses the static comparison approach mentioned above and can help students build an understanding that a variable is not just an object, but also a "container" of objects. When an equation is entered and test cases are specified and "hovered" over with the mouse, black boxes appear in the "rooms" to show a variable has taken on a value. If values correctly satisfy the equation, black boxes will fill the blocks perfectly. For a mismatch between equation and test cases (e.g., $P = 6S$, $P = 1$ and $S = 6$), empty spaces appear as a visual cue to indicate an error (see Figure 1).

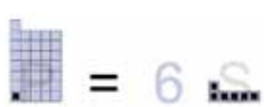

Figure 1: Feedback With Incorrect Equation And Correct Test-Case Values

To test the tool, we created five SP-type problems (see Table 2).

Table 2: Tasks Used in the Study

P1. A shoe store has four times as many pairs of leather boots as suede boots. Write an equation to represent this relationship.
P2. The gravity on the earth is six times greater than on the moon. Write an equation about this relationship.
P3. A country exports twelve times as much oil as coal. Write an equation for the relationship between the amount of oil and coal.
P4. The Yi Company produces chairs that have a seat piece and four leg pieces. Write an equation representing the relationship between the number of seats and legs.
P5. A landscape designer is designing a garden. He plans to insert six cherry trees for every ten holly trees. Write an equation using C and H to represent the above relationship.

Findings

We share our analysis of one case study, Nicole, a junior Industrial Engineering major. In P1 and P2 Nicole was able to create a correct model with little trouble. In both cases, there was evidence that she took time to think about the problem before coming up with the model. For example, she told the interviewer, "I had to figure out which side the six and one need to be on." On problem 3, however, Nicole encountered difficulty. Table 3, shows segmented transcript lines of this problem with accompanying data codes.

Table 3: Transcript Analysis of P3

Line	Transcript	Code
1	**N:** A country exports 12 times as much oil as coal. So coal equals oil	SNG
2	And if you have one coal, then you have 12 oil.(types $C = 12O$)	
3	If you have 12 coal, you have one oil (Types C=12, O=1 in test cases).	IAC
4	(Pause while she studies problem and hovers over test cases).	
5	If you have...you know you need more…. Twelve times as much oil as coal. So there is more oil than there is coal, so my equation is flipped?	MEI
6	(pauses) Yes! Ok, my equation is flipped.	SNG
	I: *Do you feel the picture helped you figure out there was something wrong here?*	
7	**N:** In this case it did help me since reading the problem, you need more oil than coal so when you see the boxes with variables, you know I wrote the equation wrong, because I was like, why do I have so many coals? (refers to picture).	IAC
8	I need more oils than I have coals.	MSP
	I: *If you had put the test case in first in this one instead of the equation, do you think it could have helped you avoid the mistake?*	
9	**N:** The test case might have helped me think about the equation, but it is a little confusing that the 12 goes...like in the equation it's 12 C, but when you do the test cases it's 12 under the O column.	MSP
		IAC
		MEI

In line 2, Nicole seems to have used a static comparison strategy where *C* stands for "coal," not amount of coal. She states that if you have 1 coal, you have 12 oil, but then in line three she uses the equation she has created rather than her previous thinking to put in the test cases *C*=12 and *O*=1. In line 4, something perturbs Nicole's thinking as she hovers over her test cases in the tool. She uses the visualization produced to understand the implications of her placement of the 12. In lines 7 and 8, she explains that the fact that there were more boxes attached to coal when she expected more under oil changed her thinking about the problem. Even though her test case and equation matched, the room metaphor helped her recognize an error in her thinking. Without this perturbation, Nicole admitted that she "probably would have just gone with her first answer."

In response to the interviewer's question in line 8, she identifies that it is counterintuitive to know that the test case value of *O* is 12, but that in the equation the 12 is attached to the *C*. This pattern that she now notices and reflects on in lines 9 represents a turning point in Nicole's thinking about these problems, making the MEI analysis a key step in her thinking.

In P4 and P5, Nicole repeatedly referred back to what she learned from her mistake in P3. In P4, she explains, "Well, for every one seat you need four legs. So since I messed this up in the last equation, I know that 4 goes with the *S*, and that equals *L*." From her discussion, it seems reasonable to assume that Nicole is not just creating the correct equation by blindly following a new procedure she developed in her work on P3, but can explain the relationship between S and L as an equivalence relationship rather than as a ratio or correspondence.

Conclusions

We feel the dynamic connection of numeric, verbal, and symbolic representations helped Nicole interact with the problem more efficiently. She used visualizations to collect useful information, allowing her to view the problem from different perspectives and further understand the need for a balanced relationship in the model. In general, when working with simple linear relationships, we need to help students move away from static comparison and direct translation habits and understand the equal sign as representing a balanced relationship. We feel this tool is one useful way of helping students reorganize their thinking on these problems.

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BROADENING STEM OPPORTUNITIES THROUGH INFORMAL EDUCATION

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This report presents the results of an interdisciplinary math and science project entitled Dinosaurs, Denver and Climate Change. Children (aged 8-13) participated in multi-aged groups to learn math within the context of paleontology and climate change. Near peer mentors (NPMs) were assigned to work with small groups of students. The goals of the project were to increase STEM content knowledge among participants and to impact STEM attitudes among NPMs. STEM professionals in geology and soil science led field-based lessons relative to their disciplines. This paper reports on the results of the fall 2012 Saturday STEM program that focused on geology and paleontology.

Keywords: Informal Education, Affect, Emotion, Beliefs and Attitudes

Introduction

STEM occupations are projected to grow by 17% from 2008-2018, compared to 9.8 percent growth for non-STEM occupations, and STEM jobs are expected to grow three times as fast as non-STEM jobs (Langdon et al., 2011). While our economy depends on the talents and skills of STEM workers, only 3% of Blacks and Hispanics hold technical or STEM jobs (Baine, n.d.). We believe preparing students of color to fill STEM jobs in the U.S. is a priority. One barrier, however, is curricular support for student-directed, inquiry-based science rarely occurs in our nation's schools (DeJarnette, 2012).

To address this problem, we developed an out-of-school community-based program called Dinosaurs, Denver and Climate Change (D2C2). Our goals were twofold: (1) to impact the interdisciplinary content knowledge of math and science for children in grades three to eight; and (2) to influence the attitudes of near peer mentors (NPMs) toward STEM and STEM careers. The NPMs in this study were high school and college students (aged 16-22) who had strong interests in STEM or education.

Theoretical Framework

Place-based education emerged in rural Alaska in the 1980s (Howley et al., 2011) and operates under the assumption that education is not *place-less* but grounded *somewhere* in order for learning to take place (Aikenhead, Barton Calabrese, & Chinn, 2006). Likewise, place is not devoid of culture. Culture and place are intertwined because *place* incorporates all of the spaces in which we live and interact—locally and globally (Aikenhead et al., 2006; hooks, 2009; McInerney, Smyth & Down, 2011). *Belonging: A Culture of Place* captures this concept well as bell hooks reclaims *place* by returning to her birthplace in rural Kentucky after an academic career in New York. The places where children reside—urban, suburban or rural—create opportunities as well as challenges when it comes to learning outcomes and educational attainment (Morris & Monroe, 2009; Tate, 2008). We use the framework of place within a sociocultural context to examine and explore student learning in authentic settings where underrepresented minority students can develop literacies in STEM (Aikenhead et al., 2006; McInerney et al., 2011).

Literature Review

Research suggests that the optimum time to create interest, awareness, and connections to STEM careers is during the elementary school years (DeJarnette, 2012). Exposure to STEM content and

inquiry in math and science at earlier ages provides a foundation for interactive problem-based learning (DeJarnette, 2012; Leonard, Barnes-Johnson, Dantley, & Kimber, 2011). Science camps and afterschool programs provide a context for younger children to develop "cognitive abilities to engage in STEM content and problem solving activities" (DeJarnette, 2012, p. 80). Activities such as gardening and soil sampling, have been used to facilitate cross-disciplinary applications (Czerniak, 2007) and to foster inquiry, science literacy, and a sense of place and belonging among young children (Rye et al., 2012). In such settings, children learn science processes by manipulating a single variable to learn how it impacts the results (Rye et al., 2012). In this study, we used place-based education as a hook to encourage STEM learning.

In addition to promoting STEM learning at early ages, research also shows that peer mentoring programs where teenagers mentor young children is a growing trend (Karcher, Davidson, Rhodes & Herrera, 2010). However, the findings are mixed concerning the impact that peer mentoring programs have on mentees as well as the mentors themselves. Cross-age peer mentors can develop the following attributes: competence, confidence, connection, character, caring, (Lerner et al. as cited in Karcher et al., 2010) and contribution (Karcher et al., 2010). In this study, NPMs worked with children in small groups to help facilitate STEM learning, which had the potential to improve their competence, confidence, character and caring.

Methodology

We recruited children from African American and Latino churches in Denver, Colorado, to participate in the D2C2 program. Ten children were recruited from the African American community in Northeast Denver, and 22 children were recruited from the Latino community in Southwest Denver. Because we wanted to keep the groups small and learn from our practice, we ran two separate Saturday science camps in each community—one in October and one in November. The instructor for both groups was a retired geologist and docent at Dinosaur Ridge. He developed the lesson plans, which included field-based activities that allowed the children to explore Dinosaur Ridge.

In addition to the instructor, NPMs were hired to help facilitate the learning. Eight NPMs were hired at each site for a total of 16 mentors. Seven were male and nine were female. The NPMs went through a four-hour training session that was led by the geologist to familiarize them with the lessons and the science concepts. Moreover, the NPMs were responsible for using flipcams to record what was going on in their groups during informal science lessons. We intended to impact NPMs' attitudes about STEM indirectly as they served as mentors and group leaders during the STEM activities.

Data Sources

Quantitative data sources included a pre-post test to ascertain participants' learning. The geologist developed the test, which contained 13 multiple-choice and two open-ended questions to measure children's geoscience content knowledge. Qualitative data sources included video clips, work samples, and structured interviews with the children and NPMs who participated in the D2C2 program.

Results

Since the children at both sites received the same treatment, the data were combined to analyze the results. If these data were analyzed separately, the results would suffer from low statistical power. The results of the fall study revealed significant improvement on the geoscience test: (t = -4.63, p < .001, two-tailed). The results are shown in Table 1.

Table 1: Results of Geoscience Test

Geoscience Test (N=20)	Mean	Std. Deviation
Pre-test	61.55	19.15
Post-test	71.05*	15.92

Analysis of the quantitative data revealed that children did better on the open-ended items (86% accuracy) on the geoscience post-test. Further item analysis revealed two math-related multiple-choice items were difficult for students. Seventy-two percent of the students missed the following question: Dinosaurs lived (A) From about 25,000 – 50,000 years ago; (B) About 2 billion years ago; (3) From 225 million – 650 million years ago; (D) Around 450 BC. The correct answer is C. Sixty-four percent missed another math-related question: The earth is about 4.6 billion years old. The dinosaurs lived on the earth (A) Near the beginning of time; (B) About halfway through this time; (C) In the most recent 5% of earth's history. The correct answer is C. Results of the STEM Semantics Survey (Kenzek & Christensen, 2008) revealed no significant differences among the interns' attitudes about STEM and STEM careers with the exception of technology (t = -2.27; p < .001; two-tailed), which was not surprising given the use of the flipcams.

In addition to the qualitative data, student work samples revealed fairly accurate measures and calculations to determine from dinosaur tracks if the dinosaur was walking, trotting, or running. Students measured the hip height (201–214 cm) and stride length (590–600 cm) of a Stegosaurus and divided the stride length (SL) by the hip height (h) (2.8–2.9) to determine the dinosaur was most likely trotting (SL/h ratio of 2.0–2.9). Other qualitative data included excerpts from a questionnaire (interns) and focus group interviews (student participants).

> I liked how all of us just interacted with the different types of fossils, and we all got our hands on 'em, and we all got to see. We didn't just see it on a picture or on a poster. We actually went to it and touched and saw it and took pictures of it. I liked how we just got to interact with that and measure by ourselves and not by looking at a certain picture. (D2C2 student, Group Interview)

> What interested me in the program is that it is not your average program. We actually got to learn while teaching. I did not know as much as I do now about dinosaurs, and I thought it would be fun to learn about them while working with the youth. (D2C2 intern, Questionnaire)

Discussion

The results of the pre-post tests suggest children in this study may have had limited opportunities to learn number sense and percent. Millions and billions of years is not an easy concept to comprehend. It is also a common misconception that dinosaurs lived near the beginning of time. Simply *teaching by telling* students about the Jurassic and Cretaceous periods, creating a timeline, and seeing a visual display at a museum are not enough. Results suggest that STEM educators should use *place* to develop rich meaningful activities to ensure students' understanding. For example, a Facebook page now has a timeline from birth to the present of significant events in a person's life. Five percent of the timeline would be an example of when dinosaurs lived on the earth. Moreover, *culture* may be used to help children to understand academic language in STEM. What cultural term may help children to understand the continuum of time? *Before zero-dark thirty, zero-dark-thirty, and after zero dark-thirty*?

The results of the qualitative data reveal the salience of place for the participants. Being outdoors and actually touching the fossils was important to several children. For the interns, both place and competencies emerged (Karcher et al., 2010). Being able to make a *contribution* and gaining *competence* and *confidence* was important to the interns.

Significance

The findings presented in this brief report show that *place* is important in the learning context, particularly in STEM education. The backdrop of Dinosaur Ridge provided the children with remarkable experiences. Mathematics teachers and educators should not minimize the value of connecting math concepts (i.e., measurement, number, & number sense) to place-based experiences. D2C2 provided the resources for underserved youth to learn about geoscience by directly experiencing it through Colorado's natural resources.

Acknowledgments

We acknowledge Cara M. Moore, Ph.D., for analyzing quantitative data. The material presented is based upon work supported by the National Science Foundation, award number (GEO-1260957). Any opinions, findings, and conclusions or recommendations expressed in this publication are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

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STUDENTING: THE CASE OF HOMEWORK

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Studenting is comprised of the behaviours that students perform or exhibit in a learning situation, such as the classroom. Student actions that do not contribute to actual learning and that subvert the intentions of the teacher are a subset of studenting behaviours that we call gaming behaviours. In the research that we present here we confirm a taxonomy of studenting behaviours around the way grade 10-12 students do homework. This paper describes these gaming behaviours and reports results showing that 65% of studenting behaviour observed subvert the intentions of the teacher.

Keywords: High School Education, Instructional Activities and Practices

Introduction

The term studenting was coined by Gary Fenstermacher to describe what students do to help themselves learn. These student activities include recitation, practice, seeking assistance, reviewing, checking, locating sources and accessing material, among others (Fenstermacher, 1986). However, studenting also encompasses behaviours that are less concerned with learning and more concerned with:

> 'psyching out' teachers, figuring out how to get certain grades, 'beating the system', dealing with boredom so that it is not obvious to teachers, negotiating the best deals on reading and writing assignments, threading the right line between curricular and extra-curricular activities, and determining what is likely to be on the test and what is not. (Fenstermacher, 1994, p.1)

The concept of studenting allows us to broaden our exploration into, and discussion about, what students do in learning situations, such as the classroom, and specifically those features of student behaviour that do not align with the goals of the teacher. Consider the following anecdote:

> With about half an hour left in class Mr. Count assigns some homework. Nadine spends the class time carefully writing out the assigned page and problem numbers, neatly writing her name and the date, and reading her notes. Later that evening she neatly copies the procedures that her tutor directs her to do. She feels that because she 'did' the problems, she understands the concepts.

From Mr. Count's perspective, although she did not work during class, Nadine has learned the material – she has completed the homework problems and shown all of the work. From Nadine's perspective, she is reducing her overall effort by waiting until she has her tutor with her to guide her. As long as she has fulfilled her obligation to 'do' the problems and has written up the solutions, Nadine feels she has met the expectations. There is a rationality to Nadine's actions, but she is not learning, at least not how Mr. Count intended. She is studenting, and in a way that subverts the teacher's intentions yet appears to achieve the required outcomes, thus beating the system.

It is exactly these aspects of studenting that we are interested in. More specifically, we are interested in the studenting behaviours that are not in alignment with the teacher's goals and expected actions, yet may be missed by the teacher during the activities of teaching. We refer to

this class of studenting behaviours as gaming behaviour, as in the students are gaming the system. The notion of gaming extends constructs such as didactic contracts (Brousseau, 1997) and norms (Cobb, Wood, & Yackel, 1991; Yackel & Cobb, 1996).

Methodology

Data for this study is taken from a larger, ongoing project studying studenting behaviour within several mathematics classrooms across a large number of contexts. A grounded theory approach is used to continually analyse data as it is collected. Over time, this analysis has revealed a number of interesting student behaviours within different contexts. The ongoing coding of the variety of studenting behaviours becomes easier as behaviours become known and eventually a saturation point is reached when new observations in a particular context reveal no new studenting behaviours.

Context

Homework is something that every teacher in our study assigned on a regular basis. In some classes homework is worth marks and in other classes it is not. After studying studenting behaviour with regards to homework in a number of grade 10-12 (ages 15-18) classrooms we had reached a saturation point around the context of how students engage in this activity.

Data

The data for what we present here comes from five different classrooms. There were two grade 10 (ages 15-16) classrooms, two grade 11 (ages 16-17) classrooms, and one grade 12 classroom (ages 17-18). In the two grade 10 classrooms and one of the grade 11 classrooms homework was checked and it was worth marks. For two of the teachers homework was checked randomly on occasion and a completion mark was assigned (or not). In the third classroom homework was handed in once a week and a number of questions were selected at random and marked. In the remaining two classrooms homework was not checked and not marked.

In each of these classrooms we interviewed 20 students selected at random. The interviews were short (1-4 minutes) and were audio recorded using a portable digital recorder. For the most part these interviews consisted of a brief answer to the questions "Did you get your homework done for today?", "If so, how did you go about getting it done?", "If not, why not?". Depending on their answers to the second question posed to them there were more questions regarding their reasons for their behaviour. These interviews occurred in the moments before the beginning of the lesson, during homework checks, within breaks in the lesson, and immediately after the lesson if the school schedule permitted it. This was not foreign to the students as the lead author had spent several lessons doing similar questioning in the same classes regarding a variety of learning contexts. In all, data from 100 interviews was collected by the two authors over the course of two lessons per classroom. Added to the above data were lengthier interviews with the teachers regarding their intentions for assigning homework prior to the lessons in which we collected the data as well as their responses to the results of our analysis. These interviews were also audio recorded.

Analysis

As mentioned, these data were analysed immediatiately after collection. Initially these were analysed using a framework of grounded theory. However, as the study progressed, and a taxonomy of behaviours began to emerge, subsequent data was analysed using a framework of analytic induction (Patton, 2002). “[A]nalytic induction, in contrast to grounded theory, begins with an analyst's deduced propositions or theory-derived hypotheses and is a procedure for verifying theories and propositions based on qualitative data” (Taylor & Bogdan, 1984, p. 127 cited in Patton, 2002, p. 454). In this case, the theory informing the analysis was the emergent

taxonomy of studenting behaviour with regards to homework. This is not to say that we were not open to the emergence of new themes, because we were.

Results

From the analysis of the data a taxonomy of four behaviours emerged: *didn't do it*, *cheated*, *got help*, and *did it on their own*. In many ways, these behaviours are obvious. However, beneath each of these obvious studenting behaviours lay less than obvious results revealed following deeper investigation and our analysis. Due to space constraints we include here only the table summarizing the distribution of behaviours (table 1).

Table 1: Distribution of Behaviours

Discussion

	Marked (n=60)	Not Marked (n=40)
Didn't Do It	**15**	**16**
I forgot	5	3
I was busy	4	2
I tried, but I couldn't do it	3	3
I took a chance	3	0
It wasn't worth marks	0	8
Cheated	**14**	**1**
Copied	7	1
Faked	5	0
Half homework risk	2	0
Got Help	**18**	**12**
Felt they would fail quiz	6	1
Felt they would pass quiz	3	3
Felt they would excel at quiz	9	8
Did it On Their Own	**13**	**11**
Mimicked from notes	4	5
Did not mimic from notes	6	6
Mimicked but completed	3	0

Although the sample sizes were too small to make any broad generalizations we did notice some very stark contrasts between the classrooms where homework was marked versus those classes where homework was not marked. In particular, there was a clear difference in the number of students not doing the homework and cheating. This prompted us to consider the behaviours across the marked/not marked divide (see table 1).

Looking at these numbers it is clear that a larger percentage of students in the classes where homework was not worth marks didn't do homework (40%) in comparison to the classes where homework was worth marks (25%). This is not surprising. We know that marks are an incredibly powerful tool for motivating student behaviour. But does it motivate the behaviour that we want? Cheating and getting help with homework only for the purposes of collecting marks are behaviours that all five teachers agreed subvert their intentions, albeit in ways that are much less visible to them. Taking these three behaviours together (not doing homework, cheating, and getting help only to get marks) provides quite a different set of results. For the classes where homework is worth marks these three behaviours account for 58% of student behaviour vis-à-vis homework, whereas in the classes where homework is not worth marks they only account for 45% of the behaviours.

Present within both the marked and unmarked classes was evidence of mimicking. Mimicking occurs when the students rely not on understanding as much as simply following the

solution pattern laid down by the teacher in an example problem. We know from prior research (Liljedahl & Allan, under review) that this behaviour may subvert a teacher's intentions. So it was in this study. The five teachers in this study all stated that the purpose of homework was for students to test their understanding of, and abilities with, new content. Mimicking, they felt, does not provide students with an accurate indication of either their understanding or their ability. As such, if we factor these into the above analysis then subversive behaviours account for 70% and 58% of the behaviours for the classes where homework is worth marks and the classes where homework is not marked respectively. Combining the two groups of participants we get that 65% of the students exhibited studenting behaviour that was viewed by both us and the teachers as subverting the intentions of the teachers.

Conclusions

These findings are consistent with our research in other contexts as well. Across the board students are finding ways to subvert the expectations of the teacher in ways that the teacher is not aware of. In many cases these behaviours are centred on proxies for learning and understanding, such as mimicking and overemphasis on note-taking, that are not actually conducive to learning – but appear to be in alignment with the teacher's goals.

From the perspective of the student, however, there is a certain rationality to their actions that we are trying to understand using theories from behavioural economics, such as minimisation of effort, economy of action, bounded rationality (Simon, 1955), loss aversion, and risk aversion.

From the perspective of the researcher we are also exploring ways to think more deeply about the classification of some of these behavioral categories as subversive. For example, although there were two students who were playing the system by completing only half of the assigned homework it is not entirely clear to us if they are subverting the teachers' intentions, or the opportunity to learn, or both. Likewise, the three students who used mimicking in their homework extensively but then extended their learning from these solutions in order to complete questions not demonstrated in class were perhaps not subverting the teacher or their learning.

Finally, it is worth noting that since we brought to the attention of the teachers in this study the results of the analysis and the emergent taxonomy they have all begun to explore alternatives to their current practices vis-à-vis homework. It seems as though the kind of knowledge generated by research into the gaming aspects of studenting behaviour can be a powerful catalyst for initiating teacher change.

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RACE-PLACE IN MATHEMATICS PARTICIPATION: LATINA/O UNDERGRADUATE STUDENTS' NEGOTIATIONS OF RACIALIZED SPACES AND IDENTITIES

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This study investigates four Latina/o undergraduate students' perceptions of how their negotiations of racialized spaces throughout their academic and mathematical lives impacted their identity development as mathematics learners. Racialized spaces were analyzed in: 1) broader societal, community, family, and school contexts, 2) a required calculus I course, and 3) an optional Emerging Scholars Program, calculus I workshop. Participants revealed that powerful racialized experiences tied to broader contexts both hindered and supported their strengthened mathematics identity development. While (re)negotiating and resisting a subset of these experiences in racialized spaces in the workshop contributed to strengthening participants' mathematics identities, negotiating racialized spaces in the required calculus I course did not have this same impact.

Keywords: Equity and Diversity, Gender, Post-Secondary Education

Purpose and Background

In mathematics education scholarship, additional information is needed regarding the complexities underlying Latina/o students' mathematics participation and learning processes that contribute to their mathematical success. Although there is evidence that Latina/o students' experiences in broader sociopolitical contexts (e.g., societal, community, school, family) impacts their mathematics achievement levels (Ortiz-Franco & Flores, 2001), additional in-depth explorations of how Latina/o students manage experiences within broader contexts in relation to their mathematical development are needed. Although sociopolitical constructs (e.g., race, gender, and class) influence underrepresented students' mathematics participation and learning (Martin, 2009), there is a lack of information about how Latina/o students' negotiations of racialized, gendered, and classed spaces in mathematics classrooms influences their mathematical development. Viewing mathematics classrooms as comprised of racialized, gendered, and classed spaces and situating them within broader sociopolitical contexts can aid in uncovering information about how and why Latina/o students succeed in mathematics.

This study investigates four Latina/o undergraduate students' perceptions of the racialized spaces they negotiated throughout their academic and mathematical lives. I explored how participants managed racialized spaces in: 1) broader societal, community, family, and school contexts, 2) a required calculus I course (lecture and discussion section), and 3) an optional Emerging Scholars Program (ESP), calculus I workshop. I aimed to understand how their negotiations of racialized spaces within each context and across all contexts influenced their mathematics identity constructions. This analysis contributes to a deeper understanding of linkages among Latina/o students' negotiations of racialized spaces, their mathematics participation and learning, and their success in culturally diverse, collaborative math classrooms.

Perspective

A deeper understanding of the mechanisms underlying Latina/o students' mathematics participation and learning that contribute to their mathematical success can be achieved by using an equity lens that embraces the sociopolitical turn in mathematics education. Advocates of the sociopolitical turn in mathematics education (Gutiérrez, 2010) support the use of critical race theory (CRT) and Latina/o critical theory (LatCrit) to examine marginalized students' mathematics experiences. In this study, applying CRT and LatCrit (Solórzano & Delgado Bernal, 2001) allowed for capturing Latina/o students' perceptions of the historical, sociopolitical, and situational nature of racialized spaces, including the salience of gender, class, and power.

In this study, spaces refer to "intersections of symbolic (personal, social, ideological, political) and physical dimensions, where issues of power, hegemony, agency, structure, control, and knowledge define boundaries that have varying degrees of flexibility and permeability" (Varelas, Kane, & Pappas, 2010, p. 4). Since school spaces are racialized (Barajas & Ronnkvist, 2007), the calculus contexts examined in this study are seen as comprised of racialized spaces (while also embracing how gendered and classed experiences may function within these spaces). Research has documented how Latina/o students manage racialized (Fernández, 2002), gendered (Jilk, 2006), and classed (Ortiz-Franco & Flores, 2001) experiences in mathematics classrooms.

Method

This study is part of a larger study on Latina/o students' identity and participation in an ESP calculus I workshop at Hall University, a predominately White university in Chicago, Illinois (Oppland, 2010). In this larger study, I analyzed how participants co-constructed mathematics and racial identities, how their workshop participatory trajectories developed over time, and how their mathematics and racial identities strengthened their participation. This study expands upon prior findings by examining how four Latina/o students negotiated racialized spaces throughout their academic and mathematical lives. Participants include Vanessa[1] (18-year-old female), Lilliana (19-year-old female), Immanuel (19-year-old male), and Dante (18-year-old male).

Participants were simultaneously enrolled in a required calculus I lecture and discussion section, and an optional ESP calculus I workshop. The workshop was modeled after Uri Treisman's doctoral dissertation work, which aimed to discover explanations for why African American students were struggling to learn calculus at the University of California, Berkley in the mid-1970's. Drawing on this model, the optional mathematics workshops at Hall run parallel to students' standard mathematics courses and encourage culturally diverse peer groups to collaboratively solve challenging worksheet problems. Workshops are held for two hours either once or twice per week, and are worth 1 credit hour (satisfactory or unsatisfactory). The workshop in which this study took place consisted of 27 students. Class demographics were approximately 41% Latina/o (4 females and 7 males), 30% Asian (5 females and 3 males), 22% White (2 female and 4 males), 7% African American (1 female and 1 male), and 44% female.

This qualitative study used multi-case study analysis (Yin, 2009) and counterstorytelling methods (Solórzano & Yosso, 2002). Three semi-structured interviews were analyzed to construct an in-depth case study for each participant. Data analysis involved coding interview data, constructing thematic memos, and writing narrative summaries that described the racialized experiences participants negotiated in broader contexts, the racialized spaces participants managed in the mathematics classroom contexts examined, and how this management of

[1] Vanessa and the names of all people and places used in this article are pseudonyms.

racialized spaces within and across these contexts impacted their identities as mathematics learners. A cross-case analysis was also conducted on the four case studies.

Results

Each participant negotiated racialized academic and mathematical experiences in all the broader contexts examined. Table 1 displays examples of racialized academic and mathematical experiences participants managed in broader contexts that acted as possible avenues or barriers for their strengthened mathematics identity development. Participants tended to describe more potential barriers than avenues for their strengthened mathematics identity development in *societal*, *community*, and *school* contexts. Participants also tended to describe more avenues than barriers for their strengthened mathematics identity development in *family* contexts. These racialized experiences also intersected with gendered and classed experiences in some instances.

Table 1: Examples of Racialized Academic and Mathematical Experiences Negotiated

Broader Sociopolitical Context Examined	Examples of Racialized Academic and Mathematical Experiences Negotiated (Possible Avenues for Strengthened Mathematics Identity Development)	Examples of Racialized Academic and Mathematical Experiences Negotiated (Possible Barriers for Strengthened Mathematics Identity Development)
Societal	**[Immanuel and Lilliana only; Vanessa and Dante did not provide examples for this category]** Had an awareness of broader societal messages that supported the notion that fair educational, mathematical, and/or occupational opportunities existed for all students, regardless of their cultural backgrounds	**[Vanessa, Immanuel, Lilliana, Dante]** Had a keen awareness of broader societal messages that supported the notion of a racial hierarchy of academic and math ability that positioned Latinas/os (and African Americans) as inferior and other cultural groups as superior (e.g., Asians)
Community	**[Vanessa, Immanuel, Lilliana, Dante]** Encountered community members who resisted negative societal messages about their academic and/or mathematics abilities	**[Vanessa, Immanuel, Lilliana, Dante]** Believed they experienced a lack of academic and math resources and/or role models
Family	**[Vanessa, Immanuel, Lilliana, Dante]** Believed their parents motivated them to succeed in education, mathematics, and life	**[Vanessa, Immanuel, Lilliana, Dante]** Believed their families' socioeconomic status levels contributed to creating academic and mathematical disadvantages
School	**[Vanessa, Immanuel, Lilliana, Dante]** Viewed their interactions with culturally diverse peer populations in math classrooms as a positive experience (this allowed some participants to challenge math stereotypes about certain cultural groups)	**[Vanessa, Immanuel, Lilliana, Dante]** Experienced lower-quality educational and math experiences in K-12 settings, including due to weak curriculums, less-qualified teachers, lack of funding, tracking practices, and/or racialized experiences involving teachers and/or peers

Racialized experiences participants negotiated in broader contexts that seemed to act as significant barriers for their strengthened mathematics identity development included: (a) negotiating socially constructed meanings projected onto the Latina/o marker related to academics and mathematics in societal, community, and school contexts and (b) negotiating racialized school experiences. Racialized experiences participants negotiated that appeared to serve as important avenues for their strengthened mathematics identity development included: (a) resisting negative racialized experiences in societal, community, and school contexts and (b) negotiating positive racialized experiences in family, community, and school contexts.

Participants revealed that a subset of racialized experiences they negotiated in broader contexts became manifested in the workshop context, including experiences that seemed to act as significant barriers for their mathematics identity development. However, they provided powerful accounts of how they (re)negotiated and resisted harmful racialized experiences tied to broader contexts mainly through peer interactions, which strengthened their identities as mathematics learners (including as Latina/o). However, in their lecture and discussion sections, where participants encountered fewer opportunities to interact with peers, they did not describe their negotiations of racialized experiences that were disjoint from their workshop experiences as playing a significant role in strengthening their mathematics identity constructions.

Conclusion

This study raises important considerations regarding the significance of understanding *what it means to be* a *Latina/o* with respect to mathematics participation, learning, and success. Participants' counterstories reveal how Latina/o students might (re)negotiate and resist racialized experiences through peer interactions in diverse, collaborative mathematics learning settings situated in broader sociopolitical contexts. This study suggests that providing Latina/o students with opportunities to engage in mathematical group work may offer them additional opportunities (as compared to within traditional mathematics learning contexts) to negotiate meanings for race and racism in ways that supports their strengthened mathematical identity development. However, this study also suggests that even if ESP-type learning environments assist marginalized students in overcoming racialized barriers and experiencing mathematical success, additional support systems within the broader sociopolitical environment, including within larger mathematics department settings, are needed to minimize or eliminate the inequities they experience as mathematics students in our schools, societies, and world.

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USING CLINICAL INTERVIEWS TO UNDERSTAND STUDENTS' LEARNING PROCESSES IN A TECHNOLOGICAL ENVIRONMENT

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The goal of this study was to better understand student learning when using TI-Nspire to study quadratic functions. We conducted clinical interviews with pairs of students who worked on three activities aimed at understanding the effects of each coefficient in the quadratic function. Data analysis revealed various ways students interact with multiple dynamically-linked representations and how they interpret the results of their actions on those representations.
]

Keywords: Technology, Clinical Interviews, Multiple Representations

Introduction

The National Council of Teachers of Mathematics (NCTM, 2008) asserts "technology is an essential tool for learning mathematics in the 21st century… When technology is used strategically, it can provide access to mathematics for all students" (p. 1). Technology can help students visualize abstract mathematical concepts and enable them to explore and conjecture. *Common Core State Standards* (2011) echoes this view: "When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data" (p. 7). These statements bring important questions to the forefront: Do students using technology (e.g., TI-Nspire) require scaffolding to support their mathematical learning? If so, what sort of scaffolding do they need?

A change in individuals' mathematical thinking as a result of activity/experience supported by technology is a theme that emerges from research (Heid & Blume, 2008). Moreover, they cite the effective use of multiple dynamically linked representations as one of the primary features of mathematical technology enabling this procedural-to-conceptual shift.

The main goal of this study was to better understand students' learning processes when TI-Nspire is used to conduct an activity on quadratic functions. Therefore, our research question is: When students use dynamically linked multiple representations, what limits their ability to bring mathematical concepts to their consciousness (Heid & Blume, 2008)?

Conceptual Framework

In the mathematics education literature, we see the dichotomy of external representations, the signs and symbols we create for ideas, versus internal representations of those ideas that exist only in our minds (Janvier, Girardon, & Morand, 1993). Similarly, NCTM (2000) confirms this dichotomy by defining representation as referring "both to process and to product—in other words, to the act of capturing a mathematical concept or relationship in some form and to the form itself." (p. 67). Technology changes the forms and capabilities of external representations.

Zbiek, Heid, Blume, & Dick (2007) view technology as a cognitive tool enabling students to access, experience and study mathematics at new levels. Dick (2008) posits that effective use of technology in an educational setting should have pedagogical, content, and cognitive fidelity. According to Dick, when students learn mathematics, they act on objects, observe the results of doing so, and reflect on its meaning. He also argues that student reflection feeds a prediction-conjecture-testing cycle which may lead to acceptance or rejection of the conjecture. From this perspective, Dick links what it means to be a pedagogically faithful technological tool to student

perceptions of the tool as enabling the creation of objects and actions on those objects, as well as providing evidence of the consequences of those actions.

Chua and Wu (2005) introduced a similar framework for constructivist learning using a technological tool. Their framework includes exploring, conjecturing, and verifying, followed by generalizing. When exploring, students inquire about and investigate a given task. Next, they conjecture, making inferences and judgments using intuition or evidence. When verifying, students test and check “the appropriateness and reasonableness of their conjectures” (p. 389). Finally students are asked to generalize their solution, thereby moving the task to higher ground.

Combining frameworks, experimenting in a technological environment is cyclic (see Figure 1). It starts with a carefully designed task. How we scaffold student work is as important as the affordances of the technology. Before students use the technology, we may ask them to predict what will happen. Asking for predictions prepares their minds for the task and makes use of prior mathematical knowledge. Next, they interact with the technology. Doing so offers them opportunities to create and act on external representations. Observing the consequences of those actions, students might see a pattern leading to a conjecture. Then they might repeat several cycles of exploring, conjecturing and testing, especially if their conjecture does not hold. Once they solidify the conjecture, they can generalize via proving or refuting the conjecture.

Figure 1. Framework

Data Collections and Analysis Methods

The participants in this study were 7th or 8th grade students enrolled in pre-algebra or algebra in a public school near a large city in the Midwest. The study was conducted in the early spring of two consecutive school years. The capabilities of TI-Nspire calculators make novel classroom activities possible. Students can dynamically manipulate a graph and observe the effects of that manipulation on its symbolic form. We created activities using TI-Nspire in which students used dynamically-linked representations to drag and bend the graph of a quadratic function to study the relationship between each of its coefficients and the graph (see Özgün-Koca & Edwards, 2008, Edwards & Özgün-Koca, 2009). The students had previously studied quadratic functions.

We conducted clinical interviews of pairs of students. For most of the students, this was the first time that they had used TI-Nspire. Interview questions mirrored the activity. Follow-up questions elicited students’ thinking. Each interview was video-recorded. The interviews took about 50 minutes each. Investigator field notes and students’ worksheets were also collected.

Video recordings of the clinical interviews were transcribed. To analyze these data, the investigators developed a set of codes used to reveal patterns and themes. Then each investigator independently coded the data with about 75% agreement. When they disagreed, final codes were negotiated. If only one of them coded an item, they negotiated an agreement. Using the conceptual framework, we sought evidence of students exploring, conjecturing, testing, or generalizing. Investigator triangulation (Denzin & Lincoln, 1994) and code checking (Miles & Huberman, 1994) were used to ensure the trustworthiness of the data analysis.

Results

We saw clear patterns related to four points in our conceptual framework: task, conjecturing, testing, and exploring. When students formed conjectures, we observed conjectures that were viable, semi-viable or naïve. We also saw signs of how much scaffolding during a task was needed to support exploration. In this paper, we focus only on conjecturing, and within that, we only share results related to naïve conjectures. Students' conjectures were coded as naïve if they were based only on superficial visual features, were limited by constraints of the technological design, or were obstructed by the presence of multiple dynamic linked representations.

Superficial visual features such as a parabola appearing larger or smaller capture students' attention first. In one of the activities, students studied the effects of changing *b* in $y=ax^2+bx+c$, while keeping *a* and *c* constant. When studying the effects of changing *b*, most students talked about the parabola becoming larger or smaller when the value of *b* changed.

Male Student: When you move to the right the …, the parabola goes to the left …
Investigator: Ok, just left….
Male Student: It goes up, too ... it gets smaller….

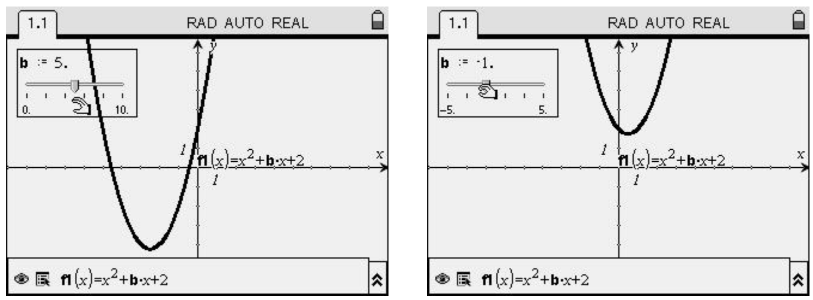

Figure 2. Studying the Effects of b

This is not wrong in a fixed window. However, we observed this repeating pattern in the data analysis. Participants generally started with one visual conjecture that is easily noticeable. Having made that one conjecture, some of them did not go any further. The main question is how to help our students to ask more "what if?" questions.

Technology can be both supportive and problematic. TI-Nspire makes it possible to grab a parabola from its vertex and translate it. The algebraic form then automatically changes to the vertex form and is continually updated. When studying how moving the graph of a parabola up and down would affect the algebraic form, the interface did not allow the parabola to be easily dragged in only the vertical direction. While trying drag up and down, there was usually also a slight movement right or left. This caused problems when we asked them to conjecture about changes in the algebraic form that resulted from dragging the parabola. Even though this is more of a technologically instigated problem, it shows the importance of the design of the task.

When there are more representations, there are more places on the screen to look. But the question is whether students are looking where we would like them to look. To study the effects of changing *b*, we designed a slider to change *b*. The goal was to create a dynamic environment. Moreover, we intended to help students connect the *value* of *b* and the location of the graph. Many made the connection between the graph and the *position* of the slider or some just focused on the slider itself (see Figure 2). Our observations confirmed the notion of Zbiek et al. (2007) that creating another representation such as a slider to control a parameter creates some added challenges. Our challenge was how to focus students' thinking on the changes in the graph and the value of *b*.

Discussion

Studying learning in a technological environment is complex. Many features of an activity greatly affect students' learning experiences in a technological environment. Our analysis of these clinical interviews showed us that students' often make naïve conjectures that interfere with their ability to bring mathematical concepts into their conscious awareness.

Clinical interviews helped us to see things we had missed in earlier whole class situations. Although each student had a calculator in the whole class situation, even with two of us in the room, it was impossible to observe how individual students were exploring and conjecturing. Moreover, most of the models explaining learning in technological environments include notions similar to exploring, conjecturing, generalizing, and so forth. However, we need to dissect these models and investigate in more detail what is happening when students make conjectures. In this study, we were able categorize some students' conjectures as naïve and suggest potential reasons that led students to make naïve conjectures. However, we had a limited number of clinical interviews conducted within a 55-minute period. Moreover, we were not able to record students' actions on the screen. More studies are necessary to better understand different characteristics of the task and technological environment that help or hinder students' experimenting processes.

The tendency of many students to make naïve conjectures clearly implies that students need scaffolding when learning in a technological environment. It appears to us that issues related to the capabilities and limitations of the technology can be at least partly mollified by acknowledging them up front. In this way, not only the technological, but also pedagogical design of the activity becomes very important. We believe that we have started to uncover some crucial aspects affecting students' experiences in technologically rich environments. However, more studies with a range of various technologies for studying different mathematical concepts are necessary to improve our conceptual framework, as well as to increase our understanding of students' learning processes in technologically rich environments.

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DOCUMENTING SUPPORT FOR CHILDREN'S MATHEMATICAL THINKING IN LOW-INCOME, AFRICAN-AMERICAN CHILDREN'S HOMES

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The purpose of this study was to explore ways in which low-income, African American mothers supported their children during an in-home activity that had the potential to elicit mathematical thinking. Six mothers were given recipes for playdough, necessary supplies, and audio recorders to capture interactions with their children while making and playing with the playdough. Analysis of transcripts showed a great deal of support offered, including eliciting mathematical thinking and talk, making connections between the activity and other contexts, and extending children's participation. These findings suggest that educators need ensure that early mathematics curricula and interventions elicit and build on parent strengths.

Keywords: Early Childhood Education, Equity and Diversity, Informal Education

Within the mathematics education community, a great deal of concern has been expressed about differences in mathematical performance across demographic groups that appear in assessments given relatively early in children's schooling (Jordan, Kaplan, Olah & Locuniak, 2006; NRC, 2009). There is some disagreement in the field about the factors that contribute to these early differences. Some researchers have argued that the quantity and quality of mathematical interactions in the homes of children located differently in terms of race, ethnicity, language, family-education, and class are central to understanding early performance gaps on mathematical assessments (e.g., Starkey & Klein, 2004; Saxe, Guberman & Gearhart, 1987). Others have argued that there are few important differences in children's homes in relation to early mathematics and that these differences appear primarily when children enter school and are then exaggerated by differential schooling practices (e.g., Ginsburg & Russell, 1981; Street, Tomlin & Baker, 2008). Understanding the role of early mathematics experiences in diverse homes is central to planning preschool and kindergarten programs, family outreach efforts, and early intervention around mathematics. The purpose of this study was to contribute to this line of work. We asked the following research question: What mathematical events occur when parents engage with their children in a guided play activity designed to elicit mathematical thinking?

Mathematics and Play

Recent research suggests that early mathematical play does lead to future learning, although there have been far fewer studies of mathematical play in homes than in schools. Levine, Ratliff, Huttenlocher and Cannon (2012) found that children who played with puzzles during observations of their at-home play performed better on a test of spatial skills given months later. Similarly, Ramani and Siegler (2008) found that playing Chutes and Ladders at home correlated with performance on tests of numeracy (unlike play with other kinds of games). In addition, a great deal of work has demonstrated that play with various kinds of blocks supports later skilled performances in mathematics (e.g., Wolfgang, Stannard & Jones, 2003).

This research suggests that we need to know more about the mathematical play children engage in while at home. However, few studies have examined mathematical play in this

context. One exception is Tudge and Doucet's (2004) study of the kinds of play engaged in by 39 diverse preschoolers, which found that about 40 percent of the children engaged in some kind of mathematical play during observations, and that while there was a great deal of variation in the amount of mathematical play observed across families, these differences were not explainable by race or class. In an ethnographic study that followed children between home and school, Anderson and Gold (2006) found that many of the mathematical practices children engaged in at home were not leveraged in school.

Interventions designed to increase the frequency or quality of mathematical engagement at home have tended to emphasize teaching parents how to do mathematical activities with their children. For example, in describing the home component of a broader study on early mathematics, Starkey and Klein (2004, p. 107) write "in a typical class, four mathematics activities were presented and parents were taught how to engage in these dyadic activities with their children." Few studies have sought to document strengths parents bring to mathematical interactions in the home – particularly in minority and low-income families – or to leverage these strengths in work with children and their families.

Mathematical Events and Practices

In this study, we draw on Street, Baker & Tomlin's (2008) work examining numeracy practices at home and at school in Great Britain. Street, Baker and Tomlin see numeracy practices as embedded in "context, values and beliefs, social, and institutional relations" (2008, p. 17). This means that when examining differences in mathematical practices enacted in homes or at school, analysis must include a recognition of the purposes the mathematics serves in the contexts and the social positions participants are seeking to occupy (knowledgeable teacher, good mother, etc.). This perspective encourages researchers to see multiple practices as productive, even if those practices may not be sensible in other contexts. Following Street, Baker and Tomlin, we talk about *events* rather than practices in this paper to differentiate between events, which are observable by anyone, and practices, which require knowledge of the intent of the participants. Although parent interviews are planned as part of this study, they have not yet been conducted. Street, Baker and Tomlin refer to "numeracy" practices and events, largely in response to particular conversations in the UK. However, we chose to use "mathematical events" because we want to be explicit that our focus includes geometry, measurement, and spatial relationships, as well as number.

Methods

The participants in this study were invited from a cohort of children that the research team has been following through ethnographic observation at school for three years. The current cohort (numbers and demographics have varied as children have moved in and out of the school) includes 16 children, six girls and 10 boys. The class includes 13 African American boys and girls, one European American girl, one Hispanic girl, and one Indian-American boy. All children in the class are eligible for free- or reduced-lunch prices. Families whose children attend the school live in one of the most rural counties in the state and many parents are raising their children in the community in which they were raised. Nearly all of the children in the cohort have siblings or cousins in the PK-12 school, which serves fewer than 300 students and has one classroom for each grade level in the elementary band.

Our study used a project reported by Anderson, Anderson and Shapiro (2004) as a model. These researchers sent home mathematics-related children's books with four dyads of middle-

class parents and children and asked parents to audio record their reading of the book to their children. For our study, all parents in the cohort were invited to participate in the playdough activity during a family math night or through a letter sent home. Seven mothers volunteered. These mothers were given a recipe for making playdough, the necessary supplies (including measuring cups and spoons), an audio recorder, and a disposable camera. The written directions asked parents to make the playdough with their children and to audio record the process as well as their children's play with the dough afterward. As part of the IRB process, parents knew the research team was interested in informal mathematics, but received no instruction about how to incorporate mathematical thinking into the activity.

Audio files were transcribed by the research team and transcripts of the activity were coded using a qualitative software analysis program. To the extent possible, the team identified mathematical events in the transcripts by attending to mathematical content (number, measurement, etc.) and mathematical thinking (reasoning, problem solving). The identified episodes were then read more closely to name and classify different kinds of events.

Supporting Mathematical Events

As with the Anderson, Anderson and Shapiro (2004), we found a great deal of variation in the mathematical events engaged in by various parents and children. However, looking across all seven families, we did find some kinds of events in common across multiple families, including creating connections between the activity and other experiences, naming mathematical vocabulary and teaching content, and encouraging active participation.

Creating Connections

Several parents connected making the playdough to other experiences cooking and eating food. For example, when starting the activity, Harris's mother said, "You know how papa watches those chef shows and you see them measure stuff out and they do it like that?" Harris responded: "I'm on a chef show!" Both Harris and his mother returned to this idea during their time cooking and playing. This context not only connected the mathematical activity to a real world example, but also gave Harris's mother a way to emphasize the importance of making an exact measurement. In other families, mothers encouraged children to create various kinds of food, which created opportunities for children to engage in visualization and geometric reasoning, as they had to hold the image of the discussed food in their minds and then use the playdough to create a three-dimensional representation.

Naming Vocabulary and Teaching Content

All of the mothers and children engaged in mathematical events that required the children to count, and typically for a necessary purpose such as counting the drops of food coloring that needed to go into the playdough, rather than for a created context such as counting the number of objects made. Parents also used a variety of words related to shape, space, and measurement, including circle, square, longer, shorter, bigger, cup, half, tablespoon, and teaspoon. Some mothers seemed to make particular efforts to repeat words that may have been unfamiliar to their children. For example, when measuring the oil for the playdough, Steven's mother said: "Where is the teaspoon? We need one teaspoon of baby oil." And then, apparently after the oil was poured added, "This is one teaspoon of baby oil."

Beyond using mathematical vocabulary, several parents also made an effort to teach mathematical concepts to their children. Several parents carefully introduced the measuring cups and spoons to their children naming each one and asking children to point out which was needed.

Some parents also invited their children to think with them about how many scoops would be needed to create 2 teaspoons or about how to create a variety of shapes with the playdough.

Encouraging Active Participation

Most of the mothers worked to actively engage their child in the activity, whether through assigning tasks in the cooking, such as finding the ½ cup measure and filling it with salt, or by encouraging the children to extend or deepen their play. Mothers asked their children what else they could make or suggested ideas like "making shapes" or "an omelet." Some mothers also seemed to play with their children and their own creations offered new ideas to their children, and also opportunities for comparison. For example, Tasha said to a playmate "oh my mom made a flat pancake and I made a bigger pancake than that." And after Harris made two French fries his mom commented: "Those two fries are just hanging out by themselves. They look so lonely. They don't have no friends." This comment encouraged Harris to continue his engagement with the activity. He replied: "I am going to put a long one. A long one."

Discussion

Documented differences on early measures of mathematics achievement have led some to speculate that minority and low-income children may not be receiving mathematical support in homes. However, ethnographic studies have suggested that children may be receiving support, but that this support may not align with school mathematical practices. This study supports the later viewpoint and suggests that educators in designing early mathematics curricula and parent workshops could seek to build on the productive support already present in children's homes.

Acknowledgement

This material is based upon work supported by the National Science Foundation under Grant No. 84445.. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of NSF.

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MATHEMATICAL SOCIALIZATION THROUGH GAME ACTIVITY: COMMON CORE STANDARDS FOR MATHEMATICAL PRACTICE IN AN AFTER-SCHOOL CLUB

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In this qualitative study, we examined the interactions and language tensions of three Spanish/English bilingual fourth graders and an adult facilitator in an after-school mathematics club as they played The Tower of Hanoi, a mathematical game. We mapped participants' actions to the Common Core State Standards for Mathematical Practice to identify the potential for mathematical socialization through game activities. We found that participants engaged in the following mathematical practices: breaking the problem into simpler versions; asking useful questions; communicating strategy with precision; evaluating peer solutions, explanations, and strategies; using counterexamples; and persistence in problem-solving.

Keywords: Elementary School Education, Informal Educational, Standards (broadly defined), Problem Solving

Introduction and Theoretical Perspective

This study provides research on the intersection of mathematics socialization with game activities that have not been sufficiently addressed in research. Although there is general agreement that children's mathematical content learning is enhanced through game activities, it is not clear how children engage with mathematical thinking and practice during game activities. This investigation provides a first step towards building an understanding of the possibilities for enhancing language minority children's development of mathematical thinking and practice through game activities.

We focused our analysis on video data of one group of four Spanish/English bilingual Mexican American females, three 4th graders and their undergraduate facilitator, as they engaged in a game activity based on the *Tower of Hanoi* (TOH). This vignette is part of a larger data set of video, field notes, interviews, and student artifacts captured over a four-year period during an after-school program. Educational activities were organized around playful mathematics enrichment activities in a bilingual atmosphere using games, puzzles, and projects in small groups and one-on-one interactions.

Our research question explores social aspects of cognitive processes in the transition from play activities to more abstract thinking. We ask, what is the value of game activities for socializing children in mathematical thinking and practice? Our study aims to uncover styles of interaction that contribute to the socialization of mathematical thinking and practice at a microgenetic level. The quote below gives background to the basis of our question, which is based on sociocultural (Vygostsky, 1987) and cultural historical activity theories (Leont'ev, 1981).

> Learning how to use a symbolic tool, play a game, deploy a strategy, carry out a complex intelligent action, and understand social and psychological functioning are cognitive attainments. These types of skills and understanding can only be obtained through social

> processes and are not available through solitary activity or introspection. The reason is that symbolic representations, like language, and cultural practices are social constructions; rules of play are social conventions; and modes of informal and formal learning by observing and interacting with others are social processes. Complex intelligent actions often entail preferred ways of doing things, which include behavioral sequences, symbolic and material tools, and a formulation and interpretation that reflect the language, values, and practices of the group. (Gauvain & Perez, 2007, p. 607)

Identifying and disaggregating the acts and actions participants use in pursuit of goals is a fundamental objective in analysis according to cultural historical activity theory.

There are three main reasons for our focus on mathematical practices as they are defined in the Common Core State Standards. First, there is a high demand for research that maps the Common Core State Standards to mathematics pedagogical practices and learning activities for classroom and out-of-school contexts. Second, the standards succinctly summarize principles of mathematical thinking and practice found in mathematics education research (Harel, 2008; Schoenfeld, 1992). Third, there is a need to explore and expand on the value of activities that counter traditional back-to-basics trends by uncovering alternate and effective routes to development and learning (Fisher, Hirsh-Pasek, & Golinkoff, 2012). It is important to note that we are not advocating for game activities to supplant classroom instruction. We view mathematical game activities as an important enrichment to mathematical learning and socialization.

Data and Methods

The participants in the vignette *What's the Point?* used a competitive participation style for playing TOH. The three children and facilitator were all bilingual Spanish/English speakers with varying degrees of proficiency in each language. All the children attended the same school where the after-school program took place. The after-school sessions occurred twice per week for 10 weeks of every school semester for 90 minutes in a classroom at the children's school.

The title of this vignette comes from one child's repeated use of the phrase, "What's the point?" Maritza, the author of this phrase, was the only one of three children who reached a successful solution. During the game set up sequence, Maritza demanded a more precise explanation of the game from Gabriela who introduced the game to the group. As time passed, the other two children gave up trying to solve TOH and Maritza occupied herself with another game. Thirty minutes later, the adult facilitator took up TOH, reached an impasse, and announced to Maritza that they should try another game. Instead, Maritza dropped her solitary game activity and announced that she would try TOH again. She reached a solution.

Two researchers independently viewed video data of a group playing *Tower of Hanoi* and coded for language tensions and mathematical thinking practices outlined in the Common Core Standards. Researchers resolved differences in coding through iterative viewing of the data in consultation with each other. Language tensions were used to index speech acts that were involved in processes of meaning making. Questions, negation, repair practices, etc. are some of the most common building blocks of mathematical thinking practices.

Results and Discussion

Children's explanations during game activities did not on the surface appear *mathematical* because they were focused on procedure, game rules, fairness, and strategies for winning.

However, their explanations did involve understanding relations among multiple strategies. Children also marked errors in their peers' strategies, providing opportunities for reconceptualizing problems contained in the game structures. They actively explored contradictions in solutions and pursued alternative strategies. Their collaborative activities involved individual accountability and though they managed to reach consensus on many occasions, they did not reference formal mathematical concepts. Considering that this after-school program was not created to be a remedial tutoring establishment or precisely aligned with this school's classroom mathematics curriculum, children's explanations during game activities were surprisingly *mathematical* when viewed with respect to the Common Core State Standards for Mathematical Thinking. We have mapped children's actions during game activity to mathematical thinking practices in Table 1 below. All quotes are from the video data.

Table 1: Summary of Mathematical Practice Coding References

Observed Activity	CCSSI Mathematical Practices
Adult labels student "expert" and asks her to explain the game.	Try to communicate precisely to others.
Student:"If she had done it another way..."	Evaluate peer solutions and reasoning.
Student challenges peer's explanation 7 times saying "What's the point?" & "I don't understand".	Evaluation of explanation.
Student restates explanation of game rules after being challenged.	Try to communicate precisely to others.
Adult asks question using game pieces and models incorrect procedure.	Modeling how to use counterexamples.
Student asks question using game pieces, "Can I do it this way?"	Recognize & use counterexamples.
Group uses only 6 out of 7 game pieces.	Break problem into simpler version.
Students interrupt another student's turn and move game pieces differently.	Evaluate peer solutions.
Student fails to solve problem 3 times but executes correct solution (31 minutes).	Persistence in problem solving.
Student asks "…you cannot move a pile like that?" while moving game pieces.	Ask useful questions and use counterexamples.
Student says "Let me practice!" & rehearses solution using game pieces.	Persistence in problem solving.

Persistence in problem solving and resilience to error correction from other participants are two factors that we believe are crucial for successful solutions in this case. Maritza, who demanded precise explanation from Gabriela and persisted in the face of contentious repair practices from her peers, appeared to thrive on challenge. Faced with occupying the role of passive observer, she retreated to her own choice of a different game and was content to play on her own. When she observed that the adult facilitator had given up on the challenging Tower of Hanoi game, she returned to the game and persevered to reach a successful solution while negotiating her peers' relentless repair practices.

Bandura (1993) reviewed four major processes influenced by students' perceived self-efficacy; cognition, motivation, affect, and selection. He outlined how students' belief in their self-efficacy to regulate their own learning determines their aspirations, level of motivation, and academic accomplishments. This may partially account for Maritza's persistence in solving the Tower of Hanoi. Maritza's motivation to defy the facilitator's suggestion to try another game may also be explained by her sense of self-efficacy, supported by the after-school environment which is responsive to children's needs and allows them to exercise their capabilities without restraint (Bandura, 1986).

Mathematical thinking practices are not easily measured or observed in informal contexts where activities are spontaneous and child-directed. These types of activities do not resemble formal or traditional classroom models and can be easily dismissed as having no value. However, we have found important components of social, emotional, and cognitive development in this informal after-school context. As a result of our study, we believe that there is a potential for amplification and reinforcement of mathematical thinking practices in game activities.

Acknowledgements

This research was supported by the Center for Mathematics Education for Latinas/os (CEMELA), a Center for Learning and Teaching supported by the National Science Foundation, grant number ESI-0424983. Any opinions, findings, and conclusions or recommendations expressed in this paper are those of the authors and do not necessarily reflect the views of the National Science Foundation.

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THE STRUCTURE OF MATHEMATICS MISCONCEPTIONS IN ALGEBRA, GEOMETRY RATIONAL NUMBER, AND PROBABILITY

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The present study compared possible relationships between content area misconceptions in algebra, geometry, rational number, and probability to develop a conceptual framework of mathematics misconceptions. The sample consisted of 1,133 middle and high school students. We found that the explanations given by students for all tasks, regardless of the content area, were rooted in their understanding of five mathematical structures: absolute/relative comparisons, rational number meaning, additive/multiplicative units, variable meaning, and spatial reasoning. We therefore consider these five foundational structures to be critical for operationalizing mathematics understanding and developing understanding of more complex mathematical ideas. We also found that the notion that a particular misconception maps to a particular content area can often paint an incomplete picture of the misunderstandings that comprise a misconception.

Keywords: Algebra, Geometry, Rational Numbers, Probability

Researchers typically classify misconceptions by the content area in which the errors are observed. For example, Lamon (1999) considered misconceptions inherent to rational numbers while Watson and Shaughnessy (2004) focused on probability and statistics misconceptions; Kalchman and Koedinger (2005) studied algebra misconceptions; and, Clements and Battista (1992) examined geometry misconceptions.

The present study identifies links between student answers and explanations on mathematics problems to misconceptions in algebra, geometry, rational number, and probability. We also compared possible relationships between misconceptions in algebra, geometry, rational number, and probability. Our investigation was organized by two research questions:

1. What misconceptions are evident in students' explanations on algebra, geometry, rational number, and probability tasks?
2. What is the most valid representation of the relationship between misconceptions in algebra, geometry, rational numbers, and probability?

Literature Background and Theoretical Framework

Rational number misconceptions. Behr, Harel, Post, & Lesh (1992) found that difficulties with fractional meaning and equivalence lead to rational number misconceptions. Specifically, they suggested that such difficulties arise because students attempt to assign addition properties in situations when multiplication properties are required. Similarly, Steen (2007) described interpretation as a more fundamental problem for students than the computation of rational numbers.

Algebra misconceptions. Researchers have repeatedly found that the abstract nature of algebra increases its difficulty over arithmetic (e.g., Kieran, 1989). Specifically, abstractness poses an impediment that can be seen when students attempt to construct multiple representations of algebraic objects (Kieran, 1992). The primary abstract concept that confuses students in algebra is the meaning of variable (Kalchman & Koedinger, 2005), compounded by the multiple ways algebraic language is used (Kilpatrick, Swafford, & Findell, 2001).

Geometry misconceptions. Student orientation toward geometry is quite different from that of algebra: Students are often intrinsically motivated to study the properties that govern the shapes

encountered in daily life (Freudenthal, 1973). In spite of this motivational factor, students still struggle with errors and misunderstandings in geometry due to limited spatial reasoning (Clements & Battista, 1992).

Spatial reasoning begins with the differentiation between objects and representations. Objects are abstractions, ideas considered through reasoning (Battista, 2007). Representations, on the other hand, are used to signify objects other than themselves. For example, a line drawn on a piece of paper only represents a geometric line, defined in Euclidean geometry as having infinite length and no thickness.

Probability misconceptions. Probability misconceptions are tightly intertwined with algebra, geometry, and rational number misconceptions. For example, Freudenthal (1973) found that misconceptions that lead to difficulties with conditional probability arise from reliance on the representativeness heuristic, which in turn arises from over-reliance on linear proportions in algebra, geometry, and rational numbers. Falk (1992) noted that probability misconceptions such as representativeness and uniformity beliefs influence how students approach problem solving situations and misconceptions about probability distributions influence student understanding of non-linearity.

Methodology

Structural equation modeling was used to compare possible relationships between misconceptions in algebra, geometry, rational number, and probability. The present study was conducted with 1,133 students enrolled in 53 algebra or geometry middle and high school classes, taught by 17 mathematics teachers in three school districts. Because the sample is cross-sectional, causal inferences are not warranted. Conclusions are therefore limited to associations. An instrument was compiled for the present study from National Assessment of Educational Progress (NAEP) released items. All 17 items remained as given by NAEP; a prompt was included for each question asking students to explain how or why they chose their response, following the advice of Zawojewski and Shaughnessy (2000).

Data Analysis

To explore the nature of relationships between misconceptions in algebra, geometry, rational numbers, and probability, six structural equation models were compared (Figure 1).

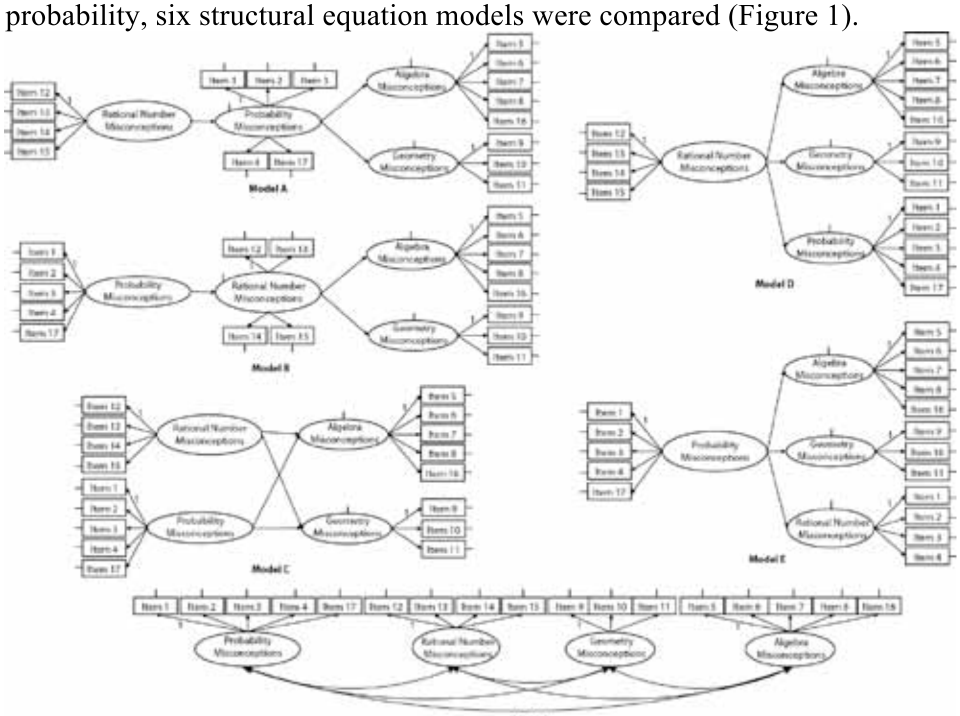

Figure 1: Six Structural Equation Models

Results

Comparisons of the six models in Figure 1 were based on the goodness of fit indices shown in Table 4. Error covariances were added based on modification indices and theoretical soundness. We based theoretical soundness on whether the same types of misconceptions were exhibited and/or the items measured the same content, with the misconception criterion taking priority. Additionally, the six data sets were randomly split into two data files each, one for calibration of the structural models and the other for validation of the resultant models to minimize the threat of over-fitting the model to the sample.

Table 4: Structural Equation Model Results

Model Description	DF	χ^2	CFI	RMSEA (90% CI)	SRMR	ECVI
Model A	**114**	**864.79**	**.98**	**0.032 (.022, .040)**	**.084**	**1.56**
Validation Model	114	764.63	.98	0.029 (.019, .038)	.077	1.34
Model B	**115**	**897.26**	**.97**	**0.035 (.026, .043)**	**.086**	**1.68**
Validation Model	115	763.73	.98	0.028 (.018, .037)	.076	1.33
Model C	**114**	**936.63**	**.97**	**0.039 (.031, .047)**	**.087**	**1.81**
Validation Model	114	745.12	.98	0.028 (.017, .037)	.076	1.31
Model D	**115**	**896.93**	**.97**	**0.035 (.026, .043)**	**.086**	**1.68**
Validation Model	115	757.73	.98	0.028 (.018, .037)	.076	1.33
Model E	**114**	**864.79**	**.98**	**0.032 (.022, .040)**	**.084**	**1.56**
Validation Model	114	764.63	.98	0.029 (.019, .038)	.077	1.34
Model F	**112**	**892.61**	**.97**	**0.036 (.027, .044)**	**.085**	**1.67**
Validation Model	112	742.57	.98	0.028 (.018, .038)	.076	1.32

Note. For the sake of space, only the final exploratory and validation models are shown.

All six models fit very well, based on benchmark criteria defined by Byrne (1998). No model fit better than the others.

Discussion

We found that the amount of overlap in students' misconceptions across content areas resulted in a high degree of multicollinearity, so we concluded that the best among the hypothesized models was Model F (see Figure 1), which modeled content area misconception factors as covarying at a single level. We concluded, however, that none of the hypothesized models represented the observed patterns of misconceptions well, primarily because none of them accounted for the observed overlap across content areas. We believe, therefore, that mathematics misconceptions are more complex than

the models indicated by previous research and that future research should investigate more complex models of misconceptions. For example, misunderstandings about the five foundational structures seemed to be at the heart of student explanations that indicated misconceptions. We therefore consider these five foundational structures to be critical for operationalizing mathematics understanding and developing understanding of more complex mathematical ideas.

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THE IMPACT OF SPATIAL ABILITY AND PREFERENCE ON PERFORMANCE IN SINGLE VARIABLE INTEGRAL CALCULUS

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The present study examines the relationships between preferred mode of thinking (analytical or graphical), spatial reasoning ability, ability to use graphical reasoning on calculus tasks, and performance in single-variable integral calculus. Data was collected from eight students. The results suggest that there may be correlations between these spatial reasoning ability and performance, preferred mode of thinking and spatial reasoning ability, as well as preferred mode of thinking and ability to use graphical reasoning on calculus tasks.

Keywords: Post-Secondary Education, Geometry and Geometrical and Spatial Thinking, Advanced Mathematical Thinking

Background

Researchers have long attempted to identify various factors that influence success in calculus. Recently, Ubuz (2011) considered factors of gender, age, major and prior achievement on success in second semester calculus and found that both gender and prior achievement impacted success. Others have considered students' preference for the use of different types of reasoning when solving calculus tasks and have found that students prefer analytical reasoning to graphical reasoning on certain integral tasks (e.g., Comacho & González-Martín, 2002; Sevimli & Delice, 2011). Following Presmeg (1985), Suwarsono (1982), and others, we believe that a student's preference for using graphical reasoning on mathematical tasks falls on a spectrum, from strongly preferring graphical reasoning to strongly preferring analytical reasoning. Combining measures of preference and performance, Haciomeroglu and Chicken (2011) looked at the relationship between performance on the AP Calculus exam, spatial ability, and preference for using graphical reasoning, and found that spatial visualization is strongly correlated to performance on the AP Calculus exam, as is preference for using graphical reasoning on calculus tasks. Differential calculus, with content parallel to AP Calculus, contains a mixture of analytic and graphical topics; the same is true of integral calculus, although one can argue that a larger portion of the content is abstract in nature (e.g., sequences and series). We are not aware of any studies that consider preference for using graphical reasoning, spatial reasoning ability, and performance in integral calculus. Are trends similar to those found by Haciomeroglu and Chicken (2011) also present when considering integral calculus students?

The goal of the present study was to examine the relationships between preferred mode of thinking (analytical or graphical), level of spatial reasoning ability, ability to use graphical reasoning when solving calculus tasks, and performance in integral calculus.

Methods

Participants

The participants were 8 undergraduate engineering students enrolled in integral calculus at a small Midwestern university. The average age at the beginning of the study was 19.1 years, with the students ranging in age from 18 to 21. The students had no integral calculus experience prior to participation in the study and were chosen from a single instructor's sections, with the

exception of one student who was added late as a replacement.

Materials and Procedure

The students met with the first author 10 times over the course of the semester. In four of the sessions, students individually completed pencil and paper assessments including measures of spatial reasoning ability and preference for using spatial reasoning. To assess preference for using spatial reasoning, students were given the Mathematical Processing Instrument for Calculus (MPIC) (Haciomeroglu & Chicken, 2011). The MPIC consists of two parts: a set of mathematical tasks and a questionnaire. The graph of a function or the equation for a function is presented and students are asked to draw the graph of either the derivative or an antiderivative. The questionnaire presents possible graphical and analytic methods of solution to each task and the student selects the solution that most closely matches their method of solution. The MPIC consists of 20 tasks, split evenly between derivatives and antiderivatives. In scoring the MPIC, students received 2 points for using a graphical solution, 0 points for an analytic solution, and 1 point if it was a mixed solution, for a total of 40 possible points (MPIC-Graph). In addition, each task was scored as correct or incorrect, for 1 or 0 points respectively, with a total of 20 possible points (MPIC-Perf). Both scores are included in Table 1.

To assess a student's level of spatial reasoning ability, four tests from the Kit of Factor-Referenced Cognitive Tests (Ekstrom, French, & Harman, 1976) were given: cube comparisons (CC), card rotations (CR), paper folding (PF), and form board (FB). These tests measure spatial orientation (CC and CR) and spatial visualization (PF and FB). Since we lack a larger population to compare these scores to, we used the means and standard deviations from Kozhevnikov, Motes, and Hegarty's (2002) study with 60 undergraduate students to compute z-scores for each of our students. Following them, we averaged the z-scores for each of the four tests to get a spatial reasoning score (SpReas). This enabled us to label students as having relatively high, medium, or low level of spatial reasoning abilities. Engineering students typically have a higher level of spatial reasoning so it was no surprise that most of our students scored above the mean.

Scored copies of the midterm and final exams were also collected. A weighted exam score was computed according to the course rubric (15% for each of the three midterms and 25% for the final exam) and a z-score was computed to compare the students' performance with the 101 total students in the class. The z-scores are recorded in Table 1.

In the remaining six sessions, the students worked in pairs to complete a series of routine and non-routine mathematical tasks centered on limits, definite integrals and approximation, improper integrals, and infinite series. Although the limit concept is differential calculus content, we included it so that connections might be made between a student's understanding of the integral calculus topics and their understanding of the limit concept. Non-routine tasks included finding connections between an infinite series and the related improper integral, for instance, $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$ and $\int_0^\infty \frac{1}{2^x}dx$ or $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$ and $\int_1^\infty \frac{1}{x}dx$. Students were paired based on performance on their first midterm exam, with the following pairings: students A and G; B and D; C and F; and E and H. Students were paired based upon performance on the first midterm exam. Each session was videotaped and coded for use of graphical reasoning as "Correct and helpful reasoning from graph" (CH); "Correct but unhelpful reasoning from graph"; "Incorrect reasoning with graph"; and "Unable to reason with graph." We summed the tallies from the last three codes to obtain an Incorrect Reasoning Summary (IR-Sum) score. We included the "Correct but unhelpful reasoning from graph" in this even though the reasoning was correct, it was not applied in an appropriate context and did not lead to a correct solution. Both the CH and

IR-Sum are recorded in Table 1. Because students were interviewed in pairs, the reasoning was tagged as coming from either student X, student Y, or both students X and Y. The authors independently coded a subset of the videos, discussed any discrepancies, and recoded until agreement was above 90%.

Results and Discussion

Each student's level of preference for using graphical reasoning when solving calculus tasks, as measured by the MPIC, is recorded in Table 1 (MPIC-Graph). We categorized two students as having a high level of preference for using graphical reasoning: students A and D; four students as having a medium level of preference: students B, C, E, and G; and two students as having a low level of preference: students F and H. Performance on the MPIC (MPIC-Perf) also varied, with three of the students performing at a very low level. The general low performance on this assessment was surprising since the graphs of derivatives and antiderivatives are differential calculus material, and we hypothesize that it is due to the amount of time since students have used these skills. The students in the study took differential semester calculus in the prior academic year and this assessment was given approximately four weeks into the fall semester.

We noticed students with lower performance on the MPIC-Graph tended to have lower levels of spatial reasoning. To investigate this further, we computed the Pearson Correlation and got a non-significant correlation (r=.535, ρ=.171). Student F is an exception to this trend and when we recomputed the Pearson Correlation after removing Student F, we got a significant correlation (r=.789, ρ=.035). We are currently extending this study with a group of 12 additional students and we hope to confirm this relationship with a larger population. Student F can be characterized as a high-ability student who is capable of using graphical reasoning, but prefers to use the more efficient analytical thinking on the derivative and antiderivative tasks. When solving tasks in interviews that required the use of graphical thinking, Student F showed an efficient use of graphics and often used only a sketch or even a portion of a sketch to gain the necessary information. This is in contrast with students like Student E and Student H who painstakingly created detailed graphs and often lost track of the goal of the task.

Another trend we noted was that students with a higher level of spatial reasoning ability (SpReas) tended to perform better on the exams (as measured by the Exams z-score). There again was not a significant correlation when we computed the Pearson Correlation (r=.437, ρ=.280), but when we removed Student B and recomputed, we got an almost significant correlation (r=.752, ρ=.051). Student B performed above the mean on the exams and had a relatively lower level of spatial reasoning ability and a higher level of incorrect reasoning on the interview tasks. The high level of incorrect reasoning on the interview tasks can be attributed to this student's reluctance to say $\lim_{x\to a} f(x) = L$ unless $L = f(a)$ (instead saying $\lim_{x\to a} f(x) \approx L$). We plan to investigate this trend further once we have a larger data pool. In particular, we would like to compare performances on the analytical versus the graphical problems on the midterm and final exams with spatial and logical reasoning abilities.

A third trend we noticed was that students with a relatively lower level of spatial reasoning (SpReas) tended to have higher IR-Sum scores. Again, this is not a perfect trend and may be impacted by the interview pairings as the scores for group answers apply to both students. When we computed the Pearson Correlation for the IR-Sum with the group answers and the IR-sum without, we had a significant correlation between the two (r=.884, ρ=.008). We found no significant correlation between these two factors (SpReas and IR-Sum) (r= -.534, ρ=.172), but, as before, we have one student who is an outlier, Student E, and if we remove that student, we do

have a significant correlation (r= -.786, ρ=.036).

In conclusion, we believe the trends that we have noted here warrant further investigation: if confirmed with a larger data pool that performance is correlated to spatial reasoning ability, this would have direct implications for instruction. While we have no evidence yet that spatial reasoning ability impacts performance on non-spatially loaded mathematical tasks, such results have been found in chemistry (Carter, LaRussa, Bodner, 1987). Furthermore, if we confirmed the correlation between spatial reasoning ability and preference from the MPIC, we can help establish the validity of using the MPIC as a tool for assessing integral calculus students' preference for using graphical reasoning.

Table 1: Performance on Measurements of Ability and Preference

	Student A	Student B	Student C	Student D	Student E	Student F	Student G	Student H
Preference:								
MPIC-Graph (40)	26	12	19	28	16	4	16	1
MPIC-Perf (20)	16	3	16	11	11	10	4	4
Ability (z-scores):								
Exams	0.94	1.16	1.18	-0.01	-1.12	1.42	0.91	-1.95
SpReas	0.73	0.04	1.16	0.96	0.88	0.98	0.85	-0.06
Reasoning in interview tasks:								
CH	29	19	33	36	30	27	23	25
IR-sum	7	17	4	5	20	4	15	14

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STUDENTS' MATHEMATICAL BEHAVIORS FOLLOWING INSTANCES OF MATHEMATICAL CONFLICT: AFFECTIVE/SOCIAL INFLUENCES

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Distinguishing viable arguments from those which are flawed is a critical skill for mathematics learners, and especially for those participating in collaborative learning environments. When students encounter mathematical ideas that conflict with their own perspectives during group problem solving, affective/social interactions may be key in shaping their mathematical responses. Through analysis of two cases, this paper highlights several affective/social considerations, such as students' personalities, classroom context of the mathematical conflict, and mathematics ability of those engaged in the conflict, which may interact to influence students' mathematical behaviors when their incorrect ideas are challenged by their peers during collaborative mathematical problem solving.

Keywords: Problem Solving; Affect, Emotion, Beliefs, and Attitudes; Middle School Education

Theoretical Framework/Purpose of Study

Mathematics goals for elementary and middle school students emphasize the importance of students' developing skills not only in constructing valid mathematical arguments, but also in "critiquing the reasoning of others" (Common Core State Standards Initiative, 2010). In order to evaluate others' reasoning, students must be able to consider another's mathematical perspective. Research, however, has demonstrated difficulty students may have in understanding others' mathematical reasoning, and especially in adopting others' mathematical view points. Students' resistance to accepting others' mathematical perspectives may be due to a lack of relevance or meaningfulness of others' ideas and the need to be actively interested and engaged in order to completely integrate others' ideas into one's own schemata (Maher & Martino, 2000). Research has highlighted complex ways in which affect may interact with students' cognitive processing in general (Schutz & Pekrun, 2007) and specifically in influencing students' mathematical thinking (Goldin, 2002; Ma, 1999; McLeod, 1994). When students are involved in collaborative mathematical problem solving activities, affective and social issues may become more salient. In particular, when students' ideas are challenged by peers, students' emotions can be critical in understanding their mathematical behaviors (Goldin, Epstein, Schorr, & Warner, 2011). This research examines two cases, both involving urban middle school students working collaboratively on the same task in different classrooms, in which students' incorrect mathematical ideas are challenged by peers. The two students' subsequent mathematical behaviors are then analyzed and compared in light of the social and affective classroom context.

Research Questions and Methodology

The questions that framed this research study were: 1) What types of mathematical behaviors do middle school students demonstrate when their ideas are challenged by peers during collaborative mathematical problem solving? 2) How may affective or social classroom contexts influence the students' mathematical behaviors during those instances?

Data for this study included videotapes collected from classrooms of two middle school mathematics teachers, Ms. B. and Ms. S., in one large urban district. These teachers were selected for a broader research project based on their participation in professional development aimed at helping teachers create emotionally safe classroom environments (Schorr, Warner, Gearhart, & Samuels, 2007). During classroom sessions for both teachers, students worked on the same mathematics task in groups of 3-4 students, spanning two class sessions for Ms. B. and three class sessions for Ms. S. Ms. B. considered her class to be a high ability group, while Ms. S. considered her students to be of low mathematics ability. The students worked on a problem adapted from a task entitled *Building Blocks Dilemma*. A worksheet distributed to the students contained the following diagram:

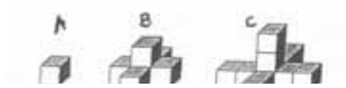

Figure 1: Building Blocks Task Diagram

Students were asked to find the number of cubes needed to build a 5-block high tower and a 10-block high tower and to generalize how many blocks would be needed for any size tower. The task was selected for use in the larger study because of its potential for high cognitive demand (Stein, Smith, Hennigsen, & Silver, 2000) and was designed to help students learn to make connections between patterns and equations. During the sessions interlocking snap cubes were available to students who wished to model the problem. The general solution preferred by students in the episodes investigated here involved noticing that each of the four branches in every tower of height n has one block less than the blocks in the height, leading to the solution method of multiplying 4 by n-1 to obtain the total number of side blocks and then adding n, the number of blocks in the tower's height.

Episodes involving students' mathematics difficulty were selected from the class sessions described above for analysis in a broader study (Seeve, 2012). These episodes were transcribed using audio and video data, and qualitative descriptions of the students' behaviors relating to the students' mathematical engagement following the instances were recorded, including descriptions of students' behaviors relating to their mathematical thinking, emotional feelings, and social interactions. Two episodes which involved students encountering mathematical ideas from peers that conflicted with their own ideas were selected for qualitative analysis here.

Results

Case 1: Marcos

Case 1 involves Marcos, a Latino student in Ms. B.'s class who appears even-tempered, self-assured, and sociable. On the first day of their work on the Building Blocks task, Marcos and his groupmates Lashanna and Larent have found correct solutions for towers up to 10- blocks high. Case 1 begins on Day 2 of the students' work, when Marcos proposes an incorrect solution of 460 blocks for the 100-block high tower.

When Ms. B. asks Marcos how he got his solution of 460 blocks, Marcos says he multiplied the number of blocks in a 10- block high tower, 46, by 10. Ms. B. suggests Marcos convince Lashanna of his solution and then walks away from the group. Lashanna tells Marcos that she got 496 as her solution for the 100- block high tower. Marcos keeps repeating his solution of 460 blocks. While Marcos speaks, he plays with his transparency. Lashanna explains that she got 500

and took away 4, one for each branch of the tower. While Lashanna speaks, Marcos continues playing with the transparency. He says, “it’s not 500” but does not address Lashanna’s justification for her solution. He continues playing with items on his desk and sings and whistles.

Lashanna later provides another explanation to Marcos for her solution, stating that since the 10-block high tower has 10 blocks in the height and 9 blocks in each of its branches, the 100-block high tower will have 100 blocks in the height and 99 blocks in each branch. When Lashanna asks Marcos to explain his method, Marcos simply repeats his earlier, incorrect reasoning. Again, he does not address Lashanna’s justification. Ms. B. suggests Marcos test Lashanna’s method by having Lashanna find the total number of blocks, using her method, for a tower Marcos already solved. Marcos asks Lashanna to solve the tower of height 5, for which he previously found a solution of 21 blocks.

Ms. B.: Let’s see. *(to Marcos)* What do you think?

Marcos: (puts his hands up.) I think…I think she probably won *(smiles)*.

Ms. B.: So you think she will? *(to Lashanna)* Ok, so now, can you walk us through your process again?

Lashanna: Eh…this is 5 *(pointing to the height in her tower)*…so in these 4 *(pointing to the tower’s branches)…*so 4 times 4 is 16, and then you just add the 5, and it would get you 21.

Marcos: (smiles at Lashanna) So you were right. *(to Ms. B.)* The same thing *(smiles)*.

Analysis, Case 1. This event occurs in a small group setting and involves Marcos, who presents confidence in his mathematics ability and works in a group with others who appear to be of similar ability. In this episode, Marcos continuously rejects Lashanna’s correct solution. He does not appear to consider her reasoning and simply keeps repeating his own solution, until Ms. B. engages him in a process of determining whether Lashanna’s solution is correct. Marcos then concedes, acknowledging that Lashanna’s, and not his own, solution method is correct. Marcos’s concession does not reflect embarrassment or loss of face.

Case 2: Leo

Leo, a reticent African American student in Ms. S.’s class who appears to have difficulty with mathematics, works on the Building Blocks task with groupmates Ordena and Ta’keisha. Ta’keisha is assertive, demonstrates stronger mathematics skills, and often dominates during the group members’ conversations. On Day 2 of the group members’ work on the Building Blocks task, the students in Leo’s group disagree on the number of blocks in a 100-block high tower. Ta’keisha arrived at a correct solution of 496 blocks by multiplying 4 by 99 (for the 4 branches) and adding 100 (for the height). Leo believes there would be 100, not 99, blocks in each of the four branches of the tower. Ta’keisha explains her method to Leo, but Leo still does not agree. Ms. S. suggests the students present both methods to the class during their presentation on Day 3.

During the group’s presentation, Ta’keisha tells the class that she obtained a solution of 496 cubes for the 100-block high tower. Leo comments, “So I disagreed with her because why is it 99 on each side, not 100?” At Ms. S.’s suggestion, Ta’keisha justifies her solution:

Ta’keisha: On this paper for example, A is only one block and it’s one cube. And then for the B, it’s a 2- block high tower...It’s only one going on each side...And then for C…it’s a 3-block high tower but you only see two. So it’s 2 on each side.

Ta’keisha continues, “You know how you don’t see the middle cube? I’m saying the amount that you see is the amount on each side.” During the remainder of the group’s presentation, Leo is quiet. Towards the end of the presentation, Ms. S. asks the class if the 5-block high tower Ta’keisha has built, with 5 blocks in the height and 4 blocks in each branch, is a valid tower. Most, but not all, students agree that it is a valid tower. When the students in Leo’s

group return to their seats following their presentation, Leo says to Ta'keisha, "See, I was right. See, ha."

Analysis, Case 2. Leo demonstrates a reserved nature and is of weak mathematics ability. He works in a group with Ta'keisha, who is assertive and of significantly stronger mathematics ability. On Day 3, Leo's disagreement with Ta'keisha occurs in the context of the full class. During the group's presentation, Leo appears to recognize that Ta'keisha's solution method, and not his own, is correct. Leo, nevertheless, seems to hold on to his own original idea in order to save face in front of his classmates. Leo's comment to Ta'keisha at the conclusion of his group's presentation indicates feelings of inferiority.

Cross-Analysis, Cases 1 and 2. In both cases analyzed here, students initially resisted adopting their peers' correct mathematical viewpoints which conflicted with their own incorrect mathematical ideas. In Case 1, Marcos eventually expressed agreement with Lashanna's solution without indication of embarrassment. Leo in Case 2, however, in an apparent attempt at avoidance of loss of face before his classmates, does not express agreement with Ta'keisha's method, even though he apparently begins to recognize his solution is not correct.

Discussion/Conclusions

Cases 1 and 2 support evidence found in earlier research regarding students' resistance to adopt mathematical perspectives of their peers. The cases examined here further demonstrate affective and social classroom contexts that may influence whether, and how, students modify their mathematical thinking to accept ideas proposed by their peers when those perspectives conflict with their own. Considerations include personalities of students engaged in the mathematics conflict, mathematics ability of those students, and whole class vs. small group context of the conflict. Limitations of the study include the small sample size and the inference of students' emotions from their observable behaviors on videotape. Results of the analysis have implications for mathematics education practitioners and researchers interested in maximizing opportunities for students to advance their mathematical knowledge by considering conflicting perspectives posed by their peers.

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THE DIRECT AND INDIRECT EFFECTS OF SELF-EFFICACY BELIEFS AND SELF-REGULATED LEARNING STRATEGIES ON MATHEMATICAL MODELING

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In this study, associations were examined between self-efficacy beliefs, self-regulated learning strategies, and students' performance on the modeling tasks. In all, 225 eighth- and ninth-grade students from a local research developmental school reported on a modeling self-efficacy scale, a modified version of the MSLQ, and a modeling test. The modeling self-efficacy scale has been developed and pilot tested, and the modeling problems were adapted from the PISA 2003 problem-solving assessment. The relationships between the variables were depicted in the form of a statistical model, which was tested using structural equation modeling techniques. The model fits the data reasonably well with self-efficacy plays a significant role in predicting students' performance on the modeling tasks. The correlations between cognitive strategies, metacognitive strategies, and modeling task success were not significantly different from zero.

Keywords: Beliefs, Cognition, Metacognition, Modeling

Introduction

Mathematics education currently emphasizes engaging students in mathematical modeling to prepare them for 21st century global workforce (CCSSO, 2010; English & Sriraman, 2010; Lesh & Zawojewski, 2007). Modeling with mathematics is the process of using knowledge and skills from across and within the curriculum to solve problems arising in everyday life, society, and workforce (CCSSO, 2010). The purpose of the present study was to investigate factors that may influence students' modeling outcomes. Due to a lack of literature informing which contextual factors impact students' success in solving real-world tasks, the study drew upon problem-solving literature to explore the degree to which effective problem-solving behaviors are associated with students' modeling outcomes.

Researchers have asserted that effective problem-solving behaviors such as setting appropriate goals, controlling one's actions, monitoring one's progress, trying alternative solution paths, and perseverance with challenging academic tasks align very closely with self-regulated learning (SRL) behaviors (Pape & Smith, 2002; Zimmerman & Campillo, 2003). Self-regulated students control and regulate their thoughts, actions, behaviors, and motivation in order to achieve a targeted goal. In the field of mathematical problem solving, self-efficacy beliefs and SRL strategy use profoundly influence students' engagement and persistence on complex mathematical tasks (De Corte, Verschaffel, & Op't Eynde, 2000; Pintrich & De Groot, 1990). The present study built upon and extended past literature by examining these associations in the context of mathematical modeling and real-life problem solving.

Theoretical Framework

Social cognitive theory is a useful framework for understanding SRL behaviors that enhance students' problem-solving skills (Zimmerman & Campillo, 2003). The theory states that proactive, self-reflecting, and self-regulated learners have the capacity to take control of their thoughts, feelings, and actions. According to Bandura (1997), people display this sense of personal agency because of the beliefs they hold about themselves and their capabilities. Such

self-beliefs, which represent individuals' perceived confidence to accomplish goals, are called self-efficacy beliefs. In academic settings, particularly school mathematics, self-efficacy refers to students' judgments of their abilities to solve mathematics problems, perform mathematics-related tasks, or engage in mathematical activities (Pajares, 1996). Self-efficacy beliefs are task- and situation-specific judgments, which are reported in relation to a goal. These beliefs reflect students' judgments of performing a task in future rather than their actual performance level. In actual reporting, students may underestimate or overestimate their judgments about their own competence (Pajares & Miller, 1994; Schunk & Pajares, 2009). Much research shows that self-efficacy beliefs correlate positively with students' problem-solving achievement (Pajares & Miller, 1994, Pintrich & DeGroot, 1990). These studies provide evidence that students' beliefs about their competence are a significant predictor of their problem-solving or academic performance even after controlling for mental ability and mathematics anxiety.

In addition to self-efficacy, SRL students control their cognition by employing several learning strategies. The present study focused specifically on the use of cognitive and metacognitive strategies. Cognitive strategies are learning strategies that influence students' processing of information, for example elaboration, organization, and critical thinking (Pintrich & De Groot, 1990; Pintrich et al., 1991). Elaboration strategies, such as paraphrasing, summarizing, and note taking, support students' acquisition of information by integrating new material with existing knowledge. Organizational strategies help students in building connections between different ideas and in arranging the material meaningfully. Finally, critical thinking strategies support students to make logical decisions. Metacognitive strategies are typically comprised of three different types of knowledge including planning goals, monitoring actions, and regulating strategies or methods. Planning strategies assist students in analyzing the task and setting appropriate goals. Monitoring strategies focus students' attention on the task and prompt them to keep track of their strategies and actions. Regulating strategies help students to improve their problem-solving skills by modifying strategies or methods of inquiry, acquiring more information or reviewing previous material.

Similar to academic self-efficacy, cognitive and metacognitive strategies directly influence students' problem-solving achievement (Pintrich & De Groot, 1990; Zimmerman & Martinez-Pons, 1988). These studies found that students who report using more cognitive and metacognitive strategies solved more mathematics problems correctly and received higher grades. Further, Pintrich and De Groot reported that students who believe in their abilities are more cognitively engaged as well as display greater use of cognitive and metacognitive strategies in solving mathematics problems. Other studies also confirmed that self-efficacy influence achievement directly as well indirectly through personal goal setting (Zimmerman & Bandura, 1994) or monitoring academic tasks (Bouffard-Bouchard, Parent, & Larivee, 1991).

Given the literature in the field, three research questions were investigated: (a) what are the direct effects of modeling self-efficacy beliefs on students' performance on modeling tasks? (b) What are the direct effects of students' use of cognitive and metacognitive strategies on their performance on modeling tasks? (c) What are the indirect effects of students' self-efficacy beliefs for modeling tasks on their performance on modeling tasks through its effects on their use of cognitive and metacognitive strategies?

Method

Participants

The current study recruited 225 eighth- (n = 88, 39.11%) and ninth-grade (n = 137, 60.8%) students from a local research developmental school affiliated with the researcher's university. The average age of the participants was 14.22 with a standard deviation of 0.85. Participants reported their ethnicity as White (n = 111, 49.3%), African American (n = 46, 20.44%), Hispanic (n = 33, 14.6%), Asian (n = 12, 5.33%), and Native Hawaiian (n = 1, 0.44%). The remaining students reflected their ethnic background as either a combination of these categories (n = 19, 8.4%) or as "others" (n = 3, 1.33%).

Measures

Three instruments were used to measure the desired constructs including: (i) a self-efficacy scale that has been developed and pilot tested to measure students' efficacy judgments on the modeling tasks, (ii) a modified version of the Motivated Strategies for Learning Questionnaire (MSLQ) as developed for *Connected Classroom in Promoting Mathematics* (CCMS) project to assess students' use of cognitive and metacognitive strategies, and (iii) a modeling test adapted from PISA 2003 problem-solving items to measure students' modeling competencies.

Results

The hypothesized relationships were estimated using Mplus statistical software. The measurement model provided an adequate fit to the data ($\chi 2$ [390 df, N = 225] = 595.97, p = 0.00, CFI = .94, TLI = .94, RMSEA = .048 (with 90% CI lower bound= 0.041 and upper bound= 0.056). The full structural model also fits adequately to the given data with regard to CFI, TLI, and RMSEA fit indices ($\chi 2$ [390 df, N = 225] = 595.97, p= 0.00, CFI = .95, TLI = .94, RMSEA = .048 (with 90% CI lower bound = .041 and upper bound = .056)). Perceived modeling self-efficacy (β = .50, p = .000) directly positively predicted students' performance in solving modeling problems correctly. In other words, students who reported greater self-efficacy for solving modeling tasks were more likely to correctly solve modeling problems. On the contrary, organization strategy use ($\beta = -.62$, p = .004) had a significant negative direct effect on students' performance on modeling test. This means that students who reported using more organization strategies tended to get lower scores on the modeling test. Further, direct effects of students' use of critical thinking ($\beta = -.59$, p = .08), elaboration (β = .40, p = .41), and metacognitive strategies (β = .46, p = .16) on their performance in solving modeling tasks were non-significant. As such, the data did not provide any evidence of the direct effects of critical thinking, elaboration, and metacognitive strategies on their performance in modeling test. As expected, the indirect effects of students' self-efficacy for modeling on their performance in solving modeling tasks through its effect on their use of critical thinking ($\beta = -.225$, p = .10), organization ($\beta = -.006$, p = .91), elaboration (β = .128, p = .41), and metacognition strategies (β = .15, p = .18) were non-significant.

Discussion

The present study provided evidence that self-efficacy is an important factor impacting students' performance in solving modeling tasks. Therefore, teachers should build or improve their students' self-efficacy beliefs for solving complex modeling problems by providing them with vicarious learning experiences, effective feedback, and engaging them in self-evaluative processes. Furthermore, creating positive and supportive learning environments such as encouraging students to participate in classroom discussions, explaining their thought processes, and focusing on the process rather than the correct answer may positively impact students' self-efficacy beliefs.

The structural model of the study, however, did not provide enough evidence to support that use of elaboration, critical thinking, and metacognitive strategies was directly associated with students' modeling outcomes. One of the possible explanations for finding results inconsistent with the past literature would be that participating students might not be aware of cognitive and metacognitive strategies they used during mathematical learning and problem solving activities in school. Data also indicated that students who had successfully solved modeling problems rated themselves low on the use of cognitive and metacognitive strategies, and students who were unable to solve modeling problems correctly had both under- and over-estimated their use of strategies on the MSLQ scale. The low reliability estimate ($\alpha = .61$) of the organization scale further indicated that items on the scale were not measuring the required construct. Further, the modeling test included problems occurring in one's personal life, work and leisure, and in the community and society. As such, performance on the modeling test might be influenced by students' individual differences in reading, cognitive ability, their familiarity with the context of the problem, socioeconomic status, gender, and their prior mathematics achievement. The present study did not control for the influence of these factors on students' modeling achievement.

Given the findings of this study, it is possible the MSLQ might not be the best choice. Probably, a scale that specifically measures students' use of cognitive and metacognitive strategies in relation to real-world problem solving would be more appropriate. Therefore, for future studies mathematics researchers might consider modifying items on the MSLQ scale to align more with the strategies used by students when engaged in modeling kind activities. Furthermore, students' responses on the modeling test were scored in accordance with the rubric used by PISA 2003 problem-solving assessment. The PISA scoring guide might be conservative because it didn't give students partial credit for correctly solving many of the sub-questions or performing many of the mathematical steps. Thus, future investigation might include developing a more comprehensive scoring rubric for scoring students' responses on the modeling test.

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UNDERSTANDING STUDENTS' ATTITUDES TOWARDS AND PERCEPTIONS OF HOMEWORK JOURNALS

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Writing in mathematics has become increasingly more common in mathematics classes of all age levels. The purpose of this study is to explore and describe students' attitudes towards and perceptions of homework journals, a specific type of writing in mathematics. Students enrolled in a college mathematics course were interviewed about their attitudes towards and perceptions of writing in mathematics. Based on the analysis of this data, students were found to have both positive and negative opinions of homework journals, such as liking instructor feedback from the homework journals and disliking the open-ended nature of the homework journals. In addition, homework journals were found to be a valid mathematics writing task consistent with the current research literature on designing meaningful writing tasks.

Keywords: Affect, Emotion, Beliefs, and Attitudes; Instructional activities and practices; Post-Secondary Education

Purpose of the Study

Mathematical writing can challenge students to purposefully reflect on their learning (Borasi & Rose, 1989), to think critically about mathematical ideas (Emig, 1977), and to deepen their mathematical understandings (Hamdan, 2005). While writing in mathematics can serve as a powerful tool, students' perspectives on mathematical thinking and learning through writing in mathematics are not well documented. The purpose of this study is to fill this gap in the research literature by exploring and describing students' attitudes towards and perceptions of homework journals, a specific type of writing in mathematics. Specifically, we will investigate the following research question: What are students' likes and dislikes associated with homework journals?

Theoretical Framework

Writing in Mathematics

Based on the current literature related to journal writing in mathematics, there are three important characteristics of journal writing assignments in mathematics: 1) prompts (Borasi & Rose, 1989), 2) clear expectations (Hamdan, 2005; Powell & Ramnauth, 1992), and 3) responses of the teacher (Borasi & Rose, 1989).

The first important characteristic of journal writing tasks, which is common among almost all of the examples in the current literature, is the prompt. The prompt can be relatively generic (Borasi & Rose, 1989) or extremely specific (Jurdak & Zein, 1998). The prompt can also be affective or cognitive (Jurdak & Zein, 1998). Regardless of their format, prompts are a critical element of journal writing.

Another important characteristic of journal writing tasks is well-communicated expectations. Teachers may communicate expectations in a variety of ways (Hamdan, 2005; Powell & Ramnauth, 1992). However expectations are communicated, it is valuable for both teachers and students to know what the teacher is anticipating from the journal responses in order for the journal writing to be meaningful to both teachers and students.

A third important characteristic of journal writing tasks is the response of the teacher; receiving feedback from their teacher is extremely important for students (Borasi & Rose, 1989; Jurdak & Zein, 1998; Powell & Ramnauth, 1992). Teachers' responses, comments, affirmation, and challenges based on students' journals are all important aspects of journal writing.

Homework Journals

Based on the literature related to journal writing in mathematics and mathematics education at a postsecondary level, we developed a journal writing assignment, called homework journals. Homework journals, which are based on the structure of Powell and Ramnauth's (1992) description of a "multiple-entry log," are a particular type of writing in mathematics in which students select one interesting or challenging problem from their assigned homework problems. On the left-hand column of a piece of paper, students solve this one problem in detail, clearly communicating their problem-solving process. On the right-hand column of the same piece of paper, students then write a short reflection about this problem and their problem-solving process. This reflection is prompted by a number of suggested topics and questions that students choose from. Teachers read and respond to each student's homework journal with comments, feedback, and questions.

Methods

Context and Participants

The context of this study was a college calculus course, *Elementary Calculus with Trigonometry*. Throughout the semester, students were assigned weekly homework assignments. The homework assignments consisted of 8-10 homework problems. Students were also required to submit a weekly homework journal along with their weekly homework problems. Nine students volunteered to participate in an interview outside of class.

Data Collection and Analysis

The nine students were interviewed using a general interview guide (Patton, 1990). Using a general interview guide allowed the interview to be systematic and comprehensive, but also afforded flexibility with important issues that came up during the interview. One of the main purposes of the interviews was to discuss students' attitudes towards and perceptions of writing in mathematics, specifically related to the homework journals.

A phenomenological design (Rossman & Rallis, 2003) was implemented to explore and describe students' attitudes towards and perceptions of writing in mathematics. The nine student interviews were videotaped and transcribed. The transcriptions were then coded in order to organize, analyze, and interpret the interview data.

Results

For the purposes of this study, we only focus on some of the results concerning what the nine students liked and disliked about the homework journals. The students' likes and dislikes about the homework journals demonstrate whether or not they hold a favorable or unfavorable attitude towards the homework journals and whether or not their perceptions of the homework journals are positive or negative.

What Students Liked About Homework Journals

The students described a number of different aspects of the homework journals that they liked. Students liked 1) the consistency and the format of the homework journals, 2) the feedback they received on the homework journals, and 3) the ability to make choices concerning the

homework journals. Below we elaborate on each of these "likes" and provide descriptions to help understand student's attitudes and perceptions.

Consistency and format. Three students, Mia, Sophie, and Kelly, liked the consistency of the homework journals. In particular, Sophie liked the consistency of the homework journals in that it was something that she did every week and that all of the homework journals were kept together in a single location. Kelly appreciated the consistency between the expectations her instructor communicated and the way her instructor graded the homework journals. Kelly believed that the communicated expectations and the grading practices matched and her instructor followed through with those expectations. Kelly contrasted this with a previous experience with writing in her high school calculus class: "When he [calculus teacher] graded them, he would tell us one thing and then when he graded it, he would be extremely harsh and grade from things he hadn't told us. So it kind of frustrated me more than anything."

Feedback from the instructor. Two students, Lydia and Chris, liked the feedback they received on the homework journals. They both felt that this feedback was an important and meaningful component of the homework journals. Lydia said, "I really like that she always wrote feedback." Lydia went on to describe the feedback she received on her homework journals and how helpful that feedback was, both for gaining a better understanding of the mathematical content and for preparing for tests. Chris also really appreciated the feedback he received on his homework journals. He said, "I like the personal feedback from our instructor, I really like that." The feedback they received from their instructor was very important to both Lydia and Chris and was one aspect of the homework journals that they liked the most.

Making choices. Adele liked that the homework journals gave her the freedom to choose which prompt to answer. She said, "I liked the choice to write about whatever you wanted."

What Students Disliked About Homework Journals

The students described a number of different aspects of the homework journals that they disliked. Students disliked 1) the way that the expectations about the homework journals were communicated to them and how their homework journals were graded, and 2) the open-ended nature of the homework journals. Below we elaborate on each of these "dislikes" and provide descriptions to help understand students' attitudes and perceptions.

Communicated expectations from the instructor. Mia and Chris disliked the way expectations were communicated and the way their instructor graded their homework journals. Mia, for example, felt very strongly about the fact that her instructor had not communicated expectations because, more than halfway through the semester, she still did not understand what was expected of her on the homework journals. She lamented, "I didn't like all the points I missed on them! I feel like that was where I lost most of my homework points because I wasn't able to fulfill expectations in the writing section." Mia does not know what the expectations are and therefore is frustrated because she feels like she cannot fulfill the expectations. Chris also expressed frustrations about his grades on the homework journals. Both Mia and Chris disliked the way the homework journals were graded.

Not enough structure or guidance. Adele and Melissa disliked the open-ended nature of the homework journals; they did not think there was enough structure or guidance to the homework journals. Melissa did not like the freedom she was given to choose which reflection prompt to answer. She explained, "I didn't like that it is kind of open-ended. I'm someone who likes to have very specific directions. So I'll be like, 'What do you want me to write? I'll write what you want me to write if you just tell me what you want me to write!'" Melissa wanted to be given specific instructions about what she was expected to write in her homework journal; she did not

like multiple choices for the reflection prompt. Both Adele and Melissa disliked the freedom of choices they were given with the homework journal.

Discussion and Conclusions

The students' likes and dislikes of the homework journals help describe their attitudes towards and perceptions of homework journals. Most students in this study liked some aspects of the homework journals and disliked other aspects of the homework journals. Therefore, it is difficult to determine whether or not a student entirely held a favorable or unfavorable attitude towards the homework journals or whether or not a student's perception of the homework journal was entirely positive or negative. However, what the students expressed as their likes and dislikes of the homework journals does provide some insight into their attitudes towards and perceptions of homework journals.

Students' likes and dislikes of the homework journals also confirm the strength of the homework journals as a valid and valuable journal writing task in mathematics, based on the three important characteristics of journal writing in mathematics. First of all, some students in the study, like Adele, expressed how much they liked being able to choose which prompt they wanted to respond to and reflect on each week. Although other students in the study, like Melissa, did not like the choice or variety of the prompts, this seemed to be a personal preference for those students. However, both students indicate that they were away of the multiple prompts were presented to students. Secondly, some students in the study, like Kelly, thought there was consistency between what her instructor communicated about the homework journals and the way the homework journals were graded. This provides evidence that expectations were communicated to the students. However, there were other students in the study, like Mia, who did not know what they were expected to write about in their homework journals, confirming the point in the research literature that communicating clear expectations is extremely important. Finally, some students in the study, like Lydia and Chris, valued their instructor's feedback, indicating that there was an opportunity for the instructor to provide meaningful feedback. Homework journals satisfy the three important characteristics of journal writing in mathematics.

While the students' attitudes towards and perceptions of homework journals were varied, it appears that homework journals provide a useful tool for actively engaging students in doing mathematics and reflecting on mathematics and/or mathematics learning by combining problem solving and reflection in a simple two-column format. Similar to the Powell and Ramnauth (1992) study, the students in this study were both actively engaged in doing mathematics and reflecting on that mathematics through the homework journals.

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CONTRIBUTION OF THE DIALOGUE BETWEEN TWO THEORIES TO THE STUDY OF TWO VARIABLE FUNCTIONS

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APOS theory and the Anthropological Theory of Didactics (ATD) are two very different theories in mathematics education. However a dialogue between them has been started. This paper continues the dialogue between these theories starting from a research problem: the learning of functions of two variables. The problem, originally posed within APOS theory, is reformulated and results from the methodological component of the dialogue are applied to its study with the ATD.

Keywords: Advanced Mathematical Thinking; Research Methods

Antecedents and Research Questions

Obtaining information that can be useful in the design of courses or didactic materials is one of the goals of mathematics education. The potential to link effectively theory and practice requires a careful selection of a theoretical framework. Results from a project, where a dialogue between APOS Theory and the Anthropological Theory of Didactics (ATD) was started from the theoretical and the "methodological" components (Trigueros, Bosch, and Gascón, 2011), were used in this study as a way to develop the dialogue, starting from a component or mode of dialogue that was left for future research in that study, and to develop materials to teach functions of two variables.

Function of two variables is an important topic of Calculus and applied mathematics courses. Results from research on this topic (Trigueros and Martínez-Planell, 2010; Martínez-Planell and Trigueros 2012, 2013, Kabael, 2011; Weber, 2012) show the difficulty of this topic as well as specific difficulties observed in students.

Our research questions for this study are: How do the research problems change when studied from the point of view of the result of the dialogue between APOS and ATD?

How can the results from this dialogue be used in the teaching of functions of two variables?

Theoretical Framework

The initial results of a dialogue between APOS Theory and the Anthropological Theory of Didactics (ATD) considered as research praxeologies (Trigueros, Bosch, and Gascón, 2011) are used as a theoretical framework in this study. In the part of the dialogue that starts from the theoretical component of the theories, the notion of a "genetic decomposition" in APOS was specified so that it refers to a "generic student of an institution" and hence could be assimilated to the notion of "subject in position of student" in ATD. This could introduce in APOS the incidence of the institutional interpretation of a concept on its genetic decomposition. A parallelism between the notions of "genetic decomposition" of APOS and "epistemological reference model" of ATD was also found.

Another of the observations from this dialogue was that meeting points between the theories can be found since the genetic decomposition in APOS is formulated in terms of actions that people *in specific institutions* do while doing mathematics and in activities designed for instruction and research where the goal is to help individuals to interiorize actions into processes

or to encapsulate processes into objects, and in the ATD, mathematical activity in institutions *to be done by individuals* is taken into consideration in the analysis of tasks, techniques, technologies and theories which form the basis of the praxeological analysis.

From the part of the dialogue that starts from the "methodological" components of the theories, seen as research praxeologies, it was shown that the notion of institutional relativity of knowledge from ATD can be used to generalize the present role of the genetic decomposition, and the theory of the didactical moments of ATD (Trigueros et al, 2011; Chevallard, 1999) can be taken into account to enrich APOS' ACE cycle.

The institutional relativity of the genetic decomposition can be used to take into consideration the fact that this model can be adjusted to the level of abstraction at which specific concepts are to be taught. The moments of study are a useful tool to analyze the activities designed with the genetic decomposition in order to determine the role that they play in the structure of the process of study.

An Example of Dialogue Between the Theories Starting from a Problem Constructed from APOS

To be able to carry out this dialogue, left as "future research" in the previous study of Trigueros et al (2011), we took a problem that we had already studied from APOS theory and constructed a new problem that could be studied with the tools that resulted from the dialogue starting from the theoretical and "methodological" components. The problem originally formulated in APOS theory as: What mental constructions are necessary for students to learn the concept of two variable functions? Was reformulated in terms of ADT as: In a specific university, what are the characteristics of didactic praxeologies used in the generation of a local mathematical praxeology associated with functions of two variables?

To answer these questions several cycles of research were needed. We report here on the part related to sketching and interpreting graphs of functions of two variables. The first cycle consisted of the analysis of a widely used textbook and the way a representative teacher taught two variable functions at the university (Trigueros and Martínez-Planell, 2011). Conclusions from those analysis indicated that the way the didactical moments of study were transited was insufficient for the students to learn and use the techniques associated to functions of two variables and that in general the organization of the study was not effective.

A set of activities was designed with APOS theory in order to complement the teaching of functions of two variables. The activities were then analyzed independently by two researchers using the didactical moments of study (for a discussion of the theory of the didactical moments see Chevallard, 1999). They compared and negotiated their results until agreement was reached.

Analysis of the Activities Designed with APOS Theory

The design of the activities to foster the constructions included in the genetic decomposition followed a two cycle process where each cycle consisted of: design of activities, classroom implementation, and analysis of results. In each of them the activities were designed using both the didactical moments of study and the genetic decomposition previously designed and validated. In the second cycle the results of the textbook and teaching analysis were also taken into account. The results that follow correspond to the analysis of the final set of activities. All the activities were used in a classroom of the same institutions where the previous parts of the project were conducted and were reviewed in consecutive semesters. The results of the analysis show that different subsets of activities transited the different moments of study. Activities include tasks intended to start students in the construction of subspaces of R^3 which can all be

considered as part of the *moment of the first encounter*; most of the activities related to the result of intersecting fundamental planes with surfaces in R^3 can be considered as part of the *moment of task exploration*. Many of the tasks focus on the interpretation of expressions that contain free variables and that result from the intersection of surfaces with fundamental planes and the coordinate axes, and activities that promote reflection on the processes included in sketching the graph of simple two variable functions in three-dimensional space which can be considered as part of the *moment of task exploration* as well. Through activities designed to coordinate different processes students construct new processes as when the development of a technique to draw cylinders in three dimensions is introduced. These are part of the *moment of work on the technique.*

Tasks that explicitly coordinate the processes involved in the technique for sketching graphs point by point as well as opportunities to coordinate different processes included in the genetic decomposition and that require justification using sections may be considered to be part of the *technological-theoretical moment*, and can help in the encapsulation of processes into objects and in the construction of a coherent schema for R^3. Tasks requiring verification of the correspondence between the sketched graph and the algebraic expression of the corresponding function can be considered to be part of the *evaluation moment* together with tasks that requires the identification of the algebraic representation of two variable functions from a set of surface graphs; In all of them justification of the decision taken using transversal sections is required.

The analysis showed that the variety of activities in the *moment of task exploration* emphasizes the use and importance of transversal sections making explicit the *moment of first encounter with the technique* and the *moment of work on the technique*. Even though the set of activities does not include the explicit discussion of the theory, which is included in the phase of discussion of the ACE cycle of instruction, opportunities for discussion and justification of the methods employed are included and can be considered as part of the *moment of institutionalization*.

This briefly sketched analysis suggest that besides promoting the constructions involved in the genetic decomposition, there is a balance in the introduction of the moments of study and it can be expected that the use of the activity sets contributes to the learning of functions of two variables, although some improvements are still to be made: the number of activities that are part of the *moment of task exploration* needs to be reduced and more activities helping students encapsulate processes into objects may be needed.

Results from the Use of the Activities

A semi-structured interview to 9 students of the group that worked with the activities and 6 students from another group that did not use the activities was conducted. Students in both groups were selected by the respective teachers so that there were an equal number of students who were considered above average, average, and below average, in terms of their results in the course. The interview questions were reviewed by the researchers and the teachers of both groups. The analysis of responses was carried out independently by the two researchers and results were negotiated. The analysis of data showed that few of the students constructed the totality of the structures considered in the genetic decomposition. Students who used the activities did generally better showing a greater tendency to use sections to justify their decisions. Their difficulties were concentrated on the construction of the more difficult processes in the genetic decomposition, such as the intersection of fundamental planes with complex surfaces. The students from the control group showed many difficulties, all of them similar to those found in previous studies and a tendency to rely more on memory.

Taken as a whole, the results show that the use of the activities promotes an organization of the process of study where the transit though the moments of study has the potential to promote students' learning.

The results of this example of dialogue between APOS and ATD theories show that it is possible to use the genetic decomposition in the design of an organization to study functions of two variables that is more effective in terms of learning.

The use of a didactical praxeology in which there is a balance between the moments of study contributes to the construction of the structures hypothesized by the genetic decomposition and to learning the graphical aspects of functions of two variables.

Conclusions

This study contributes to the dialogue between APOS theory and ADT providing an example of a dialogue starting from research problems. The problems, formulated using APOS theory, were reformulated so that they were congruent with the tools developed as a result of the dialogue between the theories. The reformulation made it possible to use the contributions from the dialogue starting from the "methodological" components of the theories as tools of analysis.

Results of this study show that new contributions emerge from the dialogue starting from the problems. The information obtained from the analysis of the textbook, of the teacher, and of the activities developed using APOS theory, together with the refinement of these activities through several research cycles, show that results from the dialogue between the two theories can be used as a powerful tool to determine the elements that should be taken into account in redesigning the original set of activities. The analysis of the interviews from the point of view of APOS shows that most of the difficulties found in previous studies can be surpassed when both cognitive and institutional factors that impact in the learning of functions are taken into account.

The activities resulting from the analysis take into account both the institutional and the cognitive dimensions of this phenomenon and this makes possible a better adaptation to institutional conditions and an equilibrium of the tasks dedicated to the different constructions modeled by the genetic decomposition. The result is reflected in a deeper learning of students.

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THE EFFECTS OF VISUAL REPRESENTATIONS AND INTEREST-BASED PERSONALIZATION ON SOLVING PERCENT PROBLEMS

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Visual representations like illustrations and diagrams, as well as story contexts personalized to interests, may be important supports as students confront challenging mathematical topics. We presented 139 seventh grade students with problem sets on percentages that varied both personalization and visuals. Diagrams helped performance, while illustrations were helpful when used without other supports. Personalized contexts were helpful for the students who had mixed attitudes towards mathematics. This research highlights the importance of considering problem and student characteristics together when making instructional recommendations.

Keywords: Curriculum; Middle School Education; Affect, Emotion, Beliefs, and Attitudes

Theoretical Framework and Literature Review

Mathematics story problems are sometimes considered notoriously difficult to solve (Cummins et al., 1988). However recent work has demonstrated that situated contexts can enhance students' ability to access and apply mathematical ideas. Walkington (in press) found that students solve problems personalized to their out-of-school interests quicker and more accurately. Other research suggests personalization may be most effective for students with lower achievement levels (Anand & Ross, 1987). Complementary research has pointed to the importance of visual representations like diagrams and illustrations for supporting students in solving word problems (McNeil et al., 2009). Research on the *multimedia effect* has found that relevant visual representations that accompany text can increase learning (Mayer, 2005). Cognitive processing capacity, the learner's prior knowledge, and problem difficulty are important determinants of this effect (Kalyuga et al., 1998). Cooper and Alibali (2012) found that diagrams helped all groups of college students on a mathematical task while the effect of illustrations depended on student background characteristics.

Supports like visuals and personalized contexts may provide a means to *ground* mathematical formalisms in students' prior knowledge (Goldstone & Son, 2005) and elicit *interest* – the psychological state of engaging (Hidi & Renninger, 2006). However, research on cognition also suggests reasons why these supports may not be helpful. Cognitive load theory (Sweller et al., 1998) posits that elements that add *extraneous cognitive load* – mental effort stemming from activities not related to learning – may decrease performance by monopolizing available resources. Mayer's *theory of multimedia learning* also accentuates limited processing capacities for separate visual and verbal pathways (Mayer, 2005).

We examine the combined effects of visuals and personalization on solving percentage story problems. Percentages have applications in everyday life, making them a natural site for personalization, and their part-whole relationships afford diagrams. Our research questions are:

1) What are the effects of personalization and visual representations (diagrams, illustrations) on students' accuracy when solving percentage story problems?
2) How do these trends change when taking into account students' reported levels of affect towards mathematics and knowledge of mathematics?

Method

Seventh grade students (*N* = 139, 60 female) at a suburban Southern middle school participated in the study. The schools' students were predominantly Caucasian (46%) and Asian (30%), with 12% eligible for free/reduced lunch and 3% classified as Limited English Proficient.

Students had previously rated their level of interest in a variety of topics (e.g., sports, movies, shopping, food) and subtopics (e.g., basketball, Italian food) on an interests survey. We then selected 8 problems on percentages from a worksheet the teachers at the school were planning to use in their classroom (Table 1) and developed 7 personalized versions for each problem. Personalized versions used the same numbers and information order as the original (Table 2), and had similar Flesh-Kincaid readability levels. All problems expressed simple part-whole relationships using percentages.

Of the 8 randomly-ordered problems each student received, 4 were personalized to each student's highest-rated interests. We also varied whether problems contained diagrams, illustrations, both visuals, or neither visual (Table 3). Additionally, a 12-item questionnaire assessed mathematics attitudes for each student. Factor analysis (PAF) revealed 3 factors on the questionnaire – students' *knowledge* about mathematics (e.g., "I find math confusing"), *value* for mathematics (e.g., "Math is important in everyday life"), and *affect* for mathematics (e.g., "I find many math problems interesting"). Value had no significant effects and is not discussed further.

Students solved the problem set during their math class with a calculator, and their answers were classified as correct or incorrect. Data were analyzed using mixed-effect logistic regression models (Snijders & Bosker, 1999). The dependent measure was accuracy, and predictors included presence of personalization, diagram, and illustration. Random intercepts controlled for elements of the problem's mathematical structure. Students were categorized as high or low on each construct (knowledge, affect) from the attitudes questionnaire by splitting student ratings at the mean.

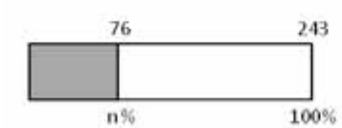
Table 1: Normal Versions of Percent Problems

Structure	Set 1	Set 2
A	A Southwest Airlines passenger plane can seat 98 passengers. On the Monday flight there were 63 passengers. What percent of seats on the plane have passengers in them?	Kris worked a total of 243 days last year. His work required that he travel out of town on 76 of those days. What percent of the time did Kris's job require him to travel and be out of his office?
B	The Pelican's Club printed color fliers for an upcoming event. They prepared 1468 fliers to distribute. 1282 fliers were passed out. What percent of the fliers are left over?	Mr. Carney is a handyman. He had a total of 130 jobs last month where he either replaced a door handle or installed a sink. Last month, he installed 77 sinks. What percent of his jobs last month involved replacing door handles?
C	Shawn has been shopping for a new office chair. The retail price for the office chair that Shawn likes is $254. On Friday the chair will go on sale for 20% off of the price. How much will Shawn pay for the office chair if he purchases it on Friday with the sale (excluding tax)?	Joey works for an accounting and tax company. The company usually charges their customers $618 per job. However, the company is having a special where all jobs are 30% off currently. What is their discounted charge for a customer who hires them (without tax)?
D	Rene is going to volunteer her time at the local library. She will spend 15% of her volunteer time shelving books. If she has committed to work at the library for 48 hours, how many hours will she spend shelving books?	Last week, Holly worked 68 hours at her office. 70% of that time was spent writing a software instruction manual. How many hours did she spend writing the manual?

Table 2: Examples of Personalized Versions of a Problem

Original	Southwest Airlines passenger plane has a capacity of 98 passengers. On the Monday flight there were 63 passengers. What percent of seats on the plane have passengers in them?
Adventure Games	In your favorite strategy game on the PlayStation, you need to gather 98 gold coins to gain a level. You have currently gathered 63 coins. What percent of the coins have you gathered?
Texting	You have sent 98 text messages on your phone so far today. You notice that 63 of the messages have been to your best friend. What percent of the messages were sent to your best friend?

Table 3: Variations in Visual Representations

	No Illustration	Illustration
No Diagram	Kris worked 243 days last year. His work required that he travel 76 of those days. What percent of the time was Kris out of his office?	Kris worked 243 days last year. His work required that he travel 76 of those days. What percent of the time was Kris out of his office? 
Diagram	Kris worked 243 days last year. His work required that he travel 76 of those days. What percent of the time was Kris out of his office? 	Kris worked 243 days last year. His work required that he travel 76 of those days. What percent of the time was Kris out of his office?  

Results

Did personalization and visual representations influence accuracy?

A diagram by illustration interaction (p = .034) indicated that the combined presence of diagrams and illustrations was equivalent to text alone. Separately, diagrams were superior to text alone (p = .007), and illustrations were marginally better than text alone (p = .06). For the sample as a whole, the effect of personalization was not significant and did not interact with either visual condition (ps > .1).

Did these trends change when taking into account affect and knowledge for mathematics?

Students were classified into 4 groups based on their ratings of knowledge of and affect towards mathematics on their questionnaire: High Affect and Knowledge (HAHK), Low Affect and Knowledge (LALK), High Affect and Low Knowledge (HALK), and Low Affect and High Knowledge (LAHK). Overall accuracy was highest for HAHK students (M = 85%, SE = 5.4%) and lowest for LALK students (M = 74%, SE = 6.6%). For all students, diagram presence significantly improved performance when there was no illustration, but had no effect when an illustration was present. The effects of illustration and personalization were more complex given the three-way interactions of personalization and illustration with reported knowledge, $\chi^2(4) = 10.28$, p = .036, and with reported affect, $\chi^2(2) = 6.55$, p = .038.

For HAHK students, illustrations improved performance when there was no diagram and no personalization; performance was not otherwise affected by personalization. The performance of LALK students was not affected by illustrations or personalization. For HALK and LAHK students, the general trend was for illustrations to be helpful when there was no personalization and no diagram. However, for HALK students, personalization combined with an illustration increased performance compared to personalization alone or an illustration alone. For LAHK

students, personalization increased performance when there was no illustration (p = .008) but the combined effects of personalization and illustration reduced performance (p = .007).

Discussion and Significance

For seventh graders solving percent story problems, diagrams provided an important support. Decorative illustrations and personalization were supportive in more limited circumstances – their effects depended on other supports present and student background. The salience of decorative illustrations and personalization may overwhelm limited cognitive processing capacity when combined with another support. Our interpretation is limited by the relatively small data set given the data complexity and by the specificity to this one type of problem. However, some of the broader patterns from the data suggest important implications. First, when students are struggling, relatively superficial modifications like personalization or illustrations may not help them engage with the mathematics – only diagrams assisted LALK students. Second, decorative illustrations can help performance, suggesting potential differences in the effects of different types of seductive details. Our data cautions against adding other potentially-distracting supports when illustrations are present, however. We conclude that the scaffolding effects of different supports are not necessarily additive and may interact in complex ways.

Students need a variety of supports to access mathematical ideas in the middle grades curriculum. Scaffolds like illustrations and personalized contexts have the potential to ground ideas and elicit interest, but we also need to be aware of some caveats when using these scaffolds. This study broadens current perspectives on mathematical learning by investigating factors that mediate the effectiveness of these supports. Being aware of trade-offs of different supports as well as students' mathematical attitudes and knowledge is important for both curriculum design and in-the-moment instructional decisions.

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STUDENTS' LANGUAGE REPERTOIRES FOR INVESTIGATING MATHEMATICS

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We identify the importance of language tools for the development of understanding of mathematics. In particular, language for expressing modality (degrees of certainty) make it possible to express conjecture and thus they support group problem solving. We note ambiguities in meaning in relation to modal verbs (which are used to express modality). Thus we wonder how students with limited language repertoires understand conjecture. We will report on students' language repertoires for conjecture in grades 3, 6, and 9, including first language English speakers in English medium and French Immersion classroom contexts.

Keywords: Classroom Discourse, Problem Solving

Mathematics is mediated through and by language. This is important to educators not only because language is necessary to convey important mathematical ideas, but also because the language used for mathematics shapes the way people conceptualize mathematics. This recursive nature of language compels consideration of students' language repertoires in relation to characteristically mathematical conceptualizations and processes.

Research and professional literature, including curricula and curriculum frameworks, point to the necessity of students learning intended outcomes through the exploration of mathematical problems. This suggests that the investigation of conjectures is one of the most important mathematical processes.

When people explore a mathematical problem together, as with mathematical investigations in classrooms, it is necessary to have a way of suggesting an idea while knowing that it may not be true. Rowland (2000) noted the centrality of such conjecture to mathematics, and coined this "space between what we believe and what we are willing to assert" (p. 142) as the Zone of Conjectural Neutrality (ZCN). Because of the recursive relationship between language and experience, the language resources available affect the possibilities for making conjectures. In our research, we are exploring the ZCN in diverse linguistic contexts. In addition to needing linguistic skill for expressing the uncertainty that is at the heart of conjecture, exploration puts students in a position of having to talk about ideas they do not have experience talking about. They have to invent language or at least invent new ways of using their current language repertoires in order to talk about something they have not experienced before.

In our research project, we are asking:

1. What linguistic resources do mathematics students use to express conjecture?
2. How do mathematics students think about the meaning of such expressions?

We are investigating these questions at a range of levels, gathering classroom and interview data from mathematics students in grades 3, 6, and 9. We are collecting data in French Immersion (FI) contexts, in which students' linguistic repertoires may be relatively narrow, and comparing these contexts to first language contexts, to understand the experience of students in each context and also because we know that the juxtaposition of linguistically diverse contexts is illuminating.

Language and Mathematics Learning

Our investigation is informed by two fields within education research – mathematics education research and second language acquisition research. Though there is diverse research on discourse practices, and language issues in mathematics learning, we are not aware of mathematics education work that focuses on specificities of lexico-grammatical repertoires as they relate to the understanding of mathematical problem solving. However, we are looking forward to a chapter by Edmonds-Wathen, Durand-Guerrier, and Trinick in the upcoming volume from the recent (September, 2011) ICMI (International Congress of Mathematics Instruction) Topic Study Conference on Language Diversity and Mathematics Education, which may open up this area of inquiry. Attention to the distinctiveness of different languages' mathematics registers can help mathematics educators in unilingual and multilingual contexts understand the way language constrains and opens up possible meaning.

The vast majority of the extensive research on French Immersion (FI) learning has focused exclusively on the language and literacy context and not specifically on FI mathematics. Our investigation builds on a small body of Canadian work investigating FI mathematics contexts. It should be noted however that with the exception of Culligan (2010) these studies have focused on achievement in mathematics rather than on particular grammatical or lexical items. With respect to particular linguistic phenomena (e.g., specific language forms; code-switching) that occur in the French immersion mathematics setting, our study expands work done in other contexts of education in an additional language medium (e.g., Setati, Adler, Reed, & Bapoo, 2002).

Our theoretical perspective research draws on the work of Vygotsky (e.g., 1962) and Wertsch (1991), which highlights the connections between thought and language, especially in the process of learning. Language is the main mediational tool in mathematical problem solving, and thus impacts how mathematical knowledge is developed, and how language contributes to that cognitive development process. Exemplifying the shaping force of one's language repertoire, Shaffer (2006) explained how deaf children with hearing parents did not develop "The Theory of Mind" (ability to separate one's self from his or her ideas) because of the absence of modality in their vocabulary. Modality refers to linguistic tools for expressing degrees of certainty; it enables negotiation among different points of view. Shaffer reported that once the children developed vocabulary for modality (in American Sign Language) they quickly developed The Theory of Mind. We note that Rowland's ZCN resembles the space opened up by modality. This suggests to us that students' facility with modality may impact their understanding of mathematical conjecture and problem solving, which depends on conjecture. Nevertheless, if students develop linguistic tools for modality later, we would wonder if their earlier experiences of problem solving without modality might impact the development of their dispositions related to mathematics.

At the ICMI Topic Study conference mentioned above, discussion demonstrated that there is an interest in linguistic variation but not much research, especially not in mathematics education contexts. This discussion also identified the challenge in this kind of research to avoid deficit models, which suggest that one's own way of speaking or thinking is superior (by evaluating whether or not others have acquired the same skills). In the study of linguistic variation for numbers, the one area of mathematics register variation that has been documented significantly, Swetz (2009) pointed out how cultures have been rated on the extent of their number systems. In our research we are not so interested in what an absence of language tools may prevent. Rather we are interested in the potential for linguistic variation to open up opportunities to understand mathematics differently – for example, how does thinking of numbers as verbs (see Lunney

Borden, 2010) change one's conception of counting and arithmetic operations? We know that students, including second language (L2) students, use grammatical structures to express modality at some point, but we wonder what other resources they might use if the usual modality tools are not in their linguistic repertoire. And how do they think about what the modulation means? It is important to investigate these questions along a range of linguistic proficiencies.

Modality and Conjecture in Unilingual Contexts

Modality has been studied in English-medium mathematics teaching and learning contexts, first by Morgan (1998). Much of this work, including our own, focuses on the way high modality (language that expresses a high degree of certainty) and root modality (language which suggests that there are no alternatives possible) can restrict discussion and obscure human agency. Modal verbs are often used to express degrees of certainty. For example, the modal verb *must* suggests more confidence (e.g., "it must be prime") than *could* (e.g., "it could be prime"). We say an expression has high modality if it has more certainty. When there is absolute confidence (e.g., "it is prime"), there is no expression of degree of certainty. Rowland (2000) called this root modality. "It must be prime" suggests a recognition that others may disagree, unlike the expression "it is prime." This recognition of multiple possible points of view allows one to put forward an idea for testing (i.e. conjecture) – the idea seems true but can be tested from another point of view.

We began to notice modality in an earlier research project that involved FI mathematics classrooms (Wagner, Herbel-Eisenmann & Kristmanson, 2011). We noticed ambiguities in English modality that could seriously impact students' understanding of mathematics. Martin and Rose (2005) identified five types of modality, which describe degrees of certainty relating to usuality, probability, obligation, inclination and ability. For example, "we sometimes add" refers to usuality, and "I might add" refers to inclination. These examples also show that there are ways of introducing modality without using modal verbs. This is why our research questions are open to a range of linguistic resources for conjecture, not just modality.

But the modal verbs carry muddy meaning because of intertextuality. For example, the meaning of the verb *can* differs depending on context: "You *can* be excused from the table" indicates a degree of obligation, "You *can* finish the race" indicates ability, "I *can* help you" indicates inclination and "It *can* be red (because there are three red marbles in a bag of marbles)" indicates probability. When students hear the word *can*, what does it mean to them? Do they distinguish between obligation and probability?

Even within obligation there are distinctions to be noted, for which we raise the same questions about students' perceptions of the meaning. For the modal verb phrase *have to*, which indicates obligation, we would expect that mathematics understanding requires a distinction between obligation due to personal role (or status) and obligation due to logic. For example, "We *have to* cancel the *x*'s to simplify the fraction" suggests an obligation attributable to logic, and "We *have to* cancel our dinner reservation" indexes obligation attributable to interpersonal positioning. When students hear *have to*, how do they understand this obligation? It is important to note that students hear *have to* very often in mathematics class, as demonstrated by Herbel-Eisenmann, Wagner and Cortes (2010) in their analysis of a large body of mathematics classroom interactions across a range of classrooms. Expressions intended to describe logical obligation may be taken by students as obligation due to interpersonal positioning. The muddiness of meaning in modal verbs exists in French too.

Methodology and Results

We are collecting a range of data from multiple points of view on students' linguistic

resources for expressing and understanding conjecture in mathematics contexts. Starting from grade 3 (when early French Immersion begins in our region) we are beginning to follow groups of students as they develop their linguistic and mathematical repertoires. We are following these groups starting from grade 3, from grade 6 (the beginning of middle school), and grade 9 (the beginning of high school). We will follow each group for three years to get a sense of students' development of their language repertoires.

In each participant classroom, we record small group problem solving and then interview students the day after the problem solving. We record audio and video because our focus is on language resources, which can include gesture and physical positioning of interlocutors. In our analysis of the recordings and transcripts, we will look for development of students' linguistic resources from year to year, but we are already interested in the snap shots of the linguistic resources in any given session. Our experiences studying language practice have made clear to us the futility of generalizing the language practice of any category of children. Instead, we are interested in illuminating the diversity of practice.

In our presentation, we will represent the range of language resources used by participant grade 3, 6, and 9 students in their mathematical problem solving and in their talk about that problem solving (in our follow-up interviews). We have recently begun our data collection. By the time of the conference, we will have collected our grade 3, 6, and 9 data and we will have had five months to analyze it.

Acknowledgments

This research was supported by the Social Sciences and Humanities Research Council of Canada, as part of a grant entitled "Students' Language Repertoires for Investigating Mathematics."

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THOMSON'S LAMP AND THE GREEN ALIEN: EXPLORING INFINITY VIA SUPERTASKS

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In previous studies paradoxes of infinity have been used in the investigation of conceptions of infinity of pre-service high school teachers and university students. In this study the Thomson's Lamp paradox and a variation of it, the Green Alien, were used to investigate the naïve and emerging conceptions of the infinity in a group of liberal arts university students and the effect of context on such conceptions. This study contributes to research on the use of paradoxes in mathematics education and to research on understanding infinity, with a focus on infinitely small quantities.

Keywords : Cognition, Advanced Mathematical Thinking

Introduction and Theoretical Perspectives

Paradoxes have been used as a lens in mathematics education research. Movshovitz-Hadar and Hadass (1990 & 1991) investigated the role mathematical paradoxes can play in the pre-service education of high school mathematics teachers. They concluded that "a paradox puts the learner in an intellectually unbearable situation. The impulse to resolve the paradox is a powerful motivator for change of knowledge frameworks." (Movshovitz-Hadar and Hadass, 1991, p. 88). Sriraman (2008) used the Russell's paradox in a 3-year study with 120 pre-service elementary teachers and studied their emotions, voices and struggles as they tried to unravel the paradox. Mamolo and Zazkis (2008) used the Hilbert's Grand Hotel paradox and the Ping-Pong Ball Conundrum to explore the naive and emerging conceptions of infinity of two groups of university students with different mathematical backgrounds.

Paradoxes such as Achilles and the tortoise, and the Ping-Pong Ball Conundrum are super-tasks. "A super-task may be defined as an infinite sequence of actions or operations carried out in a finite interval of time" (Laraudogoitia, 2011). In our study we focus on a particular super-task, known as the Thompson's Lamp, devised by Thompson (1954) to show that super-tasks are impossible.

> Think of a reading lamp with an on/off switch button. Suppose that the button can be pressed in an instant of time. Suppose at the start the lamp is off. After 1 minute the button is pressed and the lamp is on. After another 1/2 minute the button is pressed and the lamp is off. After another 1/4 minute the button is pressed and the lamp is on, and so on. That is, the on/off position of the lamp is flipped when exactly one-half of the previous time interval elapses. At the end of the two minutes, is the lamp on or off?

Thompson (1954) argued that the lamp cannot be on as it is never turned on without turning it off some time later. And similarly the lamp cannot be off. Therefore a contradiction arises and Thomson's lamp, or any super-task, is logically impossible. Benacerraf (1962) argued that the above reasoning is valid only for any instant between 0 and 2 minutes, but not at exactly 2 minutes.

In this article we study the struggles with the concept of infinity of a group of university students using the Thomson's Lamp paradox and its isomorphic version, the Green Alien, explained below. Several theoretical frameworks informed our investigation. We use APOS

theory (Dubinsky, Weller, McDonald & Brown, 2005a), reducing abstraction by Hazzan (1999), and platonic and contextual distinction by Chernoff (2011).

APOS theory (Dubinsky et. al., 2005a) suggests that an individual deals with a mathematical situation by using the mental mechanisms *interiorization* and *encapsulation* to build the cognitive structures *actions*, *processes*, *objects* and *schemas* that are applied to the situation. Researchers have acknowledged the difficulty in transitioning from process to objects conceptions. Focusing on this difficulty, Weller et al. (in press) introduced the construct of 'Totality', which is a "part of the mechanism of encapsulation, not a separate stage", but a step one goes through before the process is encapsulated to an object. Totality indicates, as its name suggests, the ability to relate to the entire process, to all the steps at once, rather than in a sequence. Another theoretical framework we use is reducing abstraction by Hazzan (1999). According to Hazzan (1999) there are three ways in which abstraction level can be interpreted: (1) Abstraction level as the quality of the relationships between the object of thought and the thinking person, (2) Abstraction level as reflection of the process–object duality, and (3) Abstraction level as the degree of complexity of the mathematical concept. We also rely on the theoretical constructs introduced by Chernoff (2011) in distinguishing between platonic and contextualized situations. Chernoff distinguished between platonic and contextualized sequences in the relative likelihood tasks in probability. A platonic sequence is characterized by its idealism. But, a contextualized sequence is characterized by its pragmatism.

While mathematical paradoxes are presented in a context, the resolution of the paradoxes requires a thought experiment and de-contextualization from the physical reality, that is, accepting the idea that a super-task can be completed in the given time interval.

The Study

Participants in our study were 38 undergraduate liberal arts students. The students were introduced to the Thomson's Lamp paradox and they were invited to express their thoughts on whether the lamp was on or off after exactly two minutes. Then the Thomson's argument that shows the logical impossibility of super-tasks was discussed. This was presented as yet another point of view, rather than an authoritative conclusion. Then an extended discussion, in which the participants had an opportunity to argue and convince each other followed. The whole session on the Thomson's Lamp was recorded. Two weeks after the discussion on the Thomson's Lamp, the students were presented with the following Green Alien problem, to which they were asked to respond in writing and justify their answer.

> A Green Alien arrived in the earth at midnight, November 5, and she has exactly 1 day to spend on our planet. That is, she MUST leave at midnight, November 6; otherwise she will change into a pumpkin. However, she learns that FBI is after her, so at noon, November 5, she instantaneously changes her color to pink (to blend in with the local girls). When the search intensifies, she changes the color back to its original Green at 6 pm, and keeps changing it at half-intervals of the remaining time, as follows: Midnight (12 AM November 5) – Green, Noon (12 PM November 5) – Pink, 6 PM – Green, 9 PM – Pink, 10h 30min PM – Green, 11h 15min PM – Pink, 11h 37min 30 sec PM – Green, 11h 48min 45 sec PM – Pink … What color is she at midnight, as she leaves the earth?

Our data consist of students' arguments in considering the Thomson's Lamp during classroom discussions, and their written responses to the Green Alien problem. Note that the Green Alien problem is isomorphic to the Thomson's Lamp paradox, but the context is changed from the familiar light switching to science fiction. In what follows we analyze students'

responses to the two versions of the super-task and the effect of context to their emerging conceptions of infinity.

Results and Analysis

In the process of halving the remaining time in Thomson's Lamp, one has to deal with an infinite number of time intervals in a finite time interval. This can create a cognitive conflict and tension. This state of cognitive conflict, according to Movshovitz-Hadar and Hadass (1991), creates uneasiness. Piaget calls it as a state of disequilibrium between assimilation and adaptation-accommodation. This state, and the tension it creates, stimulates an attempt to get out of it and achieve a new equilibrium with a more advanced mental structure (p. 80). We present here some ideas that emerged in the classroom discussion that show the cognitive conflict and various, at times implicit, ways of achieving equilibrium.

An idea expressed and accepted by several students during the extended discussion was that "it will never reach two minutes". Equilibrium is achieved with this claim, as it frees students from deciding on the state of the lamp at the end of 2 minutes. However, having recognized the absurdity of this claim, some students struggled to reconcile the idea of infinitely many time intervals within two minutes. For example, on one hand Jason says it (time) does not reach 2 minutes but on the other hand he considers the status of the lamp at 2 minutes using ideas resembling Thomson's argument. Inconsistency in Jason's response shows the cognitive conflict he is experiencing. The inconsistency in students' claims can be attributed to their inability to consider the process in its totality, which is essential according to Weller et al. (in press) in encapsulating the infinite process.

We note these students' reference to numbers, rather than time intervals. For example Jason thinks of numbers that are "too small": "there is no way of knowing whether it will be on or off because the numbers get too small, and will not reach 2 minutes". We observe here students' attempt to reduce the level of abstraction by considering numbers, rather than considering very small time intervals, which are unfamiliar and hard to conceive.

While only several students clearly expressed their views during classroom discussion on Thomson's Lamp, we wondered whether these views were shared among other students. As such, we designed the Green Alien problem and collected written responses from the 38 students in class, which we present in this section. We identified several interwoven themes in students' responses (the number in parenthesis is the number of responses): contextualization (6), halving time goes on forever (3), never reach midnight (7), impossible to determine (15), acknowledgement of paradox (9), and connection to Thomson's Lamp (3). In several works more than one theme was featured. Due to limited space, we exemplify only a few of these.

Out of 38 responses, six showed a clear contextualization. For example, one student wrote "the poor alien will be changing back and forth so many times as the midnight comes nearer that she will feel sick and wish to go home to bed anyway". These attempts in contextualization of an imaginary situation are indications of reducing abstraction from unfamiliar (instantaneous change of color) to the familiar (being tired). Some considered the situation as paradoxical. Those who expressed the opinion that halving time continues forever and never reaches midnight often concluded that it is impossible to determine the color.

Gilbert was one of three students who suggested that halving time goes on forever. But he introduced an additional idea: "since the time will continue to halve itself forever she will be both green and pink at her time of departure at midnight". The reference to "both green and pink" can be seen as a halfway color between green and pink. This view is in accord with taking ½ as the sum of Grandi's series ($S = 1 - 1 + 1 - 1 \ldots$) if green and pink are assigned 0 and 1

respectively, and turning pink is considered as adding 1 and turning green is considered as subtracting 1. Though it is highly unlikely that Gilbert thought this way, his answer shows some signs towards considering the process of halving the remaining time in its totality.

Conclusion

Paradoxes of infinity puzzled people for centuries, so it is not surprising that our students exhibited some confusion and presented arguments that lacked internal consistency. It is evident that only a few students claimed that the color of the alien at midnight was impossible to determine, presumably accepting the 'normative' resolution. However, the reasoning behind this suggestion often relied on arguments suggesting that midnight cannot not be reached.

The particular paradox discussed in this article relies on the idea of halving the time interval infinitely many times. Can something be halved indefinitely? Stavy and Tirosh (1999) explored this issue with learners in their study of intuitive rules. In fact, the answer depends on the interpretation of the question. If a line segment is divided in half, then the process can continue indefinitely, as a segment is an abstract mathematical object. If a piece of copper wire is divided, then the process must end when reaching an indivisible unit. However, what about time intervals? The added difficulty of the super-tasks that students discussed was that they were presented in a context, but were to be considered platonically, that is, disregarding the context of time. We wonder whether the fictitious context of an alien provides a more accessible avenue to consider super-tasks than a 'realistic' context of switching a lamp unrealistically fast.

The cognitive conflict and the resulting tension a learner experiences in thinking of the Thomson's Lamp or the Green Alien can be a powerful motivator to unravel the notion of 'infinitely small' and understand its formal aspects. Coupled with the Grandi's series it can provide a rich task for liberal arts students to explore, to struggle with, and to experience fascination with ideas and the formal aspects of infinity.

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MEANINGFUL LISTENING THROUGH COHERENT CONCEPTUAL METAPHORS

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Metaphors span language in many forms. Mathematics educators regularly use metaphors to relate one mathematical concept in terms of another. Embedded within these concepts are a set of shared experiences that educators use to communicate new ideas to students. However, students also use metaphors to convey their experiences, perceptions, and understanding. This paper reports on a mixed-methods study that discovered a coherent set of metaphors that high school students and teachers associated with mathematical problem solving. The existence of a coherent system of metaphors for problem solving offers a starting point for discussing problem solving experientially. It also opens new methods in student-teacher discourse. Specifically, this methodology helps teachers learn how to listen hermeneutically for student experiences and is being developed for teacher professional development.

Keywords: Classroom Discourse, Problem Solving, Teacher Education, High School Education

Introduction

This study applied Conceptual Metaphor Theory (CMT) as a means to analyze students' problem solving experiences. The results are twofold. First this paper presents the results mixed-methods study that identifies student and teachers perceptions of mathematical problem solving using metaphors. Secondly, this paper takes this study's novel methodology and suggests it is useful for teacher development in discourse (Sfard, 2009). Specifically, the researcher explores the opportunity for CMT analysis to move teachers from a constructivist paradigm of interpretive listening to a hermeneutic model of listening for conceptual understanding (Davis, 1997).

Literature

Often within education, the purpose of metaphors are to share one person's experiences through common experiences (or experiences believed to be common) (Ortony, 1993). This study identifies the shared experiences teachers and students associate with mathematical problem solving, because problem solving lacks a cohesive definition agreed upon within the mathematics education community (R. Lesh & Zawojewski, 2007; Polya, 1945; Schoenfeld, 1985; Shumway, 1982; Silver, 1985). Nonetheless, there are coherent shared experiences drawn upon by mathematics educators to work with problem solving (Chapman, 1997; R. A. Lesh & Doerr, 2003; Shumway, 1982; Silver, 1987). Thus this study identifies if metaphors associated with problem solving are a means to describe and identify a coherent set of shared experiences for discursive practices.

Fundamentally, this study's epistemological framework aligns with Sfard's (2009) work on discourse for cognition via participation and acquisition. Despite all the mediums through which one must interpret student and teacher perceptions of mathematical problem solving, the student and teacher try to communicate their personal experiences through a set of presumed shared experiences (Yee, 2012). These shared experiences are fundamental because the assumption of what students and teachers perceive as shared experiences define their interpretation of problem solving (Sfard, 2009). This study used conceptual metaphors (Lakoff& Johnson, 1980) to identify the shared experiences embodied within mathematical problem solving.

A conceptual metaphor is a mapping, an identification of the experience to be expressed (the target domain) and the experience to be shared (the source domain) (Lakoff & Johnson, 1980). For example, a student said during my interviews, "to solve it for me, it meant that I had to find it

somehow." The student used the conceptual metaphor: PROBLEM SOLVING IS SEARCHING. Thus the student was sharing how their understanding of problem solving (target domain) is perceived as searching (source domain) (Kövecses & Benczes, 2010). The student communicates this because they believe the researcher has shared the experience of searching and so relates this perception to the researcher. Due to the specifics of each domain, it is important to note that this is a unilateral relation: TARGET DOMAIN →SOURCE DOMAIN.

This study generates a unique methodology that is replicable, but more importantly, a practical means to aid teachers in listening to their students understanding.

Method

Participants for this study included 22 independent honors geometry student interviews and 6 independent teacher interviews from both honors geometry teachers in a suburban high school. The students and teachers both met with the researcher individually for 10-15 minute semi-structured interviews about specific problems from one of three recent common assessments designed and determined by the teachers. The teacher interviews focused on how teachers expect students to solve problems, while the student interviews focused on how students had solved the problems.

Results

The CMT analysis discovered some surprising relationships between teacher and student thinking. Table 1 is a list of source domains used by students, and source domains used by teachers related to problem solving.

Table 1:Teacher and Student Source Domains for Mathematical Problem Solving

Only Student Source Domains	Source Domains of Student and Teacher	Only Teacher Source Domains
ABILITIES	ACQUISITION	CHANGE OF STATES
APPROXIMATING	BUILDING	CONFLICT
CALCULATING	DISCOVERY	DOING BUSINESS
COMPARING CONDITIONAL	IMAGINING	FAMILIARITY
CONTEST	JOURNEY	GENERALIZING
CONVINCING	PARTITIONING	HABITS
EXPERIMENTING PROVING	PROCESS	RACE
THINKING	REVIEWING	RULES
VOCALIZATION	SEARCHING	SETS OF SKILLS
	VISUALIZATION	TOOLBOX
	WAR	UP

PROBLEM SOLVING IS SEARCHING was one dominant form of metaphorical discourse. The following statements from teachers and students do indicate that searching is a consistent source domain. Student5 explained, "There is probably like a missing side or missing angle or something." Student7 clearly stated, "to solve it for me it meant that I had to find it somehow." Teacher2 stated that his students would "have to go searching for them." Thus, the conceptual metaphor of searching was frequented through the perception of "finding" the "missing" solution. This structural metaphor is grounded in one's experiences searching for objects. Additionally, other source domains were regularly entailed (Kövecses & Benczes, 2010) with searching because one's experience in searching develops with other concrete experiences, such as journey.

PROBLEM SOLVING IS A JOURNEY was by far the most abundant metaphor for problem solving. Inherently, everyone travels and was identified frequently as a grounded metaphor. The most common mapping to journey occurred when student's referred to "the way" to solve a problem. In this manner, the experiential "way" is short for passage way. Student3 stated, "I couldn't really

think of a way to solve it and if I did find a different way to solve it wouldn't have been as easy". Student15 stated "I'm not sure if you could find a different way but I'm sure there are other ways". Teacher1 stated, "I don't have any issues with them doing it that way as long as they still have all needed stuff that's there." As a primary conceptual metaphor, "way" is being used as a path or trail, but as a complex metaphor, the aspects of this path or trail emphasize the journey the student is on to solve the problem.

To see these connections more clearly, notice the following other aspects of journey as described by the students and teachers. Teacher2 mentions that as the students "use the short cut conjectures that we've talked about." The source domain of journey occurred more subtly in some student's interviews. Consider Student12's statement, "I don't know, that wouldn't have gotten me too far. What is enlightening about the use of journey is that teachers encourage students to believe that the journey is equally credible in solving math problems.

The quantitative results demonstrated a strong similarity between the most popular and most frequented source domains. Table 2 demonstrates these results:

Table 2: Comparison of Teacher and Student Source Domains for Problem Solving

Students' Most Frequented Source Domains	Students' Most Popular Source Domains	Teachers' Most Frequented Source Domains	Teachers' Most Popular Source Domains
26% JOURNEY	95% JOURNEY	29% JOURNEY	100% JOURNEY
18% SEARCHING	86% VISUALIZING	17% DISCOVERY	100% DISCOVERY
13% VISUALIZING	82% SEARCHING	11% BUILDING	83% SEARCHING
12% DISCOVERY	73% PROCESS	10% VISUALIZING	83% BUILDING
9% PROCESS	68% BUILDING	7% PARTITIONING	67% VISUALIZING
8% BUILDING	68% DISCOVERY	7% SEARCHING	50% PROCESS
8% PARTITIONING	55% PARTITIONING	6% PROCESS	33% PARTITIONING
			33% RACE
			33% SETS OF SKILLS

Despite more popular source domains from the teachers, the top seven source domains remained the same in popularity and frequency for both teacher and student. These results demonstrate a coherent system of conceptual metaphors within mathematical problem solving in these high school honors geometry classrooms. PROBLEM SOLVING IS A JOURNEY, SEARCHING, VISUALIZING, A PROCESS, DISCOVERY, BUILDING, and PARTITIONING.

Other analyses were done with respect to the data, such as comparing the students' score with the frequency and popularity of the metaphors. Surprisingly, with every single student metaphor, no correlation was significant with student performance. Not a single metaphor suggested a positive correlation with a student's score on an assessment. This is remarkable and encouraging as many mathematics educators strive to demonstrate multiple representations.

Discussion

The result of this study used mixed methods with a phenomenological qualitative analysis (CMT) to identify coherent set of source domains (i.e. shared experiences) that students and teachers associate with mathematical problem solving. It is vital that this research is not misinterpreted. This study has only shown existence of a coherent metaphorical system, not uniqueness. This data is not attempting to prescribe these source domains as the only source domains for problem solving. Instead, this study validates the prescription for the methodology and the use of CMT analysis as a means to identify localized coherent conceptual metaphorical systems.

As embedded within the PME-NA theme, this study broadened through linguistic interdisciplinarity. This can be incredibly beneficial in improving how teachers listen, thus improving mathematical thinking in the classroom. As this study contained only six teacher interviews, a current study is underway to interview 30 pre-service, practicing, and master teachers' perceptions of problem solving using this methodology.

The significance to teacher listening can be perceived from the pervasive need for all teachers to see metaphors as necessary for communication at all levels of education (Petrie & Oshlag, 1993). Davis (1997) discovered through a series of vignettes that mathematics teachers listen at three levels: evaluative listening (assessment-based paradigm), interpretive listening (constructivist paradigm), and hermeneutic listening (conceptual participation paradigm). The third form of listening Davis suggests is what teachers should strive to develop, an immersed perspective of genuinely participating and sharing in the cognitive experiential development of the meaning behind the mathematical topic. Problem solving was only one means to discovering this methodology, but not its only use. This research proposes is that CMT analysis allows for this participatory conceptualization by analyzing, interpreting, and listening for the shared experiences of the students through conceptual metaphors.

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AFRICAN AMERICAN MALES AND MATHEMATICS AS THE "INTELLECTUAL PROPERTY OF WHITENESS"

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Keywords: Equity and Diversity, Policy Matters

If one were to visit an Advance Placement Calculus class, one would typically find only two African American males in the course. "In the United States, remedial mathematics classrooms contain large numbers of African American students; however, advanced mathematics classes mainly serve White students" , (as cited in Snipes & Waters, 2005, p. 107). African American students especially males do not have the opportunity to take advance math classes in high school. Early on in their schooling, many African American males experience failures in mathematics. According to Ladson-Billings, "as a group, African Americans have been told systematically and consistently that they are inferior, that they are incapable of high academic achievement" (as cited by Jackson & Wilson, 2012, p. 357). In her analysis of the cause, she thinks that it is systemic problems stemming from the remnants of slavery; where African American students fulfill the inferior prophesy. Ladson-Billings introduced the concept of mathematics as intellectual property in her research (Snipes & Waters, 2005). Ladson- Billings stated that mathematics courses are still viewed as "intellectual property" that is afforded to the dominant culture (as cited in Snipes & Waters, 2005, p. 110). This sense of owning property symbolizes power (Snipes & Waters, 2005).

Critical Race Theory, where race is used as the signifier that oppresses and symbolic interactionism, where the world is interpreted based on experience, will guide an ethnographic study of the African American male perspective of mathematics as the property of whiteness (Ladson-Billings, 1998; Roulston, 2012). In the proposed study, a group of African American males will serve as a culture that does not have access to mathematics, where their perspective will be examined to deduce their interpretation of the current practice of mathematics as the gateway course.

Current research studies by scholars such as Ladson Billings and Moses, examine the issue of the disproportionate representation from an adult point of view. However, there is not enough research of what the students think about the lack of access in mathematics and how mathematics courses determine their destiny in life. In conducting this study, academia and the public would have information from the student's perspective and could change the practice of excluding the African American students' voice from the table of policymaking.

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ONLINE OR PROCTORED: AN INVESTIGATION OF TWO SETTINGS FOR A MATHEMATICS PLACEMENT TEST

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Keywords: Post-Secondary Education, Data Analysis and Statistics, Assessment and Evaluation

Motivation for the Study and Theoretical Framework

A major driver of student success in mathematics is self-efficacy (Zajacova, Lynch, & Espenshade, 2005), which can be negatively influenced when students enroll in mathematics courses that do not match their level of preparation (Bassarear, 1986). Some post-secondary institutions have replaced a proctored mathematics placement test (MPT) with an un-proctored, online one to inform students of their preparation. This quantitative study adopts the stance that accurate information provided by the MPT positively affects student self-efficacy, so it is important to determine the accuracy of the MPT in an online setting versus a proctored setting.

Data Collection and Analysis

Data Collection

Student data on MPT scores and fall term grades were obtained from a large public university. The sample was restricted to students who entered as freshman in either fall term 2010 (online MPT) or fall term 2011 (proctored MPT) and enrolled in one of four fall term mathematics courses. Variables included MPT scores and participation rates and course grades.

Analysis

Summary statistics. In 2010, the MPT participation rate was 98.26%, while in 2011, the MPT participation rate was 96.25%. The participation rates were statistically equivalent across years. In each of the four courses, the mean MPT score was approximately 2 points higher in 2010 than in 2011, with equality of means rejected.

Hypothesis tests. Three linear regression models were employed to test the predictive power of MPT score on fall term grade. The results of the models indicate that (1) the predictive power of the MPT score is consistent across the four courses studied, (2) an increase in one standard deviation above the mean score within a course predicts a 0.33 grade point increase, and (3) the predictive power of the online MPT is statistically equivalent to that of the proctored MPT.

Discussion

The results of the analysis indicate that the online MPT predicts student success with the same level of accuracy as the proctored MPT. Therefore, the information provided to students by an online MPT is valuable to students. By using their MPT scores to help choose courses, students' preparation will be better matched to the course content and they will also likely have higher levels of self-efficacy and therefore greater success in their mathematics courses.

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RESOURCE USE OF PRESERVICE ELEMENTARY TEACHERS IN AN INQUIRY-BASED LEARNING MATHEMATICS CONTENT COURSE

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Students in mathematics content courses use resources both within and outside of the classroom. We examine how preservice elementary teachers in an inquiry-based mathematics content class use available resources for their learning. We also are interested in what impacts the ways students use these resources. We interviewed and observed three students from an inquiry-based mathematics content course for preservice elementary teachers to investigate the resources they used (e.g., family members, peers, small groups, poster presentations, instructor, readings, colored pens). By analyzing how and why students used these resources, we identified three main recurring themes, two about specific resources (individuals and small groups) and one about the rationale for using a variety of resources (motivation). In this paper we discuss each of these themes in detail.

Keywords: Teacher Education-Preservice, Post-Secondary Education

We examine undergraduate students' use of resources in order to inform those who design instruction for these students. We define resources as anything the students use to help them learn (e.g., people, Internet, textbooks). Specifically, we focus on preservice elementary teachers' use of resources in an Inquiry-Based Learning (IBL) mathematics content course. The unique nature of IBL courses provides an interesting site to study students' resources, as the course requires students to do explorations on their own by playing a more active role. While the topics covered in elementary mathematics content courses are considered to be quite basic, the focus of these courses—to develop deep conceptual understanding of ideas that students generally know in a procedural sense—causes many students to find these courses surprisingly challenging, even when they have been successful in high school and higher-level mathematics courses. It is suggested that preservice teachers need to develop the capacity to explore resources and situations in classrooms in order to develop their expertise (Edwards & Protheroe, 2003). We conjecture that the resources available to students in this environment are different from the resources that are normally available in a lecture-based class and that specific features of IBL courses may influence students' use (or lack of use) of those resources.

We proposed the following two research questions: How do preservice elementary teachers in an inquiry-based learning mathematics content class use available resources for their learning? What influences the ways students use these resources?

Analysis of the data highlighted prominent themes: resources (what), use of resources (how), and reasons for using resources (why). We came to understand the learning process as a function of how and what resources students use, which is contingent on why students make those decisions. Considering the results, educators who design instruction that incorporates IBL, especially for preservice elementary teachers, should consider the use of specific resources available to students as well as how and why they are used.

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OUT-OF-SCHOOL INFLUENCES OF KOREAN IMO WINNERS

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Keywords: Early Childhood Education, Affect, Emotions, Beliefs and Attitudes

The family environment has been found to be positively correlated with a child's academic success (Haile & Nguyen, 2008), providing stimulation for curiosity and encouragement to explore their interests with positive beliefs, attitudes, behaviors, and values. Parental involvement in educational activities as well as parent perceptions toward pursuits of learning a particular subject (Bleeker & Jacobs, 2004) has been shown to be influential to a child's academic achievement. Mentors and peers can also be a powerful influence in shaping a child's development. Advice from a mentor at the right time can have the potential to change a person's life (Muratori et al., 2006), and peers can provide the right amount of support and competition to influence academic achievement as well as influencing college and career decisions (Ryan, 2001). Competitions, out-of-school camps, and private and group tutoring such as the "Ha-Gwon" in Korea have been shown to provide additional influence in terms of academic achievement and motivation for students (Kim, 2008).

This study examined sources of out-of-school influence through interviews with five Korean International Mathematical Olympiad (IMO) winners and looked at their influence on academic achievement and development. We studied Korean students for this country's history of high performance, scoring in the top five during four of the past five years.

We conducted a series of three in-depth interviews with five IMO winners and three parents about these IMO winners' life experiences that contributed toward their academic achievement and development of talent and interest for mathematics. Interviews were conducted in Boston, MA and various cities in Korea. We started with a list of out-of-school factors, such as parents, peers, mentors, and competitions from previous research (Karp, 2003; Muratori et al., 2006) and employed grounded theory techniques to attempt to discover factors of influence common to Korean culture that have not yet been disseminated in other studies. Constant comparative methods of analysis were used to discover differences and similarities in the data as compared to the categories of influence previously mentioned.

Analysis revealed five major sources of out-of-school influence. Family members, peers, mentors, participation in mathematics camp, and the effects of mathematics competitions are influential sources that have been discussed in previous research. Mathematics workbooks and Ha-Gwon surfaced as sources of influence that were seldom previously mentioned.

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THE STABILITY OF MATHEMATICS IDENTITY AND ITS RELATIONSHIP WITH STUDENTS' CAREER CHOICE

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Purpose

Prior research has shown a relationship between identity and student persistence and engagement toward mathematics or mathematics-related fields (Boaler & Greeno, 2000; Cass, Hazari, Cribbs, Sadler, & Sonnert, 2011). In this study, mathematics identity is discussed as a way of explaining student persistence in mathematics and conceptualized as how students see themselves in relation to mathematics based upon their perceptions and navigation of everyday experiences with mathematics (Enyedy, Goldberg, & Welsh, 2006). The purpose of this study is to test two aspects of a mathematics identity framework guided by the following research questions 1) What is the relationship between mathematics identity and students' career choice as a mathematician and a math/science teacher; and 2) Does the framework adopted in this study provide a global view of students' mathematics identity?

Methods

Logistic regression was used to address the first research question using data from the FICSMath project, which was administered in the fall of 2009 yielding 10,437 surveys. A proxy for mathematics identity was used based on results from a previous analysis (Cribbs, Hazari, Sonnert, & Sadler, 2012). A Wilcoxon paired t-test was used to address the second question using data from a follow up study administered in the summer of 2012 yielding 131 surveys.

Results

Only control variables with a significance level of $p < 0.01$ were included in the final model. The first regression model predicting career choice as a mathematician indicates that Father's Education and ACT_SAT Score were significant. In addition, the results indicate that mathematics identity significantly predicts students' career choice as a mathematician ($p < 0.001$). The second regression model predicting career choice as a math/science teacher indicates that Gender, Race: Black, Race: Asian, Father's Education and ACT_SAT Score were significant. In addition, the results indicate that mathematics identity significantly predicts students' career choice as a mathematician ($p < 0.001$). Results for research question two found that mathematics identity remains relatively stable over time with a non-significant p-value.

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EXPLORING CONNECTIONS BETWEEN STORY PROBLEM TOPICS AND PROBLEM SOLVING: IS WORK HARD AND SOCIALIZING EASY?

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A mathematics story problem's topic may influence how difficult it is to solve. For example, a student is more likely to accurately solve a story problem with a topic relevant his or her interests (e.g., a "personalized" scenario on playing video games) than a matched story problem that is not (Walkington, in press). Nathan, Kintch, and Young (1992) proposed a model of story problem solving where learners construct a mental representation of the actions and relationships in the problem, a *situation model*, which is then connected with formal mathematical operations. Problems with relevant and familiar topics, like personalized stories, may promote situation model construction by allowing new mathematics concepts to be connected to prior experience, with key concepts becoming more *grounded* (Goldstone & Son, 2005). Problems with relevant topics may also elicit *interest* in the content to be learned (Walkington, in press). However, individual-level personalization may not be feasible for a heterogeneous classroom (Hidi, 1990). Therefore, we identify topics that support problem-solving across a diverse sample of students.

We used LIWC, a dictionary-based computerized text analysis program that counts words in many categories. LIWC measured problem topic by determining the percentage of the words in a story problem that belong to different categories (e.g., social processes and work). We analyzed a dataset from a diverse array of secondary students ($N = 3394$) solving 282 traditional story problems on linear functions in Cognitive Tutor Algebra (CTA). CTA is an intelligent tutoring system that provides adaptive hints and logs student answers (Morgan & Ritter, 2002). After controlling for aspects of the problem's mathematical structure, we tested for associations between the topic of the story problem, accuracy, and hints requested. Words related to social processes (e.g., talk, husband, friend) were positively associated with correct answers ($\beta = .29$, $p < .001$) and negatively associated with hint-seeking ($\beta = -.04$, $p = .03$). In contrast, words related to work (e.g., job, business, profit) were negatively associated with correct answers ($\beta = -.26$, $p < .001$) and positively associated with hint seeking ($\beta = .29$, $p < .001$). Also, words related to food (e.g., restaurant, pizza, cook) were negatively associated with hint-seeking ($\beta = -0.10$, $p = .03$). Students' situation models appeared to be facilitated if the problem was about social processes and food and hindered if the problem was about work. Future research may illuminate whether this effect is due to increased familiarity with the topic or activated interest in the story problem.

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CONTEXTUAL VISUAL INFORMATION IN MIDDLE SCHOOL PROBLEM SOLVING: A PUZZLING SITUATION

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Research findings on integrating visuals with text have indicated that students learn less when interesting but irrelevant materials, such as decorative images, are included (e.g., Harp & Mayer, 1998; Mayer, 2009; Sweller, 2005). The basic idea is that these can overload processing and disrupt student learning and performance, but evidence with math problems is mixed (e.g., Berends & van Lieshout, 2009). Based on the theories and previous research findings, we revised eight problems in an existing mathematics curriculum. We used four types of revisions within-subjects; three removed decorative images and one added relevant information to the visual.

Fifty seventh-grade students each completed eight problems. We assessed students' problem solving accuracy and strategies in addition to their math background, math attitudes, and contextual recall as these measures potentially influence the effect of visual representations. Revising the problems in concordance with the research-based principles did not have a consistent effect on performance or strategy use, nor were these effects related to students' math ability or anxiety. This is in contrast to what is often found in applications of these principles in science (see Mayer, 2009). Adding labeled dimensions to a visual increased the use of units, although not necessarily in a correct way. Of note, we found that the revisions did not affect students' opinions about the problems. The recall measures indicated that students encoded the contextual information from the text and/or visuals. Additional processing of the contextual information (as seen by correct recall) tended to be associated with lower mathematical accuracy, but this was unrelated to the presence of the decorative image. Overall, the lack of consistent effects indicates the need for further research on the influence of these principles in mathematics.

Given the multitude of visual representations in mathematics textbooks, it is critical to understand how visual information (both decorative and relevant) influences students' problem solving. The math ability level of our students was relatively high, and other individual differences may explain the lack of consistent effects. Investigating student attitudes as well as their contextual memory broadens the research base, so that researchers and educators can develop more nuanced understanding of the uses of visuals. Most clear from this research, however, is the need to further address how to enhance the integration of visual and verbal information in the mathematical problem solving of K – 12 students.

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ALLOCATING OPPROTUNITY: DISBURSEMENT OF TITLE I FUNDS AND MATHEMTICS COURSE-TAKING AMONG SECONDARY STUDENTS

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Public school children frequently take different courses, based on ability and interests. Recent national data show that low SES students enroll in rigorous coursework less frequently than their middle and upper class counterparts (Caro, McDonald, Willims, 2009; Tyson, 2011).

The Title I Program as part of the ESEA was created to provide financial assistant to schools serving large concentrations of children from low SES. van der Klaauw (2008) has shown that the use of Title I funds has done little to improve the academic achievement of low SES students. With the reauthorization of ESEA schools have more leeway in the way they spend their Title funds.

Purpose of the Study

The purpose of this study was to measure the impact that varied Title I spending may have among low SES students, in terms of level of mathematics courses taken in high school. The research question for this study is; Does the way Title I money is spent in a school influence the mathematics course-taking decisions of low SES high school students?

NAEP:09 mathematics achievement data and HSTS:09 were used to address this question. A t-test was used to compare mean level of mathematics course-taking for low SES students in schools who spend Title I funds differently.

Results

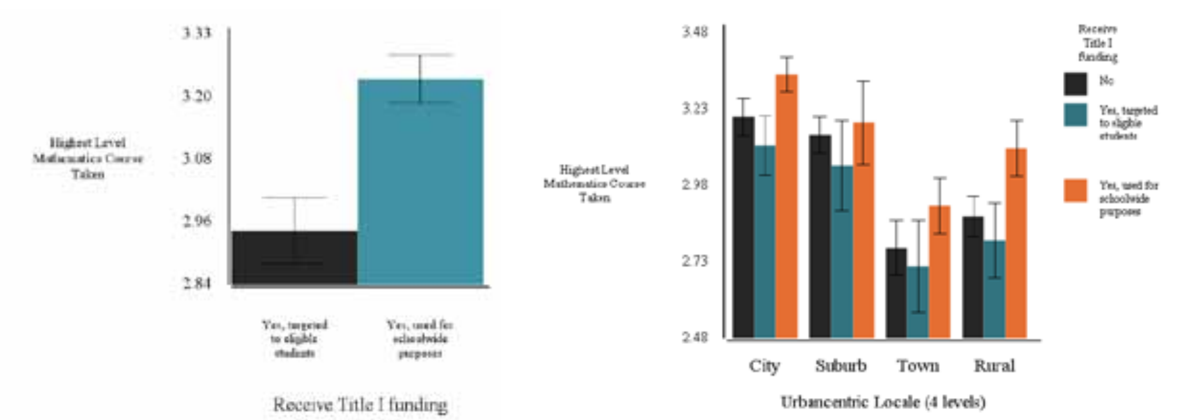

Figure 1: Course-taking Among Title I Students by Type of Title I Spending

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RELIGIOUS PARTICIPATION AND MATHEMATICAL PROBLEM-SOLVING: THE RELATION BETWEEN RISK-TAKING AND PROBLEM CONTEXT

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Research has indicated that children think about mathematics problems differently and experience different levels of success, depending on the context in which problems are situated (Saxe, 1991; Taylor, 2009). However, there has been little exploration of the relation between problem context and *intellectual risk-taking* (IRT), or "engaging in adaptive learning behaviors (sharing tentative ideas, asking questions, attempting to do and learn new things) that place the learner at risk of making mistakes or appearing less competent than others" (Beghetto, 2009, p. 210). IRT also includes a willingness to participate when one does not fully understand the material (Boekaerts, 1993). Despite its potential to promote learning, cognitive development, and identity support, IRT is atypical in schools (Clifford, 1991; Streitmatter, 1997).

In this work, I examine IRT in interviews of 27 children (aged 5-12) whose families tithe (give 10% of their earnings to the church). I ask a) *How does placing a mathematics problem in a tithing-related context influence children's IRT?* and b) *What supports might facilitate IRT on tithing context questions?* Multiple questions were presented in two problem-solving contexts: *general* (e.g. What is 10% of 10?) and *tithing* (e.g. "Let's say you have $10 and you have to tithe at 10%. How much would you give to the bishop?"). Ethnographic observations of Sunday school classes were also conducted since all children interviewed attend these classes weekly.

Results indicate that children exhibited greater IRT - or willingness to offer answers, whether correct or incorrect - when problems were posed in a tithing context than a general context. Across all questions, children responded with a guess 96% of the time in the tithing context, compared to 63% in the general context. While many factors might contribute to this difference in IRT, I hypothesize that children appropriated risk-taking patterns from Sunday school, a setting associated with tithing. Ethnographic data identified four elements of Sunday school that create a culture of IRT: 1) *Guessing activities* encouraged participation prior to complete understanding; 2) Frequent *teacher questioning* encouraged children to share ideas, even with mistakes; 3) *Talks/ prayers* allowed children to challenge themselves by speaking in front of large groups; and 4) T*eachers supports*, such as whispering words in students' ears during talks, allowed children to engage in advanced tasks by lowering entry points and difficulty of material.

This study suggests that it is crucial for teachers and researchers to carefully choose the contexts used to present problems since problem context is related to IRT. Additionally, if risk-taking patterns in out-of-school settings can be activated by problem contexts, drawing on practices in which students typically take risks might be one way to encourage IRT in schools.

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REPRESENTATIONS OF MATHEMATICS IN CHILDREN'S MEDIA

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This poster session reports on a portion of a multifaceted research project that investigated elementary students' views of and experiences with mathematics and mathematicians, and the ways that students' views may be impacted by parents' views, teachers' views, and popular media representations of mathematics and mathematicians. The overarching goal of this project is to understand how outside sources can influence children's views. Prior research (e.g., Ma, 1999) has shown that negative attitudes toward mathematics are linked to decreased achievement and participation, but it is unclear what factors impact children's attitudes toward mathematics.

This study is framed with a feminist and social constructivist stance. As such, I conceive of views of mathematics, as well as mathematics as a discipline, as socially constructed and gendered. The conceptual framework for this study involves producers and active consumers. In an interaction, messages are disseminated by a producer (e.g., media) and received by a consumer (e.g., child). Each received message is actively evaluated by the consumer, who either alters the existing schema to include the new idea or discards it (Huntemann & Morgan, 2001).

The larger research project involved questionnaires, drawings of mathematicians, and focus group interviews with Grade 4 and 8 students from Ontario; interviews with teachers and parents; and an analysis of children's media. For the purposes of this poster session, I report on the media analysis. Media play a key socializing role in children's lives (Roberts & Foehr, 2004), so exposure to media messages about mathematics may impact students' views of mathematics.

To guide the media selection, a list of the 'top' children's media was compiled from students' responses to an online questionnaire in which 156 Grade 4 and 8 students took part. For each of five media types (television shows, movies, websites, video games, and books), participants listed their three top choices. From these data, a 'top choices' list was compiled, which was comprised of 10 television shows, three movies (or series), seven websites, six video games, and five books (or series). To narrow the viewing, summaries of each individual title (e.g., an episode of a television show) were read online; if there was any indication of mathematical content, the title was selected for analysis. Detailed notes were taken for each title viewed, and these notes were analyzed for mathematical content. Mathematics-related instances were summarized and themes were established for each series, for each media type, and across all media types.

Wide variations of mathematical content and messages were found to exist across media types, although stereotypes and mathematical errors were commonly featured. A full discussion of the findings will be presented in this poster session, with the findings positioned with respect to their potential impact on children's views of mathematics and mathematicians.

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CONSTRUCTING NARRATIVES AROUND STEM ACHIEVEMENT AND MOTIVATION AMONG LEARNERS FROM NON-DOMINANT BACKGROUNDS

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Predominant narratives around STEM achievement—particularly in mathematics—often locate explanations for (un) successful experiences in schooling with an individual's motivation. These meta-narratives fail to account for the role of sociopolitical contexts in differences among opportunities for students from dominant versus non-dominant ethnic, racial and cultural backgrounds (Gutiérrez 2007). As mathematics learners from non-dominant backgrounds draw upon these ready-at-hand narratives to figure themselves within the world of mathematics education, they unwittingly participate in their own oppression (Bourdieu, 1991) This poster reports on research that examined narrative construction among underrepresented high school math and science students, and identifies features of this process that either reinforced or problematized the motivation-achievement narrative.

The study researched the question: *How do features of an activity system within which students construct narratives around achievement and motivation afford and constrain the nature of those narratives?* Narratives tend to oversimplify context in the process of identifying a "past", locating people, and putting them along a trajectory towards a future (Bamberg & Andrews, 2004). However, narrative construction is necessarily situated. Thus, analysis focus on what Ochs and Capps (2001) call dimensions of *lived narratives* such as: *tellability* (coherence and appeal of a story); *tellership* (type of involvement other people have in the narration); *linearity* (logical coherence); *embeddedness* (relation to theme and context); and *moral stance* (strength and focus). Positioning theory (Harre & Langenhove, 1999) was also utilized to attend to the way that students positioned themselves and "others" with respect to mathematics achievement and social and political processes. Desults from the study suggest that narrative construction that supports turn-taking among interlocutors, draws on everyday learning experiences, and that problematizes categories can disrupt the predominant narrative.

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GENDER DIFFERENCES: EXAMINING SIXTH GRADE STUDENTS' UNDERSTANDING OF GEOMETRIC SPATIAL VISUALIZATION

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Mental rotation is the ability to look at an object and visualize its rotation in three-dimensional space. Males, in general, have stronger spatial reasoning skills, and typically outperform their female counterparts on mental rotation tasks (Linn & Petersen, 1985). We contend students must have developed understanding in geometric spatial visualization in order to understand many astronomical concepts. The purpose of this study was to examine differences between two group of sixth grade students' geometric spatial reasoning and scientific content knowledge from pre to post implementation of an astronomy unit. Both student groups (Juniper Middle School: two experimental and one control; Butternut Middle School: three experimental and one control) studied Earth/Space concepts related to the Solar System. Students' geometric spatial content and scientific knowledge was documented before, during, and after project implementation, and their spatial and scientific understanding was assessed pre and post via survey instruments. The quantitative data sources included the Lunar Phase Concept Inventory (LPCI) (Lindell & Olsen, 2002); the Geometric Spatial Assessment, (GSA, Wilhelm et al., 2007); and the Purdue Spatial Visualization-Rotation Test (PSVT-Rot) (Bodner & Guay, 1997). For the purpose of this study, only the multiple-choice questions related to geometric spatial visualization were analyzed: LPCI (7), GSA (4) and PSVT-Rot (20). A RMANOVA was conducted with the factor being teacher, experimental/control group, and gender and the dependent variables for each factor being pre/post scores. As we compared males and females between and within the experimental and control groups at Butternut, we found experimental males and females achieved more significant gains from pre to post on geometric spatial visualization when compared to the control group. This can be attributed to teacher instruction and the implementation of the NASA-based curriculum. The NASA-based curriculum is designed to develop students' geometric spatial visualization through the use of 2D and 3D models, Stellarium (planeterium software), and moon journaling. At Juniper, the experimental males and females made significant gains, as well as the control males. Like Butternut, the gains for the experimental group can be attributed to teacher instruction and the NASA-based curriculum. However, the gains for the control males can be attributed to teacher instruction. The control teacher implemented more 2D and 3D models than the experimental teachers, and used Stellarium, and videos that enhanced students' spatial understanding. Understanding how *both* males and females developmental knowledge of spatial content differs may lead to knowing how to better support spatial reasoning for both genders.

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DEVELOPING POSITIVE STUDENT-TEACHER RELATIONSHIPS IN STEM CLASSES

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Secondary school is full of marked changes for American students. Students experience school social environments at odds with their developmental needs, shuttling through multiple subject-specific classes threatening the quality of relationships with teachers and students' sense of support and relatedness (Eccles et al., 1993). Caring student-teacher relationships, where the teacher holds students to high expectations may be particularly important for youth of color in urban schools (e.g. Duncan-Andrade, 2009). However, student-teacher relationships are often most tenuous in public schools in urban settings (e.g. Fine, 1986). Perception of the level of teacher support is correlated to students' intrinsic value of mathematics (Midgley, Feldlaufer & Eccles, 1989). Because student-teacher relationships may impact student perceptions of teacher support, this research explores the nature of these relationships. Three STEM teachers from City Performing Arts Academy,[1] a public school in Boston, and their students were collaborators in this research. City Performing Arts Academy uses an Advisory system where every teacher serves as an Advisor for students' four-year high school careers. Advisory is a small class with the goal of promoting students' sense of belonging by attending to personalized academic and social emotional needs to improve education outcomes. Because Advisory intends to build a trusting student-teacher relationship to increase student engagement, this study investigated the nature of student-teacher relationships in Advisory and how they may differ and/or impact (positively or negatively) the relationship in STEM classes. Interviews with students included those who have their STEM teacher for both their STEM class plus Advisory as well as those who have their teacher just for STEM class (and not Advisory). Student interviews and participant observation revealed that students' relationship with their teacher in Advisory differed from and supplemented their relationship with their teacher in STEM class. Similarly, teachers' viewed their responsibilities as an Advisor as greater and more personal than that of teaching in STEM class. Both student and teacher perspectives suggest ways for STEM teachers to build positive relationships with students with or without Advisory class (e.g. by taking time after school to help with homework and supporting students with challenges outside of school.)

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INTUITIONS IN MATHEMATICAL ACTIVITY: INSIGHTS FROM WORKPLACE TRAINING

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Existing research on mathematical intuitions tends to focus on intuitions related to formal mathematics and to portray intuitions as tacit mental models in the minds of individuals (e.g., Fischebein, 1987). Starting from an interest in mathematics activity and learning outside of formal academic contexts and drawing on Fischbein (1987, 1998), Kant (1998), Radford (2009), and Roth and Radford (2011), we examine intuition within workplace training from a socio–cultural perspective and attempt to answer the question, *How are mathematical intuitions related to other dimensions of the cultural–historical activity within which they arise?*

Our research examines mathematical intuitions of three ironwork apprentices during a one hour collaborative problem solving activity recorded on video during the year-two college component of their apprenticeship program. At this point, each of the apprentices had approximately one year of trade experience. In the episode reported here, they were working to determine the weight of a four-beam assembly based on the lengths and beam gauge information—weight or mass per unit length in either pounds per foot or kilograms per meter—provided for each component on a set of technical drawings for a multi-story steel structure.

Central features of mathematical intuitions identified by Fischebein (1987), namely immediacy, self-evidence, globality, and coercieveness were evident in the apprentices' discourse and actions as they complete this task. Unlike Fieschbein (1989), however, we do not see intuitions predominantly as mental models in the minds of individuals. Our analysis revealed ways in which intuitions are: evidenced through the spontaneous enactment of gestures and thus embodied, reflective of prior cognitive experience (e.g., from school mathematics) and lived sensual experiences (e.g., with construction materials and artefacts on the job as an ironworker), shared by participants in common activity, and connected with emotions related to achieving the object or goal of an activity. Implications for mathematics teaching, learning, problem solving and further research are discussed.

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COMMON TREATMENTS OF FUNCTION: WHERE'S THE RELATIONSHIP?

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The function concept is one which high school and undergraduate mathematics students' difficulties are well documented in mathematics education research. Functions are often seen as a gatekeeper in mathematics and have been given a significant focus in the Common Core Standards (2010). One of the most recent research trends attempting to support student learning in relation to function involves leveraging covariational and quantitative reasoning (Carlson et al., 2002; Smith III & Thompson, 2008). Unfortunately, several researchers (e.g., Smith III & Thompson, 2008) have illustrated that school mathematics does not include a foundational focus on covariational and quantitative reasoning. As a result, students often construct function meanings devoid of imagery that involves relationships between covarying quantities (see Oehrtman, Carlson, & Thompson [2008] for review). For instance, students conceive functions as a call to execute a calculation as opposed to a mapping between quantities' values.

In our poster, we contrast various approaches to function against the backdrop of quantitative and covariational reasoning. Using data we collected during clinical interviews with pre-service teachers at a large state institution in the Southeast United States, we draw attention to the role of these reasoning processes in conceiving functions and their inverses. For instance, our work suggests that typical approaches to teaching inverse lead to students engaging in repeated experiences devoid of quantitative reasoning. As a result, these students (who later become teachers) construct meanings that are not preparing them to support students in developing robust meanings for function. As another example, when posed with solutions that are quantitatively correct but non-canonical (e.g., graphing the variable *x* on the vertical axis), the pre-service teachers had difficulty attributing viable ways of thinking to the posed solution. Such an outcome illustrates that these pre-service teachers hold meanings that restrict their ability to interpret non-canonical but correct student work, which has significant implications for their teaching. We highlight such data and characterize implications relative to function, quantitative reasoning, and covariational reasoning.

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LATINA/O MATHEMATICS STUDENTS' PERCEPTIONS OF INTERPERSONAL RELATIONSHIPS IN COLLEGE MATHEMATICS

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Keywords: Equity and Diversity, Post-Secondary Education

It is known that interpersonal relationships are significant to the success of the Latina/o mathematics learner (Gutierrez, 1999). Unfortunately, interpersonal relationships in mathematics are understudied (Hackenberg, 2010). For this reason, this study seeks to answer the following question: How do Latina/o mathematics students perceive their interpersonal relationships with their mathematics professors, peers in mathematics, and families with respect to mathematics and what impact do their views on these relationships have on their persistence in mathematics? To explore the role of such relationships in Latinas/os experiences in mathematics, I define a *meaningful relationship in mathematics education* (MRIME) as interactions between two parties in which one party acknowledges the other party as a sociocultural being and fosters their mathematics identity. This definition draws from research on care (Hackenberg, 2010; Noddings, 1988), mathematics identity and socialization (Martin, 2000) and mathematics students as sociocultural beings (Gutierrez, 1999).

This study was done using case studies. Four Latina/o students who were mathematics majors or had a bachelor's degree in mathematics were recruited using a Facebook page. The data collection took place over a period of one year. Three hour-long telephone interviews were conducted with each participant. Analysis of the interviews was based on identifying evidence of components of MRIME including the fostering of the participants mathematics identity and acknowledgement of the participant as social and cultural beings.

The results showed that the participants' views of their interpersonal relationships had a significant impact on their persistence in mathematics. For example, Sandy perceived her relationships with her professors, peers, and family as all meaningful. She felt that each of the three groups of people recognized her as a sociocultural being and helped her to develop her mathematics identity. Sandy ended up getting a master's degree in mathematics as a result. Sonya, on the other hand, viewed most of her interpersonal relationships in mathematics negatively and dropped out of mathematics by changing her major in her last year of college.

These results imply that MRIME is a sufficient framework for studying the interpersonal relationships of Latinas/os in mathematics. Also, they suggest that meaningful relationships are powerful influences on Latina/o mathematics students and their persistence in mathematics.

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UNPACKING BROUSSEU'S DIDACTICAL MILIEU

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In this work we unpacked the construct of didactical milieu by examining ways in which interaction assisted in eliciting and evolving mathematical thinking of a child. The findings challenge the power relationship of knowledge, which is not only transferred from the more knowledgeable agent to the learner, but also conveyed from the learner to the more knowledgeable agent as guide for optimizing space of for productive didactical interactions.

Keywords: Cognition, Learning Theories, High School Education

Of particular interest to the research we will report was to identify ways in which "interaction" assisted in evolving children's mathematical thinking while engaged in solving mathematical tasks. Relying on Brauesse's Theory of Didactical Situations (1980) as a theoretical tool for grounding our methodology, and an analytical tool for interpreting interactions; we examined: *How does interaction influence mathematical practices of children?*

In analyzing the quality, content and nature of interactions and their influence on children's mathematical practice, we chose to rely on data from one specific interview (approximately 60 minutes) with one child (Dana-a pseudoname) as she was prompted to solve a pattern-generalizing task. At the time of the interview Dana was enrolled in 8th grade and taking Algebra I. This particular interview session was chosen deliberately since it provided persuasive illustrations of the impact of interactions on evolving practices of the child whilst highlighting the particular epistemological responsibilities and analytical demands that they placed on the interviewer when eliciting and responding to the child's mathematics, as previously articulated in Brousseu's theory of didactical situations. Data coding followed three stages. First, to break down the conversational exchanges and the moment-by-moment construction in the ongoing interactions, we conducted an analysis of language functions of verbal exchanges during the interview, according to turns and segments of utterances offered (Kovalainen and Kumpulainen, 2009). A second coding of data determined the specific mathematical actions or conceptual shifts that occurred during the interactive episode including both interactional agents (interviewer and Dana) as subjects of study. At the third stage, we clustered the segments according to the type of *situational* forms they seemingly resembled (Action, Formulation, Justification, Institutionalization).

A closer inspection of the participants' practices, collectively, revealed two distinct ways in which the interviewer impacted Dana's work: evoking reflection by eliciting reasoning and explaining, providing a structure to solidify theoretical knowledge through questioning and or collaboration. In turn, Dana's actions and mathematical observations forced shifts in the interviewer's position towards the task, creating a climate for collaborative mathematical sense-making.

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WHEN THE ANSWER IS NO ANSWER: ON STAYING OUT OF TROUBLE IN MATH CLASS

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Student engagement in mathematics classrooms can depend on many variables, including peer interactions and social contexts (Middleton & Jansen, 2011; Goldin *et al.*, 2011). When students work in groups, some may be more vocal, while others may say little if anything. This paper focuses on a group of students in which one girl ["Jen"] seemed to refrain from talking to her peers over the entire course of a teaching unit (8 sessions over 2 1/2 weeks), because, in her own words, she "wanted to stay out of trouble." The research question we address is: How does Jen's interaction with her peers impact her mathematical engagement and learning."

We examine Jen's engagement using the concept of "engagement structures" – which are idealized, identifiable, recurring, highly affective patterns in individual students, inferred from observed behaviors, interviews and responses to questionnaires. Our data include videotaped classroom sequences, responses to an engagement survey, mathematical pre- and posttest data, and results from retrospective stimulated-recall interviews. Our analysis indicates that Jen was both unsure of her own mathematical ability, and determined to avoid confrontation with her peers and/or the teacher. She opted to refrain from engaging in conversation with peers, neither asking nor answering questions (posed by peers or the teacher). She demonstrated many of the characteristic behavior patterns associated with the "Stay Out of Trouble" engagement structure. In this structure, as described in Goldin *et al*. (2011), a student desires to avoid interactions that may lead to conflict (e.g., a fight) or distress (e.g., embarrassment, humiliation, or anger) involving peers or someone in authority. Avoidance behavior, including striving not to be noticed, may supersede addressing the task's mathematical content.

Despite her avoidance of peer interactions, our data indicate that Jen appeared to always pay close attention to all of the mathematical activity that took place. Further, while she scored the lowest of all the participants on the math pre-test, Jen ended up with the highest score increase in the study based upon her post-test results. Moreover, in a math task-based interview, she demonstrated a very strong knowledge of the mathematics that was covered.

One main conclusion, which of course requires further study, is that it is important to broaden our perspective on the role and nature of mathematical engagement. Jen's engagement patterns appeared to be *highly complex* and *situation-dependent*, and not readily summed up on a linear scale. Further, Jen's "Stay Out of Trouble" pattern appears to have some *adaptive value* in relation to her learning of the mathematics.

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RELATIONSHIP BETWEEN NEGATIVE ATTITUDES TOWARD MATHEMATICS AND COLOMBIAN CHILDREN'S MATHEMATICS ACHIEVEMENT

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Keywords: Advanced Mathematical Thinking; Affect, Emotions, Beliefs and Attitudes

This study investigated the association between negative attitudes toward mathematics and mathematics achievement in 4th-grade Colombian children (*M*=10.4 year-old), using the large-scale data source Trends in International Mathematics and Science Study-TIMSS 2007. Scores in the mathematics test were the measure of mathematics achievement. Two indicators of negative attitudes toward mathematics were the independent variables: (a) *Mathematics is boring* and, (b) *Mathematics is harder for me than for others*. These are statements about mathematics presented to students on a Likert-type scale with four levels of response: Agree a lot, agree a little, disagree a little and disagree a lot. Students in each level were deemed as the conditions of each variable. Findings showed that scores in the Mathematics test increase as the level of agreement with both statements *Mathematics is boring* and *Math is harder for me than for others* decrease (Table 1).

Table 1: Average Scores in Mathematics Test for Degree of Agreement with Statements

	Mathematics is boring			Math is harder for me than for others		
	N	*Mean*	*SD*	*N*	*Mean*	*SD*
Agree a lot	615	329.9	82.5	1273	345.03	82.9
Agree a little	738	349.4	90.5	1310	365.02	84.2
Disagree a little	439	370.3	92.4	637	385.6	89.8
Disagree a lot	2413	393.2	80.3	1149	402.01	85.5

A one-way analysis of variance (ANOVA) indicated that the degree of agreement with the statement *Mathematics is boring* is significantly associated to the students' performance on the test, $F(3,4201) = 120.65$, $p < .001$, eta-squared = .08, $r = .28$, representing a middle-sized effect. Polynomial analyses showed a significant linear trend, $F(1, 4201) = 361.631$, $p > .001$, eta-squared = .08, $r = .28$, middle-sized effect. As the degree of agreement with this statement decreases, the means increase proportionately. A one-way analysis of variance (ANOVA) indicated that the degree of agreement with the statement *Mathematics is harder for me than for others* is also significantly related to students' outcomes on the test, $F(3,4365) = 98.85$, $p < .001$, eta-squared = .06, $r = .25$, middle-sized effect. Polynomial analyses showed a significant linear trend, $F(1, 4365) = 294.59$, $p > .001$, eta-squared = .06, $r = .25$, middle-sized effect. As the level of agreement with this statement decreases, the means increase proportionately. These findings have strong implications to educational practice and curriculum (Ma and Xu, 2004).

Acknowledgements

Supported by the grant FULBRIGHT-COLCIENCIAS-2012 and Universidad del Valle-CO

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'VALUE CREATION' THROUGH MATHEMATICAL MODELING: STUDENTS' MATHEMATICAL DISPOSITION AND IDENTITY DEVELOPED IN A LEARNING COMMUNITY

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Studies reported that most students perceive mathematics as an impediment of their success in school and barely see the value of learning mathematics in life (e.g., Ernest, 1994). It becomes a question of what learning environment enables students to engage in meaningful mathematics learning and develop positive disposition as well as self-concept. Existing literatures demonstrated that modeling tasks allowed students to appreciate the relevance and usefulness of mathematics to individuals as well as to society (Blum & Niss, 1991; Boaler, 2002). The purpose of this research is to examine how mathematical modeling activities within a collaborative learning community impact on students' perceived 'value' of mathematics. The concept of 'value' was adopted from Makiguchi's theory of "value creation"(Bethel, 1989, p6).'Value creation' concerns with human development that enables individuals to gain benefits from developing a relationship with the object (mathematics) not only at personal level but also at societal level. With respect to value of learning mathematics, researchers introduced "productive disposition"(NRC, 2001, p131) and "mathematical disposition"(NCTM, 1989, p1) as an indicator of students' appreciation of the role and value of mathematics. With a unified framework of the theory of 'value', mathematical disposition, and identity, this study identified the elements of the value-beauty, gains, and social good-with the observable evidences of mathematical disposition and identity.

A total of 60 students who enrolled in a college algebra course participated in 'LifeStyle' mathematical modeling project within a group. A mixed method approach was used for data collection and analysis in this study. The result of a two-sample t-test showed the significant changes in students' mathematical disposition between pre and post survey. Based on questionnaires, surveys, and observations data, fifteen focal students were selected for interviews. Interview data supported the evidence of how students' perceived value-beauty, benefits, and social value-has been changed while engaging in the modeling activities with peers in a group.

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EXPLORING CHALLENGES IN MATHEMATICS TRANSITIONS

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Much work on mathematical transitions has focused on challenges presented by differences in curricular materials and instructional strategies (e.g., Jansen, Herbel-Eisenmann & Smith, 2012), while transitions work outside mathematics has focused on socio-emotional experiences (e.g., McDermott, Watkins, Rovine & Rikoon, 2013). This poster links these two analyses by looking broadly at transitions across grade levels to consider the ways that socio-emotional experiences impact participation in mathematics.

In this analysis, we rely on positioning theory to explain the ways in which people are constructed differently in different contexts as a result of "positioning by whatever powerful discourses they happen to encounter" (Holland, Skinner, Lachiotte & Cain, 1998, p. 27). We augment this work with Latour's actor-network theory, which directs attention not only to the ways that interactions among human beings shape each other, but also to the ways that physical objects (books, toys, tables, etc.) shape human experiences.

For this project, we analyzed the experiences of four students over three years of mathematics learning. All four students (two boys and two girls) were African American and attended a rural school that serves low-income families. We began observing these students in prekindergarten, videotaping both formal mathematics lessons and math-related play. We continued to videotape mathematics lessons throughout kindergarten and first grade. In addition, we conducted individual mathematics assessments at the end of each year, organized and documented parent math nights after school, and held focus groups and individual interviews with parents. During the first grade assessments, we also interviewed the children about their experiences in math classes and in the school broadly.

While our analysis is still in the preliminary stages, we have found that the difficulty of students' transitions was closely related to how students were positioned as mathematics students and as learners in general. For example, as students lost freedom to move about the room as they got older, they expressed distress in a variety of ways that impacted their participation in mathematics. In addition, students identified by a particular teacher as having behavior problems spent large sections of the school year isolated in their desks, which impacted the ways in which they could participate in mathematics. One surprise was that some children's positioning as "good" or "difficult" was quite fluid across the three years. Transitions marked as difficult – either in interviews or through the observation of behavior including crying and outbursts – seemed to relate as much to changes in the amount of movement permitted and to the ways the student was positioned by the current teacher as to either the mathematics curriculum or pedagogy.

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PERCEPTIONS OF K-16 MATHEMATICS EDUCATION FROM 'LA FRONTERA': A QUALITATIVE STUDY OF MEXICAN-AMERICAN POSTESECONDARY STUDENTS

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The purpose of this study is to investigate the non-mathematical factors, such as teacher beliefs and student attitudes that influence diverse postsecondary students' learning experiences in mathematics. The study examines Mexican-American students' perceptions about mathematics instruction in the U.S. formal education (K-16) system. The majority of Latino students in U.S. higher education today start their postsecondary education experience in a community college (Green, 2006). When these students complete high school, test scores indicate that many are not "college-ready" (Flores, 2007), and community colleges provide an excellent starting point for these students for several reasons.

The first stages of the study are framed mainly by constructionism, contructivism, and critical race theory (CRT). Constructionism is based on theories very similar to constructivism, however while constructivism places the main focus on the development of individual and isolated knowledge structures (Dubinsky & McDonald, 2001), constructionism focuses on the connected nature of knowledge with its personal and social dimensions (Kafai, 2006). Critical race theory (Hylton, 2012) focuses on issues of equity, desconstructing power relationships in schools and society. Constructionist scholars describe authentic practices within schools that help students to develop new knowledge based on their prior knowledge. These authentic practices allow students to engage in authentic learning experiences through meaningful activities.

This qualitative research study uses purposeful sampling consisting of a sample selection procedure, based on the researcher's knowledge and experience of the group to be sampled using clear criteria to guide the process. Then, the researcher administered an on-line survey to all students that decide to attend the free interactive technology sessions. The next stage consists of in-depth interviews with a smaller sample of self-selected students. The transcriptions of the were coding for theoretical meaning (Charmaz, 2006). Grounded theory methods provide a unique approach to capture informants' reflections about issues in mathematics education.

This study contributes to existing knowledge about non-mathematical factors that impact student attitudes, teacher beliefs relative to the mathematics education in the U.S. public school system. This knowledge has implications for policymakers, researchers and practitioners to broaden perspectives on mathematics thinking and learning in the 21st century.

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COLLABORATION, BEHAVIOR, AND AFFECT IN A GAME-BASED LEARNING ENVIRONMENT

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Keywords: Middle School Education, Rational Numbers, Technology

Collaborative activity during game play is instrumental to learning (Stevens et al., 2008), and it is important for developing mathematical reasoning (Phelps & Damon, 1989). Yet, the majority of mathematics games are designed for single players, which could make it less likely for students to learn from each other while playing. In this poster, we present a pilot study examining middle school students playing an online, single-player fraction game called Refraction both individually and in pairs. Refraction is a puzzle game requiring a combination of spatial reasoning and fraction understanding. Players split laser beams into appropriate fractions needed to power space ships stuck in space. We compared students' behavior and affect while playing alone and playing with a partner.

Fourteen seventh grade students played Refraction once a week for thirty minutes over the course of four weeks. While students were playing, we used the Baker-Rodrigo Observation Method Protocol (BROMP 1.0) to collect data on students' behavior and their affective states (Ocumpaugh, Baker, & Rodrigo, 2012). Two researchers made observations every twenty seconds of individual students' behavior and affect. They determined whether a student was silently playing the game, giving another student help, receiving help from another student, discussing the game, or off task. They also recorded whether the student was concentrating, delighted, confused, bored, frustrated, or experiencing a moment of eureka.

During the first game play session, students took a fraction pre-test and played the game individually. The following two sessions students played the game with a partner sharing a computer. During the last session, students took a post-test and played the game individually.

We found that students behaved differently depending on if they were playing individually or with a partner. There was a significant increase in the proportion of giving and receiving help from the first individual play session to partner play, and interestingly, giving and receiving help remained high when the students returned to playing individually in the last session. In other words, once students were put in a situation where needed to collaborate, they continued to do so even when returning individual play. We also found no differences in the level of collaborative activity for students who had different levels of fraction understanding on the pre-test. In this poster, a further exploration of the students' behavior and affective states will be discussed.

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RELATING TEACHER CHARACTERISTICS, STUDENT CHARACTERISTICS, AND INSTRUCTIONAL PRACTICES TO FLEXIBILITY IN ALGEBRA

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Flexibility in mathematics problem solving is an important skill for students (Woodward et al., 2012). Flexibility refers to the ability to solve mathematics problems in multiple ways and know when it is most appropriate to apply particular solution methods for a given problem (Star & Rittle-Johnson, 2008). The goal of the present analysis was to learn more about important factors associated with students' school year gains in flexibility in algebra.

Data for the study was taken from teachers and students using researcher-created Algebra I supplemental curriculum materials that focused on explicit comparison of side-by-side worked example pairs. We compared students' flexibility gains for the upper (n = 10) and lower quartile (n = 10) of teachers (*high flexibility teachers* and *low flexibility teachers*, respectively).

Results suggest that high and low flexibility teachers were not reliably different on any demographic variables. However, high flexibility teachers generally had younger students, were more likely to teach middle school classes, and had classes with significantly lower flexibility scores at pre-test. High flexibility teachers exposed their students to significantly more of the Algebra I intervention materials, both by using the materials more often (i.e., dosage) and using them for a greater percentage of their instructional time. Using video data, 3 high and 3 low flexibility teachers were compared on the types of questions they asked students. Both high and low flexibility teachers tended to ask more *why* questions when using the materials (treatment videos) compared to instruction without the materials (general videos). However, high flexibility teachers tended to ask more *open-ended* questions when using the materials, such as "So what's the general rule?" and "What did you just learn from this?".

This analysis suggests that frequent and extended exposure to materials that explicitly promote flexibility appear to make the most impact on flexibility gains. Moreover, when teachers ask students to verbalize what they learned (e.g., by using open-ended questions), students may benefit more from such materials. Further study is needed to get a more nuanced understanding of these findings, in order to better understand the ways in which teachers' implementation and enactment of the curriculum materials impacted student learning. Such an analysis is underway.

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MIDDLE SCHOOL TEACHERS' UNDERSTANDING OF COGNITIVE DEMAND

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Keywords: Cognition, Middle School Education, Teacher Knowledge

This study examines middle school in-service teachers' understanding of cognitive demand. N=20 teachers from a number of urban middle schools at the Southwestern USA participated in the study as part of their professional development activities.

According to Smith and Stein (1998), "tasks that ask students to perform a memorized procedure in a routine manner lead to one type of opportunity for student thinking; tasks that require students to think conceptually and that stimulate students to make connections lead to a different set of opportunities for student thinking" (p. 269). This study focuses on Stein et al. (2000) cognitive demand framework. The framework separates low-level from high-level cognitive demands where memorization and procedures without connections fall at the low-level and procedures with connections and doing mathematics are at the high-level. Our research questions are the following: (1) Are middle school mathematics teachers able to recognize different levels of cognitive demand when presented by different tasks? (2) What are middle school mathematics teachers' rational and reasoning to identify different levels of cognitive demand? (3) To what extent are middle school mathematics teachers able to construct tasks at the different levels of cognitive demand?

We designed the following instruments to address the research questions:

1. Cognitive Demand Recognition (Measure 1). The purpose of this measure is to examine weather teachers recognize tasks at different cognitive levels. They were presented with four different problems involving the concept of similarity and were asked to rate the cognitive demand level necessary to solve the task. Teachers also provided a short rational for the rating.

2. Challenge Recognition (Measure 2). The second measure was presented as a task to derive the Pythagorean Theorem using the idea of similarity. After solving the task teachers were asked to rate the task according to the level of challenge on a scale from 1-5 where 1 is the lowest challenge and 5 is the highest challenge. The purpose of this task is to examine teachers' recognition of the link between cognitive demand and the level of challenge.

3. Task Construction at Different Levels of Cognitive Demand (Measure 3). This measure addresses the third research question. The purpose of this measure is to investigate whether the teachers were able to design tasks for their students at the different cognitive levels using the topic of area of a circle.

The result of the study showed that teachers are capable to recognize levels of cognitive demand and challenge of the presented tasks. However, participants had difficulty in constructing tasks at different levels of cognitive demand, most specifically, in distinguishing between tasks focused on procedures without connections and tasks that require connections among procedures. The study also presents the contextual categories teachers used in constructing tasks.

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COGNITIVE TUTORING SOFTWARE IN BASIC MATH COURSES: BROADENING PERSPECTIVES ON MATHEMATICS THINKING AND LEARNING

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A critical issue facing educators today is the large numbers of students required to take developmental math upon entering college due to traditional approaches to mathematical teaching and learning.

Cognitive Tutoring Software: ALEKS

Assessment and LEarning in Knowledge Spaces (ALEKS) is based on Knowledge Space Theory, a formal framework for assessment of knowledge (Flamagne, & Doignon, 2011). The purpose of this study is to determine whether the use of ALEKS in basic math courses helps entry level college students broaden their perspectives in mathematics learning. ALEKS uses adaptive questioning to accurately determine what a student knows and doesn't know in a course. Students practice, are assessed then repeat this process until they have mastered all or most of the concepts in a particular course.

Research on Broadening Students Perspectives on Math

Research on Developmental Math and Pre-calculus have found that the use of ALEKS has not only improved students test scores on the ACT and Accuplacer (Hagerty, $ Smith, 2005) and also improved students attitudes towards math learning. These studies found that the amount of time spent on ALEKS along with immediate face-to-face tutoring contributed to success in pass rates. Students also felt that ALEKS positively impacted their grade (Hampikian, Guarino, Chyung, Gardner, & Pyke, 2006).

Research in Progress

The researcher has worked with ALEKS over the last two years and has found that students working with peer tutoring showed greater weekly progress when compared to fully online classes. A traditional lecture was not required since students are in different knowledge spaces. ALEKS students have a higher pass rate in subsequent courses compared to non ALEKS students. The doctoral research proposal in progress is a mixed methods study, including pre and posttests (quantitative) and qualitative components (e.g., surveys, interviews, focus groups, observations) to determine effectiveness and impacts on student affect, emotions, and attitudes about mathematics thinking and learning. Preliminary qualitative and quantitative data will be presented for discussion.

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DESIGN-BASED RESEARCH, REPRESENTATIONS, AND LEARNING TRAJECTORIES: THE CASE OF FRACTION MULTIPLICATION

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Fractions, as a K-12 curriculum topic, are considered the most challenging to learn, difficult to teach, and essential to later success in mathematics (Lamon, 2007). Area models and numbers lines are two common representations used to support student understanding of fractions (Izsak, 2005; Saxe et. al., 2010). The challenge lies in extending the use of these representations to foster student understanding of fraction operations, such as, fraction multiplication (Charalambous & Pitta-Pantazi, 2006). When working with whole numbers, multiplication is often interpreted as repeated addition. Given 2/3 x 1/2, what does it mean to add 2/3 to itself 1/2 times? To support a more robust interpretation for fraction multiplication, multiplication as stretching and shrinking, we designed a learning environment centered on the Area Model for Fraction Multiplication (AM-FM) representation, a computer-based tool that combines area model and number line representations (Figure 1).

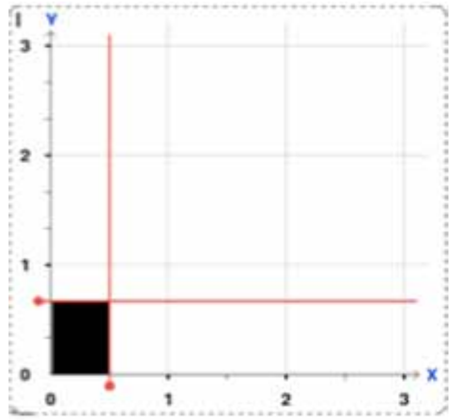

Figure 1a. The axes are partitioned into halves and thirds and the red marker lines are moved to x=1/2 and y=2/3.

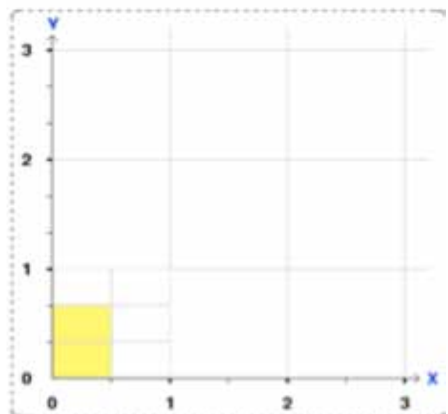

Figure 1b. The final area is tiled depicting in an area model representation of 2/6.

Figure 1. 2/3 x 1/2

We use design-based research to guide the development of the AM-FM representation and our theory about students learning trajectories specific to the designed learning environment. We present a single case study of knowledge growth and change, which supports our hypothesized learning trajectory and highlights the context sensitivity of knowledge coordination across the area model and number line features of the AM-FM representation. We end with implications for revision to the learning environment and to our local theory on student learning trajectories for fraction multiplication.

Acknowledgements

This work was made possible with support from Alan Schoenfeld, Dor Abrahamson, & Andrea diSessa.

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A LEARNING TRAJECTORY FOR DIRECTIONAL DERIVATIVE

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Keywords: Cognition, Post-Secondary Education, Learning Trajectories (or Progressions)

Models in physics, engineering, economics and mathematics inevitably involve reasoning about changes among many variables in a system. Students must grapple with the meaning of "varying the input" when there are two independent variables and determine rates when there is no natural direction or path in which change happens. However, little is known about what understanding(s) of rate of change of a single variable function allow students to conceive of the necessity of direction and path in interpreting and measuring rate. The purpose of this paper is to propose what I call the *two-change problem* as an important cognitive obstacle that students experience as they conceive of direction and path of rate of change.

Students must see their conceptions of rate of change in the plane as insufficient to describe a system in which at least three quantities are varying. Put another way, they must conceive of direction and path as *necessary* to quantify how fast a function is changing with respect to two or more arguments. Based on ten clinical interviews with both single and multivariable calculus students, I hypothesized that the students could be accomplished by the students' noticing that if a one quantity depends on at least two independent quantities; there are two rates of change occurring. I made this hypothesis because I found students' natural inclination when describing a rate of change is a single value that describes how fast the function is changing. However, often their first observation of a two-variable function at the same time they notice that the dependent quantity [f(x,y)] is changing with respect to both x and y. Thus, when there is a change in f, there can be a change in *both* independent quantities. For instance, a number of students noted in the interviews that if there is a change in both x and y, then it is plausible that there are two rates of change. However, at this point students often do not conceive of the x or y axis as a direction itself. However, as students grappled with how to form a "single" rate of change in three dimensions they puzzled over how to account for the changes in x and y. Initially students in the interviews suggested that the changes in x and y could be chosen independently of each other but some students insisted that the changes must be dependent or "there could be a million different rates of change". Indeed, the dependence of the changes in x and y specifies the direction vector in which the rate of change is considered in three-space. The independence or dependence of these changes is the core of the *two-change* problem.

Every student in the interview encountered this issue yet few could resolve it. Those who noticed the changes in x and y could not be chosen independently of each other without generating an infinite number of possibilities for the measurement of the rate of change of the function. They were able to then conceive of a fixed relationship between the changes in x and y as specifying a direction or path.

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Research Reports

Brief Research Reports

Poster Presentations

DEVELOPING MATHEMATICAL KNOWLEDGE FOR TEACHING THROUGH MENTOR-GUIDED LESSON STUDY

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Field experience can be a rich site for prospective teachers to develop the knowledge and skills they need for effective teaching, however little is known about how structures for supporting interactions between mentor and prospective teachers can maximize prospective teachers' learning. Lesson study has been shown to enhance practicing teachers' mathematical knowledge for teaching through collaborative inquiry with their peers. In this paper, we discuss the use of mentor-guided lesson study to support mentor and prospective teachers' collaboration in the field and provide evidence showing its potential to strengthen prospective teachers' mathematical knowledge for teaching. We will also share insights from the field for those interested in implementing this activity in teacher preparation coursework.

Keywords: Teacher Education-Preservice, Mathematical Knowledge for Teaching, High School Education, Lesson Study

Background

The apprenticeship structure known as "student teaching," "field experience," or "internship" is a universal part of teacher preparation, in both traditional and alternative certification pathways. Teachers often cite the field experience as the most valuable part of teacher preparation (Lortie, 1975). Over 1,400 institutions of higher education require completion of a student teaching experience for teacher candidates and 39 states in the U.S. have set minimum requirements for the length of the student teaching experience (NCTQ, 2011). Given the importance of field experience in teacher preparation, it is surprising that research on what prospective teachers learn in the field, as well as aspects that promote productive relationships between mentor and novice teachers, is so scarce (Wilson, Floden & Ferrini-Mundy, 2001). Existing research has predominately investigated how field experiences influence novice teachers' beliefs about teaching, but more research is needed to learn about *what* knowledge and skills novice teachers are learning through field experiences.

One type of knowledge, that field experiences may provide rich opportunities for prospective teachers to develop, is *mathematical knowledge for teaching* (Ball & Bass, 2003). Ball, Thames and Phelps (2008) define MKT as the "mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students" (p. 399). Over the past decade, much attention has been placed on the role of MKT in understanding the knowledge, skills, and dispositions needed for mathematics teachers to engage in ambitious practice. Lampert and Graziani (2009) define ambitious practice as mathematics instruction that provides all students with an opportunity to develop multifaceted knowledge of mathematics, participate in discussions about mathematical ideas, and solve authentic problems. Prospective teachers need opportunities to develop and deepen MKT if they are to enact ambitious practice.

For prospective secondary teachers of mathematics (PST-Ms), the mathematics coursework required as part of a teacher certification program can offer opportunities to deepen subject matter knowledge. However, these content courses are not designed to develop other aspects of

MKT, such as *knowledge of content and teaching* (KCT), which Hill et al. (2008) describe as the knowledge needed to select an appropriate representation to highlight key concepts in a mathematical investigation or choosing effective instructional examples. Opportunities to develop aspects of MKT such as KCT and KCS (*knowledge of content and students)* may occur in mathematics pedagogy coursework and in field experiences. One experience that has been shown to promote teachers' MKT is lesson study, a particular form of collaborative action research (Fernandez, 2005). The purpose of this paper is to share findings from an investigation into the potential for *mentor-guided lesson study*, where a mentor teacher collaborates with PST-Ms to do lesson study cycles, to support growth in PST-Ms' MKT. Specifically, we will discuss findings to address the following questions: *To what extent does mentor-guided lesson study promote prospective teachers' reflections on teaching related to their MKT? How does a mentor's prior experience in doing lesson study contribute to the development of prospective teachers' MKT during cycles of mentor-guided lesson study?*

Theoretical Framework

Mentor-guided lesson study (MGLS) is an abbreviated form of lesson study that involves a small team consisting of a mentor teacher and two or more PST-Ms collaborating during four phases of a lesson study cycle: 1) develop a learning goal within the context of a unit of instruction; 2) design a lesson that addresses that learning goal; 3) collect evidence of student thinking related to the goal when the lesson is taught, and; 4) discuss the effectiveness of the lesson in achieving the desired learning goal after the lesson. A wealth of research shows how lesson study as professional development contributes to the development of teachers' PCK and MKT (see Fernandez & Yoshida, 2004; Lewis, 2005), and that focusing on specific events and teaching situations is a feature of effective mentoring practice (Wang & Odell, 2002). For PST-Ms, we posit that MGLS offers rich experiences in examining student thinking and linking evidence of student learning to specific instructional decisions, both planned and spontaneous.

One aim of this study was to investigate to what extent MGLS promoted novice teachers to reflect on teaching practice in ways that would develop their MKT. The MKT framework (Hill, Ball & Schilling, 2008) provides a foundation for examining the knowledge used in the work of teaching mathematics, which has been largely achieved to this point by analyzing responses to carefully designed written assessments (e.g. Hill, Schilling, & Ball, 2004). This research shows that the MKT framework is well suited to categorize the actions PST-Ms might make in a classroom setting. In the current study, however, we are focused on PST-Ms' observations of practice, instead of their actions in response to particular tasks. The work of Sherin and Van Es (2005) explores how teachers learn to *notice*, both attending to particular events in an instructional setting and making sense of those events. Building upon this work, we examined the relation between the aspects of instructional practice to the PST-Ms noticed and reflected upon, and their developing knowledge of mathematics and of teaching. Given that the MGLS experience provided the PST-Ms opportunities to plan, enact and reflect on instruction with real students and within the context of a real curriculum, we anticipated the experience might allow them access to particular subdomains of the MKT framework, specifically *knowledge of content and students* (KCS), and *knowledge of content and teaching* (KCT). The PST-Ms would only gain access to this knowledge, however, if they attended to the features of instruction that pertained to those knowledge domains.

Using a grounded theory approach, we constructed a mapping to particular domains of MKT. Our resultant analytic framework, Figure 1, represents the relation between observations made

by PST-Ms and their corresponding MKT subdomains. In this framework, PST-Ms' observations *analyzing mathematics* suggest opportunities to develop MKT of specialized content knowledge. Likewise, PST-Ms' work *attending to student thinking* corresponds to their MKT domain knowledge of content and students. Finally, PST-Ms' work *analyzing teaching moves* provides evidence of their MKT within the knowledge of content and teaching domain.

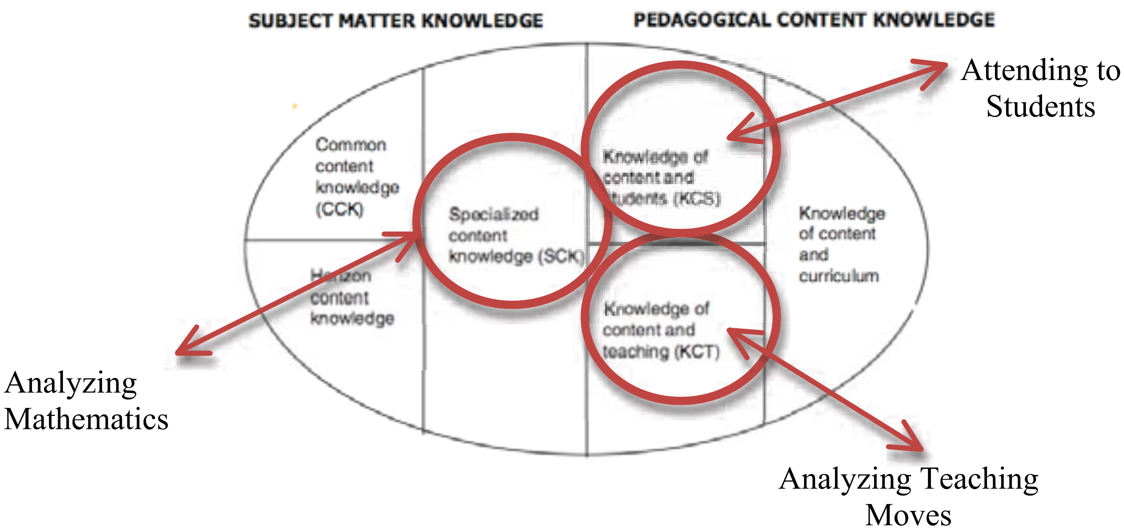

Figure 1: Mapping PST-Ms' Observations to Their MKT

Each of the three categories represented in Figure 1, is comprised of a set of subcategories of observations or analyses made by the intern teachers in this study. The descriptions below briefly describe the three overarching categories, with subcategories listed in Figure 2.

Analyzing Mathematics	Attending to Student Thinking	Analyzing Teaching Moves
Analyzing mathematics related to: • students' thinking • teaching moves	Observations of: • students' misconceptions • student understanding • mathematical procedures • students' prior knowledge • student engagement • student discourse • students' ability to apply new knowledge Questioning/wondering about student thinking	Analyzing teaching moves related to: • students' misconceptions • students' prior knowledge. • students' thinking • students' mathematical procedures • students' discourse • students' engagement

Figure 2: Composition of Categories

Analyzing Mathematics

Statements in this category focus on the mathematics of a particular lesson, perhaps reflecting upon the mathematical features of students' verbal or written responses or attending to the mathematics of teaching moves. We did not include observations regarding the pedagogical features of the lesson included in this category. Due to our primary focus on the mathematical content, we interpreted that observations that analyze mathematics may relate to development of PST-Ms' SCK. One example of a response coded as *analyzing mathematics of students' thinking*

is: “Students discussed all but one possible method to estimate the area.” Here, the PST-M specifically analyzes the mathematics content, not the students interaction or instruction.

Attending to Students’ Thinking

This category includes observations pertaining to students’ thinking, procedures or interactions. This includes observations of how students made sense of the mathematical concepts, misconceptions, or how students interact with one another or with the teacher. This category also includes the PST-Ms’ questions or hypothetical wondering about students’ thinking. Within the subcategories of this code, we also made the distinction between a focus on students’ understanding of concepts and students’ ability to execute procedures, as well as discussion of students’ prior knowledge and how students might use the knowledge they acquired from the target lesson in the future. For example, the response “Students were having trouble determining the effect of the negative leading coefficient on the shape of the graph” was coded to *observations of students’ misconceptions.*

Analyzing Teaching Moves

This category captures PST-Ms’ statements related to their teaching moves or the observations of the teaching moves of their partner or mentor teacher. Such observations include analysis of the reasoning for a particular teaching move. The subcategories group the intern teachers’ observations based on what they interpreted to motivation of the teaching move to be. For instance, responses coded to the category *analyzing teaching moves related to students’ discourse* were similar to the following: “Putting the students in small groups helped them be able to share lots of ideas together.” We interpreted this as the PST-M remarking on the impact on classroom discourse of the team’s instructional decision to use small groups.

Methods

The setting for this study was a second course of a four-course sequence of mathematics pedagogy courses taken by undergraduate prospective secondary mathematics teachers at a large, Midwestern university. The first two courses in the sequence are typically completed during the senior year of a Bachelor’s degree program and prior to full-time student teaching placement. Participants in this study (n = 17) were enrolled in the second mathematics pedagogy course, which we will call Methods II. In Methods II, students participate in: 1) a seminar component for three hours per week; 2) a lab component where students learn methods of teaching in their minor area, as well as issues related to working with students with special needs; and 3) a field placement component (4 hours per week in a middle or high school mathematics class). Two of the stated aims of the course were to explore issues entailed with enacting high cognitive demand tasks (such as issues of equity, social justice, discourse techniques, and complex instruction), and to develop prospective teachers’ mathematical knowledge for teaching geometry. The mentor-guided lesson study assignment in this course was intended to maximize students’ experiences in learning from their mentor teacher during the field placement component.

Mentor-guided lesson study

The mentor-guided lesson study activity consists of four phases, completed as a cycle, with two cycles during the semester. The team met first to set goals for the lesson study, and the PST-Ms were required to complete an online Goals Development Log, answering questions such as “Write the content goal you have for this lesson” and “Indicate any questions or concerns you

have at this point about the goals you have developed for this lesson study." The Goals Development Log was submitted prior to the next phase, and reviewed by the course instructor. The process of this first phase was the same during Cycle 2.

In the second phase, *planning*, the team met to design the lesson plan to address the chosen goals, as well as discuss how the team members who would not be teaching would observe the lesson and document students' thinking during the lesson. Because of the PST-Ms' limited teaching experience, especially with regards to planning and enacting lessons, the mentor teacher in each team was the lead teacher for Cycle 1. The novice teachers met and discussed possibilities for the lesson with the mentor teacher; however, the mentor teacher took primary responsibility for final decisions regarding the lesson plan for the first cycle. In Cycle 2, the novice teachers enacted the role of lead teachers for the study lesson, to provide a structure for scaffolded experience in leading lessons for an entire class period. Participants completed two online logs for planning: 1) thinking about the mathematical content of their lesson (Topic Study Log) and 2) planning for data collection and observation (Observation Guide Log). In addition to responding to these logs, novice teachers submitted a completed lesson plan to the Methods II course instructor (first author) in advance of teaching the lesson.

In phase 3, *Teaching*, the members of the lesson study team observed the study lesson taught by the lead teacher. In Cycle 1, the novice teachers observed the mentor teacher teach the study lesson, whereas in Cycle 2, the novice teachers taught the lesson while the mentor teacher and, in some cases other interns, observed. In the fourth, and final, phase, *the Post-Lesson Discussion*, teams reflected upon the lesson enactment and considered possible revisions to make to the lesson to better achieve the desired learning goals. Prior to meeting as a team to discuss the lesson, and no more than 24 hours after the lesson enactment, the PST-Ms completed an online Lesson Reflection Log. After teams met to discuss the lesson enactment, the PST-Ms completed a final online log, the Post-Lesson Reflection Log. In this log, PST-Ms reported on the revisions the team decided to make to the study lesson, if any, as well as reflected upon the process of collaborating with their team. Participants also submitted a revised lesson plan and reflection paper at the end of the cycle. The process of completing phase 4 was consistent across cycles. The Lesson Reflection and Post-Lesson Discussion Logs consisted of questions such as:

> "List one or two observations you would like to share with the team. Be as specific as possible about the evidence of student thinking that you observed." (Lesson Reflection)
>
> "Discuss how your team came to make decisions about the revisions. You will want to discuss what seemed to be the most important pieces of evidence collected about what students learned." (Post-Lesson Discussion)

Method of Analysis

To examine the opportunities the PST-Ms had to develop their knowledge and skills for teaching, we first examined the video conversations and written logs for two focus teams. These teams were chosen for our initial examination because the PST-Ms reported their lesson study experiences to be quite distinct from one another. This close investigation of all of the written and video data for the two focus teams revealed that the Lesson Reflection Log and Post-Lesson Discussion Logs contained significant evidence of the intern teachers' opportunities to develop their MKT, and would serve as the primary source of data for this investigation.

We used a modified grounded theory (Strauss & Corbin, 1976) approach to analyze the written logs. We identified instances in which the intern teachers noticed particular features of the lesson related to their knowledge of mathematics, students and of teaching. The unit of

analysis for this examination was the individual, as each intern teacher had opportunities to develop their knowledge independently from their team members. After the initial open coding, we generated codes for the emergent themes. The PST-Ms responses fell into three general categories: (a) analyzing the mathematics of the lesson, (b) attending to students' thinking, (c) analyzing teaching moves. Using these categories, we developed the coding scheme shown in Figure 2. The category definitions were refined through a constant comparative method. We later mapped the refined categories onto the domains of MKT, represented in Figure 1.

Results

We coded 340 instances of reflection related to MKT (see Fig. 2) in participants' responses to the lesson reflection and post-lesson discussion logs across both Cycle 1 and Cycle 2. Figure 3 shows the results across three categories of observation. A couple of results worth noting in Figure 3 are that there were more instances of codes related to attending to student thinking than the other two categories, and there were more codes overall in Cycle 2 than in Cycle 1. Although the mean number of instances of attending to student thinking coded was greater in Cycle 2 than in Cycle 1, this difference is statistically insignificant (p = 0.162).

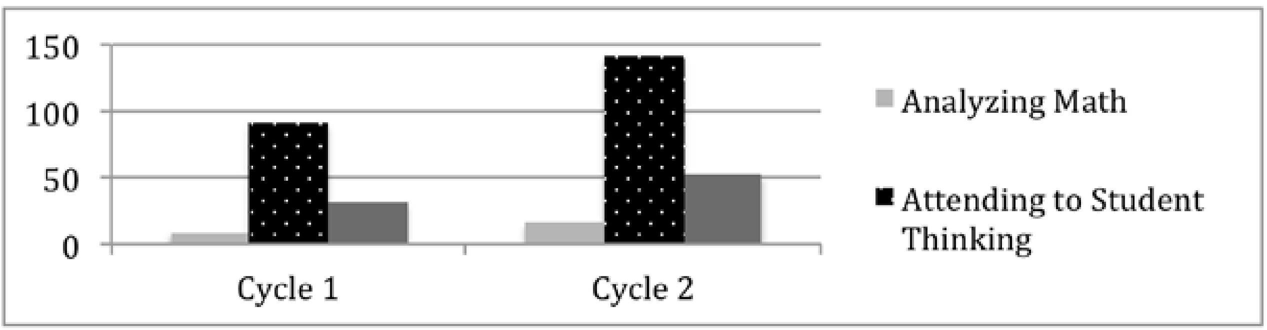

Figure 3: Frequency of MKT Codes by Cycle

There are at least a few possible reasons for the increase in responses coded into one of the 3 general categories of reflection that develops MKT from Cycle 1 to Cycle 2. First, the prospective teachers may have become more familiar with the prompts and the process of lesson study and could focus more on lesson study process and learning from the lesson observation, instead of the mechanics of completing a lesson study cycle. Another possible reason for the difference is the shift in the prospective teachers' role in the lesson study process from Cycle 1 to Cycle 2. The prospective teachers' assumed the role of the study lesson teacher in Cycle 2, and that role may have afforded opportunities to recognize important moments in the lesson related to student thinking that are not as apparent when in the observer role.

Figure 4 provides a breakdown of the kinds of reflections about students' thinking that surfaced in the responses across both the Lesson Reflection and Post-Lesson Discussion Logs. The data in Figure 4 suggest that PST-Ms were more likely to notice aspects of students' engagement in the lesson. The data also indicate that PST-Ms' responses that discussed students' thinking related to mathematical procedures were relatively low compared to other categories (such as reflections on students' understanding).

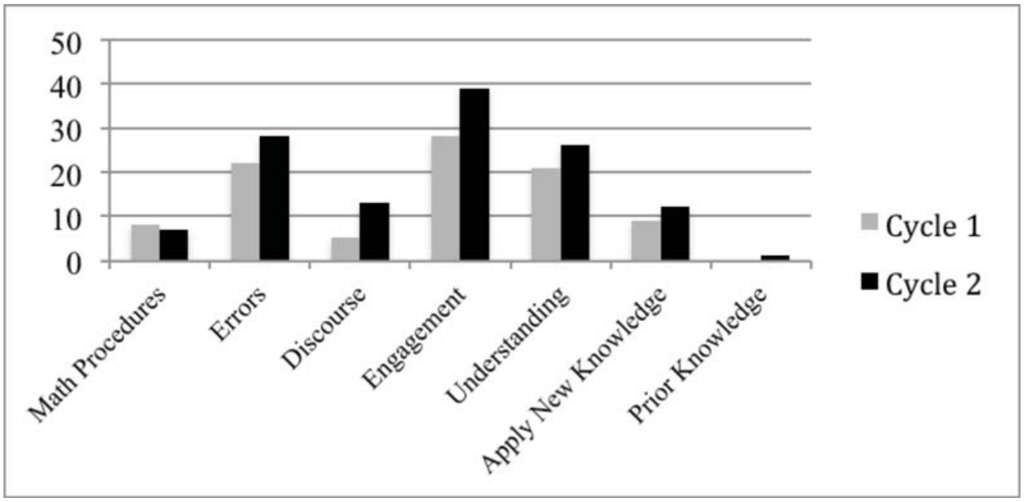

Figure 4: Sub-Codes Assigned for Responses Attending to Students' Thinking

One initial hypothesis of the study, was that the more familiar a mentor teacher was with the lesson study process, especially when they had completed lesson study with other colleagues as a part of their professional development, the more frequently prospective teachers would reflect upon the lesson in ways that could develop their MKT. Figure 5 shows the frequency of codes assigned to the three categories of observation with respect to mentor's lesson study experience.

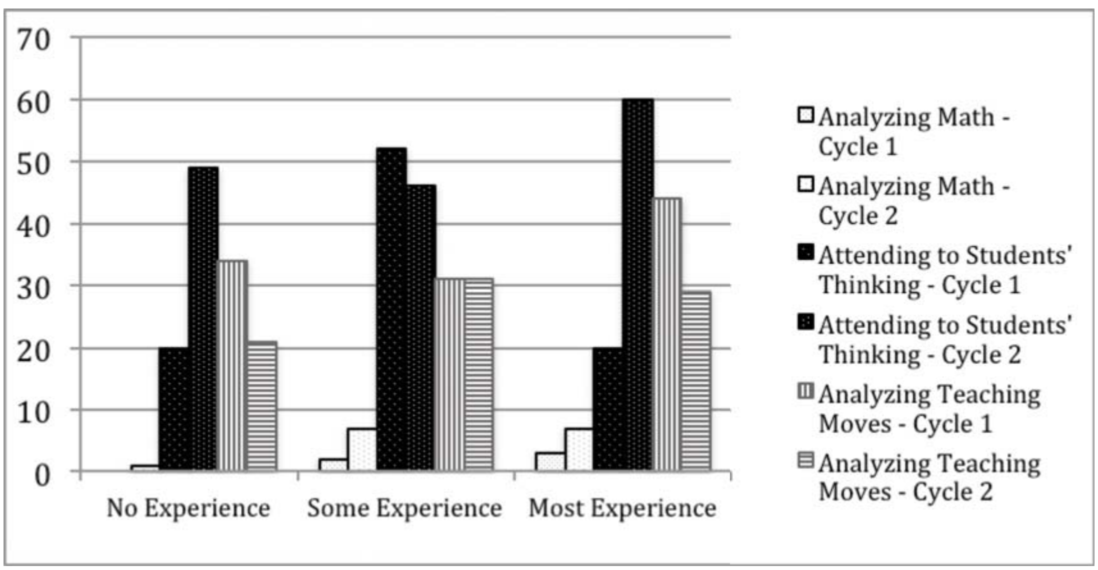

Figure 5: Frequency of MKT Codes Disaggregated by Mentors' Lesson Study Experience

As can be seen in Fig. 5, more mentor teacher experience with lesson study correlates with greater frequency in codes assigned to prospective teachers' responses to the lesson reflection and post-lesson discussion logs and level of experience is a significant predictor of increased frequency in responses coded to the three categories of observation ($F(2,13) = 4.515$, $p = 0.032$). The results also show the significant increase in responses coded for attending to students' thinking in Cycle 2, especially for PST-Ms whose mentors had the most lesson study experience.

Discussion

Although the analyses are ongoing, these results suggest that the mentor-guided lesson study experience focuses PST-Ms' attention upon student thinking during study lessons and, thus, may support the development of one particular aspect of MKT known as *knowledge of content and students* (KCS). The results also suggest that PST-Ms' capacity for noticing aspects of instruction related to students' thinking and key teaching moves improves with more experience in doing mentor-guided lesson study. Although significant differences were found in the frequency of MKT codes assigned to responses from PST-Ms to the writing prompts in the logs when aggregated by mentors' experience, we are conducting ongoing analyses to investigate how mentors' experience might be shaping PST-Ms' reflections during the lesson study process. Preliminary findings suggest that mentors' experience with lesson study supports their capacity to facilitate procedures during the lesson study process, but a more critical factor may be how mentors position PST-Ms as collaborators during Cycle 1. We are discovering that mentors who invite PST-Ms to be collaborators in designing the lesson during Cycle 1, when the mentor takes responsibility for teaching the lesson, establishes norms for future lesson study cycles that engage PST-Ms in more thoughtful reflection.

Acknowledgements

This study was funded by an MSU Lilly Fellowship awarded to the first author.

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WHY IS THE METAPHOR OF ENGLISH LEARNERS *BRINGING* RESOURCES TO MATHEMATICS CLASSROOMS NOT HELPING TEACHERS?

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While the metaphor of students bringing linguistic, cultural, and community resources to classrooms abounds in mathematics education research with non dominant students, teachers seem not to benefit from such metaphoric language as most of them struggle to figure out how such resources can be used during mathematics instruction. This paper urges a revision of this metaphor by proposing a new perspective on resources. The revised perspective is illustrated with an interaction with a third grade English learner working on a volume problem. During this interaction, the student and the interviewer recognized resources that did not pre-date the interaction but instead shared and sustained the life of the interaction. Revising this metaphor is important for helping teachers build common resources with English learners.

Keywords: Teacher Education-Inservice/Professional Development, Equity and Diversity, Elementary School Education, Measurement

Purpose

Research with English learners is replete with arguments that counter deficit views of these students. One common argument is that English learners bring resources to classrooms. Included in this argument is the metaphor of *bringing resources*. The purpose of this paper is not to question the intent of these arguments, as negative views on English learners persist even after years of counterevidence. Instead, the purpose is to revise this particular metaphor in order to construct stronger arguments in which everyone—particularly teachers—can participate. The revision—illustrated with an interaction with an English learner—is important because the metaphor has left many teachers of English learners wondering how they can begin to use such resources in classroom instruction.

Theoretical Approach

To revise the perspective that English learners bring resources to classrooms I propose a theoretical approach that shifts the emphasis on the *bringing* of resources *by* students to *recognizing* resources *with* English learners. Three interrelated ideas in this approach include: heterogeneity, togethering, and care. Each idea is elaborated next.

I begin this metaphor revision by making clear that I do not question the idea that students bring resources from their language, culture, and community. What I question is that it leaves unspecified the role of teachers in using these resources. Teachers can recognize these resources with the students, specifically in the diversity of ways in which children talk—in the *heterogeneity* of their talk (Rosebery, et al., 2010; Warren, et al., 2001)—as well as in what they do—the embodiment of knowledge that reflects the child as a linguistic, cultural, and communal being. We must start, in other words, where the child is (Mercer, 1995) and with what makes sense to them as young learners. For example, in Turner et al.'s (in review) study of English learners participating in mathematical discussion, one student said: "Cada 8 valía por 2" (Each [bag of] 8 is worth 2 [bags of 4]). This unusual way of talking about proportional reasoning—

grounded in a common saying in Spanish—was recognized by the teacher in the moment of this interaction, becoming a resource that enhanced the class understanding of proportionality.

The second idea in this framework that supports recognizing resources with students is explained by Radford and Roth (2011) in his analytic category of *togethering*: "[*T*]ogethering… accounts for the *ethical* manner in which individuals engage, respond, and tune to each other, despite their cognitive, emotional, and other differences" (p. 236). Togethering also addresses the issue that teachers and English learners too often do not share common resources (Author, in review), by urging teachers to decenter cognitively and join the students in learning processes that transcend cognitive and emotional differences. Radford and Roth illustrate how togethering works in a teacher-student interaction in which a student, Albert, is looking at a savings pattern problem by using a doubling strategy, whereas the teacher is thinking about the related algebraic formula $2n+1$. The researchers discuss the culminating moment of the teacher-student togethering process in which the teacher is about to bring the student to join her in seeing the formula: "She is tense. There is much at stake. If the interaction fails, the activity fails. It would not just be Albert's failure. It would be her failure as well." (p. 239).

Finally, if left unconnected, heterogeneity and togethering may not help teachers recognize resources with students. After all, heterogeneity emphasizes divergence of ideas whereas togethering focuses on participants transcending such differences. The category of *mathematical caring relationships* (Hackenberg, 2010) helps to conceptually connect these two categories. Defined as "a quality of interactions between a student and a teacher that conjoins affective and cognitive realms in the process of aiming for mathematical learning" (p. 237), mathematical caring relationships are evolving relationships in which the teacher continuously considers the students' heterogeneous perspectives, instantiating moments of togethering with the student aimed at understanding each other's ideas. The formation of caring relations between teachers and students in mathematics has not received enough research attention (Vithal, 2003; Hackeberg, 2010), much less in classrooms with English learners. An example is a case of a teacher who moved from unreciprocated to a responsive form of care about the students' difficulties in an estimation unit (Author, et al., in press).

Methods (Participants, Context, Data Collection, and Analysis)

Data for this paper come from a two-year professional development project aimed at supporting elementary teachers develop common resources with English learners in mathematics. The teachers learned to recognize resources with students by watching on video their own interactions with students but also by inviting me to teach some lessons that they observed. The school was located in a working class, predominantly Latino neighborhood. I videotaped an average of 2-3 days per week the interactions that these teachers and I had with Latino/a students (ELs, bilinguals, recent immigrants). Additional sources of data included analytic memos, e-mails, co-planning sessions with teachers, and interviews with teachers and students. I used these multiple data sources to triangulate findings and determine whether what we were recognizing were actual common resources between teachers and students. To illustrate the process of recognizing resource with English learners, I used a purposive sampling (Cohen, Manion, & Morrison, 2000) by selecting the case of Marifer, an English learner who participated in an interaction filled with moments in which resources were recognized.

Results

Marifer's teacher, overwhelmed by the multiple demands of her first year of teaching, referred Marifer to me because "she needs help understanding the concept of volume" (field notes). The teacher's request was based on Marifer's poor performance on volume items on a benchmark test. I approached my interaction with Marifer with two questions in mind: (1) What does she know already about volume that I can recognize as possible resources to sustain our interaction? (2) How can I join her in this knowledge of volume so she and I can construct common resources for advancing her understanding of volume? The transcript of our conversation is presented in three parts. Each part is followed by analysis that illustrates the analytical categories of heterogeneity, togethering, and mathematical caring relationships. The English translation is bracketed in italics.

Interviewer: Muéstrame una pregunta con la que batallaste mucho. [*Show me one question with which you struggled a lot*.]

Marifer: (Opens test booklet and carefully looks for one question. When she finds one, she presses on it with index finger strongly and emphatically.)

Interviewer: Pues léemela tú primero. [*Well, you read it to me first.*

Marifer: El siguiente modelo está hecho con cubos de un centímetro. ¿Cuál es el volume de este modelo? [*The following model is made with one-centimeter cubes. What is the volume of this model?*]

Interviewer: Está hecho con cubos de un centímetro. ¿Me puedes encontrar un cubo de un centímetro? [*It's made with one centimeter cubes. Can you find me a one-centimeter cube?*]

Marifer: (Puts index finder on various cubes in the model)

Interviewer: OK. Éste es uno, y aquí hay otro. ¿Cuántos cubos de un centímetro crees que hay? [*OK, that's one, and here's another one. How many one-centimeter cubes do you think there are?*]

Marifer: Eighty-two.

Interviewer: ¿Cómo sabes? [*How do you know?*]

Marifer: Because…yo los conté ayer. [*…I counted them yesterday*]

Interviewer: ¿Y cómo los contaste? [*And how did you count them?*]

Marifer: De uno por uno. [*One by one*]

Interviewer: Pero éstos están atrás, ¿cómo sabes contar si no se ven, cómo le hiciste? [*But these are in the back, how do you know how to count them if they cannot be seen, how did you go about that?*]

Marifer: Yo nomás conté todos éstos, y éstos, y éstos [*I only counted all of these, and these, and these*] (Points to each of the 3 visible faces of the model)

Interviewer: Ah, contaste…[*Ah, you counted…*] (Marifer interjects)

Marifer: Como, conté esta parte (circula cara frontal), y esta parte (circles cara superior), y éstos de al lado (cara lateral), 82 en total. [*Like, I counted this part (circles front face), and then this part (circles top face), and the ones from the side (circles side face), 82 in total*]

By asking Marifer to pick one problem that was challenging for her, I wanted her to take me back to a moment when she experienced struggle and share that moment so we could begin to engage, respond, and tune to each other's ideas as suggested by togethering (Radford & Roth, 2010). Asking Marifer to find a 1 cm cube served to recognize a common object between us, given our cultural familiarity with the metric system. This common object I predicted was going to be pivotal for finding additional resources with Marifer. These resources, in turn, would move us beyond any cognitive and emotional differences possibly related to her failure on the test. First, the 1 cm cube served to reveal her misconception of

volume as being only the three visible faces of the model. Noticing that Marifer was not seeing the intended (but hidden) three-dimensionality of the model drawn on the test, I embarked with Marifer in a new act of togethering by suggesting to co-construct a three-dimensional model using connecting cubes. In the next part of the transcript, I retake our conversation immediately after she and I had helped each other to reproduce the model on the test using connecting cubes.

Interviewer: ¿En qué se parece este modelo a éste? [*How does this model look like this on*
Marifer: Porque, está, está, uh, en cuadros, los dos están divididos en cuadritos. [*Because, it's, it's, uh, in squares, both are divided in squares*]
Interviewer: Uh, huh, en cubos de un centímetro. [*Uh, huh, in one centimeter cubes*]
Marifer: Sí. [*Yes*]
Interviewer: ¿Y en qué más se parece este modelo de los cubos al modelo del examen? [*And how else does the cubes model look like the model in the exam?*]
Marifer: Es la misma, uh (mueve ambas manos hacia arriba y abajo, con un espacio en medio) altura. [*It's the same,* (moves both hands up and down, with a space in between) *height*]
Interviewer: La misma altura, OK. ¿Cuál es la altura? [*The same height, OK. What's the height?*]
Marifer: Esto [*This*] (points to the top of the model on test).
Interviewer: Es como el techo. OK, ¿cuál es el largo? [*It's like the roof, OK. What's the length?*]
Marifer: That one. (Points to bottom of the model on test).
Interviewer: OK, aquí es el largo, y acá en los cubos, ¿cuál sería el largo? [*OK, here's the length, and over here with the cubes, which one is the length?*]
Marifer: Aquí el de abajo [*Here on the bottom*] (runs finger along the base of the block of cubes).
Interviewer: Uh-huh. Y el ancho, ¿cuál sería el ancho? [*Uh-huh. And the width, which would be the width?*]
Marifer: Aquí, como el ancho [*Here, like the width*] (points to base of block at one end).
Interviewer: OK, aquí es el ancho. Entonces tú me dices que aquí contaste 82, o sea ¿nadamás contando los que se ven? [*OK, here's the width. So you told me that here you counted 82, I mean, only counting the ones that are visible?*]
Marifer: Uh-huh.
Interviewer: Qué te parece aquí, en este modelo, ¿cuántos cubos de 1 centímetro hay aquí? [*What about here, on this model, how many one centimeter cubes are there?*]
Marifer: (Takes a careful look at front side for a while, then tilts head for an easier and closer look at one end side of the block, then announces): Cien treinta [*One hundred thirty*] (a common way among Mexican Americans to say numbers larger than one hundred; the standard way requires adding the suffix –to after the word cien, as in ciento treinta).
Interviewer: Ciento treinta. ¿Por qué ciento treinta? [*130. Why 130?*]
Marifer: Porque uh, conté uh, primero conté éstos, de esta línea, y había 10 cuadritos, entonces pensé que en cada línea había de éstos 10, y nomás conté todos. [*Because uh, I counted uh, I counted these first, in this line, and there were 10 little squares, and so I thought that in every line there were 10 of those, and so I just counted all of them*] (points to several of the sticks in descending order)
Interviewer: (Echoing Marifer as she speaks): A ver, ¿y por qué son ciento treinta? [*Let's*

see, and why there are 130?]

Marifer: Porque [*Because*], uh, I don't know if I'm right…(Points to each stick of ten as she skip counts by 10 in English): 10, 20, 30, 40, 50, 60, 70, 80, 90 (aspirates 90 as she runs out of breath, switches to Spanish after 100), 100, cien diez, cien veinte. [*One hundred ten, one hundred twenty*] Oh no, never mind, cien veinte. [*one hundred twenty*]

Interviewer: Estabas cerca, ¿verdad? (Marifer: Uh-huh). Ciento veinte. Pero cuando tú me dijiste aquí (le muestro el examen), no me dijiste que eran ciento veinte, ¿qué pasó ahi? [*You were close, right?* (Marifer: Uh-huh). *120. But when you told me right here* (show the test), *you didn't tell me it was 120, what happened there?*]

Marifer: I think, um, I counted, I count wrong.

Interviewer: You think you counted wrong (she nods) OK. ¿Qué más piensas? ¿Por qué son dos respuestas diferentes? [*OK, what else do you think? Why are these two different answers?*]

Marifer: Oh! Oh, porque, uh, en éstas (apunta a la pregunta del examen), hay como de cuatro (apunta a los 4 grupos en la parte superior del modelo con cubos) aquí 4 (apunta a los extremos de los 4 grupos en un lado del modelo con cubos) y aquí hay de diez (apunta al largo de un grupo). Como, en cada, en cada línea, como, el techo, como usted dijo, hay, están, tiene 4, y acá en el largo (ahora apunta al modelo con cubos) tiene 10, y allá (apunta al examen) tiene cuatro, como en cada línea está, como en cada lado tiene diferentes, uh, números, como así (coloca el filo de la mano en el ancho del modelo con cubos). [*Oh! Oh, because, uh, in these* (points to test item), *there are like (lines) of four* (points to 4 sticks on top of cubes model) *right here 4* (points to the ends of the 4 sticks on one side of the cubes model) *and here there are (sticks) of ten* (points to the length of one stick). *Like, in each, in each line there is, like on each side it has different, uh, numbers, like this* (puts edge of hand along the width of the cubes model)]

Interviewer: Uh-huh. Entonces ¿cuál crees que está bien, como los contaste aquí o como los contaste acá? [*Uh-huh. So, which one do you think is correct, the way you counted them here or the way you counted them there?*]

Marifer: Como los conté aquí [*The way I counted them here*] (points to cubes model).

Interviewer: ¿Por qué? [*Why?*]

Marifer: Porque…alli, aquí, uh, tenemos, la foto, y está como, aquí (redirige la atención del examen al modelo con cubos) tenemos las líneas que t-, como en cada uno está el mismo número de líneas. [*Because…there, here, uh, we have, the picture, and it's like, here* (shifts attention from test to cubes model) *we have the lines that, like in each one there's the same number of lines*]

The three-dimensional model that Marifer and I constructed served as a new resource that anchored the heterogeneity of our talk. The first evidence of Marifer's heterogeneous talk was when she began to recognize height—an important attribute of volume—first by embodying that knowledge as gesturing, then by pointing to the top of the model as the height, and finally by using the correct word *altura*. Her use of similar gestures to identify the length and the width of the model suggests that Marifer used everyday ways of engaging life experiences (Gutiérrez & Rogoff 2003) as resources to make sense of the school concept of volume. This resourceful way of making sense contrasted with her earlier isolated counting of visible faces that she reported using on the test. My initial question was, what does she know already about the concept of volume? Marifer certainly knew a lot and, perhaps more importantly, her knowledge was expressed in a variety of ways. For example, she "lifted" the attribute of width by gesturing with

her hands, leaving a space to indicate the width. This gesturing was enacted between the two models. She recognized that she had counted wrong in the two-dimensional model and was able to explain why she had counted wrong. In her explanation there was reference to "la foto" as something that prevented her from seeing all the dimensions in the test item that she was able to see in the constructed model.

Marifer's attention shifted back and forth between the two-dimensional and the three-dimensional model. I followed her as she moved back and forth between the two models, because I wanted her to decide which one was going to be our common resource. We also followed each other linguistically, sometimes talking in English, sometimes in Spanish, and sometimes bilingually, demonstrating to each other that our heterogeneous talk was a common resource. An important moment when Marifer "saw" the concept of volume occurred when she was mentally counting the connected cubes. She never touched the constructed model; instead, she was surveying the model, tilting her head to gain a different perspectival side view. Adler (2000) calls this the transparency of resources. Just like the paper version, the cubes model was not showing all the cubes that it was made of, but Marifer was seeing through it this time. For example, in her explanation of how she counted the invisible cubes, she declared: "*…and so I thought that in every line there were 10 of those*." The models clearly became transparent, and her reasoning through them became visible both for me and for her. In the final part of the transcript and as a result of noticing her miscount of 130, I asked Marifer to invent a different way of counting the cubes, one less prone to counting errors. Hearing my request, Marifer took a look at the model, pursed her lips slightly, then looked at me again and, while pointing to herself to emphasize her reaction to my request, exclaimed:

Marifer: Me?!
Interviewer: Uh-huh!
Marifer: (Continues looking at model for a while, then smiling exclaims): ¡No!
Interviewer: ¿No? Está bien, está muy bien lo que hiciste, de 10 en 10, pero… ¿Quieres pensar un poquito más, a ver si se puede más rápido? (Asiente) OK. [*No? That's alright, what you did is great, by tens, but… Do you want to keep thinking to see if you can do this faster? (She nods) OK*]
Marifer: (Looks at the corner of the block model where she can get a perspective of the three dimensions): Oh! uh, hay, en cada, en cada de éstos hay 4 uh de estas líneas, entonces, [*Oh! Uh, there's, in each, in every one of these there are 4 uh of these lines*] I think, y en cada linea hay 10 cuadritos [*and in each line there are 10 little squares*], so we can do, I think uh, ten times four, diez por cuatro.
Interviewer: Oh! Sí, está muy bien, y ¿cuánto es ten times four? Diez por cuatro, cuánto es diez por cuatro? [*Oh! Yes, that's great, and how much is 10 times 4?*]
Marifer: Forty.
Interviewer: Forty. OK.
Marifer: But I think it's not it.
Interviewer: Hu?
Marifer: I think it's not the answer.
Interviewer: Well, not the final answer, but I can see the forty right here. Can you see the 40 right here? (I lift the top layer of 4 sticks of ten, Marifer nods). OK, so you got like part of the answer, and then what? I like what you're doing! Y luego, ¿qué más harías, Marifer? [*So then what else would you do, Marifer?*] So you got 4x10. ¿Qué más? [*What else?*] *Marifer:* (Looks at model for a long time. I do not say a word. I am holding up the top layer: If she

fails, I fail, because we are doing this together.): We get another 4, (she lifts the layer of 4 sticks of ten that was in the middle) then these ones (points to the base layer).

By the time I asked Marifer to invent a different way of counting, we were already in a mathematical caring relationship (Hackenberg, 2010). We cared so much about each other's ideas and our common resources that now I wanted to move Marifer with me to a common understanding of a multiplicative way of thinking about volume, to prepare her to have a even more powerful resource that she could use in future tasks. Her surprise with my invitation indicated that she was not expecting this challenge. I asked the teacher for her interpretation of Marifer's reaction, and she explained: "She probably has not been asked that question before" (field notes). However, when students see themselves in mathematical caring relationships, they do not give up so easily when they face challenges. Marifer approached the challenge with care. She began the multiplicative counting first with one layer of cubes (4x10), but she knew that this was not the final answer. There was a final moment in our interaction when Marifer created yet another common resource: In that moment, Marifer lifted the layer that was in the middle, an action that I had initiated and that she appropriated. By doing this unprompted action, Marifer finished constructing our common resource that was helping us recognize the dimensions of the model and using these dimensions to count the one-centimeter cubes multiplicatively.

Discussion

For many teachers, certainly for Marifer's teacher, the metaphor of students bringing resources to classrooms is an esoteric idea that does not include the teacher in an active role. In this paper I have presented evidence from my instructional interaction with an English learner, to call for a revision of the metaphor in the sense that it is possible to recognize resources with students in interactions, instead of expecting that students will bring and spontaneously use such resources. The transcript of my interaction with Marifer includes multiple moments in which she and I recognized linguistic, cultural, and experiential resources that we used for advancing her understanding of volume. These resources did not pre-date our interaction. Instead, they shared and sustained the "life" of our interaction. The generative power of students' everyday experiences (Warren, et al., 2001) can be the teachers' best ally for recognizing resources with students. Finally in this paper I have suggested and used heterogeneity, togethering, and mathematical caring relationships as analytical categories that can help us recognize resources *with* our students. In revising the metaphor of students bringing resources to classrooms, I have rearranged the multiple pieces of this metaphor. First, the source of students' resources is not to be found outside the instructional interaction; instead, these resources are an important and recognizable part of any interaction. Second, teachers are not expected to know students' resources prior to interacting with students; rather, they are expected to recognize these resources with students by engaging, responding, and tuning to each other's ideas while forming caring relationships with students. Finally, this recognition of resources is urgent given that in many cases, the teachers of English learners share very little in common with the students. The revised metaphor is intended to guide teachers to recognize and construct these common resources with the students.

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FINE-GRAINED ANALYSIS OF TEACHER KNOWLEDGE: PROPORTION AND GEOMETRY

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In this study, we considered how four teachers reasoned about proportional situations involving geometry. We used pilot interviews conducted with in-service middle grades teachers around a set of proportional tasks. Analysis relied on a knowledge in pieces lens to uncover the resources that each teacher used in solving the tasks and which resources seemed to support the reasoning. Our findings suggest that this is an area of proportional reasoning in which multiplicative comparisons are important, but insufficient, for supporting reasoning. We also found that these teachers did not invoke the constant of proportionality in ways that supported their reasoning.

Keywords: Teacher Knowledge, Middle School Education. Mathematical Knowledge for Teaching

Purpose of the Study

Teacher knowledge matters. This is a conclusion reached by scholars and researchers alike (e.g., National Mathematics Advisory Panel, 2008; Hill, Rowan, & Ball, 2005). However, the nature of that knowledge has been less thoroughly studied. To date, three main approaches have dominated the landscape of teacher knowledge research. The most enduring approaches have involved using proxy measures to seek correlations between teachers' knowledge and particular aspects of their backgrounds such as courses taken in their preparation programs (e.g., Begles, 1975; Goe & Stickler, 2008). These studies have yielded inconclusive results about the relationship of teacher knowledge to student achievement. A second approach has been the use of assessments that attempt to measure the amount of knowledge teachers have (e.g., Baumert et al., 2010; Hill, Rowan, & Ball, 2005; Post, Harel, Behr, & Lesh, 1988). These studies have uncovered a host of gaps in teacher knowledge. A third approach to studying teacher knowledge, exemplified by Ma (1999), relies on qualitative approaches to understand teacher thinking. While some of these studies have focused on teacher deficiencies as the basis for developing theories of teacher knowledge (e.g., Ma, 1999), others have used the fine-grained analysis of teacher understanding to uncover important aspects of teacher reasoning with an eye toward supporting further teacher development (e.g., Izsák, 2008; Orrill & Brown, 2012). The current study continues this third tradition by carefully considering teachers' reasoning about particular proportional situations. Here, we considered how the teachers' knowledge about proportions and geometry would work across the three situations.

Theoretical Framework

Consistent with this kind of approach to the study of teacher knowledge, we relied on the knowledge in pieces theory (e.g., diSessa, 1988, 2006; Smith, diSessa, & Roschelle, 1993) to guide our analysis of teachers' reasoning. Knowledge in pieces is appropriate and useful because it posits that our understandings are stored as fine-grained knowledge resources invoked in a given situation. In the knowledge in pieces view, as in other research focused on knowledge organization (c.f., Bédard & Chi, 1992), the development of expertise (i.e., learning) involves

building connections and refinements that allow appropriate pieces of knowledge to be invoked in various situations. Thus, knowledge in pieces allows us a lens for considering *coherence* of knowledge. For those with more coherent knowledge, more knowledge resources should be readily invoked whereas knowledge resources for someone with less coherent knowledge would like be more haphazardly invoked. Knowledge in pieces allows us to consider that people may have understandings that are tacit in a given situation rather than assuming that unseen knowledge is missing. Further, it allows us to consider that any individual's knowledge is reasonable in isolation–that is, people are making sense of their world in ways that are reasonable to them given the knowledge they have invoked for the situation. This view is removed from the knowledge deficit models that choose to focus on the mathematics teachers may not know. It is also important because it allows us to begin to operationalize the notion of *coherence* which appears often in scholarly pieces about desired teacher knowledge (e.g., Ma, 1999; NMAP, 2008; Silverman & Thompson, 2008) but that has been rarely operationalized.

Methods

Participants, Context, & Data Sources

This study was an exploratory, qualitative study conducted as a pilot effort for a larger project. The participants were four middle grades teachers: one fifth-grade teacher (Aubry – all names are pseudonyms), one seventh-grade teacher (Liam), one eighth-grade teacher (Autumn), and one elementary mathematics coach who had also been a middle school teacher (Luke).

The data collected consisted of a set of clinical interviews (Ginsburg, 1997) focused on three key areas of proportional reasoning: geometric applications of proportional reasoning, the relationship of fractions and proportions, and the appropriateness of using proportional reasoning to solve a given task. All interviews were conducted by the first author with occasional questions from the second author. All of the interviews were videorecorded with two cameras, providing a record of all written work and interactions with the technology as the participants worked on each task. Participants were told that the purpose of the pilot study was to "try out" an evolving set of tasks about proportions that were being piloted for the larger study. The participants were interviewed for 60-90 minutes at a time and each participated in two or three separate interviews that spanned proportional reasoning more broadly.

Data Analysis

All data were analyzed by mapping the ideas the participants used in solving each task (e.g., Empson, Greenstein, Maldonado, & Roschelle, 2013). Then, we distilled these maps by looking for the key proportional ideas that emerged as the participants reasoned about the task. We then, created a timeline of the key events in solving the task. Timelines included key mathematical phrases from the interview with the prompts placed inline so we could determine the extent to which an idea was naturally emerging or being prompted by the interviewer. We then analyzed the solution path to determine which mathematical resources they had invoked and whether those ideas were supporting a reasonable solution of each task. To allow ourselves to understand the participants' approaches more broadly, we transferred this information into a cross-case table that allowed us to evaluate each task to see what knowledge resources were invoked and whether those resources seemed useful to the participant(s) invoking them.

The Tasks

Each task is briefly described below including a description of the task and our reasons for including it in the protocol. The first two tasks were intentionally developed so that neither was a comparison or missing value problem (Lamon, 2007) as our intentions were to understand how teachers reason about proportional situations rather than understanding whether they can

determine correct responses to procedural items. Tasks were modified slightly from one participant to the next in response to our desire to refine the items, however the essence of the questions was the same among the four participants.

The Thermometer Task is a Geometer's Sketchpad (GSP) sketch that had two parts. The first showed two lines that related through a linear relationship, thus as the user drags the slider, they increase in length at the same rate. The second set of thermometers showed two lines that were proportional to each other, thus the second is always 3/5 the length of the first (Figure 1). This item was developed to determine the extent to which teachers recognize proportional and linear situations presented visually.

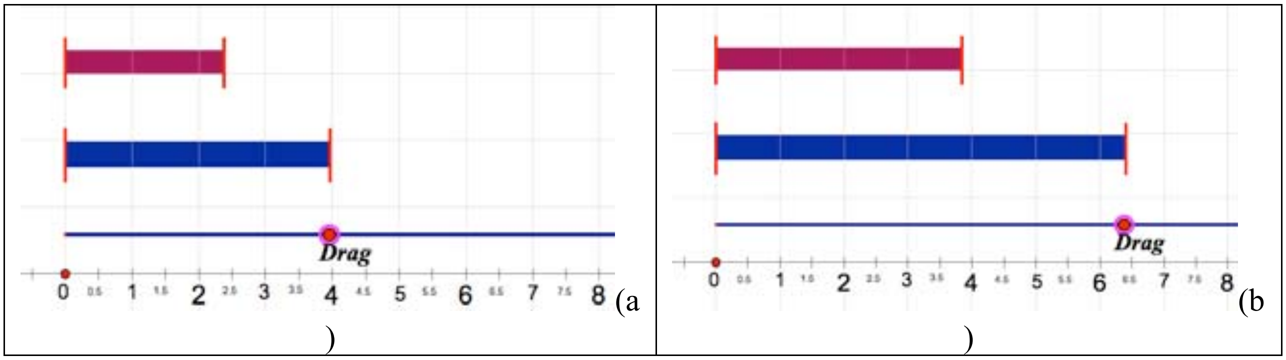

Figure 1: Proportional Thermometer Task (a) Initially and (b) After Dragging the Point.

The Smiling Guy Task was another GSP sketch that offered a fixed picture of "Smiling Guy" and a version of him that scaled up/down in a proportional way as the slider was moved (Figure 2). The item was based on a task about an imaginary character named "Wimpy" who liked to visit the hall of mirrors (Ben-Chaim, Keret, & Ilany, 2012). Based on early responses to the Santa task and to the original Wimpy task, we determined that our participants may not be attending to area as they reason about the changing size. We hypothesized that the dynamic nature of this new task would support more area-focused reasoning. Luke was not interviewed for this task.

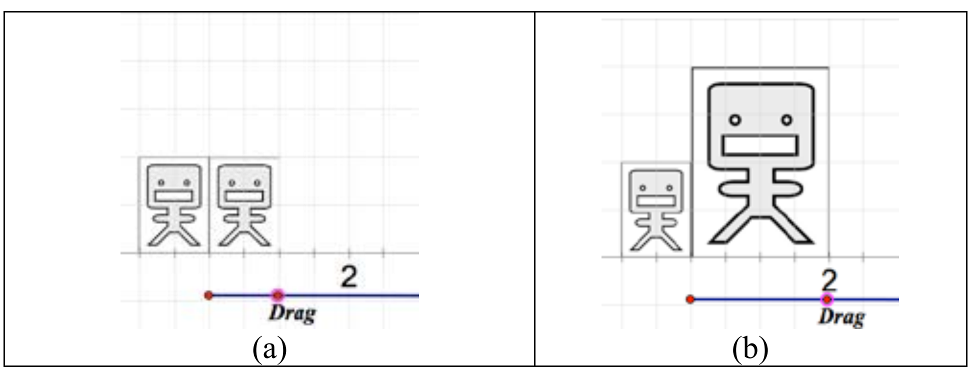

Figure 2: Smiling Guy Task (a) at Starting Point and (b) After Dragging

The Santa Task was adapted from De Bock, Van Dooren, Janssens, and Verschaffel (2002). In their study, high school students were given the task along with a set of prompts to determine

the persistence of linearity in students' thinking. Because their results about the persistence of linearity were compelling, we wondered whether middle grades teachers would perform similarly. In this paper-based task Santa has been painted on a bakery door and the task asks how much paint will be needed if the painter were to paint the same picture three times taller on a grocery store window. A series of scaffolding situations, some from DeBock et al. (2002) and some developed for our participants, were provided to support area reasoning in determining the correct paint amount.

Bart is publicity painter. In the last few days, he had to paint Christmas decorations on several store windows. Yesterday, he made a drawing of a 56 cm high Santa on the door of a bakery. He needed 6ml paint. Now he is asked to make an enlarged version of the same drawing on a supermarket window. This copy should be 168 cm high. How much paint will Bart approximately need to do this?

Figure 3: Initial Presentation of the Santa Task.

Results

In this section, we briefly highlight observations from each task. Our goal was to understand how the teachers leveraged their knowledge resources to address each of the tasks. While each participant used a unique approach to each task, we present the findings here by looking for the particularly notable commonalities and differences between the teachers.

Thermometers Task

In the Thermometers task, we noticed a split in our participant pool. Autumn and Luke both relied on multiplicative reasoning—comparing one thermometer to the other—and making and testing conjectures to develop reasonable responses to the task. Both were ultimately able to identify that the first relationship was linear and the second was proportional. They relied on equivalent ratios to make this determination. For example, Autumn described the first (linear) situation saying, "I mean, the gap is always two until the initial start. Oh, the initial start, so zero to two. So there's always just a two lead gap. So as far as proportions go, no, that would not be good." Clearly showing that one of her criteria for differentiating proportions was that the "gap" between bars should not always be the same. This idea was reinforced on the second thermometer when she immediately tested the bars to see whether each could be 0 at the same time. Yet, when she declared that the second pair of bars was proportional, she had already written down some equivalent ratios as she shared, "if these two are proportional, then the difference when you increase the temperatures would have to be proportionally the same…So if this increases by 1, this would have to increase by 3/5. If this increases by 5, this would have to increase by 3." She went on to explain that the length of one bar could be found relative to the other bar (e.g., the red bar is 3/5 of the blue bar.) To us, this demonstrated that she was invoking a number of knowledge resources including equivalence, multiplicative relationships, and a definition of proportion that included the (0,0) instance.

Like Luke and Autumn, Liam and Aubry focused on the difference between the length of the blue line and the length of the red line to come to different conclusions. Liam determined that

there were not proportional "because they're not the same". This was not an idea well elaborated as Liam moved on to describe how he would find one specific instance of the relationships between the bars. Aubry, on the other hand, felt they were proportional because "it's constant. It doesn't matter—if I went over—like if I went to 12, this would be 10. If I went to 10, this would be 8." For Aubry, the idea of constant, as it related to proportionality, seemed to refer not to a particular multiplicative constant relationship between the bars, but rather to any constant relationship.

However, for the proportional thermometers, they referred to a constant change focused on how much length was added with each move of the slider. Both participants maintained that there were not generalizable relationships between the thermometers on the proportional version. In fact, Liam pondered, "If red is 3/5 of blue, how can I talk about that as a proportional relationship?" The participants' responses indicated problematic use of knowledge resources related to the idea of constant in a proportion. Liam's response suggested that constant might mean "the same", though it was hard to know whether he meant the same length or the same rate of increase. However, his statement of not knowing how to think about red as 3/5 of blue in a proportional way was revealing in terms of highlighting that multiplicative reasoning was not being invoked by this task for him.

Smiling Guy Task

In the Smiling Guy Task, Autumn again used multiplicative language to compare the new Smiling Guy to the original. However, she rejected that the new Smiling Guy was proportional to the original because the ratio of areas changed as Smiling Guy increased or decreased in size. Thus, we assert that Autumn understood that there should be a comparison that yielded equivalent ratios, but was unable to identify the correct comparison between images. In contrast, Liam and Aubry accepted that Smiling Guy was proportional, but both again relied on reasoning that sounded additive. They explained that for every two units wider Smiling Guy gets, he also gets two units taller. We noted that this seemed almost like a formula—whatever you do to the top, you have to do to the bottom—rather than reasoning about the ratio as a coordination of two units (Lobato & Ellis, 2010). Thus, multiplicative reasoning used by Autumn did not yield a correct answer because she the understanding of constant she invoked was not adequate, whereas procedural reasoning from Liam and Aubry did yield correct answers.

When prompted about the relative size of the new Smiling Guy to the original, Autumn easily replied using multiplicative reasoning that took area into account. In contrast, Liam and Aubry struggled to name the relative size of the new Smiling Guy. They tried to make sense of the new size by relying on statements that included the phrase "for every" to explain the change. For example, when estimating how much larger the new Smiling Guy was than the original, Aubry said, "For every two units (on the slider scale) larger, it's two times as big." This further suggested that she was not coordinating units, but instead relied on application of a rule that says whatever is done to the width needs to be done to the height. It also demonstrated the problems the participants had with area. As shown in Figure 2, when the slider is pulled "over 2", the resulting Smiling Guy is four times larger than the original, but he is two units taller.

Santa Task

In the Santa Task, we again saw Autumn and Luke using multiplicative comparisons to talk about the relative size of the new Santa to the original. Consistent with the findings reported for high school students in De Bock et al. (2007), however, these participants initially attended only to the height of the new Santa rather than the area of the Santa for reasoning about the amount of paint needed. In contrast, Liam and Aubry capitalized on the appearance that this task was a

missing-value problem (Lamon, 2007) by changing from their previous additive approaches. Instead, both set up proportions and used within space reasoning to find the scale factor between the two ratios (e.g., 168 is 3 times more than 56, so they multiplied the 6 ml by 3 to find 18ml of paint needed). As with Luke and Autumn, they only attended to height.

Interestingly, Luke, Aubry, and Liam rejected the use of area throughout the scaffolds included in the task. One such scaffold presented the Santa images with boxes drawn around them and asked whether this was a strategy that might be helpful for a student to use. All of the participants were readily able to see that the smaller image tiled the larger image, yet they continued to reject area reasoning for this task. Luke explained his rejection of this strategy saying, “"Like in the original problem, they didn't mention anything about width or anything like that. It was just strictly based on height. So I was just seeing it just in that relationship - a ratio. Here, they're literally seeing how many of these copies will cover this larger copy - will fill up that larger copy.” Liam and Aubrey made similar statements. This suggested to us that the participants had knowledge resources available to them (e.g., tiling the small Santa on the larger one to determine relative size correctly) that they were unable to capitalize on in reasoning about the task.

In contrast, Autumn began to question her assertion of only attending to height when presented with the first scaffold, which presented two different solutions found by students in a fictitious class (18 ml and 54 ml). She replied,

> [The students] were looking at the… oh, just the height. Oh I could even be wrong. I mean we’re not considering maybe the whole area… I don’t know. I mean I really don’t know necessarily, I might be over thinking it, but how would they get… I’m thinking that these students thought of it in terms of proportions for something to be this high if it has to fill up that much paint, that to do this much 168, you need a proportional amount of paint and for the other students I’m honestly not sure.

By the end of the task, which included three additional prompts, including the one in which boxes are drawn around the images, Autumn concluded that area was the necessary measure rather than height. Based on our analysis of her reasoning, we believe that she realized perimeter (another option) would only allow her to know how much paint would go around Santa and that queued her to invoke her area reasoning.

Discussion

From this small-scale study, it became clear that the ability of teachers to invoke multiplicative reasoning varied, and when invoked it supported better reasoning about the situations than other (e.g., additive) reasoning. However, multiplicative reasoning was not enough to consistently yield correct reasoning in these tasks. Other knowledge resources seemed important to combine with multiplicative reasoning. In our analysis, we noted two such resources: understanding what is constant in a proportion and understanding the role of equivalence in proportions. We also noted that procedural applications of proportional problem solving came with trade-offs of their own.

Understanding the constant in a proportion seemed potentially important, but elusive for these participants. For example, Autumn provided reasoning about the Smiling Guy task that compared the original to the new Smiling Guy in reasonable ways given that she did not exhibit an understanding of the appropriate constant in the relationship. She knew that the ratios needed to be constant (i.e., equivalent), but was looking at the wrong ratios. In contrast, Liam responded, when asked specifically about the constant of proportionality, that he did not know what that was. For the four teachers in this pilot effort, the constant of proportionality was not invoked in

ways that allowed us to understand how the teachers might understand it as a concept or use it in their problem solving. There is a need for careful consideration of what teachers need to know about the constancy of proportions and how to support them in developing that understanding.

The second particularly helpful knowledge resource we noted what the use of equivalence. In these tasks, Luke and Autumn both regularly checked for equivalence as they worked. They invoked a definition of proportionality that allowed them to rely on such equivalence. Luke was very systematic about this, in that he created a t-chart for each situation and used that to reason both across and down the chart to ensure equivalence before declaring any situation proportional.

As with prior research (e.g., Karplus, Pulos, & Stage, 1983; Singh, 2010) we found that existing algorithms obscured our ability to fully understand the teachers' sensemaking in places. This was particularly true in the Santa Task in which Liam and Aubry relied on a particular algorithm focused on one particular relationship to approach the task. We ponder whether this was an invocation of the common missing-value approach. We also ponder whether they understood the mathematical similarity of Santa to Smiling Guy. This raises questions about how teachers think of an equivalence class of relationships (e.g., Lobato & Ellis, 2010) if they treat a single instance in a different manner than a dynamic relationship. The use of the apparent missing-value approach also raised questions about whether Liam and Aubry were reasoning about ratios as relationships involving coordination of units or whether they were applying a rule of 'do to the top what you do to the bottom.'

Combined, we assert that these data suggest that the teachers had understandings that were not invoked (e.g., area) as well as those that may have been underdeveloped (e.g., the constant of proportionality). However, the teachers did bring a number of resources to their analysis of the tasks. This suggests the needs for further research into this area so that we are better able to support teacher development of proportional reasoning. From this small study, we hypothesize that without perturbation, our participants did not face their weaknesses in understanding, even when scaffolds were provided to perturb their thinking, because their schemes were not challenged. Thus, being able to solve missing-value problems as in Santa may have demonstrated their ability to apply a procedural understanding in absence of conceptual grounding. This may limit the utility of the approach for a wider range of proportional tasks. Because these are preliminary data, this is a question for further exploration.

Finally, we see from these data that teachers do, in fact, have knowledge that may not be invoked in particular situations. This is important because it suggests that all of their knowledge resources may not be equally available to students in the classroom. It is also important because it suggests that more research is needed in this area. It is clear from this study that some of the knowledge not invoked was, in fact, available to the teachers more broadly. For example, they all demonstrated an easy ability to think about the Santa task using a tiling approach that allowed them to talk about area, yet they chose not to use that knowledge in thinking about the amount of paint needed for the task. Given the push for coherence in teacher understanding, it seems critical to more carefully understand what resources teachers have, which are invoked in different kinds of settings, how to support teachers in creating new connections within their existing understandings, and what understandings may need to be developed.

Acknowledgements

The work reported here was supported by the National Science Foundation through grant DRL-1125621. The findings reported here are those of the authors and may not reflect those of the NSF. We wish to thank our advisory board members—Pat Thompson, Sybilla Beckmann,

Deborah Schifter, Allan Cohen, & David Shaffer—for their valuable input into the design and analysis of these items.

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SECONDARY TEACHERS' TALK ABOUT THE MATHEMATICS REGISTER

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The mathematics register refers to the forms of meaning and styles of communication characteristic to the mathematics disciplinary community. An important role of teachers is to support students in developing facility with the mathematics register in order to support students' learning. This study focuses on the ways in which a group of secondary mathematics teachers talked about the mathematics register over the course of a year-long study group. In particular, we analyze their discourse to identify themes and shifts in the ways they collectively made sense of the mathematics register. We found that the teachers came to discuss the mathematics register as more than specialized vocabulary, and we anticipate that their understanding of the mathematics register will continue to deepen as they use this academic idea in the context of their own teaching practice.

Keywords: Classroom Discourse, Teacher Education-Inservice/Professional Development

Researchers in mathematics education have become increasingly interested in mathematics discourse and classroom discourse (Ryve, 2011). In particular, the idea of the "mathematics register" has become central to this research. Michael Halliday (1978), a sociolinguist, introduced this idea and defined it as:

> A register is a set of meanings that is appropriate to a particular function of language, together with the words and structures which express these meanings. We can refer to a 'mathematics register', in the sense of the meanings that belong to the language of mathematics (the mathematical use of natural language, that is: not mathematics itself), and that a language must express if it is being used for mathematical purposes. (p. 175)

Our review of literature utilizing the mathematics register (MR) has shown that the number of articles using this idea has more than quadrupled from the 1980s to the 2000s. The use of MR, however, seems to have stayed primarily in the mathematics education *research* community, which raises the question of how it might be taken up by mathematics *teachers*.

In this article, we build upon the existing literature to investigate how teachers, who were involved in a year-long study group focused on secondary mathematics classroom discourse, talked about and made sense of the "mathematics register." In order to make theoretical ideas useful to practice, we would argue, it is imperative to better understand how people who might use the ideas with students in classrooms talk about and make sense of the ideas. In fact, some research has suggested that teachers' implicit understandings of mathematical discourse (and the values that go along with those understandings) shape teachers' assessment practices (Morgan, 1998). Thus, it seems important to understand these implicit understandings in order to capitalize on them in work with teachers in contexts like study groups. Although we restrict our focus to how the teachers talked about the MR in the study group and not what they did with it in their

classrooms, we would argue that this remains an important step toward informed work on how the MR is taken up in the context of classroom practice, especially since the activities of the study group were grounded in artifacts of practice.

Theoretical Perspectives

Our overarching framing of this work is sociocultural and sociolinguistic. We see learning as being related to participation in discourse practices of a community (Lave & Wenger, 1991). Because participation and context are central to learning, we draw on the tools of systemic functional linguistics (SFL) in our analysis. SFL assumes that language learning is intimately related to the cultural and situational context in which the learning takes place (Halliday & Matthiessen, 2003). Halliday, who introduced the MR, was a pioneer in SFL.

Pimm (1987) elaborated on Halliday's beginning definition of MR provided earlier, pointing out that Halliday was not only talking about how mathematical terms are used but also saying that there are characteristic phrases and certain modes that are acceptable for processes such as argumentation. These modes of argument, for example, should be precise, brief, and mathematically logical (Forman, McCormick, & Donato, 1997). O'Halloran (2005) provided a detailed description of the MR by focusing on the processes, representations, symbolism, and so forth. O'Halloran highlighted, in particular, the important role that this range of meaning systems plays in construing mathematical meaning.

Schleppegrell (2007) delineated many grammatical patterns used to construe the MR, beginning with aspects related to mathematical vocabulary, such as the fact that certain terms may take on different or more precise meanings in the MR than they do in other contexts. Beyond mathematics vocabulary, Schleppegrell also described how dense noun phrases, nominalizations, logical connectors, and verbs can be challenging for students. With regard to dense noun phrases, when contrasted with students' out-of-school experiences, relatively short written and spoken phrases in mathematics may hold a great deal of meaning in very few words. Moreover, within the mathematics register, complicated processes are turned into nouns—called nominalizations—so that we can do new things with these processes. Examples of nominalizations in mathematics include "rotate the triangle 90 degrees in the plane" becoming "the rotation" and "taking the limit of a difference quotient" becoming "the derivative." In some cases the nominalization involves a new form of the original verb (as in rotation), but in other cases a new noun is introduced (as in derivative). A result of nominalization is that the human participants are removed from the statement, giving agency to mathematical objects and processes rather than to the people who are doing the mathematics. Even the verbs tend to be different in the MR. In other domains, there tends to be a prevalence of doing or thinking verbs, for example. In the MR, however, relational verbs like *be*, *have*, and *means* are more often used to express relationships between objects and processes.

The characteristics described here provide some insights about what makes the MR complex and potentially difficult for students. It also highlights the fact that the teacher's role as a "more knowledgeable other" is important in scaffolding students' facility with these meaning systems and grammatical characteristics. Yet, Morgan (1998) has provided evidence that teachers unknowingly evaluate students' work based on their differential use of characteristics of the MR. So, it is important for teachers to be aware of the particular ways in which mathematics is construed in language and to make these language choices more explicit to themselves and to students. As Schleppegrell (2007) pointed out, supporting students to develop facility with the MR takes time and serious consideration. Such consideration requires that teachers recognize

these characteristics of the MR. Here we investigate how teachers talked about and made sense of this idea in the context of a study group about mathematics classroom discourse. In this way, we sought to answer the following questions: What kinds of discourse practices did the secondary mathematics teachers use to participate in discourse *about* the mathematics register? How did they talk about this idea over time? and Which characteristics of the mathematics register garnered the most attention?

Method

Setting

The participants in this study were middle school and high school mathematics teachers from three different school districts. The nine participating teachers had varying educational backgrounds, teaching experience, and classroom settings. Most of the teachers had little or no prior professional development (PD) experience focused explicitly on classroom discourse, though some had completed teacher preparation programs that included work on discourse-related ideas and one teacher was also certified to teach English. The teachers volunteered to participate in the pilot of a set of PD materials focused on classroom discourse (see Herbel-Eisenmann, Steele, & Cirillo, 2013), thus indicating at least some level of interest in classroom discourse and a potential desire to make changes in this area of their practice. The facilitation team comprised two faculty members and four graduate students. Almost all had classroom teaching experience and had worked with prospective and practicing teachers previously. The facilitation team was comprised of two men and four women and all facilitators were involved in writing and revising the PD materials that we were piloting.

The PD materials were designed to support teachers in becoming purposeful about developing productive and powerful discourse in their classrooms, where *productive* refers to discourse that supports students' access to mathematical content and ways of meaning and *powerful* refers to discourse that positions students as legitimate knowers and doers of mathematics. One particularly important idea from sociolinguistics that is explored throughout the materials is the mathematics register. The first unit focuses primarily on students and characteristics of mathematics classroom discourse. It launches an investigation of the mathematics register by looking at student written work and textbook pages. In the second unit, particular "teacher discourse moves" (TDMs) are introduced and teachers are encouraged to consider their use of TDMs with respect to developing productive and powerful discourse. The remaining units include a variety of activities, many of which directly involve the mathematics register as the teachers analyze and discuss their mathematics textbooks, cases of classroom discourse, and artifacts from their own teaching practice.

Data

Study group sessions were approximately three hours in duration and occurred every 2–3 weeks throughout an academic year. Project team members video recorded and took field notes at each session and collected teacher artifacts (e.g., written responses to discussion questions, notes on hand-outs). For this analysis, we identified study group activities that related directly to the mathematics register. These activities either involved the mathematics register by name after it had been introduced and defined or were from early sessions in which various ideas related to communication in mathematics classrooms were discussed. Through this process, fourteen activities were identified for further analysis spanning nearly the entirety of the PD pilot.

Analysis

To analyze each of the fourteen PD activities related to the mathematics register, we employed a form of thematic discourse analysis (Herbel-Eisenmann & Otten, 2011; Lemke, 1990). First, we reviewed the videos and formed lexical chains that mapped the flow of the discussion over time. Second, we identified the segments of discussion from the lexical chains that involved the mathematics register most directly. For example, segments of the discussion pertaining to the characteristics that made one explanation "more mathematical" than another were marked for further analysis, whereas the segments from the same discussion in which the teachers talked about types of tasks that they had enacted in their classrooms were not. Third, with these focused segments, we generated thematic maps of the terms used by teachers and the semantic relations construed between the terms (Herbel-Eisenmann & Otten, 2011). This process illuminated the ideas that arose in the discourse related to the MR and also provided a representation of how teachers were making sense of those ideas in each particular interaction. Finally, we identified recurring themes in the ways that the teachers engaged in the discourse as they discussed the MR and also examined shifts in the content of the teachers' discourse related to the MR. Specifically, our analysis highlighted two particular discourse patterns that the teachers used to make meaning of the MR: providing particular examples of characteristics associated with the mathematics register and were sometimes able to connect these examples to broader categories (e.g., audience, who the actors were, dense phrases, use of logical connectors) associated with the MR; and utilizing comparison and contrast as a means for talking about what they noticed with respect to the mathematics register. In the findings, we illustrate these two meaning-making themes related to the language practices the teachers used to make sense of this idea.

Findings

The two salient themes that emerged from the teachers' talk about the MR were comparison and contrast (comparison/contrast) as a meaning-making device in the discourse and the use of examples to highlight broader categories related to the MR (examples/categories). We use these two discourse patterns to organize our presentation of the findings of how the teachers' discourse about the MR shifted over time. We thus organize the examples chronologically and highlight only a few representative excerpts in each section below to illustrate some of these themes and to show some of the subtle differences we saw over the course of the PD.

Early Professional Development Sessions

Several times across the sessions, teachers identified specific mathematics terminology or vocabulary as examples related to the MR. They provided examples of these words and also named it by the category "math terminology" or "vocabulary." In early sessions, they sometimes used the category label of "language" to be synonymous with vocabulary. An early excerpt from the PD, in which the teachers identified the category of "math verbs" and listed examples, is given below. The teacher here was sharing aspects of her definition of discourse. The other teachers then added their own examples.

Kelly: Using and just understanding the other math verbs appropriately. Calculate, justify, analyze.
Xander: Construct, draw.
Diedre: Describe.
Kelly: Estimate.
Diedre: [laughing] Find, solve.

Generally speaking, we found that the participants often focused on mathematical terminology. In another example, a participant described how her students were using the "right vocabulary" and she listed "perimeter," area," and "angles" as examples of this vocabulary. We noticed that this emphasis on mathematical terminology persisted across the sessions, but it tended to be a sole focus in the early sessions.

In this next excerpt, both meaning-making themes (example/category and comparison/contrast) are illustrated. After each participant said something about his or her own definition of discourse (shown in the previous example), the group began to discuss their definitions and a reading they had just completed about the definition of discourse underlying the PD materials:

> *Diedre:* I kind of like the paragraph here [in the reading], where it said, you know--, it's related to the fact that students are studying *math* and what we use in our connections and our words have totally different meanings than when they come in and out of other classes. And that's really hard to get across to other people that are coming in and watching, you know, there is a specific meaning to "if-then", you know, and it means something very different and not just, well, that *and* that.
>
> *Xander:* But then there is also showing the similarities that there are. Like, you know, when you have a word like intercept, showing them that it's not just some new math word. OK, what does intercept mean in the real world when you are talking, when you say that word intercept a pass in football or intercept your path? OK, it means the same thing in math, just on a graph context. So, yeah, some are different, but some are the same and having that discussion of where you see these words and where we get them in math.

In this excerpt, Diedre stated that connections and words in mathematics contexts have different meanings than in other contexts. The category Diedre exemplified may be characterized as "connections and words with different meanings in math" and she provided the example of if-then. Xander provided another example, intercept, but shifted the category to similarities in meaning and in doing so drew a contrast with Diedre's example, marked with the word "but" in the first and last sentences of his turn.

We see Xander's contribution as illuminating an issue of theory and practice at play. Xander is someone who tried to develop meaning for mathematical ideas by building on students' prior knowledge from outside of the classroom to make aspects of mathematical language and communication meaningful. These connections can be valuable in a practical sense and are important with respect to developing classroom discourse, yet these connections also emphasize similarities between different contexts rather than differences that help to illuminate particularities of the MR. Having access to such nuances are important for students, especially students who have been historically underrepresented in mathematics classrooms (e.g., see Moschkovich, 2007).

Later Professional Development Sessions

In an interaction two PD sessions later, teachers discussed excerpts from mathematics textbooks related the ideas of area and perimeter. Whereas the previous (and earlier) excerpts showed the teachers focusing primarily on mathematical terminology rather than other particular characteristics of the MR, this excerpt contains the beginning of a more nuanced understanding of the MR. In this case, however, one of the facilitators, Beth, played a central role in the interaction by asking the teachers to reflect on uses of the word *is* in the textbooks they were examining. The uses of *is* in the excerpt is readily articulated by Diedre and other teachers'

contributions indicate that this information is new to them. The topic, however, is not taken up further by the group or elaborated on:

> *Beth:* I was noticing in what you [Diedre] just read [from the textbook], "the base is changed to twice its current measure." And then the other one in the upper right hand, "line segment BF is the perpendicular bisector." How is *is* being used differently in those two [sentences]?
> *Diedre:* Yeah, one's being equal and one's an attribute.
> *Beth:* So they're indicating different relationships between the noun, or what is before *is* and what follows it, which I think could make it [the meaning of *is*] confusing, too.
> *Xander:* That's true.
> *Beth:* Because if I thought base *is* change, that those two [phrases] are the same thing.
> *Xander:* I always preach that *is* means equals.
> *Donna:* I do too.
> *Maggie:* That's very confusing.

Here, Beth pushed participating teachers to consider an example of a characteristic of the MR by re-reading two lines from the textbook a teacher had just read and drawing a contrast between the two. In these lines, there are differences in the uses of the word *is* that are common in the mathematics register but often go unnoticed. Diedre quickly identified the functions for these two uses, but did not quite provide a category for this instance. Beth pointed out that these are "different relationships" between "the noun" and "what follows it," locating the example as being about relational verbs (e.g., *being* and *having* verbs). Such a category might also include the fact that relational verbs are typically more common in the MR than in other kinds of texts. For example, many non-mathematical texts would describe what people are doing, thinking, or talking about, rather than indicating abstract relations between objects, which is common in the MR. Although the teachers did not name the category in this interaction, we see the instance as representing a shift in attention to characteristics of the MR beyond mathematical vocabulary.

Three sessions later, the teachers were asked to analyze new textbook excerpts, following an activity in which they examined textbooks from their own classrooms. They were given a set of questions about the text to consider. Xander reported back to the group by sharing what he noticed about his mathematics textbook. In particular, he compared what his experience was like reading a mathematics textbook as compared to when he reads a novel.

> *Xander:* The interesting thing for me when I was going through my textbook is how I read a novel versus how I read a textbook and then incorporating that with how I teach. When I read a book--, when I'm reading a novel, the narrator's talking to me like I'm having a conversation with the narrator. You know, I'm hearing what they're saying. So when I read a textbook it's almost like you associate it as the teacher's voice saying what you're reading, you know what I mean? Because you could talk to somebody in class but when you read it in a textbook at home and you're trying to think about, what did the teacher say to relate it to? But when I teach stuff up at the board I say a lot of, "You do this," "We do this," "What do we do next." And it's a lot of you's and we's in it. And in the textbook it's none of that. It's all commandments at you. [others laugh] And it's because in the textbook they don't get responses back. And so the textbook is just, "Simplify this," "Next this." It never says "you do this" or "we do this." It doesn't relate it in with the subject like I do when I read a novel, that it relates it back to the characters and in me. Because you do associate the reading. When I read this, I want to have my teacher--, I want to connect it to what my teacher said in class. And I kind of expect that same kind of language, and when I have my teacher saying things

> like "you" and "we" and "us" and all of a sudden there's none of that in the textbook, I kind of see how there's a disconnect there.

In this excerpt, Xander draws both comparisons (i.e., between reading a novel and talking with people such as teachers and students) and contrasts (i.e., between reading a novel or talking with people and reading a mathematics textbook) as he makes sense of certain features of the MR. He also has moved beyond vocabulary to issues of voice in mathematical discourse. Although Xander did not mention the MR by name, he articulated the distanced and authoritative voice described in some other analyses of mathematics textbooks (e.g., Herbel-Eisenmann, 2007). Xander provided specific examples of the grammatical features that contribute to this voice, for example the bald imperatives (*simplify this* versus *we do this*) and lack of human actors that occur in mathematics textbooks. This excerpt provides evidence of an expanding attention to the construct of the MR as a genre rather than particular attention primarily on mathematics vocabulary, as in many of the early PD sessions.

Discussion

In this paper, we extended the exploration of the MR in the context of mathematics classroom discourse and written work to an investigation of how teachers make sense of the idea in the context of a study group focused on mathematics discourse. We do so to set the stage for informed work with teachers on this idea. We found that the process of unpacking the MR was mutually constructed through two particular discourse patterns: providing examples and (sometimes) locating them in categories associated with characteristics of the mathematics register; and comparison and contrast. Each of these discourse patterns appeared throughout the study group and teachers used them to unpack the idea of MR throughout the study group. As they engaged in these discourse practices, we have evidence that the teachers moved from focusing almost exclusively on mathematics vocabulary or terminology to describing and engaging with more nuanced characteristics of the MR.

Although we did not follow the mathematics teachers into their practice to see how they took up the MR ideas, we see this kind of investigation as providing background knowledge for teacher educators in order to work with teachers as they support students in developing facility with the MR. As designers of PD, we have learned about how the teachers made sense of the MR and then interpreted the MR in the context of talking about their practice. In fact, we have revised parts of the PD materials to shape them in terms of these meaning-making themes. For example, we described the kinds of examples and categories teachers might suggest when they first look at student work and suggest ways in which a facilitator may use examples and categories to probe for more nuanced noticings. We have also reflected, through this analysis (e.g., the *is* excerpt and pushing for more nuance), on our roles as facilitators in a discourse community of teachers, which is in many ways analogous to the roles of teachers as they support students with respect to the mathematics register.

As Gibbons (2009) argued, when teachers attend to the mathematics register explicitly and carefully consider how they scaffold language learning, they can provide access to every student because the 'rules' of the game are made apparent to everyone. As teachers engage with ideas from discourse literature, we have found that the ideas shift and change as they then try them out in their classroom (Herbel-Eisenmann, Drake, & Cirillo, 2009). Thus, it is important, as a next step, to continue this investigation by building on the meaning-making discourse patterns the teachers employed as they talked about the MR in order to support their exploration of how to best use ideas from the MR to scaffold their students' learning.

Acknowledgements

This project is supported by the National Science Foundation (grant 0918117). Any opinions, findings, and conclusions or recommendations expressed in this article are those of the authors and do not necessarily reflect the views of NSF. We thank our colleagues and collaborators (Michael Steele, Alexandria Theakston Musselman, Heejoo Suh, Jillian Cavanna, and Kathleen MacAneny) and the teachers who participated in the pilot for their contributions to the project.

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THE TIMING OF TEACHING PRACTICE: TEACHER KNOWLEDGE AND THE CASE FOR CHILDREN'S MATHEMATICAL THINKING

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This study explores the hypothesis that experience with children's mathematical thinking facilitates the development of teachers' professional knowledge. Multilevel models of a data set representative of US public institutions preparing K-6 teachers from the Teacher Education and Development Survey in Mathematics (Brese & Tatto, 2012) suggested that the timing of some teaching practice prior the last year of the preparation program (i.e., concurrent with or preceding some coursework) had a positive effect on prospective teachers' content knowledge and was moderated by the length of teaching practice. Teachers' pedagogical content knowledge was not associated with the timing or length of teaching practice. The findings corroborate prior experimental work at a single institution (Philip et al., 2007) and raise questions about how pedagogical content knowledge develops during teacher education.

Keywords: Teacher Knowledge, Teacher Education-Preservice, Elementary School Education

Recent policy recommendations for teacher preparation in the United States (e.g., National Research Council [NRC], 2010; National Council for the Accreditation of Schools [NCATE], 2010) have focused on the promise of school-based field experiences for producing desired outcomes for preservice teachers. Those recommendations echo an international consensus on the importance of clinical experiences for teacher education (e.g., Musset, 2010; Wang, Coleman, Coley, & Phelps, 2003) and an international trend over the last few decades among teacher educators of increased emphasis on clinical experiences (Maandag, Deinum, Hofman, & Buitink, 2007; Ronfeldt & Reininger, 2012). Policy recommendations call for increased field-based teacher education by scheduling earlier clinical experiences and extending their duration (Goodson, 1993; Villegas-Reimers, 2003). At the same time, descriptive studies of clinical experiences in the United States suggest that they can be poorly aligned with teacher education program goals and that placements in schools can be haphazard, with little university oversight (Wilson, Floden, Ferrini-Mundy, 2001). Some researchers argue that earlier or longer clinical experiences may be ineffective or even detrimental if the quality is poor, for example, by leading to beliefs about mathematics teaching and learning that are inconsistent with university course work (Zeichner & Gore, 1990).

Rather than examining student outcomes directly, this study focuses on teachers' knowledge outcomes. In recognition that student teaching may not have the same effects across the wide range of content areas and grade levels for which teachers are prepared and in response to urgent calls to find ways to improve mathematics teacher education (e.g., National Mathematics Advisory Panel, 2008), this study focuses on the preparation of elementary (grades K–6) teachers to teach mathematics. I hypothesized that *teaching practice that is timed early in the preparation program and thus preceding or concurrent with some content and methods courses will enable otherwise comparable prospective teachers in otherwise comparable programs to develop greater content knowledge and pedagogical content knowledge.*

Framework

Teacher Knowledge

Historically, as measures of mathematics teachers' knowledge have increasingly focused on mathematical knowledge that is used in practice (rather than advanced disciplinary knowledge), the

strength of the observed relationship with student achievement has increased (Hill, Sleep, Lewis, & Ball, 2007). Several measures of teacher knowledge have been framed in light of Shulman's (1986) notion of pedagogical content knowledge, and these measures are more predictive of student achievement in large-scale studies than covariates such as gender, race, and poverty (e.g., Baumert et al., 2010; Hill et al, 2005).

The Length Of Teaching Practice

There is less empirical support for the recommendations for longer student teaching; the relevant research has been primarily descriptive and frequently lacked adequate controls for selection bias. Some studies have provided evidence for positive effects of extended field experiences on teacher outcomes (e.g., Andrew, 1990, Andrew & Schwab, 1995; Silvernail & Costello, 1983). By contrast, large-scale studies (also lacking adequate controls) have compared teachers completing one versus two semesters of student teaching and found no difference in teaching self-efficacy beliefs (Chambers & Hardy, 2005; Spooner, Flowers, Lambert, & Algozzine, 2008). Only two studies of which I am aware estimated pseudo-causal effects for student teaching: Boyd, Grossman, Lankford, Loeb, and Wyckoff (2009) used a robust set of controls and found that estimates of the effect of no student teaching on teachers' value added to student achievement was unstable across models; Ronfeldt and Reininger (2012) used similar controls and concluded that the length of student teaching had no effect on teachers' preparedness to teach, but that the quality of student teaching had significant positive effects.

Children's Mathematical Thinking And The Timing of Teaching Practice

Teachers' experience with children's mathematical thinking is an important theme in the research on teachers' knowledge and beliefs about mathematics and learning, and a key marker of quality for field experiences. A randomized experiment (Philipp et al., 2007) compared the change over a semester-long mathematics content course in the beliefs and content knowledge of prospective teachers who were assigned to guided experiences that focused on children's mathematical thinking with that of prospective teachers assigned to clinical experiences that lacked such a focus. A key design feature was the early timing of the clinical experience to be concurrent with a content course. The authors hypothesized that experience with children's mathematical thinking would promote the prospective teachers' development of beliefs and content knowledge, and they found significant differences between the groups with respect to changes in beliefs and greater (although not statistically significant) increases in content knowledge among the students who focused on children's mathematics. They did not use an instrument to measure pedagogical content knowledge. Silverman and Thompson (2008) have also argued that teachers' experience and knowledge of children's thinking is critical for developing mathematical knowledge for teaching, a domain of teacher knowledge that is closely related to pedagogical content knowledge.

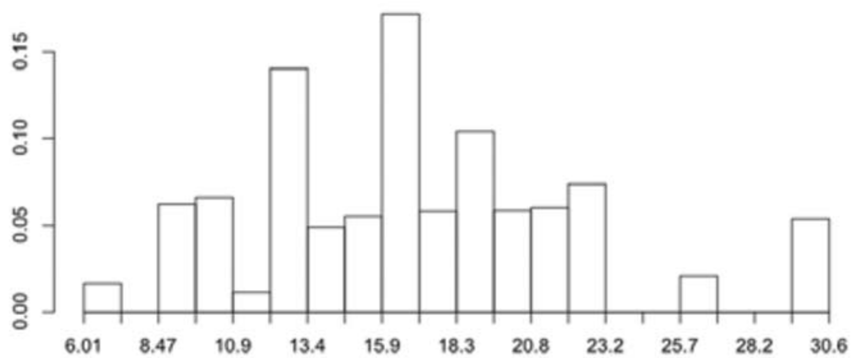
Data

This study used data from prospective teacher and teacher preparation program surveys conducted by the Teacher Education and Development Study in Mathematics (TEDS-M, Brese & Tatto, 2012). The analytic sample was restricted to US teachers who participated in primary (K–6) programs as identified by TEDS-M. The data set included scales of content knowledge and pedagogical content knowledge, and the U.S. sample was designed to be nationally representative of the public institutions that prepare teachers. The large-scale data sets used in prior research have not included teacher knowledge scales and have been restricted to single school districts (Boyd et al., 2009; Ronfeldt & Reininger, 2012) or states (Goldhaber & Liddle, 2011; Harris & Sass, 2007).

The TEDS-M study distinguished introductory field experiences (e.g., observation) from extended teaching practice. All teacher preparation programs reported both kinds of field experiences, but programs varied with respect to the timing of teaching practice, with many reporting teaching practice prior to the final year of the program. For this study, I defined *early teaching*

practice as attending a teacher preparation program that involved teaching practice prior to the final year of preparation (i.e., preceding or concurrent with other preparation activities such as content and methods coursework). About 41% of the US institutions in the sample (20 of the 49) scheduled teaching practice before the final year of the teacher preparation program, but only an estimated 35.7% of prospective teachers experienced early teaching practice after adjusting for the survey sample population weights.

The length of teaching practice was operationalized as the total number of contact hours taken as the product of days of extended teaching practice per year and the corresponding annual estimate of the average number of hours per day. To ease interpretation of the analysis, I divided the number of contact hours by 40 to obtain the number of contact weeks. The length of teaching practice ranged from 240 contact hours (6 weeks) to 1224 contact hours (30.6 weeks). There was a clear peak in the distribution around 16 weeks—approximately 1 semester—see Figure 1. Note that some programs achieved this amount of teaching practice over one calendar semester and others spread 16 weeks of teaching practice over one calendar year or more.



Weeks (40 contact hours)

Figure 1: Histogram Of The Length of Total Teaching Practice

The outcomes for the study were operationalized using IRT scales developed for TEDS-M that are well matched to the underlying constructs and well aligned with the research questions and framework of the study. In particular, the TEDS-M scale for pedagogical content knowledge is explicitly based on Shulman's work (1986).

Methods

Multiple Imputation

The TEDS-M sample for the United States consisted of 1,119 prospective primary school teachers in 49 concurrent U.S. preparation programs operated by public institutions. Four programs (75 teachers) were missing institutional data on teaching practice and these were excluded from the sample, leaving an analytic sample of 1044 prospective teachers in 45 programs. Of these teachers, 25.3% were missing data on the outcome measures. I used the R program Amelia II (which employs a bootstrap expectation-maximization algorithm; see Honaker, King, & Blackwell, 2011) to create 50 imputed data sets for all 1,119 individuals in the TEDS-M sample using 70 individual- and program-level variables. Observed data from the 75 individuals with missing institutional data and who were excluded from the analytic sample nevertheless contributed to the imputation of missing data for other individuals. The missing data on the variables used for imputation ranged from 0% to 28%. Because of the large number of variables in the data set related to the participants' prior experience with mathematics, education, and their teacher education programs, I was confident that the distributions of imputed values for the missing teacher knowledge outcome measures were

reasonable.

Covariates To Mitigate Selection Bias

Selection bias is a concern when comparing teachers in different programs; individual characteristics may be responsible for observed outcomes rather than the teacher education program. Individual and program-level covariates similar to those used in earlier research (Boyd et al., 2009; Ronfeldt & Reininger, 2012) to mitigate bias in estimates of the effects of teaching practice duration and quality are available in the TEDS-M dataset and were used in the proposed analysis. Estimates of the effects of teacher programs can conflate selection into the program with program features (Goldhaber & Liddle, 2011; Harris & Sass, 2007), so I also included program-level variables related to selectivity and individual-level high school achievement to control for self-selection bias. The sets of pretreatment covariates at the program and individual levels also included many of the variables used to predict college choice (e.g., Cabrera & La Nasa, 2000) and teacher knowledge (e.g., Hill, 2010). Thus, these covariates likely reduced bias from omitted variables.

Multilevel Modeling With Survey Weights

The complex sampling design of the TEDS-M data was addressed by using multilevel modeling and by incorporating sampling weights into the analysis. For each outcome variable I used the statistical software MPLUS (Version 6.11 for Mac) to estimate a multilevel model (prospective teachers nested within preparation programs) across the 50 imputed data sets with standard errors for testing individual regression coefficients. Weights should not be used without appropriate scaling because unscaled weights can bias estimates (Carle, 2009). Both scaling methods recommended by Carle (cluster sample size and effective cluster sample size) were available in MPLUS, and I used both methods and compared the results. I also ran the analyses without weights. The results across all three methods were very consistent with each other, and I report results from the cluster sample size method.

The multilevel model for this study was adapted from VanderWeele (2008) and is appropriate for estimating neighborhood effects—effects at the program rather than individual level. This model accommodates the expected homogeneity among prospective teachers in the same program (Gelman & Hill, 2007). The first equation in Figure 2 expresses the individual level of the model. The model predicts the outcome *Y* (prospective teachers' content knowledge or pedagogical content knowledge) with *i* indexing individuals and *j* indexing programs. The matrix *X* represents the individual-level covariates for individual *i* in program *j* (see Table 1); and e_{ij} is the random error term associated with individual *i* in program *j*. The vector of coefficients β_1 are estimated by fitting the model to the observed data; these terms provide estimates of the relationships between these variables and the outcome *Y*. The remaining term in the first equation is the intercept term μ_j. It represents the average outcome for each program after accounting for differences in the individual level covariates.

$$Y_{ij} = \mu_j + \beta_1 X_{ij} + e_{ij}$$
$$\mu_j = \alpha + \gamma_1 T_j + \gamma_2 L_j + \gamma_3 T_j L_j + \beta_1 Z_j + u_j$$
$$e_{ij} \sim N(0,\sigma_1);\ u_j \sim N(0,\sigma_2)$$

Figure 2: Multilevel Model For Neighborhood Effects.

The second equation in Figure 2 predicts μ_j, the mean outcome of program *j* using program-level variables. The variable α is the overall mean outcome. The binary indicator variable T_j represents whether program *j* has early teaching practice; the variable L_j represents the length of total teaching practice in program *j*; and the interaction term T_jL_j expresses the possibility of an increase in the mean program outcome for timing and length beyond that accounted for by each variable independently. The matrix *Z* represents the program level covariates (see Table 1). Finally, u_{ij} is the random error term associated with program *j*. A fitted model provides estimates of the coefficients γ_1, γ_2, γ_3 and of the coefficient vectors β_1 and β_2. The last line of Figure 2 indicates the assumption that

the random error terms at each level are normally distributed.

Results

After estimating the full model for both outcomes (content knowledge and pedagogical content knowledge), I compared the results with the corresponding null models (i.e., no individual or program level predictors, just random program level intercepts). The focus of the analysis for this study was on program level differences in outcomes, and I found that the full models explained a large portion of the variance in outcomes at the program level in addition to a moderate portion of the variance in outcomes at the individual level. The full model for content knowledge explained 8% of the individual level variance and 71% of the program level variance that was not explained by the null model, and the full model for pedagogical content knowledge explained 6% of the individual level variance and 74% of the program level variance. Thus, these models explained to a large degree how observed program outcomes differed in relation to program characteristics including the focal variables describing teaching practice.

The estimated coefficients for the full models are reported in Table 1. Overall, the covariates predicting knowledge outcomes appeared to have functioned as expected to mitigate selection bias. Many of the program and individual level covariates in models are significant, and all significant predictors have the expected sign. At the individual level, for example, socioeconomic status and secondary school achievement were statically significant and positive predictors of each outcome as expected. Results also reflected a substantial gender gap in content knowledge, with men (n = 111) more knowledgeable.

Very different patterns of significant coefficients linked the development of content and pedagogical content knowledge to different features of teacher preparation programs, and provided evidence of different pathways for these two knowledge outcomes. Prospective teachers were asked how many topics related to continuity and functions they had studied in their tertiary coursework (e.g., limits, sequences, derivatives), and the TEDS-M data set included a Rasch scale based on 5 such items. The tertiary mathematics variable was statistically significant and positively related to content knowledge but not significant in the model of pedagogical content knowledge. The TEDS-M data set also included a Rasch rating scale of program coherence based on 6 rating-of-agreement items (e.g., "Later courses in the program built on what was taught in earlier courses in the program.") The program coherence variable as well as the number of mathematics and mathematics pedagogy classes in the program were statistically significant and positively related to pedagogical content knowledge, but were not significant in the model of content knowledge.

Table 1. Models of Content Knowledge (CK) and Pedagogical Content Knowledge (PCK)

Term	Model 1: CK *B (SE)*	Model 2: PCK *B (SE)*
Intercept	510.08 (5.13) ***	537.30 (4.13) ***
X – Individual level covariates		
Age (years)	-0.04 (0.40)	7.10 (8.02)
Gender (male)	25.29 (8.38) **	-0.64 (0.36)
SES (e.g., mothers' education, number of book)	2.03 (0.99) *	2.28 (1.14) *
Hindering circumstances (e.g., need loans, need to work)	-0.51 (2.80)	-1.35 (3.06)
Secondary school achievement	14.40 (2.11) ***	11.96 (2.51) ***
Tertiary math topics studied - continuity & functions	5.42 (1.83) **	2.87 (2.09)
Coherence of preparation program	1.19 (1.06)	2.73 (1.04) **

Z – Program level covariates		
Length of program (years)	1.83 (5.35)	4.18 (3.78)
Number of math & math pedagogy classes	-0.25 (2.14)	8.08 (3.25) *
Average secondary school achievement of students	13.27 (4.17) **	1.90 (2.20)
Importance of standardized tests for selection	3.84 (2.81)	-1.66 (1.41)
Teaching practice variables		
T – Early timing (0 or 1)	13.65 (10.23)	6.67 (7.13)
L – Length (40-hr contact weeks)	2.62 (0.93) **	0.62 (0.59)
T × **L** – Timing & length interaction	-3.80 (1.53) *	-

* $p < .05$; ** $p < .01$; *** $p < .001$.

I next examined the estimated coefficients of principal interest: those corresponding to the length and timing of teaching practice. The most notable result from Model 1 (content knowledge) pertaining to teaching practice was the significant interaction term for timing and length of teaching practice (see Table 1). The length of the teaching practice moderated the effect of early timing on the teachers' content knowledge. Using the simple slope method (Preacher, Curran, & Bauer, 2006), I found the region of significance (α = .05). Early teaching practice had a statistically significant effect when the length of total teaching practice was less than 13.7 contact weeks or more than 28.1 contact weeks. Very little data were in the upper region of significance, but the lower region of significance was informative because that region included all programs with short teaching practice—more than 30% of the sample. For these programs, the shorter the length of teaching practice in the program, the greater the estimated effect of early timing of teaching practice on content knowledge. The estimated effect ranged between 54 points at 5 weeks of teaching practice (2 *SD* below the median length) and 14 points at 16 weeks of teaching practice (the median length). Thus, early timing of teaching practice had an average effect size of approximately .30 *SD* in the region of significance.

By contrast, none of the predictors related to teaching practice in Model 2 (pedagogical content knowledge) were significant, meaning that the preparation programs achieved similar outcomes with respect to pedagogical content knowledge regardless of the features of student teaching after controlling for other covariates. This result was surprising because pedagogical content knowledge—even more than the content knowledge—was hypothesized to develop in the context of teaching practice. A possible explanation is that the preparation programs did little to influence the prospective teachers' pedagogical content knowledge because the classes and teaching practice had not been designed for that outcome. Given the widely cited research about the deficits in elementary teachers' mathematical knowledge (e.g., Ma, 1999) and the large number of other content areas that K-6 teachers must be prepared to teach, mathematics teacher educators may spend the available time focused on content knowledge instead.

Discussion

This study provided evidence of the critical role that experience with children's thinking may play in the development of teachers' knowledge. In particular, the finding that early teaching practice has a significant and positive relationship with teacher knowledge in programs with less than 13.7 contact weeks of teaching practice echoed the relative knowledge gains of prospective teachers who focused on children's thinking (Philip et al., 2007). One limitation of this study is that the teaching practice was not directly observed, whereas explicit discussion of student thinking was one of the variables that was manipulated experimentally in Philip et al.'s (2007) study. The inferences in the present study are based on the assumption that teaching practice includes opportunities to attend to student thinking. Yet in spite of this limitation, the signal from these data remains clear, and the

findings confirm (on a national scale) the hypothesized relationship between the experience of children's thinking and teachers' content knowledge.

Philip et al. (2007) hypothesized that the early experience with children's mathematical thinking would have an even stronger effect on pedagogical content knowledge than on content knowledge (pp. 468-469). Instead, the results of this study showed that the timing and length of teaching practice was not related to pedagogical content knowledge, and they challenge mathematics education researchers to provide a clearer theoretical account of how teacher knowledge develops in teacher education programs.

Acknowledgements

This research was supported by a grant from the American Educational Research Association which receives funds for its "AERA Grants Program" from the National Science Foundation under Grant #DRL-0941014. Opinions reflect those of the author and do not necessarily reflect those of the granting agencies.

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TALKING ABOUT PEDAGOGY, STUDENTS AND MATHEMATICS

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When student teachers and cooperating teachers have conversations during the student teaching experience, research indicates that they spend considerable time talking about classroom management. In an effort to focus student teaching on the practice of eliciting, understanding and using student mathematical thinking, we altered the structure of student teaching. In this altered student teaching experience, the frequency of conversations involving mathematics was significant (almost 60%) and the frequency of those involving classroom management was correspondingly quite small (about 4%). The nature of the mathematical conversations focused on how students were making sense of the mathematics and how teaching facilitated that process. The management conversations that did occur also focused on strategies that could facilitate students' mathematics learning.

Keywords: Mathematical Knowledge for Teaching, Teacher Education-Preservice, Teacher Knowledge

The conversations that take place between student teachers (STs) and their cooperating teachers (CTs) constitute a significant portion of the "enacted curriculum" of student teaching. As explicit occasions for reflection and collaboration, these conversations represent much of what STs have the opportunity to learn. Research has shown, however, that these ST/CT conversation sometimes focus more on classroom management than on mathematics and facilitating student mathematical learning (Peterson & Williams, 2008). As one CT put it,

> I still don't think that the challenge of teaching mathematics, at least at the junior high level, is mathematics. It's not having a better understanding of what we need to teach. I think its understanding how to control them, and I feel that is the key to helping the mathematics at this level in the junior high. (Peterson & Williams, 2008, p. 470)

The traditional structure of student teaching actually seems to support such a focus on classroom management (Leatham & Peterson, 2010a). CTs often decide much of the curriculum of student teaching, and they tend to view the purpose of student teaching as STs experiencing real classrooms with real student behavior (Leatham & Peterson, 2010b). This paper presents results from a study that investigated ST and CT discussions in a nontraditional student teaching structure—one that had been purposefully designed to change the focus of student teaching away from students' behavior and onto students' mathematics (Leatham & Peterson, 2010a). As part of this structure, STs and CTs observed each other teach and then participated in post-lesson reflection meetings. We analyzed these reflection meetings to see how much and in what ways these participants talked about mathematics, students, pedagogy and classroom management. The results of this analysis show that STs and CTs are indeed capable of talking a great deal less about classroom management and a great deal more about mathematics and their students' mathematics. We also show that the nature of these conversations better aligns with mathematics teacher educator expectations and desires with respect to STs' opportunities to learn.

Theoretical Framework

We view mathematics teaching as the purposeful facilitation of student mathematics learning. For us the essence of mathematics teaching lies in the interactions between teachers, students and mathematics (cf. Cohen & Ball, 2001). As teachers and students interact around mathematical ideas, students provide evidence of their current understandings and teachers seek to provide experiences that are likely to help students build on those understanding. We thus see as a foundational teaching practice the elicitation, understanding and use of students' mathematical thinking. As described previously, we designed the student teaching structure under examination with the express purpose of facilitating STs' understanding and development of this practice.

In defining *mathematics teaching* in this way we do not mean to discount the myriad other responsibilities mathematics teachers have as part of their job. As we have discussed elsewhere (Leatham & Peterson, 2010b), we have found the metaphor of a shoe store apprentice useful when distinguishing among these aspects of a teacher's work. There are two main arenas of knowledge an apprentice cobbler needs to learn: 1) how to make shoes and 2) how to run a shoe store. Similarly, an apprentice teacher needs to learn 1) how to facilitate student learning and 2) how to manage a classroom. Although in each case the apprentice needs to learn both aspects of the job, the former is far more important in general and, we would argue, should take precedence over the latter. What good is having a well-run shoe store if you cannot make quality shoes? What good is creating a well-run classroom, if you cannot facilitate student learning? For us, "teaching" is facilitating students' learning (making shoes), not running the classroom (running the shoe store). Thus our emphasis in moving the conversations during student teaching away from classroom management toward facilitating student mathematics learning is to focus teachers on what we believe to be the most important aspect of teaching.

We find *opportunity to learn* a useful construct for describing the enacted curriculum in a given classroom—the content students have the opportunity to learn as evidenced by the content with which they interact and the ways in which they do so (Schmidt, Cogan, & Houang, 2011). The content of student teaching for our study is "mathematics teaching" and STs interact with this content through observation, teaching, discussion and reflection. We view the content of these activities as the content STs had the opportunity to learn. In this paper we do not seek to provide evidence of what STs actually learned. Rather we seek to provide evidence of what they had the opportunity to learn. Our intent is thus to explore in what ways a particular structure of student teaching seems to provide the opportunity for STs to learn the practice of eliciting, understanding and using student mathematical thinking.

Methods

In order to provide an appropriate background for this study, we first described the setting—in this case the structure of the students teaching experience in which these STs and CTs were participating. We then discuss the how the data were collected and analyzed.

Structure of Student Teaching

As mentioned earlier, we defined the primary purpose of student teaching to be learning how to elicit, understand and use students' mathematics thinking. We then reorganized the student teaching experience so that the structure would support this purpose. See Leatham and Peterson (2010a) for greater detail on the structure and the rationale behind it. The primary facets of the structure are as follows: Two STs were paired with a single CT. Two or three pairs of STs at neighboring schools (consisting of at least one high school and at least one middle or junior high

school) were joined to form clusters. During the second week of the 15-week experience each CT taught a lesson for all the STs in the cluster to observe. The university supervisor (US) also observed this lesson. After the lesson the STs, CT and US participated in a reflection meeting (similar to those held in a lesson study model). During each of the next three or four weeks each pair of STs prepared a single lesson. On the assigned day the pair of STs each taught their lesson to a different class while the US, CT and the other STs in the cluster observed. Again they all held a reflection meeting soon after the second lesson. During these first five weeks the STs also participated in other learning to teach activities that focused their attention on students' mathematics—conducting focused teaching observations, writing reflection papers, interviewing students, and prompted journaling. During weeks 6-13 the pairs of STs took primary responsibility for the CTs' classes, each taking on about half of the CT's regular teaching load. During week 14 the STs turned their classes back over to the CT, again prepared just a single lesson that was observed by all in the cluster, and held associated reflection meetings.

Each reflection meeting was structured in a similar way. In general, one observing ST facilitated the meeting by first asking the teaching STs to describe the mathematical goal of their lesson, how the lesson was designed to accomplish that goal, and how they felt it played out in their two classes. The STs then asked each other questions and shared comments about the lessons. After this conversation seemed to have run its course then the facilitator invited the CT to share their questions and comments, then the US was invited to do the same. Although the CT and US could participate in the earlier discussions, they were encouraged to wait until the end of the meeting to initiate conversation.

Participants and Data Collection

One cluster of six STs student taught under this structure during the Fall 2006 semester, with one pair at a high school and two pair at neighboring junior high schools. Eight STs participated during the Fall 2007 semester. They were divided into two clusters, each consisting of one pair at a high school and a second pair at a neighboring junior high or middle school. The participants in this study include these 14 STs, their 6 CTs (one of the teachers was a CT both years) and the three USs (one for each cluster). The 2006 cluster held 5 early reflection meetings and a sixth near the end of the semester. Each of the 2007 clusters held four early reflection meetings and a fifth near the end of the semester. We video recorded each of these 38 reflection meetings.

Data Analysis

We transcribed each of the 38 reflection meetings and then coded the transcripts according to the content of the conversations. Because the designed purpose of the student teaching experience was to learn how to elicit, understand and use students' mathematical thinking, and the structure was designed to facilitate that focus and lessen focus on classroom management, we coded for these fundamental aspects. That is, we looked for evidence of participants talking about pedagogy (P), mathematics (M), students (S), and behavior (B) (the latter being the way we tried to capture "classroom management"). Statements were coded as P if they referred in any way to the actions or words a teacher might or did say. Any reference to specific mathematical language, procedures or concepts was coded as M. Reference to students in general or by name were coded as S. Statements were coded as B if they referenced behavioral expectations or problems or referenced school-related issues outside of or tangential to the act of teaching. It is important to note that all B codes were applied as additional codes to statements that already had some combination of the PSM codes.

Our unit of analysis was an idea unit, which was typically comprised of one or two sentences. Thus, throughout the data set, we basically coded each sentence for whether it referenced in

some way these four basic themes. Multiple researchers coded trial transcripts until intercoder reliability reached 80%, at which point each transcript was coded by a single researcher.

Results

Approximately 11,000 statements across the 38 reflection meetings received some combination of the PSM codes. Approximately 4% of these statements also received a B code. Thus, on average there were about 12 statements related to classroom management in a given reflection meeting, which on average contained almost 290 statements. These broad stroke findings suggest that the focus of these conversations was *not* classroom management. To further examine how discussions of classroom management did and did not occur in these discussions we show the distribution of the behavior codes across the PSM codes (see Table 1).

Table 1: Distribution of Behavior Codes across the PSM Codes

	% of all comments	% of these statements coded as Behavior	% of Behavior-coded statements coded here
Pedagogy	17.9	2.4	12.3
Students	6.9	18.1	36.0
Mathematics	6.5	0.0	0.0
PS	16.1	10.2	47.0
SM	15.9	0.3	1.3
PM	19.8	0.1	0.5
PSM	17.0	0.6	2.9

Behavior and mathematics are seldom combined as a focus. There are no instances of M-coded statements also coded as behavior and very few among the other M-related statements. Taken together, less than 5% of all B-coded statements also received an M code. When one considers the nature of mathematical conversation, perhaps this is not surprising. But the literature has repeatedly found that STs and CTs *do* talk about behavior and *do not* talk about mathematics. These results provide some explanation as to why—the two topics do not often go together. When STs, CTs and USs focus on students' mathematics they talk a lot about mathematics and not as much about classroom management.

Almost half of the B-coded statements are coded as PS, with the remaining half going primarily to S (about 1/3) and the rest to P (about 1/6). So, STs and CTs are more likely to talk about behavior when they are talking about both students and pedagogy and less likely when talking about simply pedagogy (although such statements exist). Also of interest—it is the B-coded statements that were coded as S and PS where there are significantly more B codes than the normal percentage. That is, although only 6% of all statements were S-coded, 36% of the B-coded statements were S-coded. Similarly, B-coded statements were much more likely to be PS-coded than were statements in general (47% compared to 14%). The middle column in Table 1 further illustrates this pattern. B-coded statements make up about 18% of the S-coded statements and about 10% of the PS-coded statements. B-coded statements make up a very small portion of any of the other categories. Discussions that focus solely on students or students and pedagogy are much more likely to focus on classroom management.

Almost 60% of all statements that received some combinations of the PSM codes were mathematical in nature (combining codes M, SM, PM and PSM). Thus the participants in these reflection meetings were directly talking about mathematics in a majority of their conversations.

We now discuss what it looks like for STs and CTs to talk about mathematics. We present these results for two main reasons. First, given that the literature seldom reports significant mathematics conversations between STs and CTs, it is important for the field to get a feel for the kinds of mathematical conversations that can indeed happen. Second, we end this section by discussing statements that involve both mathematics and behavior. Although rare, these examples illustrate how even the nature of discussions about behavior can be altered when the underlying focus of conversations is students' mathematics.

Although the frequency of individual PSM codes provides some general indication of *what* reflection meeting conversations focused on, the individual statements, taken out of context, do little to help one understand the nature of these conversations. We have found it helpful to look at excerpts of conversations and then to discuss how the various statements (and their associated PSM codes) help us to capture the nature of the overall conversation.

The following excerpt is taken from a pre-algebra lesson toward the end of the 2006 student teaching experience. The goal for the lesson was to formalize the rule that "adding a negative is the same as subtracting a positive" (ST Emily). This excerpt contains examples of all of the PSM codes that involved mathematics—M, SM, PM and PSM. (The codes are included in the transcript following each statement.) ST Christina and ST Emily planned together and taught individually the lessons that were observed. As was often the case in these reflection meetings, this discussion begins with a statement about students' mathematics. (Note: It is helpful to know that the student thinking under consideration in this excerpt actually came up in both classes—by a girl in ST Christina's class and by a boy in ST Emily's class.]

ST Ashley: When you did your True/False questions at the beginning—I don't know whose class it was; I think it was yours [points to ST Christina]—one of the students came up and it was like 15 = 7 + 8. She came up and rewrote it 7 + 8 = 15. (SM) Is there a reason you didn't focus on that? I mean, did you want to make a statement about that at all, because I think that's an interesting idea that a student would bring up: you can just switch the things on the equals sign and it means the same thing. But then also that she has to rewrite it for her to make sense. Does that make sense? (PSM)

ST Christina: Yeah, no, I thought that her rewriting it just solidified even more that these are both the same thing. (PSM) I don't know, maybe I didn't understand that she was thinking that you had to rewrite it. (PSM) Is that what you were thinking, that she thought that you had to rewrite it to make sense? (SM)

ST Megan: I was thinking more about [the] commutative property that they did in the one right before—that they were just building on that. (PSM)

ST Jennifer: It's not the commutative property though; it's the property of the equals sign. (M)

ST Ashley: It's the property of equality. (M)

ST Jennifer: Equality. (M)

ST Megan: What am I thinking? (M)

ST Christina: I think the commutative was before that—it was the 3 + 4 = 4 + 3. (M)

ST Megan: Right, and that's what she did. She switched it because she knew that adding could commute and give you the same answer, right? Am I doing that wrong? (SM)

US Karl: Well, what you're talking about is moving something from left and right to right and left across the equals sign, and the commutative property is the across the operation. (M)

ST Megan: Well, I can see where they're coming from, thinking that when you've just talked about the commutative property. (SM)

ST Christina: Oh, right.
ST Jennifer: I think his understanding was the fact that, "Okay, I work from left to right." You know, "I do the operations on the left and then I write them on the right side of the equals sign." (SM) So I think that's where his understanding was. It's like, "I have to have the 8 + 7 on the left first and then it equals 15". (SM)
ST Emily: And didn't I ask—I said, "Is 7 + 8 = 15 the same thing as 15 = 7 + 8?" But I don't know if I really got them going through the discussion. (PM)
ST Christina: Well, I think the simple fact that they rewrite and they say it's true—that they understand that it's equivalent already. (SM)
ST Jennifer: Have you emphasized the fact that—. (PM)
ST Christina: No, we haven't. Like, we just barely did commutative and equalities now. (PM) And I thought about, "Should I state that?" But it was not a focus, a goal we had for the day, so I didn't state it. (PM)

ST Ashley begins her contribution to the discussion by presenting a particular instance of student mathematics that she noted during the day's lesson (coded SM). Her purpose in presenting the student' mathematics, however, was to pose a question about a pedagogical decision ST Christina made with respect to that student mathematics (coded PSM). ST Christina responds by first answering the question—explaining how she interpreted the students' mathematics in the moment (coded as PSM)—then questioning that interpretation (coded as PSM) and finally asking ST Ashley if she had correctly understood her interpretation of the student's mathematics (coded as SM). This return to talking about the students' mathematics prompts a mathematical discussion about the mathematics embodied in the student's actions. The next few turns focus on making sense of the mathematics itself, no longer in reference to the student (coded as M). ST Megan then ties the mathematical conversation back to the student's mathematical work. At this point US Karl chimes in to give his take on the two different properties that are under discussion—the symmetric property of equality and the commutative property of addition. ST Megan indicates that she sees how students might confuse the two properties (SM). It is in this context that ST Jennifer redirects the conversation back to the original analysis of ST Ashley—namely that this student might think that one actually *should* write the addends on the left and the sum on the right, an analysis that is related to the student not fully understanding (or at least appreciating) the symmetric property of equality. ST Emily then analyzes her own teacher move when a student did the same thing in her class (coded as PM). ST Christina provides her own interpretation of the student's mathematics (coded as SM), although there seems to be an implied pedagogical purpose in her statement—that ST Emily's teacher move would be justified with this particular interpretation of the student's thinking. ST Jennifer raises the question of whether that particular aspect of equality had been emphasized in their class (coded as PM), which seems to imply that ST Christina may have been justified in not pursuing this student's mathematical thinking. ST Christina follows up by explaining that these ideas are new and that she had not intended for them to be a focus of that day's lesson (coded as PM), providing yet another rationale for not pursuing the thinking during her lesson.

The SM statement that began this discussion provided inroads to discuss mathematical ideas (e.g., the symmetric property of equality and the commutative property of addition), students' mathematics (e.g., possible confusion about the symmetric property of equality), and pedagogical issues related to students' mathematics (e.g., why one would or would not pursue this particular students' rewriting of a number sentence). In this portion of the reflection meeting the CT and US were mostly silent, allowing the STs to work through their ideas. These ideas,

however, were on the table now and could become fodder for discussion later in the meeting. The purpose of this paper, however, is not to analyze what the STs did or did not learn in these conversations. Rather, the purpose is to describe the nature of these conversations as a means of illustrating the content of the experience—what these STs had the opportunity to learn. We posit that conversations such as these gave the STs opportunities to learn mathematics, to learn about the nature of students' mathematical thinking, and to learn about effectively using students' mathematical thinking for mathematically-relevant pedagogical purposes. As mathematics teacher educators, these are the kinds of opportunities to learn we want for our STs.

We conclude this discussion by illustrating what behavior-related conversations can look like when they occur in the context of focusing on students' mathematics. In this meeting one of the 2007 clusters of STs was reflecting on lessons planned and taught by ST Katie and ST Jake.

ST Jane: For [the] Tony's pizza [problem], ST Jake kind of had them work on the equation in a group, whereas you did it as a whole class. And I was just wondering—. (PSM)

ST Katie: Our plan was to have them do it as a group, but again, to me, it seemed like when they go into their groups, it takes a long time to get them back, and I was already pressed for time. (PS, B)

ST Jake: And there wasn't really a big diversity in what students came up with. They were all pretty much the same. (SM)

ST Jake: We kind of talked about it in between [lessons] and said, "You know what, if we just let them go, most of them are going to be right on anyways." (PSM)

ST Katie: So that's why I figured I could probably do it as a class, because we had just done a problem where they had written the equation, so I thought that maybe as a class, they could see this, and most of them would get it anyways. (PSM) So it was just—. For me, it was just another way to save time. It takes too much time to get their attention back, I think. (P, B)

This discussion begins with ST Jane questioning a difference in how ST Katie and ST Jake engaged their students in working on a particular mathematics problem (coded as PSM). ST Jake taught the lesson first and, as planned, had the students work on the problem in groups. All of the STs, including ST Katie, had observed that lesson. When it came time for ST Katie to teach the lesson, she altered the plan and solved the problem in a whole-class discussion. ST Katie explains that student behavior is often an issue when it comes to transitioning from group work back to whole class discussion and that she did not feel that she had time to deal with those behavior issues (coded as PS, B). ST Jake adds that the two of them had discussed the lack of diversity in the students' mathematics that had emerged in the group work in his class (coded as SM) and thus that they might get the same mathematical thinking out there through a whole class discussion (coded as PSM). ST Katie then expounds on this rationale, adding that the sequence of tasks seemed to be a contributing factor to the lack of diversity in solving the pizza problem (coded as PSM), then returns to her original behavior-based justification (coded as P, B).

Although the actual student mathematics is not visible in this discussion, clearly the STs consider students' mathematics as they discuss these lesson variations. The STs are able to discuss students' behavior (and how to deal with it) while connecting it to concerns about the mathematics learning. The point we wish to make here is that the focus on students' mathematics not only increases the frequency of mathematics related conversations, but also alters conversations about student behavior to be less about "controlling" students in general and more about managing that behavior for more productive mathematical discussions. These STs had the opportunity to learn about effective ways to elicit, understand and use students' mathematical thinking in addition to learning how classroom management might influence this practice.

Conclusion

Mathematics-related conversations among STs and between STs, CTs and USs were prevalent in this student teaching structure, whereas classroom management conversations were infrequent and were typically related to facilitating students' mathematics learning. These results provide evidence of movement toward the goal of making the student teaching experience more about making shoes and less about running the shoe store. More important than the frequency of mathematical conversations is the nature of them—conversations that, as mathematics teacher educators, we long for our STs and CTs to have.

Several aspects of our student teaching structure seemed to contribute to this prevalence of mathematics-related conversations. First, the presence of multiple observers lessened the likelihood of classroom management issues arising during the lessons, providing space for reflection on other topics. Second, the STs were engaged on a daily basis with learning-to-teach activities that focused their attention on student mathematical thinking. It seems likely that this focus carried over into these reflection meetings. And finally, the structure of the reflection meeting itself, wherein the leading questions asked the STs to reflect on the goals of the lesson and how the lesson tasks were designed to accomplish those goals of the task, seemed to set the stage for a conversation that began and remained focused on student mathematical thinking.

Because of how seldom mathematics-related conversations seem to take place during student teaching, STs and CTs may not understand just what such conversations might entail. Peterson and Williams (2008) reported that one ST in their study "felt that because he taught comparatively low level mathematics (Algebra and Pre-Algebra), he wasn't learning anything about mathematics, nor did he and [his CT] need to discuss it" (p. 470). He explained, "Especially since we teach pre-algebra and algebra, we rarely talk about mathematics content" (p. 470). This ST did not seem to have a clear image of how mathematics teachers can productively talk about students' mathematics. The present study demonstrates that conversations about how students were making sense of the mathematics and about how a teacher might help students make mathematical connections, regardless of the level of mathematics being taught, can be very productive. As STs and CTs discuss their students' mathematics they come to realize that such conversations are not a rehashing of "simple" mathematical procedures but rather involve in-depth analysis of mathematical connections and sense that students are or could be making. Such conversations also tend to lead to mathematical conversations that provide the opportunity for STs, CTs and USs to deepen their own mathematical understanding. In our opinion, such analysis and conversation is at the heart of learning to teach and should be an integral part of every student teaching experience, thus providing STs the opportunity to learn not just how to run the shoe store, but how to make shoes.

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PRE-SERVICE TEACHERS' MATHEMATICAL HORIZON: THE CASE OF AN IRREGULAR HEXAGON

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This paper analyses pre-service secondary mathematics teachers' responses to a hypothetical situation concerning a student's struggle to determine the area of an irregular hexagon. The analysis focuses on individuals' Knowledge at the Mathematical Horizon, as it relates to philosophical notions of inner and outer horizons. Specifically, the paper attends to common features in participants' understanding of the mathematical task, and explores the interplay between their personal solving strategies and professed preferences when advising a student. Implications for teacher education and further avenues of research are suggested.

Keywords: Mathematical Knowledge for Teaching, Teacher Education-Preservice, Geometry

This research contributes to on-going conversations about teacher knowledge and the interplay between knowledge of mathematics and pedagogical decisions. It focuses on pre-service teachers' responses to a learning situation regarding a hypothetical student's struggle to determine the area of an irregular hexagon. The aim of the study was to investigate the connection between participants' personal strategies and preferences for solving a novel (for them) problem and their expectations and recommendations for student learning. Specifically, the paper considers the question: *What are pre-service secondary mathematics teachers' preferences when considering recommendations for how to determine the area of an irregular hexagon, and what are the bases for these preferences?*

Participant responses were analysed using the construct of Knowledge at the Mathematical Horizon (KMH), which was introduced by Ball and colleagues (e.g. 2008, 2009), and extended by Zazkis and Mamolo (2011). The analysis intends to shed new light on how different facets of KMH may manifest in a pre-service teachers' address of a teaching situation, and what consequences this might have for student learning. The paper concludes with pedagogical implications for pre-service education, as well as suggestions for future avenues of research.

Survey of Literature on Teacher Knowledge

Knowledge required for teaching mathematics has been widely discussed from a variety of perspectives. Attention has focused on what knowledge is required in teaching, for teaching, and of teachers (e.g. Ball, Thames, & Phelps, 2008; Davis & Simmt, 2006). Shulman's (1986) classic categorization of Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK) made an early distinction between teachers' knowledge of mathematics and knowledge of student learning, while more recent studies have sought to both refine these categories and connect them. For instance, Ball, et al. (2008) elaborate that SMK is knowledge required by teachers for the tasks of teaching, "which require a host of other mathematical knowledge and skills - knowledge and skills not typically taught to teachers in the course of their formal mathematics preparation" (p.402). They refined Shulman's categories by distinguishing between (for example) knowledge of subject matter specific to teaching (such as how to explain rules and procedures) and knowledge of subject matter that is common to other uses of mathematics. They

also identify as necessary aspects of teachers' PCK: a familiarity with students' ways of thinking, as well as knowledge of effective examples or teaching sequences.

While this research was on distinguishing facets of teacher knowledge, Ball et al. (2008) acknowledge there are interconnected relationships. They suggested that "Teachers who do not themselves know a subject well are not likely to have the [pedagogical content] knowledge they need to help students learn this content" (2008, p.404). Similarly, Potari et al. (2007) observed that robust subject matter knowledge allowed teachers to interpret and develop student ideas with greater ease and effectiveness. They also suggested that teachers' ability to connect different mathematical areas and their awareness of the relevance of these connections were part and parcel to their ability to effectively integrate SMK and PCK to create a rich mathematical learning environment. Ball, Lubienski & Mewborn (2001) highlighted that: "It is not only what mathematics teachers know but also how they know it, and what they are able to mobilize mathematically in the course of teaching" (p.451).

Extending on these studies, this research considers pre-service teachers' address of a student's struggle to solve a non-routine problem, which can be addressed in a variety of ways with connections to core concepts in secondary school curricula. Participants' SMK is analysed via the sub-category Knowledge at the Mathematical Horizon, which is discussed in the following section. The intent is to offer a refined look at how this specific aspect of individuals' SMK can manifest in, and influence, a teaching situation.

Theoretical Framework

One of the facets of teachers' Subject Matter Knowledge, as introduced by Ball and Bass (2009) is Knowledge at the Mathematical Horizon (KMH), which is described as a structural, connected, and robust understanding of mathematics that goes beyond what is taught in school curricula. Ball and Bass present KMH as a teacher's knowledge of students' mathematical horizon, while other perspectives, such as that of Zazkis and Mamolo (2011), look to teachers' horizon. This study also attends to teachers' horizon and what lies "in and out of focus" as they consider the mathematics in a hypothetical teaching situation.

In line with the description of horizon knowledge as connected, robust, and beyond school curricula, Zazkis and Mamolo (2011) extend the construct of KMH to focus on teachers' horizon by connecting it to Husserl's philosophical notion of a (conceptual) object's horizon. Husserl's description of horizon relates to an individual's focus of attention – in particular, when an individual attends to an object (conceptual or physical), the focus of attention centers on the object itself, while the 'rest of the world' lies in the periphery (Follesdal, 2003). With this perspective, a teacher's horizon knowledge is dependent on the specific mathematical object under consideration – how that object is understood, what aspects of the object lie in focus or in the periphery, and what connections the teacher is able to make between the in-focus and peripheral facets. What lies in the periphery is in Husserl's perspective the object's horizon. Thus, Zazkis and Mamolo (2011) interpret teachers' horizon knowledge with respect to a specific mathematical object as knowledge of that object's horizon, which according to Husserl can be partitioned into an *inner* and an *outer* horizon.

Husserl's partition allows an analysis of different features in an object's periphery. An object's inner horizon refers to specific attributes of the object itself which are not (at that moment in time) in focus for the individual. With respect to a mathematical object or entity (say, a polygon), these attributes might relate to the polygon's symmetry or colour or area, if what is in focus for the individual is the polygon's number of edges and vertices, for example. Thus there is a reflexive relationship between what lies in the inner horizon of an object and what lies

in focus for an individual. In contrast, an object's outer horizon refers to the "greater world" in which the object exists, and as such is not dependent on the individual's focus of attention. The outer horizon includes features which embed the object in a greater structure, and it consists of generalities which are exemplified by the particular object. For example, the fact that algebraic equations may be used to express or determine measurements of the polygon (such as its perimeter or size of interior angles) would lie in the outer horizon, exemplifying structural connections between strands (e.g. algebra and geometry) and between concepts (e.g. ratios of lengths and angles).

Teachers' KMH is thus interpreted as their knowledge of, and ability to access, elements of a mathematical object's inner and outer horizons. Zazkis and Mamolo (2011) use this construct to explore several examples of teachers accessing their KMH to inform their decisions in teaching situations. Extending on this work, this present study explores what mathematical knowledge is accessed or connected to the concepts "in focus" for pre-service teachers as they addressed a non-routine problem concerning the area of an irregular shape. The analysis attends to specific instances of inner and outer horizon knowledge being mobilized by participants and how this knowledge shaped and influenced their preferred recommendations for a hypothetical student.

Methodology

The participants in this study were 20 pre-service secondary school mathematics teachers enrolled in a teacher education program. Each participant had up-wards of three university level mathematics courses and professed a high level of confidence with secondary school material. Participation was voluntary and included responding to two written questionnaires, administered one week apart, and taking approximately 30 minutes to complete. Participants were informed of the scope of the questionnaires, which sought to explore their mathematical and pedagogical knowledge given a hypothetical teaching situation, but they were not made aware of the specific content in advance, aside from being told that the second questionnaire would follow up on ideas raised in the first session. Both questionnaires were administered with the instructions to answer honestly and reflectively, and that there was no "right answer".

> Imagine you are a teacher in the following situation: Delia, a high school student with good grades, is working on an extra-curricular math problem and approaches you for help. Here is the problem:
>
> You are given a hexagon ABCDEF, where the lengths of the sides are equal to AB = CD = EF = 1 and BC = DE = FA = $\sqrt{3}$, and AB is parallel to DE, BC parallel to EF, and CD parallel to FA.
>
> 1. What is the measure of each interior angle?
> 2. What is the area of the hexagon?
>
> Delia has found that all of the interior angles are of equal measure, but is unsure how to find the area. *How do you recommend Delia go about finding the area?*

Figure 1: The First Questionnaire

The first questionnaire, depicted in Figure 1, was designed to uncover participants' personal strategies and approaches when advising a student, Delia, on how to compute the area of an irregular hexagon. Delia was presented as a "good student" with the intent to encourage participants to reason freely with the problem, without a perception that they were being "tested" on their knowledge of specific curricular content (e.g. if Delia were in one class or another) or of special needs or math-anxious students. As mentioned above, this hexagon was chosen because of the many applicable concepts and solving strategies, which would (i) allow participants access

to solving by various means, and (ii) provide information on what relevant (and irrelevant) mathematical concepts were evoked and remained "in focus" for participants. During this stage in the data collection, a diagram of Delia's hexagon was deliberately omitted since how (and whether) participants constructed their own diagrams would provide further insight into how they were interpreting the problem and its key features.

Since the second questionnaire followed up on the emergent themes in participants' recommendations to Delia, it is helpful to quickly summarize the observed trends and initial analysis before presenting the task, as they motivate its scope and intentions.

Of the 20 participants, 18 drew diagrams and all 20 described "deconstructing" the hexagon into smaller "easier" shapes. Of these 18 participants, 15 drew regular hexagons, with the most common diagrams depicted in Figure 2 below. The two most prominent trends in participants' recommendations for Delia were: (i) based on broad ideas, such as "put in lines to break up the hexagon into shapes which we have established rules and laws to work with" (Sophia); and (ii) giving step-by-step procedures of how to solve, such as "she can solve the area of the two triangles and the rectangle in the middle using formulas for the areas. Once this is calculated, she can just add the area of the rectangle and the two triangles" (Abigail).

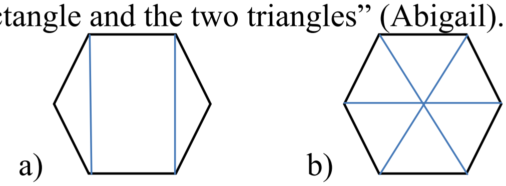


Figure 2: Decomposing Delia's Hexagon into a) Two Triangles and a Rectangle, and b) Six Equilateral Triangles Meeting at the Center Point

The initial analysis suggested that participants relied too heavily on the regularity of their depicted hexagons, using strategies that were either inappropriate (e.g. Figure 2a) or incomplete (e.g. Figure 2b) to generalize to the irregular case. As such, the follow up questionnaire (Figure 3, below) included: (i) a recommendation with a diagram of a regular hexagon, and one with an irregular hexagon, (ii) recommendations that could apply to both regular and irregular hexagons without introducing any additional mathematics, and (iii) recommendations that reflected and contrasted participants' inclination to decompose the hexagon.

In the following section, participants' responses to the second questionnaire are analysed in depth. For the purposes of this proposal, the focus is on identifying specific instances of participants' KMH via the refined lens of inner and outer horizons. Participants' preferences and recommendations are connected to their abilities to access these different components of horizon knowledge, with the intent to shed new light on what, and how, mathematical knowledge may be mobilized in a teaching situation.

Results and Analysis

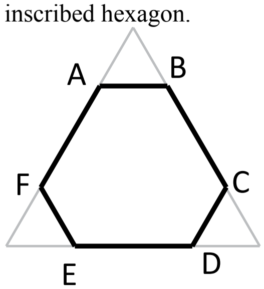
Due to space limitations, the analysis in this section focuses on participant responses to the second questionnaire which exemplified the trends and themes observed more generally in the data. Before turning to an in-depth analysis, a summary of these trends is presented.

Briefly, of the 20 participants, 11 preferred recommendation A and 9 preferred recommendation B. A common feature in participants' justification of their preferences for both recommendations was the ease and clarity of the approach, yet differences emerged in why the recommendations were considered "easy." Specifically, participants who preferred A attended to structural features and consequences of the provided diagram, while those who preferred B attended to surface features of the solving proves and their prior personal experiences.

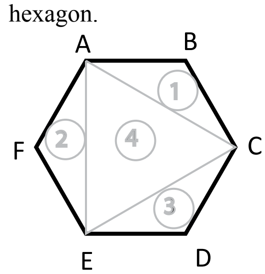
Recall Delia's hexagon ABCDEF, with sides lengths AB = CD = EF = 1 and BC = DE = FA = $\sqrt{3}$.

To determine the area, Delia was given a variety of different recommendations. Here are two of them:

Recommendation A:
Extend the hexagon into an equilateral triangle as in the figure below. Then use the areas of the large triangle, and small outer triangles, to determine the area of the inscribed hexagon.



Recommendation B:
Decompose the hexagon into three triangles (1, 2, 3, which are all equal), and an equilateral triangle 4, as in the figure below. Then sum the areas of the inscribed triangles to determine the area of the hexagon.



Which approach do you prefer, and why?

Figure 3: The Second Questionnaire

Considering first the respondents who preferred rec. A, their focus on structural rather than surface features has a direct connection to Knowledge at the Mathematical Horizon, via Husserl's notions of inner and outer horizons. For instance, Sarah explained:

> "I prefer rec. A because the image makes the idea very clear, where rec. B visually seems more complicated. Also with A, you're only using equilateral triangles… I know the smaller triangles are equilateral because the larger triangle is equilateral, making its interior angles 60 degrees by definition, which means the remaining two angles [of the small triangles] must be 60 degrees because they are all similar and the sum of all angles is 180 degrees."

Accompanying her explanation, Sarah drew the following diagram (Figure 4), noting lines of symmetry and illustrating her conclusion that the small and large triangles were similar:

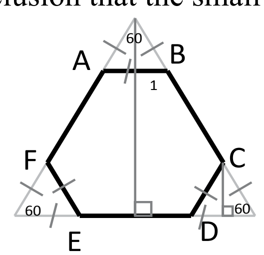


Figure 4: Sarah's Diagrammatic Reasoning

Sarah's attention focused on equilateral triangles and included properties of the formal definition and structural features which allowed her to assess the level of difficulty of this recommendation as opposed to the other. The markings on her diagram suggest Sarah's attention was at first focused on the encompassing triangle and its symmetries and angles. Her deductions regarding the smaller triangles can be interpreted as stemming from her knowledge of an equilateral triangle's outer horizon – e.g. the congruency of angles for all equilateral triangles, and the general property that all interior angles of any triangle must sum to 180 degrees, are part of a greater structure in which these specific triangles are embedded. Sarah further identified the area of the hexagon as the difference between the areas of the large triangle and three small ones,

which again may be viewed as an instance of KMH at the outer horizon – that a shape's measurements may be determined indirectly through knowledge of other shapes' measurements. In addition, her attention toward the encompassing triangle suggests that the embedded hexagon shifted to Sarah's periphery, and as such became a part of the triangle's inner horizon (since Delia's hexagon was specific to this triangle). Sarah's continued focus on triangles, rather than the hexagon, suggests a certain comfort and flexibility in her horizon knowledge – she seemed to know that she need not explicitly address the hexagon to determine its properties.

In addition, Sarah also accessed these aspects of her KMH to critique the alternate recommendation. She contrasted the "easier" approach in A with the approach in B, which she identified as "unclear" since "I don't think I could find the side lengths [of the triangles] in B. It seems as though there is a lot more work to finding the areas of the triangles in B." Sarah's judgement that "there is a lot more work" may be seen as awareness that some important structural features of rec. A were not present in B. Thus, it seems as though what was in focus for Sarah – equilateral triangles – and the associated KMH she was able to access as she reasoned with rec. A, also influenced what was in focus for her as she considered rec. B.

Similar themes emerged from Miles, who focused on the specifics of the diagrams provided and the related reasoning. In explaining his preference for rec. A, he noted:

> "Negative space thinking is something that is rarely cultivated and can prove very useful in fields extending beyond pure mathematics. While method B is applicable for most polygons, method A offers a different approach that will get the student to begin thinking of alternative methods to exact the same end."

In this excerpt, Miles seems to be accessing aspects of his outer horizon knowledge, both in terms of situating "negative space thinking" in a world "beyond pure mathematics," as well as in acknowledging that "method B" is broadly applicable. There is also evidence of Miles accessing knowledge at the inner horizon when he attends to the particulars of the two diagrams:

> "It must also be noted that both figures shown are only one possible configuration. In fact, figure B is further from an accurate scale representation than figure A (the sides are not accurately proportioned)."

Miles' focus of attention seemed to be on the specific structure and shapes of the diagrams – in fact, Miles was one of only four participants who alluded to the inaccuracy of the hexagon in rec. B. His observation that there are other possible configurations suggests that while these alternate configurations were not in focus, they were certainly in his periphery. This led him to note that even "if the internal angles aren't equal, figure A's approach can still be used. The triangle form though, may not be equilateral, but it will be isosceles." In contrast to prior research which observed that teachers' images of hexagons tended to be restricted to regular prototypes (e.g. Ward, 2004), Miles was considering hexagons more broadly. His response suggests that he was able to reason with these shapes without having them directly in view. His consideration of how the encompassing triangle would differ depending on the specific hexagon and of how "A's approach can still be used" more generally, instantiate both inner and outer horizons, respectively. Miles also noted that "both methods should be shown and Delia should be encouraged to question both approaches.... Finding these different approaches will allow a student to choose a method that works well for him, and will deepen their understanding of the concept being taught / explored."

Miles' reflection on the value of both approaches is noteworthy as it contrasted with common responses that preferred rec. B as more familiar, comfortable, and more closely connected to strategies participants used when they were students. For these participants, personal preference

and comfort seemed to take precedence, and also seemed to influence what participants "allowed in view" and what was kept to the periphery. For example, Abigail claimed that "rec. B is the approach I would take because of the way I learned geometry. The hexagon divided into triangles is the approach I learned in school" and because the "subtraction method [is] confusing to me, but adding small shapes to make a big shape is easy." Abigail's desire to use the approach she learned in school indicates a reluctance to access KMH – that is, knowledge that goes beyond school curricula. This reluctance also seemed to focus her attention toward superficial features of the recommendations (such as adding versus subtracting), which were not considered in relation to the specific entities that would be added or subtracted. It is unclear from Abigail's response whether she lacked appropriate KMH, or whether she viewed such knowledge as inapplicable. What can be said is that with a robust KMH, and a willingness to access it, then Abigail could have analysed the level of difficulty of the arithmetic with respect to the specific features of Delia's hexagon (inner horizon), and provided justification for her preference that spoke to the general validity or applicability of the proposed recommendation (outer horizon). The idea that adding is easier or more straightforward was echoed by the majority of participants who chose rec. B.

Comfort level was also important for Lexi, who chose B because "automatically I can recall the Pythagorem Theory [*sic*] to solve for the individual triangles 1, 2, 3, which are all equal" and "I prefer formulas in mathematics, and checking that my work is correct, as opposed to guessing what number relates to what side, which is probably how I would go about solving approach A." It should be noted that the Pythagorean Theorem is not applicable to the triangles in rec. B, thought it can be useful in determining the area of the equilateral triangle from rec. A. Lexi's strong preference for formulas and Abigail's preference for the approach she learned in school suggest that a desire to stay within a familiar comfort zone limited what mathematics they were willing or able to consider and mobilize in analysing the recommendations. For Lexi this led to an inappropriate approach, and for Abigail it led to a superficial consideration of what would be "easy" or "confusing." While personal comfort and familiarity are certainly valid reasons for choosing one approach over another, teachers must be able (and willing) to reason beyond their comfort zones in order to adequately meet the diverse needs of their students. In the data from this study, individuals who were reluctant to go beyond their comfort zones drew inappropriate conclusions about the solving methods and their levels of difficulty, and showed a limited access to knowledge beyond what was emphasized in their prior school experiences. In contrast, participants who looked beyond school curricula and made connections between and across ideas (KMH) were able to appreciate aspects of the recommendations on a deeper level, and also considered how these ideas related to a student's broader learning experience.

Concluding Remarks

What influences pre-service teachers' preferences and recommendations when advising a student on how to determine the area of an irregular hexagon? There were several factors. In accord with research done with children and elementary teachers (e.g. Ward, 2004; Walcott et al., 2009), participants with strong mathematics backgrounds also relied on prototypes of regular hexagons (first questionnaire). This resulted in recommendations that were inappropriate or incomplete for the problem at hand. Personal comforts with formulas or previously learnt school material were also key features in participants' preferences for how to advise a student (e.g. Abigail, Lexi, second questionnaire). While Delia's hexagon was new for all participants, some were more willing to 'dig in' and explore the problem on a deeper level, making connections beyond what was previously learnt in school to attend to structural features of the approaches

(e.g. Sarah) or potential benefits to students' broader learning (e.g. Miles). These connections were interpreted as instances of Knowledge at the Mathematical Horizon, as it related to inner and outer horizons of the mathematical entity in focus. The analysis suggests that comfort and flexibility in what horizon knowledge is accessed and mobilized are important features in assessing and recommending appropriate solving strategies for learners, and it makes a case for developing such flexibility in teacher education programs.

Ball and Bass (2009) note that "knowledge of the horizon does not create an imperative to act in any particular mathematical direction" (p.10). This research suggests that while KMH may not provide an imperative, it can provide an invitation to act in ways that consider mathematics in a more structural and connected way, even if this may diverge from prior school expectations. Also, participants who accessed their KMH provided an analysis of the two recommendations that included more depth and accuracy than those who did not. This suggests that teachers with a more robust KMH (or one which is more readily accessed) are in a better position to guide and interpret student thinking. Participants in this study who were so able seemed more flexible in how they viewed the different recommendations, and were more willing to set aside their personal and initial preferences for decomposing the shape when interpreting the benefits and draw backs of each. This is particularly significant when considering the potential effects on a teacher's response to a student's unconventional or unexpected solutions or approaches.

Finally, this research opens the door to exploring a possible connection between a teacher's KMH and her Knowledge of Content and Students (KCS). Specifically, Miles' KMH seemed to influence what he believed would be important for student learning (e.g. "negative space thinking"), and this in turn influenced what horizon knowledge he was willing to mobilize when considering the alternate recommendation. Additional research regarding the relationship between KMH and KCS, how one might influence the other, and how one might develop in conjunction with the other may provide further insight into the experiences necessary for preparing future teachers of mathematics.

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THE USE OF AUTHENTIC HIGH-SCHOOL STUDENT SOLUTIONS AS A CHANGE AGENT WITH PRE-SERVICE TEACHERS

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This paper presents the results of a study designed to facilitate teacher candidates' shift from students of mathematics to teachers of mathematics. We develop a series of tasks each of which encourages candidates to reconsider rich tasks from increasingly teacher-centric points of view. The results show that using authentic high school student solutions, in particular Livescribe™ videos, provides a powerful experience in seeing solution paths they had not considered and in helping teacher candidates think about mathematics from someone else's point of view.

Keywords: Teacher Education-Preservice, Mathematical Knowledge for Teaching, Teacher Beliefs

Introduction

Successful teacher preparation programs provide learning experiences that scaffold a trajectory for teacher candidates in making the shift from "student" to "teacher" i.e. ushering them along a continuum of professional development. This transition is a crucial goal: to be a successful teacher a candidate must learn to see the tasks of school mathematics from other people's perspectives. However, this is not an easy transition, particularly for future school mathematics teachers, complicated, as it often is, by the fact that candidates' interest in mathematics and self-efficacy regarding mathematics arises from their personal success in solving complex problems rather than in their ability to understand others' solutions. In the course of attempting to scaffold this transition for teacher candidates and reflecting on what we have found to be helpful, a five-step approach has evolved whereby we feel we can effectively facilitate this transition for teacher candidates in the context of examining rich content tasks. The approach, which we refer to as the Mathematics as Teacher Heuristic (MATH), is designed to provide a set of experiences, in a two-week period at the beginning of a prospective secondary mathematics teachers' Methods class, that gradually require candidates to shift their mathematical view from a "learner/doer of mathematics" orientation to one embracing teacher-oriented perspectives. The MATH process requires candidates to engage with rich mathematics tasks but from points of view rarely considered in content-oriented coursework that, by their nature, are concerned with candidates' attempts to do their own mathematics rather than consider another learners' mathematics. In a two-week project at the very beginning of a Methods class, candidates complete a five-step process that consists of:

(1) solving a rich task as a "doer";
(2) assessing student work samples associated with the same task;
(3) constructing a set of questions for students who are stuck in the problem;
(4) developing scaffolded instructional materials addressing student challenges, difficulties and misconceptions (gleaned from earlier analyses); and
(5) reflecting on the process.

Each step of the MATH process encourages candidates to consider the rich tasks from increasingly teacher-centric points of view. (This is not to be confused with a "teacher-centred" approach to pedagogy, rather in this case, teacher-centric means thinking about mathematics as a teacher of mathematics needs to.) Assessing authentic student work and developing scaffolded instructional materials are activities that require candidates to consider interpretation of a learner's

work and guidance for a learner (teacher-oriented tasks) rather than solving the problem on their own terms and presenting it for consideration (a learner's perspective).

In an earlier study (Meagher, Ozgun-Koca, Edwards, 2010) using the MATH process, we showed the particular impact of Task 2 above i.e. the power for teacher candidates of engaging with authentic high school student solutions in facilitating the transition from "doer of mathematics" to "teacher of mathematics." In that study we showed that this is perhaps the most crucial juncture for teacher candidates in the five-step MATH process i.e. the step that caused the greatest dissonance and resulted in the greatest level of reflection on the transition from "doer" to "teacher". In order to amplify this juncture and to study it more closely in this iteration of the MATH process we included a set of Livescribe™ videos along with the static solutions. Video technology allows the teacher candidates to see each step of the student work as well as being able to hear the students' description of their own thinking. As we will see below having a combination of video and static solutions is effective as there is great value to the teacher candidates trying to work out what the high school students were thinking, particularly when they took solutions paths other than the abstract algebraic approaches the teacher candidates were expecting.

The research questions guiding the research and the analysis are:

(a) To what extent does the MATH process help teacher candidates' transition from doers to teachers of mathematics?

(b) What is the particular impact on teacher candidates of engaging with authentic high school student solutions?

Literature Review and Relationship to Research

Many models for understanding the negotiation of the transition along the continuum of professional development from "student/doer" to "teacher" focus on dissonance and motivation in school settings, factors which occur regularly for in-service teachers in their day-to-day practice (Clarke & Hollingsworth, 2002; Edwards, 1994). Clarke and Hollingsworth (2002) and Loughran (2002) stress the importance of self-reflection in the development and evolution of teacher knowledge, beliefs, and attitudes. We believe that candidate teachers need to experience similar dissonances in university coursework order to accept the need for change and reflect on their change process. Providing such opportunities for change and reflection is, perhaps, more difficult in the case of candidate teachers at the very beginning of their course of study since they have fewer authentic teaching experiences to draw upon. Studies show that two main sources for dissonance in initiating candidate growth are in methods courses and student teaching.

Ball (1989) studied the role of a methods course for elementary mathematics teachers in helping candidates learn to teach. Ball highlights the importance of content knowledge and the experiences of candidate teachers as learners of mathematics. "Unless mathematics teacher educators are satisfied with what prospective teachers have learned from their experiences as students in math classrooms (and most are not), this highlights a need to interrupt, to break in, what is otherwise a smooth continuity from student to teacher" (p. 4). In this sense, we see Ball's call to break with experience, the "interruption," as an opportunity to create a dissonance.

Of particular relevance to the kind of shift we are asking teachers to make is the notion of "unpacking" discussed in the work of Adler & Davis (2006). In one of their examples of unpacking they present five different student responses to a standard question requiring finding solution(s) of a quadratic and note that, after seeing at a first level of analysis that all the students have found a correct answer, "The teacher will need to unpack the relationship between a mathematical result or answer and the process of its production" (p. 274). Adler & Davis note that the teacher is also faced with the challenge of interpreting the specific strategies used by each student and consider how those strategies, some of which are incomplete or problematic, will be orchestrated in a classroom setting to consolidate the learning of all students. Engaging in such understandings which are part of the

Mathematical Knowledge for Teaching (MKT) (Bass, 2005; Ball, Hill and Bass, 2005; and Hill, Rowan & Ball, 2005) construct involves the creation of a dissonance whereby teacher candidates are challenged to think about mathematics in ways that are not their own and displace them from the role of "doer" of mathematics.

Loughran (2002) explored the development of knowledge through effective reflective experience from teacher candidate to experienced teacher, comparing candidates' views of teaching and learning with those of practitioners. He noted that candidates typically equated learning "with gaining right answers" (p. 41). Loughran's study illustrates the importance of giving candidates opportunities to face their views, reflect and reconsider them. Swafford et al. (1999) echo these findings for inservice teachers. Swafford et al. recommend the creation of environments for teachers that improve their content and pedagogical knowledge through reflection and collaboration.

Finally, the efficacy of using authentic student work in teacher development has been established (e.g. Kazimi & Franke, 2004), albeit often for inservice teachers in lesson study-like settings. Hiebert et al. (2007) have demonstrated how the use of student work can be effective in the preservice setting.

Methods and Methodologies

Participants

Participants in this study (n=38) were candidates enrolled in a methods course designed for prospective secondary mathematics teachers. At the very beginning of the methods class the teacher candidates worked in pairs for a two-week period on the five-step MATH process.

Candidate Tasks

Candidates were given the following set of tasks building from the problem in Figure 1 below:

> **The Bridgewater Problem.** At 1:00 p.m., two hikers began walking, the first from Amityville to Bridgewater, the second from Bridgewater to Amityville along the same path. Each walked at a constant speed. They met at 4:00 p.m. The first hiker arrived at Bridgewater 2.5 hours before the second hiker arrived at Amityville. When did the second hiker get to Amityville?

Figure 1: The Original Rich Problem

Task 1: the teacher candidates solve the Bridgewater problem using any method. This puts the teacher candidates firmly in the position of doer of mathematics and produces examples of their own mathematics that the students can then compare to other people's mathematics.

Task 2: the teacher candidates review forty-five authentic high school students' attempts at the Bridgewater problem and review five Livescribe™ videos of high school students (unsuccessful) attempts at the problem. The purpose of this task is for the teacher candidates to be confronted with many examples of how the problem can be solved including perspectives totally different from their own. In undertaking this task the students can begin the shift to thinking about the problem from another person's perspective.

Task 3: using the Livescribe™ videos the teacher candidates construct questions they would ask those students to understand their thinking and help them make progress in solving the problem. This task requires teacher candidates to consider the incomplete work of a student and understand how that student was thinking about the problem. The purpose of this task is to require student candidates to continue their shift from doer to teacher by thinking about how they would work with students engaged in the task.

Task 4: the teacher candidates revise the task to construct a more scaffolded experience for students solving the task. The purpose of this task is to require candidates to engage in task design in such a way that they have to think about how someone else would do the problem.

Task 5: the teacher candidates are asked to reflect as in Loughran (2002) on the process they have undergone and to articulate their thoughts on the shift they are making from doer to teacher.

Each step of the process encourages candidates to reconsider rich tasks from increasingly teacher-centric points of view i.e. reconsider the tasks in terms how a teacher has to understand learners' thinking and how to engage learners in a task.

The data collected consists of all teacher candidate work on each of the five tasks above. In our analysis here, we focus on Tasks 1, 2, 3, and 5 of the MATH process assignment. Due to the qualitative nature of data, our analysis concentrated on looking for patterns, categories, and themes. Building on the previous study's codes and literature review, we looked, in particular, for instances of dissonance, unpacking, and rich reflections.

Results

Working through the rich problem on their own terms and then working with the combination of video and static solutions resulted in three major elements which moved teacher candidates to think about their transition from "doer of mathematics" to "teacher of mathematics": (i) understanding the multiple solution paths students will bring to a problem, (ii) understanding the thinking process of students as they work through the problem, and (iii) changing their expectations of the level of work of which high school students are capable. The first of these was particularly powerful in creating the kind of "dissonance" (Edwards, 1994) and "interruption" (Ball, 1989) that can prompt teacher candidates to reflect. The videos were particularly powerful in the second of these as the teacher candidates undertook the task of "unpacking" (Adler & Davis, 2002) the student solutions as they sought to understand exactly how a high school student worked through a problem and exactly how that student got stuck.

Multiple Solutions

Our previous study (Meagher, Ozgun-Koca, Edwards, 2010) showed that in their own solutions to the problem the teacher candidates strongly privileged abstract algebraic methods. This finding was replicated in this study whereby all 17 of the 17 pairs of students who presented a solution used exclusively algebraic methods, i.e. they set up an algebraic expression for each walker and solved the quadratic arising from setting the expressions equal to one another.

The privileging of abstract algebraic methods was further underscored by the fact that while many used a graphical representation such as that in Fig. 2 below, 0 of 17 pairs used a Cartesian graph and 17 of 17 solved the quadratic equation by factoring rather than graphical methods.

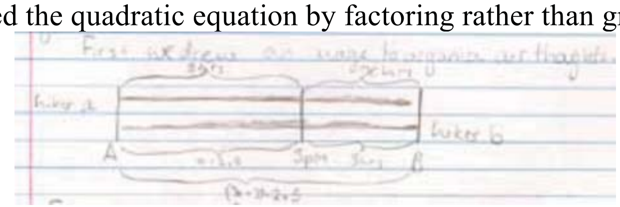

Figure 2: A Teacher Candidate's Graphical Representation of the Problem

The dissonance and/or surprise caused by the multiple solution methods used by the students was explicitly mentioned by approximately half of the pairs in their reflections: "The ability of students to solve high-demand problems in a number of ways was truly a surprise to us" (Student pair B8). Another pair commented: "In grading the student answers, we were particularly amazed at the variety of approaches. Some students performed a series of "guess, check, and revise," others created functions, some seemed to rely solely on number lines or graphs. Despite the variety, many of these attempts resulted in successful solutions" (Student pair A2). Note here the extra surprise that the teacher candidates felt that non-algebraic methods resulted in successful solutions. It was further interesting to observe how deeply the privileging of algebraic solutions (Cooney, 1999) is held for some students: "I knew that students could use guess and check, or solve things graphically, but I

never knew it had been formalized" (Student pair A5); "In fact, many of the students were able to arrive at an answer with much less computation and mathematical rigor" (Student pair A7); and "Since we have had a lot more experience in problem solving than many of the high school students and have learned much higher mathematical concepts, using a method such as guess, check, and revise or reasoning through the problem with pictures was not our initial instincts for coming up with a solution." (Student pair A8). We see here that after some exposure to authentic multiple solutions produced by high school students, some of the teacher candidates are attached to the notion of abstract algebraic solutions as the best and most mathematical.

In our pilot study (Meagher, Ozgun-Koca, Edwards, 2010) we saw that the high school students often took a more embodied approach to the problem thinking of the hikers in real terms as actual people moving, whereas the teacher candidates tended to immediately think of the problem in terms of setting up equations to be solved. This approach is exemplified by the following comment: "At our level of math, we generally start with equations with a goal of solving them, whereas the students use less abstract math applications and more straightforward thinking and assigning values for items" (Student pair B6).

The teacher candidates acknowledged that, as they transition to being teachers, they need to move away from their own predilections for how a problem should be solved "One big idea … was that there are multiple ways to solve the same problem. It is important that we as teachers are open to letting students explore different methods and find their own strategy that works best for them" (Student pair B6), and "The big idea that we can take away from math project one is that every student's thought process is unique. In order to respond to this, it is important for us as educators to acknowledge the various ways in which students think to help them arrive at a solution" (Student pair B3). We see here that the examination of multiple solutions has facilitated the teacher candidates in moving from a "doer" perspective to a "teacher" perspective.

Livescribe™ Videos

As part of the assignment the teacher candidates watched five Livescribe™ (http://www.Livescribe.com/en-us/) videos of high school students attempting to solve the Bridgewater problem. Livescribe™ is a digital platform whereby a user, by employing a smartpen with an optical and audio recorder, can record a video of each of their written steps in solving a problem and a simultaneous synchronized audio of their thought process. The video can be played back with a "shadow" of what the user wrote permanently visible so the viewer can see everything the user ended up writing while they view the student writing in real time.

In order to provide extra insight into student thinking the teacher candidates watched videos of a number of students who struggled with the problem. The teacher candidates were then asked to provide questions they would ask the high school students to help them past their impasse. This task provided a useful challenge for the teacher candidates as they were forced to think about the problem from another person's point of view and to formulate questions that explicitly used what the student had already shown they knew to move then forward. The teacher candidates had many interesting responses as they struggled to negotiate the line between helping the high school student reflect but not telling them too much or telegraphing responses.

Many of the questions were unsuccessful: (a) too leading "What if you set up 2 equations? It might be helpful to use x and y, showing the distances before and after the hikers meet" (Student pair B4); "Do you remember the d=rt formula?" (Student pair B5); (b) Too vague: "What different approaches are there to solving the problem?" (Student pair A5); "What can you assume about the time that the hikers meet? What can you not assume?" (Student pair A10); or (c) not really questions "Can your picture be improved?" (Student pair A2).

Others were more successful in finding questions that might prompt students to reflect on their work thus far: (a) general questions "Have you used or represented all of the information given in the question?" (Student pair A1); "Why do you think we are told they maintain a constant speed?"

(Student pair B9); (b) and more specific questions related to the student's work up to getting stuck "What do your variables stand for in the context of the problem?" (Student pair B6); "What units would you give each of your variables?" (Student pair B8); and "Are the rates of the hikers the same?" (Student pair B5)

Thinking of good questions is challenging and, as we saw above, many of the student teachers found it difficult. Nonetheless watching the videos and thinking of questions to pose to the students was a powerful exercise. In the free-response reflection 11 out of 19 of the pairs explicitly mentioned the power of the smartpen videos in helping them think about student thinking and student approaches to the problem rather than thinking as a "doer."

The key value of the videos was to help teacher candidates in the transition from thinking about the problem for themselves to thinking about it from a student's point of view. An important part of this is for them to hear and understand a student's thinking process, i.e. to "unpack" (Adler & Davis, 2002) the student's thinking and to use that student thinking as a basis for helping the student move forward in their thinking. Listening carefully to students and understanding their thinking is a crucial element in teacher development. The value of this part of the experience was evident in the teacher candidate reflections. "Being able to put ourselves in the mindsets of our students is a key skill in helping our students figure out solutions on their own. In connection to this, brainstorming scaffolding questions for the students using the smartpen videos was one of the most useful tasks of this project. It helped us to see how students approached the problem and what things they struggled with." (Student pair A8). It is interesting to see here the teacher candidate focus on the student approach and struggles. This idea of how the student got to their impasse was noted by others, e.g. "Viewing the Livescribe™ files and watching the students work out the problem was a whole new experience. It allowed us to see and hear what the child was thinking while they were working out the question. It allowed us to see what went wrong and why it happened." (Student pair B5).

We can see that the use of video not only facilitates the transition to thinking from a student's point of view but also pushes teacher candidates in the direction of another crucial step: what questions can they ask which will move the student forward. The teacher candidates recognised this opportunity as evidenced by comments such as "some of those students had the right idea and with a little push … they would have reached a solution." (Student pair A2) and "it was nice to think about what questions we could ask to help them out. We didn't want to be too leading in our questions, so at first it was difficult to find questions." (Student pair B9). This transition along the continuum of professional learning begins to activate the MKT of teacher candidates, most particularly the combination of subject content knowledge and knowledge of how students think about and engage with content knowledge called "knowledge of content and students (KCS)" (Hill, Ball and Schilling, 2008, p. 373).

We should be careful to say that the smartpen videos could certainly have been replaced by "traditional" videos of students solving the problems, but it is clear from the evidence above that a video has a different effect on the candidates than static solutions or even solutions accompanied by a transcript. Furthermore Smartpens arguably make video data easier to collect.

Future Expectations for their Students

A significant advantage of having the teacher candidates work through the rich problem themselves is that they had considerable experience with the problem and had developed specific expectations both for what they expected to see in the student work and of how they expected high school students to work with the problem. As well as the "dissonance" (Edwards, 1994) caused by the variety of solutions offered by the high school students it was clear that the teacher candidates were impressed by the quality and ability shown by the high school students. This led many to comment explicitly that they have learned to expect more of high school students.

Many of the teacher candidates struggled with the Bridgewater problem and expected that ' students would find the problem extremely challenging. They were surprised to see that the high

school students were able to solve the problem using less abstract methods than they themselves had used but also that many were capable of using abstract algebraic solutions. “One ‘big idea’ that we can take away from this project is that we should have higher expectations of our high school students.” (Student pair A3); “I really had never thought that high school students would ever write anything so clear. They expressed their thought processes clearly, and thoroughly” (Student pair A5) and “The ability of students to solve high-demand problems in a number of ways was truly a surprise to us” (Student pair B8).

The expectations the teacher candidates had for their prospective students was changed by the quality of the work they saw: “Another aspect of the students’ work that was interesting was how much the honors algebra students wrote for each problem, with the exception of one or two students” (Student pair A10). Perhaps more importantly, the exposure to multiple solutions helped the teacher candidates in the transition they must make to differentiating tasks and to thinking about how the same task can be framed differently for different populations. This idea, common among the responses is typified by the following reflection: “We’ve learned about how problems can be accessible and understandable by students at earlier ages with the help of scaffolding and technology. For instance, the Bridgewater problem would typically be a Algebra 2 level problem. However, there are geometric (visual) solutions as well as algebraic and numerical solutions. Students in Level 1 were able to set up the problem and some students even had a thought process that could have lead them to the right answer, thus even 2 mathematical levels below the suggested level for this problem students can find ways to achieve a solution” (Student pair A6). The experience prompted teacher candidates to reflect on how they would frame tasks “A ‘big idea’ that I can use in my future teaching endeavors is that when I create a lesson, I need to think not only about how I would solve the problem but the multitude of different ways that my students could solve the problem” (Student pair B2). They also learned that they can expect a lot of their students if they themselves understand their role as a teacher “One ‘big idea’ that we can take away from this project is that no problem is too hard for students to complete with the right guidance and scaffolding … When we were assigned this project, we thought that no high school student would be able to complete this problem because it involves deep thinking, but we were surprised by the results in the student work … Thus, we think a big idea we can take from this project is to never be too afraid to give our students a challenging problem because they may completely take us by surprise with the level of thinking they are capable of.” (Student pair A8).

Conclusion

As teacher candidates negotiate the continuum of professional development it is crucial for them to transition from being “doers of mathematics” to being “teachers of mathematics.” The study presented above shows that the MATH process, in particular the engagement by teacher candidates with authentic high school student work provides the kind of “dissonance” (Edwards, 1994) and “interruption” (Ball, 1989) that can facilitate that transition. Moreover, the use of Livescribe ™ video can engage teacher candidates in the kind of “unpacking” (Adler & Davis, 2002) that aids them in understanding mathematics from someone else’s point of view and how to move that student’s thinking forward. Of course the candidates’ work analysed above takes place in a two-week period at the beginning of a methods class and represents a first intervention to shift the teacher candidates’ thinking. Further research is needed to examine any lasting and transferable effect of the transition we see beginning here. In particular, as our research continues we will examine what happens in subsequent semesters when these candidates go into the field for student teaching: Are they able to enact effective questioning techniques? Do they encourage multiple approaches to solving problems? Do they encourage multiple approaches to problems but still privilege algebraic solutions as “better” or “more mathematical”?

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TEACHERS' POSITIONING OF STUDENTS IN RELATION TO ABILITY/ACHIEVEMENT IN A PROFESSIONAL DEVELOPMENT SETTING

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This paper reports on elementary teachers' positioning of students as learners of mathematics related to their perceived ability level or previous achievement in mathematics. This work, which is a part of a larger design experiment on teacher learning of mathematics learning trajectories, examines the ways in which LTs influenced teachers' positioning of students and the changes we observed in teachers' discourse as teachers engaged in various professional learning tasks throughout the professional development.

Keywords: Teacher Beliefs, Learning Trajectories, Teacher Education-Inservice/Professional Development

Research has demonstrated that professional development providing teachers with opportunities to examine students' mathematical thinking can lead to productive changes in teachers' mathematical knowledge and classroom practices (Franke, Carpenter, Levi, & Fennema, 2001; Kazemi & Franke, 2004; Sowder, 2007). These experiences promote teacher learning by providing teachers with specific knowledge of students' strategies and bringing attention to students' underlying conceptions (Little, 1999). Moreover, using student work as a tool for professional development can assist teachers in shifting from focusing on students' mistakes to attending to the processes students are using and levels of sophistication in students' reasoning (Kazemi & Franke, 2004).

In this paper, we focus on a professional development project that uses student thinking in the form of learning trajectories (LTs). Researchers developing LTs use clinical interviews with students to empirically develop and refine pathways of student knowledge in particular content domains that progress from less sophisticated to more sophisticated (Clements & Sarama, 2004; Confrey et al., 2009). The emerging research on teachers' uses of LTs suggests that trajectories can support them in attending to student thinking during task selection, classroom interactions, and in assessing students' knowledge (Sztajn, Confrey, Wilson, & Edgington, 2012). Our work uses design experiment methodology to examine teacher learning of LTs to allow for the study of the processes of change in teachers' discourse about students' mathematical thinking. In particular, we are interested in the ways teachers talk about students' successes and failures in mathematics as they position themselves and their students within their professional community.

We begin this paper by briefly reviewing teacher learning of students' mathematical thinking and presenting our research question. We describe the context of the professional development and research methods. We share findings with respect to one particular aspect of teachers' discourse related to students' achievement and ability and conclude with a discussion of the role of LTs in teachers' positioning of students in their discussions.

Background

Fennema and colleagues (1996) developed Cognitively Guided Instruction (CGI), a well-

known research program based on the idea that providing teachers with a framework for listening to and understanding children's problem solving strategies and mathematical thinking will promote instruction that meets the needs of all students. The results of their work with teachers "suggest that giving teachers access to research-based knowledge about students' thinking and problem solving can affect teachers' beliefs about learning and instruction, their classroom practices, their knowledge of their students, and most importantly, their students' achievement and beliefs" (Carpenter, Fennema, Peterson, Chiang, & Loef, 1986, p. 530).

Battey and Chan (2010) reported on their efforts to use CGI as a vehicle to challenge teachers' beliefs about their students. They note that teachers often describe students by metanarratives and that these stories position African-American students and other populations as inferior to their White and Asian counterparts, casting them as "other." In this multi-year project, they worked to counteract these metanarratives by drawing teachers' attention to what students can do as opposed to what they cannot do and use this *new* way of talking to re-structure the discourse in their professional development program. Their findings indicated that focusing on student thinking did change teachers' discourse about students, and subsequently, teachers rooted their claims about students in evidence instead of assumptions. Additionally, they found that CGI helped teachers focus on individual students' thinking and caused them to shift from larger notions about the group to which they belong to their individual needs. They posit that these behaviors can help dispel the deficit orientation teachers have towards students (Battey & Chan, 2010).

When analyzing teachers' discourse and participation in a professional development organized around student work, Kazemi and Franke (2004) reported changes in the ways teachers talked about student mathematical thinking. They described teachers' focus early on as emphasizing students' mistakes and being unable to provide detailed explanations on how students completed the problem posed to them. However, overtime, teachers provided more detail regarding their students' mathematical work and highlighted various levels of sophistication in students' mathematics reasoning.

Our work engages teachers by using students' mathematical thinking in the form of LTs. Initial studies of teacher learning of LTs indicate that they support teachers in focusing on the process of students' thinking and provide a framework for making instructional decisions (Edgington, 2012; Wilson, 2009). We build from both Kazemi and Franke's (2004) observation of the changes in teachers' conversations around students' work and the emerging research on teacher learning of LTs to investigate the ways teachers discuss their students as mathematical learners in a professional development setting. We use van Langenhove and Harré's (1999) notion of *stereotyping* to consider the set of representations that teachers use to talk about students in their professional community. That is, when teachers talk about students, they use acceptable social representations that conform to the norms of the group. Such discourse positions teachers and students in a variety of ways. Analyzing teachers' discursive patterns in a professional development setting can clarify the types of representations present within a particular professional community.

Methods

This study is part of a larger design study entitled Learning Trajectory Based Instruction (LTBI) and seeks to examine the themes that emerged in teachers' discussions of students during a yearlong professional development. The research question under investigation is: *How do teachers represent students as learners of mathematics in a professional development setting*

focused on students' LTs? The LTBI project is a multi-year design experiment conducted with elementary school teachers in a professional development setting with the goal of studying teacher learning of students' learning trajectories and an instructional model where LTs provide guidance for teachers' instructional decisions. Design experiments are iterative in nature and provide researchers with a means of studying learning in context. They are used to develop "a class of theories about both the process of learning and the means that are designed to support that learning" and they "entail both 'engineering' particular forms of learning and systematically studying those forms of learning within the context defined by the means of supporting them" (Cobb, Confrey, diSessa, Lehrer, and Schauble, 2003, p. 9).

Early on in our work with teachers, we noted the ways teachers talked about students' successes and failures in their mathematical work, revealing some of the acceptable representations of students as learners available to the group. In particular, teachers often described students as "low" or "high" achievers, attributing students' work to a fixed quality of students' aptitude. We contend that these types of statements position students as learners whose capacity for learning is largely predetermined and beyond their influence.

Context

In our project, we initially define Learning Trajectory-Based Instruction (LTBI) as instruction that situates LTs at the center of teaching practice. In the first cycle of this study, teachers spent 60 hours learning about the equipartitioning learning trajectory (EPLT, Confrey, 2012) and how this trajectory can be used in their instruction. Confrey (2008) defined equipartitioning as the cognitive behaviours that lead to the creation of equal sized groups from a collection, or equal sized pieces from a continuous whole, and which result in fair shares. The EPLT is organized by levels of cognitive proficiency; with the earliest levels addressing the ways students accomplish equipartitioning tasks. Confrey's (2012) definition of equipartitioning requires that three conditions be met; 1) students must create equal-sized groups or pieces, 2) students must create the correct number of groups or pieces, and 3) students must use the entire collection or exhaust the whole. These three criteria are represented in the initial levels of the LT, which address dealing collections and splitting a single whole. In addition to changes in what is shared (e.g., a collection of objects, a rectangle, a circle), the trajectory also addresses sharing for different numbers of recipients. The structure of the trajectory represents the ways that the task parameters affect the difficulty of the task. Other ideas contained in the EPLT proficiency levels are: justifying, naming, qualitative compensation, quantitative compensation, and transitivity (Confrey, 2012).

The first portion of the professional development (30 hours) was conducted in the summer of 2010 with a focus on supporting teachers in learning about the content and structure of the trajectory. To do this, teachers viewed clinical interviews of students as well as videos of whole class instruction that exemplified the various task parameters and proficiency levels found in the EPLT. This learning was ultimately formalized by providing teachers with the LT. During the 2010-2011 school year, teachers participated in monthly meetings where the goal was to consider the trajectory in relation their students and instructional practices. In the fall, the focus of the professional development was assessment, and teachers were asked to assess students' understanding of equipartitioning concepts. The focus of the spring meetings was instruction, and teachers were asked to think about how they could use the EPLT in planning and implementing a lesson in their classroom. The professional development concluded with a two-day follow up session the following summer.

Participants

Our partner school for this study was mid-sized elementary school located in the southeastern United States. The school had approximately 600 students, 35% Caucasian, 29% Hispanic, 25% African American, 7% Asian, and 4% other; 54% of the children qualified for free or reduced lunch. Twenty-two elementary teachers participated in this study. Of these teachers, nineteen were full-time teachers, and the other three served in various instructional support roles. The following grade levels were represented in the project: kindergarten (5 teachers); first grade (4 teachers); second grade (5 teachers); third grade (4 teachers) and; fourth grade (1 teacher). Participation in this study was voluntary, and teachers received a stipend for participation.

Data Sources and Analysis

Data for the study consists of video of all whole group discussions and audio of all small group discussions from the professional development, as well as the research team's field notes. Our initial analysis followed a grounded theory approach (Strauss & Corbin, 1989) in which we used open coding of the field notes to identify ways that teachers represented students as learners of mathematics in their discourse. Eight codes were identified: ability/achievement, age/grade, effort, luck/random, out of school contexts, tasks, teaching, and math. Following the recommendation of DeCuir-Gunby, Marshall, and McCulloch (2011), a codebook was developed to provide names, definitions, and examples to illustrate each code. Five independent coders were trained using the codebook to code the audio and video data with 85% reliability. We used our definitions to code every turn teachers made in all whole group and small group discussions that were about students' mathematics using one or more of the codes defined in the codebook. For the purposes of this paper, we focus on one code, "ability/achievement", to examine themes present in these data and how the presence of the LT affected the ways teachers positioned students as they talked about their achievement or ability. In our codebook, we defined ability/achievement to be "a personal trait of the student, a characteristic that defines the student as a person, a fixed quality that relates teachers' views of students' aptitude in mathematics due to either an innate capacity or previous work."

To facilitate the data analysis, we divided the data into four time periods: Quarter I (the first summer institute—30 hours); Quarter II (the Fall monthly meetings—8 hours); Quarter III (the Spring monthly meetings—10 hours) and; Quarter IV (the second summer institute—12 hours). After data from all sessions were coded, further analysis of the achievement/ability code was conducted by looking across all coded talk turns. Open coding allowed for the creation of sub-themes that emerged within this set of data. In the next section, examples illustrate one of these themes along with examples of the influence of the LT.

Results

While examples of teachers' use of ability/achievement to represent students were present in all four time periods, by the end of the professional development, some teachers appropriated this representation in different ways. From our analysis, we noted several patterns within the data related to this evolution, including the nature of teachers' surprises, the role of language, and task differentiation. For this paper, we present one theme along with evidence of the ways the LT was incorporated into the use of the ability/achievement representation in the professional development community.

The Nature of Teachers' Surprises

One trend related to teachers' reference to ability/achievement was the issue of what teachers found surprising about students. In Quarter I, many of the teachers were surprised when students that they typically considered "low," "struggling," or "weak" in mathematics were successful at

completing a task. Consider the following statement made early on in the professional development. After teachers we asked to reflect on when students' mathematical work surprised them, one teacher (G1) stated:

> *G1:* When kids who seemingly struggle with math in general have something that they get just like that, it is surprising. And you definitely have those moments where you want to run out of the classroom. For whatever reason they didn't get nothing else you said all week and they might can't connect it to nothing else you said, but they got that one.

In this example, G1 used the representation available within the discourse of their school community, which we infer to be that students who "struggle" are not successful. Likewise, some teachers described the opposite of this -- when a "high student" or "gifted student" experienced difficulty with a task. For instance, when discussing her students' work on an assessment, C4 states:

> *C4:* The high kids, I really thought they were going to fly through it, but they didn't…But then like my top kid, he gave some out and counted it and then was like guessing and checking…adding one, looking at it, erasing, and he had no strategy.

As with G1, C4 appropriated a representation that students' ability and/or achievement levels predict students' work on mathematics tasks.

Across the duration of the professional development that focused on supporting teachers in learning the EPLT, some of the teachers' uses of ability/achievement related to "low" and "high" students changed to incorporate language from the EPLT. When asked what teachers learned about their students' understanding of equipartitioning, E3 stated:

> *E3:* I taught some of the lower kids in math. At the beginning of the year I thought…wow we've got our work cut out for us. And one of the first tasks that I had kids do was sharing a region and one of the little girls used parallel cuts and got the right number. Cause a lot of times they often think if you're making four shares, they make four lines. But she knew to make three lines…and so I asked her, "If you wanted to make this many equal parts, what would you do?" And she said, "If it's four equal parts, then I'm gonna make one less cuts." It's just amazing…it totally challenged this preconceived idea and notion I had of what my students could do and couldn't do and had knowledge of.

Here, E3 references the ideas of "parallel cuts" and the misconception relating the number of splits to the number of parts produces from the EPLT when discussing what she found surprising for the "lower kids." In a similar manner, teachers expressing surprise about "higher" students also referenced ideas from the trajectory, in particular the proficiency level concerning the naming of results from equipartitioning. During the same discussion as above, C2 stated:

> *C2:* We split our classes and I took the higher of the kids for math. And the naming…they still don't get that piece for as high as they are, they still don't get it. That piece that says, "What would you call this share?" They come up with crazy things so that is something that I see they still need practice.

In addition to being surprised by "low" or "high" students that deviated from their expectations, teachers also attributed students' success or failure in mathematics to their number sense. Although these examples differ slightly from those presented above, teachers' still expect that students with "low number sense" will have difficulty with or not be able to complete mathematical tasks. In Quarter II, teachers were asked to use an equipartitioning task to assess their students. The 2nd grade teachers had students share a collection of 24 items with different numbers of people. When reporting on one student in her class, D2 expresses her surprise:

> *D2:* His number sense is not high. But when we were doing the sharing activity, he was like "I know how to do this! One for me, one for you, one for me, one for you." And he said "it could go faster if we did two for me and two for you." And I was like "God!"

Here, D2 is surprised not only that her student was able to fairly share the objects, but also that he recognized that sharing by two is more efficient that sharing objects one at a time as highlighted in the EPLT. This initial way of representing the student in relation to his number sense was problematic because when teachers initially spoke of students' number sense, it was addressed as a fixed characteristic of students.

While teachers' statements from early in the professional development demonstrate that they recognize instances where students differ from the accepted notions of ability and achievement, the statements lack the mathematical specificity useful for building future instruction for either "group" of students. Later as teachers learned about students' mathematics as represented by the EPLT, their dialogue and the nature of their surprise included language about students' mathematical thinking and activity that was more productive for their community when thinking about teaching and learning. This suggests teachers' learning of the trajectory provided additional information for teachers when using this particular representation to talk about students.

Discussion

Our goal for this paper was to provide evidence of the ways teachers' discourse about students related to ability and achievement changed as they learned about one LT. When teachers position students as "low" or "high" achievers or as having "low number sense", there exists an implicit idea that students' ability is beyond the control of the teacher and thus, these statements are not useful for teachers' to consider their instruction. One conjecture we made during this analysis was that teachers' use of language that positions students in this way would decrease over time because teachers would come to use the LT to orient their representations of students as learners. While we did not find this to be the case, we did find evidence that teachers began to use the language and ideas from the trajectory in addition to the "low" and "high" labels.

When teachers represent students as a "low student" or as someone with "poor number sense", they take a deficit approach towards these students by focusing on what they do not have as opposed to what they do have However, when teachers use the LTs to talk about students, they begin to acknowledge what mathematical knowledge the student brings to instruction and discuss their mathematical successes or failures using language from the LT. Because the trajectory provides teachers with a progression of how students' thinking develops over time, teachers are able to situate students' understanding and use the LT as a referent when designing future instruction. The following statement made by a teacher at the conclusion of our professional development speaks to the value of LTs and their usefulness in changing how teachers talk about students:

> *D2*: I think from what we've learned…learning is not just a linear thing where we can pigeonhole kids and put them in specific groups, particularly because we know they learn through experiences with their peers and all of those things.

As our analysis of these data continues and through our next cycle of design, we are continuing to examine the ways in which teachers represent students as learners of mathematics. With our current cohort of teachers, we have deliberately challenged their positioning of students from the onset of the professional development as we seek to change the discourse of the community. When discussing their students, we ask teachers to describe what students "can" do

as opposed to what they cannot do and to use evidence to ground their claims. Although we did not eliminate the deficit-oriented language from teachers' discourse in the first cohort and we continue to observe this language with the current cohort, we have caused teachers to think more carefully and purposefully about how they talk about students. van Langenhove and Harré (1999) suggested that instead of attending to individuals holding particular stereotypes, one should attend to the act of *stereotyping* within the discourse of a group and that stereotyping is a speech act that builds upon the "acceptable" social representations held by that community. While contend that LTs can be a useful tool in disrupting what is deemed as "acceptable" in a professional development community, we recognize that LTs are not sufficient in eliminating stereotypes. One conjecture is that the language teacher's use in their professional learning teams, grade level teams, and even at the school level is still acceptable and thus continues to re-surface in our professional development group. Findings from this work indicate that while LTs can cause teachers' to focus on individual students and their mathematical thinking, they are not enough to re-write the metanarratives that teachers hold about groups of students. Therefore, future professional development projects should include learning about LTs with a specific focus on unpacking deficit orientations and equity.

Acknowledgement

This report is based upon work supported by the National Science Foundation under grant number DRL-1008364. Any opinions, findings, and conclusions or recommendations expressed in this report are those of the authors and do not necessarily reflect the views of the National Science Foundation.

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TASKS TO SUPPORT PRESERVICE TEACHERS' UNDERSTANDING OF DECIMAL QUANTITIES

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In this article, we present some mathematical tasks designed to support pre-service teachers (PSTs) in obtaining a broader, more connected understanding of decimal numbers and quantities. We describe classroom scenarios from a content course in which PSTs experienced these cognitively demanding decimal tasks, enacted in ways that maintain high demand. We present evidence of shifts in quality of representations and explanations from pre- and post-test responses of PSTs to unique decimal comparison tasks. Prior to a unit on decimal fractions, several PSTs utilized incorrect decimal notation for amounts such as "0.16" for "16-tenths," relying on symbolic representations they may not fully understand. Written explanations of decimal comparisons commonly referred to procedural rules. After the decimals unit, responses to comparison tasks showed an increase in use of meaningful, connected representations and richer, more robust justifications.

Keywords: Teacher Education-Preservice, Rational Numbers, Mathematical Knowledge for Teaching

Teaching mathematics for understanding requires teachers to know the mathematics they teach with depth and flexibility (CBMS, 2001; CBMS, 2012; Ball, Hill, & Bass, 2005). However, studies have shown that elementary teachers in the U.S. appear to lack profound mathematical understanding (Ma, 1999). To support preservice teachers (PSTs) in experiencing mathematics as a subject that relies on reasoning and in learning mathematics with understanding, we have been developing and modifying curriculum materials to be used in content courses for PSTs.

Based on the recommendations of Stein and colleagues (2009), these curriculum materials feature cognitively demanding tasks, which require learners to solve problems in multiple ways, to make connections among representations and solution methods, and to explain their thinking. Critical in the use of such tasks to develop understanding in children is the teacher's ability to maintain a high level of cognitive demand during classroom implementation. Building on students' prior knowledge, allowing sufficient time to explore, and pressing students to justify their reasoning are all factors associated with the maintenance of high-cognitive demand of the tasks in classrooms of children. We wondered: Do these same features of mathematical tasks and their enactment in cognitively demanding ways also support a diverse group of adult students (PSTs) to learn mathematics in deeper, more connected ways? Can we measure PST improvement in their abilities to represent mathematical ideas in multiple ways, to make connections among those representations, and to use them to help justify why the mathematics works? These were questions we set out to study.

In this report, we present some of the tasks designed to help our PSTs develop meanings and representations of decimal fractions, content shown to be challenging to both children and their adult teachers (D'Ambrosio & Kastberg, 2012; Martinie & Bay-Williams, 2003; Putt, 1995; Stacey et. al., 2001). In particular, we examine some of the features of decimal tasks and their enactment that seem to play a role in broadening PSTs' conceptions and representations of

decimals. To measure shifts in quality of representations, depth of reasoning, and validity of justifications, we analyzed PST responses from a pair of items on pre- and post-tests that challenged PST reasoning about decimal quantities.

Theoretical Framework

An understanding of decimal numbers and quantities requires knowledge of both base-ten place value and fractions, two conceptual fields that are known to be difficult for learners (Fuson, 1990; Kamii, 1986; Luo et al., 2011; Post et. al, 1991; Ross, 1989; Siebert & Gaskin, 2006; Sowder et al., 1998; Thanheiser, 2009; Zazkis & Khoury, 1993). Limited understanding of base ten relationships in decimal notation and of connections between fractions and decimals leads to patterns of decimal comparison errors that have been well studied (Resnick, et al.,1989; Stacey et al., 2001; Steinle & Stacey, 2004). Because the Common Core State Standards for Mathematics introduce decimal fractions in grade 4 as, “Understand decimal notation for fractions, and compare decimal fractions” and build on this knowledge in later grades (CCSSI, 2010), elementary teachers need to possess a deep and well-connected understanding of the base-ten and fractional nature of decimal numbers. Due to the diversity present in nearly every classroom, teachers should be able to use a variety of representations and approaches to support this understanding.

To address shallow and incomplete understandings, several researchers have suggested activities to help with making connections among multiple representations (Suh et al., 2008). Ten-by-ten grids, that embody the base-ten structure of decimals, have been proposed to help learners visualize the order of magnitude represented by place value positions (D’Ambrosio & Kastberg, 2012; Martinie & Bay-Williams, 2003) and Cramer and colleagues (2009) emphasized making connections between and among representations (symbolic, pictorial, and verbal). In an earlier report, (Rathouz & Cengiz, 2012), we found that introducing PSTs to such diagrammatic representations as well as appropriate verbal re-namings (e.g. “two-tenths” rather than “point two” for 0.2), helped them to make sense of the relative sizes of decimal quantities.

Presenting representations imbedded in tasks that require a high level of cognitive demand has been shown to further support children’s learning. Stein and her colleagues (2009) found that asking learners to solve problems in more than one way, to make connections among solution methods, and to explain their thinking, all contribute to the cognitive demand of the task. However, maintaining a high level of demand during classroom instruction was found to be just as important in children’s learning. What does this mean for our courses where adult PSTs are trying to make sense of prior mathematical knowledge that they themselves will need to teach? In this study we investigate whether enactment of mathematical tasks in ways that have been shown to support children’s learning also foster deep, connected understanding in adult learners. We conjecture that additional task features, such as reflecting on and reasoning about others’ mathematical thinking, seem to contribute further to the demand while introducing PSTs to work they will be doing to support learning in diverse populations of their future students.

Methodology

The data for this study were gathered from the first of three mathematics content courses for preservice elementary and middle school teachers. The subjects were 160 PSTs enrolled in this course during one of five semesters (Winter 2008 – Fall 2010). The authors were the instructors for the course sections during this time. The data include pre- and post-tests on PSTs’ knowledge of decimals, PSTs’ written work, and videotapes of lessons from the decimals unit.

In this report, we present data on one item from pre- and post-tests, describe mathematical tasks implemented during the unit on decimals, and provide sample classroom discussion to give insight into how shifts in thinking about decimals might happen in such environments.

Pre- and Post-Test

The pre-test was administered before the decimals unit began and the post-test was given at the end of the unit. PSTs were given 15 minutes to complete the items and neither test was used in determining course grade. Unlike typical decimal comparison items where numbers are written in decimal form (Stacey et. al, 2001), this pair of corresponding items, listed in Table 1, required PSTs to compare two amounts that are written in numerals and words, necessitating the coordination of at least two representations. Further, PSTs were asked to provide explanations for their comparisons. Many used diagrams to justify their thinking.

Table 1. Items About Comparing Decimals.

Pre-test Item	Post-test Item
Which one is bigger: **16-tenths** or **134-hundredths**? Show all work and briefly explain how you decided which is larger.	Which one is bigger: **18-tenths** or **172-hundredths**? Show all work and briefly explain how you decided which is larger.

The analysis of PSTs' representations and written work on these decimal comparison items was an iterative process, briefly described here and in more detail in a prior report (Rathouz & Cengiz, 2012). First, we categorized the types of representations that were used: symbolic; verbal; everyday life context; and diagrams; and the correctness of the representations. Next, we determined strategies for **renaming the quantities**, including (a) finding a common denominator; (b) making a whole with some leftover; and (c) positioning the right-most digit of the numeral in the named place value position (i.e. for 134 hundredths, the "4" would be placed in the hundredths place); (d) using division. The strategies (including both correct and incorrect ones) for **comparing the quantities** were categorized as (a) using common denominators or same length decimals; (b) using a benchmark; (c) comparing matching place value positions; (d) comparing the denominators or place value names/positions (i.e., tenths are larger than hundredths); (e) comparing the numerators or digits; (f) shorter is larger (e.g., 0.6 is closer to the decimal point); (g) longer is larger; (h) erroneous place value comparison; and others.

PSTs' explanations for both renaming and comparing decimals were evaluated using the following levels: explicit and valid explanation (3); partial explanation (2); description of the process or rule without explaining why (1); no explanation, only computation or representation (0); and confusing/incorrect (-1). The structure of the item revealed PSTs' strategies of representing, renaming, and comparing decimal quantities. Thus, PST responses on these items were used as one measure of effectiveness of the decimal curriculum materials and their classroom implementation.

Decimal Tasks and Enactment

Lessons in the content courses are 100 minutes long and focus on two or three tasks during one session. These tasks are designed to provide PSTs with opportunities to explore and make connections among important mathematical ideas. PSTs typically work on the tasks first individually; then share their thinking in small groups. Finally there is a discussion where solutions are shared and synthesized as a whole group. The focus of in-class discussions is not only on how problems are solved, but, importantly, why the solutions make sense. The expectation of justifying solutions and mathematical claims challenges PSTs' thinking.

The unit on decimals is designed to support PSTs' developing knowledge about meanings and representations of decimals (such as currency,10-by-10 grids, fraction notation and place-value language) and connections among these meanings and representations. Throughout the unit PSTs are encouraged to use both meanings and representations to support their justifications. See Table 2 for a selection of tasks from the decimals unit.

Table 2. Selected Decimals Tasks From Coursework

Task A: Renaming 0.1	Rename this number [0.1] in more than one way. You could use numbers, words, drawings, and/or contexts.
Task B: Representing fractions as decimals	Find four different ways to show 1/4 on a 10 x 10 grid by shading in that portion of the grid. Try to choose ways that allow you to clearly see the result is 1/4 without requiring that a person count every shaded box. Explain how your ways also show the decimal for ¼.
Task C: Analyzing students' re-namings	Several students were discussing different ways to name 245-hundredths. In evaluating the following students' claims, first create your own representation of 245-hundredths using 10 x 10 grids. Bradley said that 245-hundredths could be represented as $\frac{2450}{1000}$. Use both words and a diagram to validate Bradley's claim. Amira said that she could write 245-hundredths as 0.245. Use both words and a diagram to persuade Amira that these two expressions are <u>not</u> equivalent.

Results

Implementation of Decimals Tasks

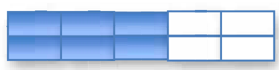
As a formative assessment of PSTs' conceptions of decimal fractions and notation at the beginning of the unit and to reinforce the course emphasis on making sense of numbers and operations, we pose the following for the whole class to consider: "Rename this number [0.1, written on the board, not verbally read] in more than one way. You can use numbers, words, drawings, and/or contexts." The task of finding ways to rename 0.1 [Table 2, Task A] provides opportunities to raise PSTs' awareness of their dependence on previously learned, but not understood, place-value rules and to make sense of the rules by pressing PSTs to think about the structure of our number system. Their challenge is to produce arguments that convince others that their renamings are equivalent to 0.1.

The PSTs begin by working independently to give individual time to think and to generate a greater variety of responses. They share briefly their renamings in small group discussions. This gives reluctant participants a chance to try out their ideas and to begin to challenge each other to explain how they know that their renaming is equivalent to 0.1. The small group discussions also provide the instructor with a window into the kinds of thinking present in the group as a whole. She can choose, from among the PSTs' renamings she sees and hears, those she wishes to pursue in the whole-group discussion. The sample discussion below exemplifies the kinds of reasoning we have heard from PSTs in our courses. Such typical dialogue is also provided in the lesson plans to aid in instructor preparation. All PSTs have been given pseudonyms.

Instructor: We have a lot of different ways you thought of to rename this number. Is everyone comfortable with all of these or are there ones you are questioning?

Katy: Well, at first I wrote ".6 + .4 = .10," but I think it's wrong now.

Instructor: What made you change your mind about that one, Katy?
Katy: I was thinking….that's "point six and point four equals point ten." Then Jason reminded me that point six is really six-tenths and that's already bigger than point one…I mean, one-tenth.
Instructor: Something we all need to practice is using language that is meaningful. Instead of using "point" language, how did it help to say "six-tenths?"
Amanda: It helped me when I wrote ".01 + .09 = .1" as my renaming. 'Cause I thought back to fractions like one-hundredth and nine-hundredths: [writes on board $\frac{1}{100} + \frac{9}{100} = \frac{10}{100}$]. And I think you can write that fraction like this ".1" as a decimal.
Instructor: What do others think about Amanda's justification? What else might someone say to help understand her way of renaming this number [0.1]?
Asma: You could think about it as, like, money. I wrote, "0.01 x 10 = 0.1," like one penny is a hundredth of a dollar and I've got ten pennies, so that's a dime. Amanda just did one penny and nine pennies.
Jason: I wanted to see if I could use division, so I wrote, ".6 ÷ 6 = .1" and drew this diagram.



Instructor: *Hold on a sec… don't tell us about the diagram yet, Jason. Can someone besides Jason figure out how his diagram might help to show his renaming equation?*
Samantha: *I guess it's showing a whole of something, maybe a pan of brownies. Then there is only point-six or six-tenths left of the pan and you are sharing that amount with six people, so everyone gets a tenth.*
Instructor: *That was very nice, Samantha, to connect Jason's equation back to a sharing meaning of division.*
Zeinab: *We were talking about that one [0.025 x 4] at our table….None of us felt comfortable with how to explain it.*
Nichole: Now, when I see something like that, I always multiply 25 times 4, which is 100 and then remember to move the decimal point three places for decimal places in 0.025 so that give you .100. But I guess that sounds kind of like a rule.

Even in this seemingly trivial initial task, many mathematical ideas are introduced, representations shared, and class norms re-established. For example, Katy acknowledges an incorrect idea first and shares thoughts on editing. Amanda makes connections to a course unit on fractions recently completed. Asma considers currency as a real-world use of decimal numbers and Jason reminds the class of area models and a meaning for the operation of division. Typically, the instructor will prompt the PSTs to comment on others' solutions and to use more helpful fraction language rather than "point" language. Zeinab's question about a number with three decimal places provides an opportunity for others to enter a discussion regarding the structure and language of the base-10 place value system. Nichole's computation shows that, at this early stage in the unit, some PSTs' explanations are based on rules. As students attempt to show decimals in the thousandths place, they require the discussion of very small pieces. This fact motivates the use of 10-by-10 grids and of more descriptive place-value language.

A second task [Table 2; Task B] continues to help connect familiar decimal numbers to their fraction equivalents using 10-by-10 grids as well as students' own representations. Representing ¼ several ways on the 10-by-10 grids, PSTs visualize 0.25 as both "2 tenths plus 5 hundredths"

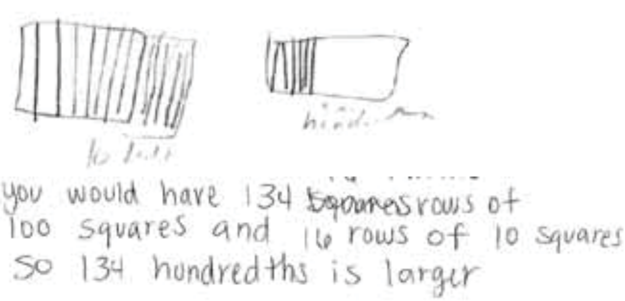
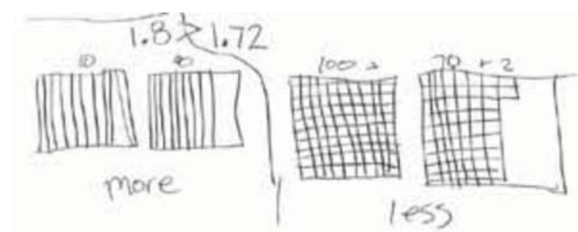
and "25 hundredths." As they move to producing 1/8 on the 10-by-10 grid, they run up against further renaming issues. Some use the relationship between ¼ and 1/8. They figure that 1/8 is half of 25 hundredths and arrive at 12 ½ hundredths as their equivalent. Others write the decimal as 0.125. It is then the work of the entire class to justify why 125 thousandths (0.125) is the same as 12 ½ hundredths. This is non-trivial work and involves using base-10 language and ideas. For example, one hundredth must be viewed as both split in half and split into ten equal thousandths with five of those thousandths shaded.

A third task involves renaming a decimal quantity and evaluating other students' reasoning about equivalence [Table 2, Task C]. The PSTs work on the task as homework and then share ideas during a whole-class discussion. As they evaluate Bradley's and Amira's reasoning, PSTs confront their own incorrect ideas about renaming decimal quantities, use representations to clarify their thinking, and work on arguments to convince others that quantities are equivalent. For example, Amira's solution provides a typical error similar to one PSTs made on the pre-test. Evaluating the correctness of other students' renamings requires the use of 10-by-10 grids or other diagrams. The introduction of fraction notation and verbal representations (such as 24-tenths + 5-hundredths; a third renaming suggested) provide fodder for discussion of connections to the base-ten structure of decimal numbers.

Shifts in Representations

To document the types of learning that PSTs experience during tasks in the decimals unit described above, brief pre- and post-assessments were administered containing items that required PSTs to represent, compare, and order decimal quantities. The most dramatic changes we noted in PSTs' representations were shifts from incorrect decimal notation [0.16 for 16-tenths and 0.134 for 134-hundredths] on the pre-test to correct decimal notation on the post-test. Additionally, PSTs began to utilize 10 x 10 grids to represent the quantities (see Table 2) and to successfully compare those quantities (Rathouz & Cengiz, 2012).

Table 2. Sample PST Responses for Using 10-by-10 Grids

An incorrect response from the pre-test	A correct response from the post-test
16-tenths vs. 134-hundredths  	18-tenths vs. 172-hundredths 

Strategies and Explanations

The explanations for and reasoning about why one quantity was larger than the other also shifted in two critical ways from the pre- to the post-test. The first change we documented was a decrease in the use of incorrect or incomplete strategies. For example, on the pre-test, eighteen PSTs (11%) held the misconception that numbers with fewer digits past the decimal place were always smaller (the so-called "shorter is larger" misconception). Such PSTs claimed, "16 tenths is bigger because tenths is closer to the decimal, meaning it's closer to being a whole number."

On the post-test, only five (3%) of the PSTs used this incorrect idea. Interestingly, these five PSTs had used other incorrect strategies on the pre-test to explain their comparison.

Another 11% of PSTs on the pre-test held a related, incomplete conception that any number of tenths is larger than any number of hundredths: "When you have 16 tenths, it's bigger because each piece is bigger. With hundredths, you are basically chopping the tenths up into hundredths." Again, observation of this strategy decreased on the post-test to 5% (8 PSTs), the majority of these PSTs having used other erroneous strategies on the pre-test. Twenty-two PSTs (nearly 14% on the pre-test) were unaware of the distinction in language between "tens" and "tenths" and, therefore, compared the quantities incorrectly. This error was completely absent on the post-test.

The other shift we noted was from use of rules-based justification or claims without justification to use of justifications that were based on reasoning. For example, on the pre-test, several PSTs calculated, "$\frac{16}{10}\times\frac{10}{10} = \frac{160}{100}, \frac{160}{100} > \frac{134}{100}$," without any explanation. A typical rules-based responses was, "16 tenths is bigger than 134 hundredths. I decided that because when they have the same number of places, 160 is bigger than 134." On the post-test a PST compared 18-tenths and 172-hundredths, "If we rename the 8/10 left over into 10 times smaller pieces, there would be 80/100 pieces. Compared to 72/100 pieces, there are more with 80/100." Some PSTs referred to 10-by-10 grids in justifying why 18 tenths is the larger quantity: "180 hundredths vs. 172 hundredths. 18 columns with 10 boxes in each column. That gives you 180 boxes shaded, which is 180 hundredths because the columns are tenths and the boxes in the tenths are hundredths. 180-hundredths is more than 172-hundredths." There were also a few PSTs who made reference to everyday life contexts, such as currency: "I looked at it as 18 dimes, because a tenth of a dollar is a dime so if I had 18 dimes I have $1.80. If I had 172 hundredths that would be 172 pennies because a penny is a hundredth of a $1. So 1.80 is more than 1.72."

Conclusion

This study contributes to efforts to understand better how to support future elementary teachers in obtaining deep, connected knowledge of the mathematics they will teach. Prior to our decimals unit, future teachers struggled to understand the meanings of decimal numbers and to use those meanings to solve problems of equivalence and comparison. Whole group discussions about decimal renaming and introduction of representations such as 10-by-10 grids provided tools to help learners reason about the relative sizes of decimal quantities. Evidence from the post-test shows that PSTs began to visualize that 18 tenths makes more than a whole (10/10 + 8/10) and to explain why 8 tenths could be renamed as 80/100.

Throughout the course, we challenge our future teachers to analyze others' solutions and representations, both correct and incorrect. Pressing PSTs for justifications during whole group discussions encourages them to explain their reasoning and provide arguments convincing to others. Discussions that allow PSTs to examine and reflect on their own, often incorrect, procedural knowledge provide them with an opportunity to practice precise mathematical language they will use later in teaching.

The work described here helps to document changes in PST mathematical reasoning and representations, as they experience high-demand tasks and classroom implementation similar to those recommended by Stein et al. (2009) for classrooms of children. The results of this study suggest that these same strategies may support future teachers' development of mathematical knowledge they will need for teaching diverse groups of children.

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LEARNING TO PROFESSIONALLY NOTICE: PRESERVICE ELEMENTARY TEACHERS' ATTITUDES TOWARD MATHEMATICS IN CONTEXT

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The goal of this study is to better understand preservice elementary teachers' changes in attitudes towards mathematics in connection with their participation in a module aimed at developing professional noticing capacities. This module, typically implemented in the mathematics methods course, involves practice with the three interrelated components of professional noticing – attending, interpreting, and deciding. Pre- and post-assessments of participants' mathematical attitudes and professional noticing capacities were administered to measure change in these areas. Participants demonstrated significant growth in their professional noticing capabilities and mostly positive attitudinal change; however, there was no significant correlation between the changes on the respective measures.

Keywords: Affect, Emotion, Beliefs, and Attitudes; Learning Progressions; Teacher Education - Preservice

Introduction

This study focuses on preservice elementary teachers attitudes towards mathematics after participation in an instructional module developed with the intent to increase PSETs' abilities to professionally notice within the context of early numeracy. The instructional module, titled *N3: Noticing Numeracy Now*, was implemented in elementary mathematics content and pedagogy courses at five universities. This research is informed by the literature on preservice teachers' attitudes toward mathematics (Philipp, 2007), Jacobs, Lamb, and Philipp's (2010) definition of professional noticing, pedagogies of practice (Grossman, Compton, Igra, Ronfeldt, Shahan, & Williamson, 2009), and the progression of early numeracy (Clements & Sarama, 2009; Steffe, 1992; Steffe, Cobb, & von Glasersfeld, 1988; Steffe, von Glasersfeld, Richards, & Cobb, 1983). More specifically, we aim to investigate the following research questions:

1. To what extent can PSETs develop the capacity to professionally notice children's mathematical thinking in the context of early numeracy?
2. To what extend do PSETs' attitudes towards mathematics change after participating in a mathematics methods course where professional noticing skills were developed?
3. To what extent does PSETs' professional noticing performance correlate with PSET attitudes towards mathematics?

Theoretical Framework

Preservice Elementary Teachers' Attitudes Toward Mathematics

Research on attitudes towards mathematics has become an increasingly prominent area of study. Attitudes are often defined as a component of *affect* (Philipp, 2007), which has various

meanings in the field of psychology (Chamberlin, 2010). For the purpose of our study, we draw upon Philipp's (2007) definition of attitudes as "manners of acting, feeling, or thinking that show one's disposition or opinion. ...Attitudes, like emotions, may involve positive or negative feelings" (p. 259). Research on attitudes is critical because students and teachers often develop negative attitudes towards mathematics, which can later result in anxiety (Quinn, 1997). PSETs tend to view mathematics as a system of rules and procedures that must be transferred to students (Ball, 1990; Foss & Kleinsasser, 1996). Moreover, they tend to view mathematics negatively or with neutrality, but rarely positively (Ball, 1990; Bekdemir, 2010; Quinn, 1997), which has the potential to influence the practices they adopt. Research in mathematics education has shown that attitudes often influence teachers' classroom practices (McLeod, 1994; Wilkins, 2008). Thus, it is important to understand the attitudes PSETs hold as an aspect of the types of practices PSETs may develop. Ambrose (2004) argued that too often the view of PSETs' attitudes and beliefs is negative and stereotypical and suggested that mathematics educators focus on the range of strengths PSETs bring, such as the PSETs' view of teachers as nurturers of children. Jong and Hodges (2011) have also found that it is possible for PSETs to experience positive changes in attitudes as a result of completing a mathematics methods course. Our work focuses on PSETs and their development of attitudes towards mathematics by engaging them in professional noticing of children's mathematical thinking in the context of an early numeracy progression.

Professional Noticing

The construct of professional noticing, as defined by Jacobs et al. (2010), is "a set of three interrelated skills: attending to children's strategies, interpreting children's understandings, and deciding how to respond on the basis of children's understandings" (p. 172). The first skill, attending, contains physical evidences observed from the student and teacher, such as eye movements, finger counting, and touching objects to count, just to name a few examples. The second skill, interpreting, is determining how those observations in the attending category can inform the observer on the mathematical abilities of the students. Finally, deciding involves the next steps in the process, which can include diagnostic or instructional decision-making. Jacobs et al. (2010) found that teaching experience alone does not contribute to an increase in professional noticing skills; professional development in the area of professional noticing is needed to adequately develop these skills, especially in the deciding component. Several studies have found that closer attention to children's mathematical thinking can significantly impact student learning (Carpenter, Fennema, Franke, Levi, & Empson, 1999; Kersting, Givven, Sotelo, & Stigler, 2010); however attention to the three individual components of professional noticing is missing from much of the previous research.

Pedagogies of Practice

Defining professional noticing as three components is an example of the decomposition of practice, or breaking the complex practice of teaching into smaller parts to better focus the teaching and learning of practice (Grossman, 2011; Grossman et al., 2009). Decomposition of practice is one of three pedagogies of practice, a framework for describing the teaching of practice. Representations of practice, another pedagogy of practice described by Grossman et al. (2009), are artifacts that make practice visible to others. Video is one type of representation of practice, but representations can also include lesson plans, student work, and teacher reflections about their practice. Inexpensive video equipment makes video an increasingly accessible and rich representation of practice, providing for "opportunities to investigate the complexity of teaching offline" (Grossman, 2011, pp. 2836-2837). Video has been used successfully to focus and hone teachers' attention to children's mathematical thinking (Carpenter et al., 1999; Schifter,

Bastable, & Russell, 2000; Seago, Mumme, & Branca, 2004) and to features of a classroom (Star, Lynch, & Perova, 2011; Star & Strickland, 2008). Attending, however, is just one of the three components of professional noticing. Intentional collections of video representing children's work along a mathematical progression, in this case, early numeracy, enriches the representation to allow for development of the other two components of professional noticing, interpreting children's thinking and deciding next diagnostic or instructional steps. Such detailed analysis of representations of practice provide support for PSETs' participation in the third pedagogy of practice, approximations of practice, described by Grossman et al., (2009) as those activities that closely resemble authentic practice but might take place in less intense environments than an authentic classroom. Roleplaying is one example of an approximation of practice.

Early Numeracy Progressions

As a portmanteau of "numerical literacy", the term "numeracy" is typically invoked to describe an understanding of number and arithmetic operations. This area of mathematical learning has been the subject of considerable study over the past four decades (Clements & Sarama, 2009; Fuson, 1988; Gelman & Gallistel, 1978; Siegler & Robinson, 1982; Steffe, 1992; Steffe et al., 1983; Thomas & Harkness, 2013; von Glasersfeld, 1982; Wright, Martland, & Stafford, 2006). From the differing progressions of numeracy development available, we chose to focus on the developmental perspective put forth by Steffe and his colleagues (Olive, 2001; Steffe, 1992; Steffe et al., 1988; Steffe et al., 1983). This progression, referred to as the Stages of Early Arithmetic Learning (SEAL), is the product of extensive teaching experiments aimed at determining authentic student actions and practices when dealing with problematic arithmetic tasks (Steffe & Thompson, 2000). SEAL provides a highly detailed explication of children's changing understanding of quantity and serves as the mathematical backdrop for PSET learning activities in this study.

Supporting Attitudinal Improvements

Situating the professional noticing of children's mathematics in the context of mathematics progressions, illustrated through video representations of children's work, capitalizes on PSETs' nurturing attitudes about teaching but also reveals to them the complexities of the mathematics content. The content of early numeracy, on the surface, seems simple for PSETs to understand, because it encompasses such skills as forwards and backwards counting, skip counting, and addition and subtraction of numbers within 100. As PSETs view video vignettes of children engaged in mathematical thinking along the early numeracy progression, they are exposed to the idea that counting, for example, is not an all or nothing skill. The children in the videos display nuanced understandings and skills that demonstrate the incremental, but important steps through which children progress. Thus, we purport that engaging PSETs in professionally noticing children's mathematical thinking through representations of practice of real children engaged in early numeracy reveals the complexities of the content, and will result in a positive shift of PSET attitudes toward mathematics.

Methodology

Participants

The participants were PSETs enrolled in an elementary mathematics methods or content and methods blended course at one of five participating public universities in a south central state. The module was a component of the methods or blended course at each institution. One hundred twenty-three PSETs completed the pre- and post-assessment of professional noticing and the pre- and post-assessment of the Attitudes Toward Mathematics Inventory (ATMI).

Module Description

The module consisted of multiple in-class sessions in which professional noticing was developed in the context of early numeracy, specifically, SEAL. The decomposition of professional noticing into three interrelated skills allowed for the skills to be progressively nested throughout the module sessions (Boerst, Sleep, Ball, & Bass, 2011). The first two sessions focused solely on the development of attending. Subsequent sessions further developed attending with interpreting and deciding. SEAL was nested within the development of professional noticing and integrated through video cases as representations of practice. The culminating experience was an assignment that required the PSETs to conduct at least one diagnostic interview with a child, an approximation of practice, one of the three pedagogies of practice proposed by Grossman et al. (2009).

The researchers intentionally chose early numeracy as the context through which to develop the PSETs' professional noticing skills. Videos cases of teacher/student exchanges provided opportunities to explicitly attend to and discuss salient features of children’s mathematical thinking often unnoticed by novice teachers in real-time classrooms. Because the video clips represented children's thinking along the progression of SEAL, interpreting the nuances of the children's thinking was supported by hallmark examples of each stage of SEAL. Decision-making was also supported by the common progression of children's thinking outlined by SEAL. Using video of real children as a precursor to engaging in their own diagnostic interviews with children provided opportunities for PSETs to anticipate and plan more specifically for what might occur in real-time classroom events.

Data Sources

Professional Noticing Assessment

A pre- and post-assessment was used to measure the changes in professional noticing at the beginning of the semester and again at the end of the semester. A video of a diagnostic interview with a child completing a comparison, difference unknown task (Carpenter et al., 1999) was used and both pre- and post-assessments were identical in prompts and video. The brief, 25 second, video shows an interviewer presenting a first grade student with a partially screened task that extends beyond finger range. The screened component consists of eleven seashells hidden by the interviewer’s hand and the visible component is seven red counting bears in a row. The student is asked to determine how many more shells there are than bears. Counting the bears from one and continuing the count on his fingers until he reaches eleven, the student then glances at his raised fingers and correctly responds, “I’m gonna have four left over.” A screenshot from the video showing the setting and the child’s use of fingers is in Figure 1 (Schack, Fisher, Thomas, Eisenhardt, Tassell, & Yoder, 2013).

Figure 1: Child Completing the Comparison Task (Schack et al., 2013)

After the PSETs watched the video, they were asked to respond to the following prompts and questions: 1) Please describe in detail what this child did in response to this problem, 2) Please explain what you learned about this child's understanding of mathematics, and 3) Pretend that you are the teacher of this child. What problems or questions might you pose next? Provide a rationale for your answer. These prompts were drawn from the work of Jacobs et al. (2010) and each prompt addresses one of the components of professional noticing.

To develop scoring benchmarks for each of the professional noticing prompts, key response details for each prompt were outlined and coupled with emergent themes identified through examination of PSET responses. This process was used for each prompt and resulted in four potential rankings for attending, three for interpreting, and three for deciding. The high rank of four for attending represents an emergent theme from PSET responses that represented an elaboration beyond the salient attending features. A similar elaborating theme did not emerge for the remaining two professional noticing components. Teams of two scorers ranked all data using the scoring benchmarks. Discrepancies in ranks were resolved through discussion and/or a third scorer.

Attitudes Toward Mathematics Inventory

The Attitudes Toward Mathematics Inventory (ATMI) was also administered as a pre- and post-assessment approximately at the same time as the professional noticing assessment. The ATMI is an instrument consisting of 40 Likert-scale items used to measure the following four factors associated with attitudes toward mathematics: value, enjoyment, self-confidence, and motivation (Tapia & Marsh, 2005). The ATMI is a reliable instrument with content and construct validity (Tapia & Marsh, 2004). It has been primarily used with both secondary and college students.

Data Analyses and Results

Professional Noticing Assessment

Our research questions sought to determine changes in PSET professional noticing and attitudes toward mathematics along with the correlation of changes between these areas. The professional noticing data were analyzed using non-parametric statistics because of the ordinal nature of the rankings. Wilcoxon Signed Ranks Tests were employed to determine if there was growth in each component of PSETs' professional noticing. The results, indicating statistically significant increases in all three components, are displayed in Table 1. The larger z-score in deciding can be attributed to the greater frequency of rank 1 in deciding, relative to attending and interpreting, on the pre-assessment, allowing for greater aggregate growth.

Table 1: Pre- and Post-Assessment Results of Wilcoxon Signed Ranks Tests

	Scale	N	z	p-value
Attending	1-4	123	-3.466	< .001
Interpreting	1-3	123	-3.841	< .001
Deciding	1-3	123	-5.378	< .001

Attitudes Toward Mathematics Inventory

To examine PSET change in attitude toward mathematics from pre-assessment to post-assessment paired t-tests were applied to ATMI data. There were significant increases in the enjoyment, self-confidence and motivation factors. Table 2 summarizes the paired t-test results for the 123 paired samples. There was not a significant change in the fourth factor, value, when

all 123 cases were included. However, when the 15 cases that achieved the maximum possible score on the pre-assessment were removed, there was a significant increase in the value factor from pre- to post-assessment ($t = 2.181$, $p = 0.031$).

Table 2: Pre- and Post-Assessment Results of Paired t-tests

	N	t	p-value
Value	123	-1.543	.125
Enjoyment	123	-3.070	.003
Self-confidence	123	-5.057	<.001
Motivation	123	-2.733	.007

For each of the four factors, more than 50% of the PSETs increased from pre- to post-assessment. The self-confidence factor had the greatest percent of PSETs increasing with 65% increasing. The 95% confidence interval for the “improvement” rate for this factor was 55.9% to 73.4%.

Correlation Between Professional Noticing Assessment and ATMI

Spearman’s rho correlations were employed to examine the correlation between the change in professional noticing and the change in attitudes toward mathematics as measured by the ATMI. While both PSET professional noticing and PSET attitudes toward mathematics, when analyzed independently, showed significant increases from pre- to post-assessment, there was no significant correlation between changes in professional noticing and changes in attitudes toward mathematics. The results of this comparison are found in Table 3 below.

Table 3: Spearman’s rho Correlations Between Changes in Professional Noticing Components and Attitudes and Beliefs Factors

		Change in Value	Change in Enjoyment	Change in Self Confidence	Change in Motivation
Change in Attending	Correlation Coefficient	.127	.045	.136	-.009
	P-value	.163	.621	.134	.924
Change in Interpreting	Correlation Coefficient	-.151	-.081	-.127	-.052
	P-value	.097	.376	.161	.566
Change in Deciding	Correlation Coefficient	.023	.102	.033	.091
	P-value	.799	.261	.715	.316

Discussion and Implications

We remain cautiously optimistic that the significant growth in the professional noticing skills and three of the four attitudes factors are indicators that the modules used in these courses are contributing to the significant growth of PSETs’ professional noticing and attitudes towards mathematics. Other factors can contribute to this growth so further data using control sites should be collected as a means for justification of the module. The significant increase in professional noticing skills reveals that PSETs can develop professional noticing skills within the context of early numeracy (research question #1). Additionally, the significant increase in three of the four factors of attitudes towards mathematics inventory (enjoyment, self-confidence, and motivation) reveals the possibility that components of their attitudes can increase when experiencing a course where professional noticing skills are explicitly taught, modeled, and

enforced (research question #2). Unfortunately, the correlation between the growth in attitudes and the growth in professional noticing were not found to be significant. We suspect that this might be the case due to the assessments being on unmatched ordinal and interval scales. In addition, we realize that the ATMI focuses on attitudes towards mathematics as a discipline, rather than attitudes towards teaching and learning mathematics, which might be more closely connected to gaining pedagogical skills, such as professional noticing. Further investigation on this lack of correlation will be conducted in future studies to better understand this result (research question #3).

As our research focus has been primarily professional noticing (Schack et al, 2013), the data provided for the attitudes toward mathematics is still in the preliminary stages. Additional and more sophisticated statistical tests should and will be conducted using the ATMI data in order to better understand the lack of significant correlation with the professional noticing increase.

Acknowledgments

This research was supported by a collaborative grant from the National Science Foundation (TUES # 1043667, 1043656, 1043831). The authors wish to thank Dr. Lenore Kinne, Northern Kentucky University, for her advice as evaluator of this project.

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PROSPECTIVE TEACHERS' ANALYSIS OF CHILDREN'S ERRORS USING CLINICAL INTERVIEWS

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Prospective teachers (PTs), enrolled in a mathematics methods course, were asked to analyze children's mathematical thinking by viewing videotaped clinical interviews over the course of an entire semester. They were also asked to conduct two of their own interviews with children as the semester progressed. While the type of analysis varied, all of the PTs commented on the errors that the children made during the interviews, often providing explanations for them. We describe these explanations and how they evolved over time. We note in particular that the PTs were better able to provide a more in-depth analysis of the videotapes than they were of their own interviews.

Keywords: Teacher Education – Preservice

Introduction & Theoretical Framework

Exposing prospective teachers (PTs) to situations in which they can observe and/or interact with children can provide them with the opportunity to better understand and appreciate the ways in which children think about, interpret, internalize, and represent various mathematical concepts (Ginsburg, Cami, & Schlegel, 2008; Author, 2000). One way to accomplish this is through the use of clinical interviews. Clinical interviews, as used by Piaget (1952), involve a flexible style of questioning which allows the interviewer to observe children's problem-solving behaviors as they work on tasks, and then ask questions that are tailored to the child's observed behavior. Ginsburg (1997; see also Mast & Ginsburg, 2010) notes that teachers' understanding of children's mathematical thinking can be enhanced through the use of clinical interviews—whether by actually interviewing the child, or critically observing another doing so.

As PTs engage in the practice of analyzing mathematical thinking through the use of clinical interviews, they often encounter situations in which the children think about the mathematics differently than they do. They may also be confronted with cases in which the children make errors. Without understanding the underlying reasons for the errors, the PTs may feel that the best way to deal with the situation is to redouble efforts to re-teach the concept, as interpreted by the PT or teacher, and not as understood by the child. In such cases, the PT remains unaware that the child may have a reasonable and logical rational for his thinking (Author, 2000).

Understanding errors can help teachers, both prospective and in-service, understand how children think about mathematical ideas. Ball (1991) notes that it is important for all teachers to consider the information that can be obtained by attending to children's errors, including whether the errors stem from simple calculation inaccuracies or more nuanced ways of thinking about a particular concept. Moreover, errors can be used by teachers to provide opportunities to stimulate mathematical discussions that further learning for the child making the error, as well as many of the other children in the class (Ball, 1991; Borasi, 1994; NCTM, 2000).

If PTs can learn to view children's errors as opportunities to learn more about children's

thinking, they may enter their teaching experiences better prepared to interpret and respond to children's complex ways of thinking. Consequently, one main goal of this study was to better understand when and how the PTs, all of whom were enrolled in an elementary/middle grade level mathematics methods course, identified, interpreted, and responded to children's mathematical thinking in general, and errors in particular (in both videotaped clinical interviews and interviews that they conducted), and how these interpretations changed over time. This relates directly to the conference theme that emphasizes broadening perspectives on thinking and learning. In this report, we focus on one PT in order to illuminate findings that are representative of the class as a whole.

Methods

Background

VITAL (Video Interactions for Teaching and Learning; Inoue, 2009, and Lee, Ginsburg, & Preston, 2009) was used in order to provide an online opportunity for the PTs to view and reflect upon archived clinical interview videos illustrating young children's mathematical thinking. It also allows for PTs to support their conjectures concerning children's mathematical thinking with video and text-based evidence.

Subjects

The subjects for the study were enrolled in the mathematics methods course for a full semester (September through December). Six PTs were chosen for more in-depth analysis based upon several factors: their attitudes towards mathematics and mathematics teaching as measured in a survey (White, Way, Perry & Southwell, 2006); their mathematical background (the number of mathematics courses taken prior to the class); and, their willingness to participate in the research study. Tania, the PT whose work is highlighted in this paper, was a female in her early 20s majoring in psychology. She had taken only the required mathematics courses (college-level algebra), and expressed anxiety about teaching math at any level (as per her survey results).

Data

The qualitative data for this study came primarily from the PTs' written work and audio-recorded classroom discussions. Their written work consisted of lesson plans, field observations, reflections on the archived clinical interview videotapes, and their reflections on their own clinical interviews with children. In all cases, the course instructors provided feedback to the PTs regarding their work.

Design

The VITAL reflections were assigned weekly, totaling 11 throughout the semester. In each, the PTs were asked to provide commentary on the mathematical behaviors observed in the videos, how the clinical interviews helped to provide information about children's thinking (covering topics including: number, numerical operations, geometric thinking, and patterns and algebraic thinking), and what questions or tasks they (the PTs) would have posed to a child had they been the interviewer. The PTs were also asked to support their claims with evidence from the videos and from assigned course readings. Of the 11 weekly VITAL assignments, there were seven in which the prospective teachers had the opportunity to identify and respond to children's errors. In addition to the weekly VITAL assignments, the PTs conducted their own clinical interviews at least twice (a third as extra credit). The first clinical interview assignment occurred approximately six weeks after the PTs began to work with the VITAL software. This was purposely done to allow the PTs to view others engaged in the practice of clinical interviewing, read about particular techniques used, and then prepare and conduct their own interviews. The

second took place during the last three weeks of the semester.

Results

All six PTs commented on children's errors right from the outset. The level at which they reflected on the errors became increasingly sophisticated as the semester progressed, as will be highlighted below. Early on, their reflections consisted of simple descriptions of the child's error, noting when in the interview the child made an error. No further explanations or analysis were provided. As the semester progressed, the analysis increased to include possible explanations for the errors made by children including the following: confusion with learned algorithms; calculation mistakes; the child's understanding relative to a developmental level as described in the readings; procedural versus conceptual knowledge; confusion with the interview questions; difficulties with language and/or communication; desire to please the interviewer; and, incorrect or inappropriate use of manipulatives. Several of the PTs also posed follow-up questions or instructional strategies, which, if used, might provide evidence for the cause of the error, or help the child to self-correct.

In order to illuminate the above, we focus on one PT, Tania. We share some excerpts from her VITAL assignments to illustrate some of the ways in which she attempted to analyze children's errors.

In the excerpt that follows, Tania reflected on a third grade child, Henry, working on finding the solution to the problems of 8 + 7 and 13 – 7.

> Henry adds these two numbers perfectly in his head and even has his own unique strategy for finding the answer. Henry describes his strategy for adding the two numbers quite well. He says, "because 8 plus 8 equals 16 and 7 plus 7 equals 14 and that if he were to add 1 more to one of the 7's he would get 15". If he were to "add one less" to one of the 8's from the 8+8=16, he would get 15. Henry is less successful when he has to subtract 13 and 7. I don't understand how Henry was unable to do this because he described his technique for adding two numbers really well, and when it came to subtracting, Henry said that 13-7 was 5. Even when Henry explains his strategy he does not really apply it when he has to subtract. He does begin saying that "14-7 gives you 7 and because you have one less than I think there should be two less". And when he says I think there should be 2 less you can see that to him it does not even make sense because he makes a face like he's unsure.

Of the six case studies, Tania was the only one who made reference to a child's facial expression as an indicator of a child's discomfort with a response. Her explanation of Henry's thinking, while not extensive, did provide evidence of her attempt to understand his thinking. In the excerpt below, we note that Tania quoted the course textbook as part of her analysis:

> This made me think of chapter 9 of Van De Walle's book [the course text] … "Though the concept of less is logically equivalent to the concept of more, the word less proves to be more difficult for children than more". However, I also felt that perhaps Henry just made a mistake of trying to recall subtraction problems he was already taught to just memorize in school.

Note that Tania thought that Henry might just be recalling memorized facts. She also pointed out that Henry decided to do this "*by himself in Kindergarten*", a comment that surprised her.

In the fifth VITAL assignment, Tania's reflection provided a description of a first grade child's reasoning when writing down numbers that involved two or three digits.

> Tarik does very well writing a couple of numbers down… For instance, she writes down twelve and fourteen correctly as 12 and 14. However, when Tarik is asked to write twenty-one down she writes 201. I believe Tarik is just listening to the standard number word of

> twenty-one and writing exactly what she hears. Tarik hears twenty so she writes 20. Tarik knows the words but has not thought of them in terms of tens and ones. I do believe Tarik knows that the larger number like the 20 is on the left and the smaller number like the one in 21 is on the right, but has not exactly pinpointed where they go when we consider them in terms of their tens and ones place.

What Tania did next was to detail a plan that could have provided Tarik with an opportunity to develop a more accurate understanding of numbers and place value:

> I think Tarik should have been first introduced to the base ten language. So instead of saying twenty-one, the interveiwer [*sic*] could have said two tens and one ones. This might have helped her consider the tens and ones place when writing numbers. It would have also helped if Tarik used base ten blocks while developing an understanding of the oral names. Once Tarik became familiar with the base ten language, I might have given her a task where she would have to arrange some ones and tens I wrote down on the paper using the square-stick-dot method to create a number. I'd then allow her to say the number in the base ten language and the standard name and then have her write it down. This might be a good way for Tarik to connect all three components of the relational understanding of place value.

Tania detailed strategies that she believed could be used to develop relational understanding of place value, referring to the work of Richard Skemp (1978), who describes relational understanding as "*knowing both what to do and why*" (pg. 2), and which is referenced in the course text (Van de Walle, 2007). Tania also uses ideas from the course text for her teaching plan above which, in this case, describes the components of a relational understanding of place value as including base-ten concepts and the oral and written names for numbers.

Over time, Tania became increasingly interested in trying to understand the underlying nature of the errors that she saw. Consider the following example (VITAL assignment 6) in which Eddie, a first grade boy obtained an incorrect solution of two when subtracting 9 from 12. Eddie was asked to demonstrate his answer using chips. Despite the fact that Eddie ended up with three chips after removing 9 of the 12 chips, he still maintained that the answer was two, which was his paper-and-pencil solution.

> I believe Eddie tried to justify his answer of 2 by matching it to the answer on his paper despite that subtracting 9 from 12 left him with 3. I believe he tried to reason that since 12 and 9 were both large numbers subtracting one from the other would result in a small number because he tells the interveiwer [*sic*] that 12 and 9 are high numbers.

In this part of her reflection, Tania was intrigued by Eddie's insistence that the answer was two and not three, noting that Eddie said, "*12 and 9 are high numbers*". She also added another possible explanation for his answer:

> This clinical interveiw [*sic*] helped us to see that perhaps Eddie had falsely memorized what 12-9 was and even when he was asked to prove it he still believed it was 2. I think it also clarified that sometimes kids do conceive mathematics as a bunch of facts and rote memorization and despite giving Eddie an opportunity to prove it, he wanted the answer to just match the answer on his paper. I think from this interveiw [*sic*] we can see the importance of and having the "thinking curriculum".

Tania believed that Eddie may have just memorized the answer to 12 – 9. She tried to see the strengths in Eddie's thinking (he estimated very well what the answer was going to be). She also commented on the fact that his paper-and-pencil solution was the one that he used, regardless of the fact that he had just used his counters to come up with a different solution. This, Tania noted, was quite surprising.

In the ninth VITAL assignment, Tania cited course readings to support her claim:

> The first shape Chidera [a pre-K child] is shown is a triangle and she correctly identifies it. When Chidera is presented with the next shape she is unsure of its name. The interveiwer [*sic*] then gives her some possible answers, one of which is a square. Chidera says that it is a square. When Chidera is given the third shape, she seems to depend on the orientation of the shape to name it. The third shape Chidera is given is a diamond, when one of its corners is on top and one is at the bottom, Chidera says its [*sic*] a dimaond [*sic*]. When the diamond is turned about 90 degrees, she says its [*sic*] a square. Like Dillion [a child in another video], Chidera seems to rely on visual prototypes for identifying shapes. I would also consider Chidera to be at level 0 [referring to the Van Hiele levels] because she relies on the appearance of the shape to define it and for Chidera the shape can change as it is rotated.

Tania's analysis was focused on when and how Chidera identified the shapes. She also related her observations to van Hiele's levels of geometric thought (as referenced in the course text), noting that Chidera was able to identify shapes based on appearance, but her ideas relating to the properties of the shapes were not fully developed. To Tania, this was evident when Chidera identified the square as such when it was placed on one of its sides, but called the same shape a diamond when it was rotated 90 degrees.

In the last VITAL assignment of the semester, Tania identified the difficulty one pre-Kindergarten child was having when asked to extend a color pattern with bears. She mentioned that she expected the child (Genesis) to be able to continue the pattern based on her behavior earlier on in the interview video:

> Genesis does not extend on the pattern correctly. She chooses orange and green bears to put into the pattern. When he [the interviewer] asks her what the green bear will match with, she say [*sic*] "nothing" and that it would be pretty if it was there. In the beginning of this video, I felt that Genesis was going to be able to extend on the pattern because when the interveiwer [*sic*] asked her what is this (referring to the pattern he created) she said its blue and yellow and blue and yellow. It appeared that she had noticed the sequence and would know what would come next. However, after reading the article, "Economopoulos" the author mentioned that to generalize and predict students must move from looking at a pattern as a sequence of what comes next to analyzing the structure of the pattern meaning to see that it is made up of repeating units… She [Genesis] doesn't understand the predictability and repetition that patterns imply because she would have been able to extend on the pattern correctly.

Tania used her understanding of one of the assigned course readings in her analysis of Genesis' response. Tania alluded to the situation in which a child may be able to identify a pattern but not be able to extend it.

We now share excerpts from Tania's own interviews with children to illustrate some ways in which she attempted to analyze children's errors in her own interviews. Tania's first interview involved a third grade boy named Billy. Tania's goal was to understand how Billy solved addition and subtraction problems. She began by posing several problems involving two digit numbers. She then shared a story problem in which Billy needed to find the total cost of buying a pencil for fifty-nine cents and a notebook for three dollars and twenty-five cents, as well as find the change someone would receive for the two items when paying with a five-dollar bill.

In her reflection, she noted that Billy tried to apply the traditional algorithms for addition and subtraction, but made some mistakes when subtracting. He also had difficulty in solving the subtraction part of the story problem above. She described, in the excerpt below, some of her observations, and noted in particular, how Billy's solution changed when she asked him to

describe his process using actual money.

> When Billy subtracted and came up with the wrong answers I didn't think Billy was unable to subtract. I just thought it was because he did not know how and when to apply borrowing or trading using the traditional algorithm for subtraction because he had said he made a mistake and was supposed to change both 0's into 10's. I knew Billy was capable of subtracting because when I asked him if 5 minus 3 was actually 4, he demonstrated using the money that it was in fact 2. I don't feel Billy saw the connection between what he had just done and what he could have done for subtracting 3 dollars and eighty-four cents from 5 dollars.

As can be seen in the above excerpt, Tania was able to recognize the occurrence of errors in the application of the subtraction algorithm, but she did not provide an in-depth analysis or description of the possible root causes for the errors. She also did not pose follow-up questions that might have provided her with more information about Billy's thinking. After watching a replay of her interview, she noted that she should have asked Billy different questions. She wrote that she should have said, "*how can you prove to me that 5 dollars minus 3.84 would equal 4 dollars and 84 cents instead of saying show me this in a different way*". Tania felt that this would have allowed Billy more of an opportunity to explain his thinking.

In her second clinical interview (also with Billy), Tania decided to use a task that she had seen in one of the VITAL videos. The problem involved dividing 12 cookies amongst two children and then three children. In her analysis, Tania reported that Billy used a strategy of doubling or tripling numbers in order to calculate the answer (see below). She liked his solution strategy noting that it resulted in correct answers. She then decided to pose additional problems in order to find out how Billy would solve the problem when the numbers did not lend themselves easily to his strategy.

> …the next question I asked Billy was now to pretend that he only had 5 cookies and he wanted to share those 5 cookies evenly with 2 boys, how many would each person get. Since I saw that Billy was finding his solutions by doubling and tripling numbers that he picked, I thought I might give him another problem where he wouldn't be able to do this, but would still allow for the 2 boys to get an even amount of cookies. Billy thought awhile about this problem and then said it is impossible. I said, "impossible", and he said "yes, because nothing equals 5, like 1+1 = 2, 2+2 = 4, and 3+3 only equals 6." Then I said to Billy, so you can't distribute 5 cookies evenly among 2 people? He said no again. Therefore, I thought awhile about how I could rephrase it to Billy so that he might see at least two and a half cookies could go to each person. Then I said to Billy, "Well, what about if they are not even, how much could each person at least get, because if you have 5 cookies at least some of them can go to two people." I don't know if saying what about if they are not even were the right words to say, because Billy then said well one would get 2 and the other would get 3.

From the excerpt above, we see that Tania attempted to interpret what Billy said in terms of both how he thought about the problem, and the words that she had used in posing it to him. She noted, in particular, that the word 'even' was problematic. As she concluded her reflection of this interview, she compared some of Billy's thinking with what she had seen him do with the subtraction problem from the first interview she conducted. Tania noted a strategy she believed Billy used when thinking about the division for sharing cookies—the "*think-addition*" strategy that Van de Walle (2007) describes as "*What goes with the part I see to make the whole?*" (p. 148-149)—and suggested that he may have had less difficulty subtracting if he applied this same strategy to the subtraction problems from the first interview. At the end of her reflection, Tania

mentioned that she wished that she had asked Billy to draw a picture for the scenario of five cookies split between two children, explaining that maybe with a picture Billy might have realized that the extra cookie could be split in half.

We next share comments that Tania made about the utility of conducting interviews:

> I liked the VITAL assignments because unlike the field observations where you're watching an entire class in progress, they allow you to focus in on one student and his/her thinking… The VITAL assignments and readings were also helpful because they gave me ideas for what questions I wanted to ask the child I interviewed. The clinical interviews then allowed me to compare and contrast what I saw in the videos and readings. The interviews allowed me to put the readings into practice and that is something you cannot experience just from reading the book… I also had to listen attentively to Billy when I asked him to justify his solution so that I could elicit or challenge his thinking. In my second clinical interview, I noticed a pattern in Billy's thinking when he tried to find out how many cookies each person would get if they were distributed equally. Billy would begin by picking a number and doubling or tripling it. Therefore, I quickly came up with a follow up question I could ask Billy to challenge the way he originally thought about distributing the cookies. I would not have been able to do this if I had not listened carefully to Billy when he explained to me how he got his answers.

Discussion

As can be seen from the above accounts, Tania was able to identify the errors that occurred, and was better able to analyze them over time. She and several other PTs noted that it was easier to reflect on the errors that occurred in the VITAL videos than those that occurred during their own interviews. There are many possible reasons for this. Perhaps, as Tania and the other PTs noted, it is often easier to objectively analyze someone else's work than your own. Perhaps it is because the selected VITAL videos were both more professional nature, and chosen specifically to reveal certain aspects of children's thinking. Nonetheless, the PTs also noted that interviews performed by others appeared far easier to do than those that they actually did themselves. In particular, they noted that they were often confounded when a child made an error. They felt the pressure to make 'in the moment' decisions, which often led to comments and questions to the child that, in retrospect, did not address the error in a way that seemed productive. The PTs' post facto reflective analysis of their interviews indicated that at times, they did not have enough information to analyze children's errors because they had not asked the right questions or understood what really was happening during the interview. This highlights their realization of the skill needed, and importance of knowing how to respond to a child's thinking, especially when it is different than the way that was anticipated, or understood at the time it was being communicated. This is another of the many critical insights that the PTs gained as a result of their experiences.

The level and type of analysis presented in this report underscores the difficulty, yet importance, of having PTs reflect on clinical interviews. It further suggests the importance of using both videotaped interviews as well as personal interviews with children.

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ELEMENTARY PRESERVICE TEACHERS READING OF EDUCATIVE CURRICULUM: ATTENDING TO EXAMPLES OF STUDENTS' MATHEMATICAL THINKING

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We present research on how PSTs attended to and made sense of educative features of Standards-based curriculum designed to support teachers' knowledge of students' mathematical thinking. Data analyzed from 47 PSTs' written responses to five mathematics lessons demonstrate PSTs attended to these features roughly 25% of the time, but often made sense of them in an educative manner when doing so.

Keywords: Teacher Education-Preservice, Curriculum, Elementary School Education

Introduction

In 1996, Ball and Cohen called for the design of curriculum materials that could "contribute to professional practice" (1996, p. 7) by supporting teacher learning in addition to student learning. In their response to this call, Davis and Krajcik (2005) presented a series of five "high-level guidelines" describing the roles educative curriculum could play in promoting teacher learning. These guidelines posited curriculum materials could be designed to: 1) support teachers' development of pedagogical content knowledge (as in Shulman, 1986); 2) support development of teachers' subject matter knowledge; 3) "help teachers consider ways of relating units during the year" (p. 5); 4) "make visible the developers' pedagogical judgments" (p. 5); and 5) "promote teachers' pedagogical design capacity" (p. 5) (as in Brown & Edelson, 2003).

Several curriculum developers within elementary mathematics have attended to the suggestions proposed by Ball and Cohen and Davis and Krajcik. *Standards*-based curriculum series such as *Math Trailblazers* (UIC, 2008), *Investigations in Number, Data, and Space* (TERC, 2008) and *Everyday Mathematics* (UCSMP, 2007) are examples of curricula that have been explicit in their intent to be educative for teachers and students alike by including educative features. However, little research has been conducted to examine the ways in which pre-service teachers (PSTs) read and interact with these materials. In this paper, we report findings related to PSTs' noticing of educative features intended to support the development of PSTs' knowledge of students' mathematical thinking.

Theoretical Frame

Effective mathematics teaching requires teachers to employ a variety of knowledge, skills and dispositions. In her handbook chapter on teacher development, Sowder (2007) suggested one goal of teacher development should be to develop teachers' understanding of how students think about mathematics. Research suggests that teachers who understand how students think about particular mathematical ideas will be better positioned to recognize, interpret and support these ideas in their instruction. Cognitively Guided Instruction (CGI) research has demonstrated teacher knowledge of student thinking, reasoning and strategies can lead to gains in student achievement (Carpenter & Fennema, 1992; Carpenter, Fennema, Franke et al., 2000). More recently, within the Mathematical Knowledge for Teaching (MKT) framework (Ball, Hill, &

Bass, 2005; Ball, Thames, & Phelps, 2008; Hill, Sleep et al., 2008), Ball and her colleagues highlighted Knowledge of Content and Students, "the amalgamated knowledge teachers possess about how students learn" (Hill, Sleep et al., 2007, p. 133), as an important facet of pedagogical content knowledge. Research has also demonstrated a positive correlation between teachers' mathematical knowledge for teaching and student achievement (Hill, Rowan, & Ball, 2005). In preparing PSTs, it is paramount mathematics teacher educators address the development of this knowledge base. One way to aid PSTs' understanding of student thinking is by leveraging the educative features of *Standards*-based curriculum materials.

Remillard (1999; 2000), Collopy (2003), as well as Davis and Krajcik (2005) have posited that curriculum materials can be educative for teachers as well as students. In their first high-level guideline, Davis and Krajcik (2005) suggested educative curriculum materials could support teachers' knowledge of student thinking in two ways. Educative curriculum could be designed to: 1) help teachers anticipate and interpret how students might respond to instructional tasks; and 2) inform teachers of common student conceptions and how they may be addressed (Davis & Krajcik, 2005). Empson & Junk (2004) found that 13 teachers using *Investigations* developed integrated knowledge (knowledge of concepts, procedures, mathematical practices, and children's thinking) of multiplication. Curriculum developers have taken up Davis and Krajcik's (2005) guidelines in a variety of ways -- making different design decisions concerning how to incorporate this information into their materials for teachers. In this study, we were curious to examine how different *Standards*-based curriculum materials responded to and operationalized these particular guidelines, as well as how our PSTs interpreted this information when reading the curriculum materials.

In order to help frame our understanding of PSTs' reading of curriculum materials, we employ the construct of teacher noticing. Teacher noticing encompasses two main processes – *attending* to particular events in an instructional setting and *making sense* of those events (Sherin, Jacobs, & Philipp, 2011). Although teacher noticing has been mainly used to describe the active process a teacher engages in while in the act of teaching, we view these two aspects as applicable to teachers' reading of curriculum materials, particularly those materials designed to be educative as they contain myriad visual information for teachers to process and make sense of. In our study we use the teacher noticing framework to see if PSTs are, in fact, attending to these particular educative features and, if so, if they are making sense of them in ways that promote the development of their knowledge of students' thinking.

Methods

Data for this study came from 47 PSTs from two different university sites. Each PST was enrolled in an elementary mathematics methods course jointly designed by the three authors and taught by either the first or second authors. One of the big ideas of our methods course is the idea that children's mathematical understandings emerge from solving problems and teachers can use questioning to scaffold the development of children's mathematical understanding and sense making. We employ the use of *Standards*-based curriculum materials as a way to develop PSTs' knowledge of students' mathematical thinking, which contributes to our big idea. PSTs were asked to read a series of five lessons, selected to represent a variety of topics, grade levels and educative curriculum series (2 lessons from *Investigations*, 2 from *Math Trailblazers*, and 1 from *Everyday Mathematics*). After reading each lesson, PSTs were asked to answer three questions based upon their reading. Here we report on the second of the three questions, which was same in each assignment and was designed to elicit to what and how PSTs were attending in their reading of the lesson materials. The question asked, "How does this lesson plan help you teach

this particular concept? In other words, what information does it give you that is most important or helpful? Please be specific." The reading of the five lessons and their responses were homework assignments given over the course of the semester. We provided multiple opportunities for PSTs to read and respond across the course. On the day a lesson and response was due, a portion of class was devoted to a discussion of the educative features of the lesson.

Analysis

Prior to our analysis of the PSTs' responses, the first two authors jointly identified and labeled the features within each lesson plan (educative as well as non-educative). Next, we individually coded the educative features with one of Davis and Krajcik's five high-level guidelines (2005). The established labels for the features of each lesson were used to identify which features, both educative and non-educative, PSTs attended to. For this report, we then identified the educative features in each lesson that could support teachers' understanding of students' mathematical thinking in some manner.

PSTs' responses to the second question were first unitized into differing ideas, separating each response by the various features of each lesson to which the PST was attending. These responses were then double-coded by the first two authors. We first used the labels to code each unit for *what* feature of the lesson PSTs were attending to. Inter-rater reliability for this coding was 90% and discrepancies were resolved by discussion.

We also coded each unit according the manner in which PSTs made sense of the identified lesson facet. We developed an initial series of five codes using the literature related to teachers' use of curriculum materials. After several passes through the data, the codes were refined and operationally defined. The first two authors independently assigned all data units one of five codes (descriptive, interpretive, evaluative, adaptive, and educative) through a process of focused coding (Charmaz, 2004). A *descriptive* response described the lesson feature, with no indication as to why it was important, only that it was included. In an *interpretive* response, PSTs described the feature and interpreted the lesson feature using their own or prior knowledge (not information given in the materials); indicating to us that they did not learn anything new from the lesson materials. An *evaluative* response described the feature and included a value judgment as to whether the PST believed feature would be effective or not, of if the PST "liked" the feature or not. An *adaptive* response described how the PST might modify or adapt the feature using her own knowledge or ideas. Responses coded as *educative* showed evidence of the feature informing the PSTs' knowledge in some way (as per Davis & Krajcik, 2005), beyond directions of what the teacher should do in teaching the lesson. Inter-rater reliability for this coding was 87%. Discrepancies were resolved by discussion.

For this specific report, we examined PSTs' data that attended to the educative features identified as supporting teachers' understanding of students' mathematical thinking.

Results

We examined data from 47 PSTs to answer our research question, "How do PSTs read educative features of curriculum materials designed to support teachers' knowledge of students' mathematical thinking?" In the following sections we identify and explain these educative features within the five lessons PSTs were assigned to read, which features were attended to by PSTs and the ways in which PSTs made sense of these features.

In our analysis of the curriculum materials, we determined each of the five lessons provided information for teachers that would support their knowledge of students' mathematical thinking. These supports were presented in various formats across curriculum series, and were often

presented differently within the same lesson. We saw these features as educative, however in our analysis we found PSTs did not always attend to them, and even when doing so, did not always make sense of them in an educative manner.

Stickers: A Base Ten Model

In the 3rd grade *Investigations* lesson, "Stickers: A Base Ten Model", students explore what happens when 10 is added to an existing two-digit number. At one point in the lesson, the teacher is to ask students to count the number of stickers represented on an overhead sheet (4 strips of 10 and 6 single stickers) and to tell how they counted. The materials then present the phrase "Students might say:" (TERC, 2008, p. 28) followed by photos of two individual student faces and possible ways students might count the stickers. Later in the lesson, after the teacher and students have created a chart with various representations of the problems they have completed, the teacher is directed to ask students, "What digit changed every time Pilar bought more strips of 10? How did it change?" (TERC, 2008, p. 30) In this case, there are no examples of specific student talk presented. Rather, the lesson reads,

> The students may notice that the digit in the 10s place increases by 1 each time another 10 is added and increases by 2 when two 10s are added. Use the class pocket 100 chart to help students understand that the 1 and 2 represent 10 and 20 being added to the starting number of stickers. (TERC, 2008, p. 30)

These two facets of the lesson were educative in supporting teachers' understanding of student thinking, albeit in slightly different ways. The example of student talk serves to help teachers anticipate what counting strategies students might use. The second facet informs the teacher that some students will have picked up on the pattern at this point in the lesson and suggests a scaffold to use in addressing those who do not.

Of the 47 PSTs, 7 (14.9%) attended to the "Student might say:" feature in their written responses. All seven who attended to this feature addressed it in an educative manner within their response. Madison's (all student names are pseudonyms) answer was representative of these responses, "Another helping tip is that it gives me exact questions to ask the students and some sample responses. This will help me to be better prepared and know how to answer students' questions before they ask them".

Eleven PSTs (23.4%) attended to the "Student may notice" piece of student thinking; seven PSTs attended to the feature in an educative manner, while four did so interpretively. Ellie attended to the information in an educative manner, as evidenced in her response:

> One of the key things for the students to understand place value is for them to be able to see where the changes are. With this chart they are able to see how the number in the 10s place continually changes, but the 6 in the ones place remains constant. This pattern is a great way to explain to them what the 10s place is and that the number is not changing by 1, but rather each change of 1 in that place is actually a change by 10.

Ralph's answer was one of the responses coded as interpretive, as he interpreted the information on student thinking as something to assess against and teach directly if needed.

> [T]he teacher will expressly ask "What digit changed every time…?" and draws the student's attention to the work they have been doing. (Specifically, the tens digit, reinforcing/teaching the concept of place value.) Some students will most likely already "get it" by that point, but this is some direct teaching that must be done for this concept.

Field Day Refreshments

The 5th grade *Investigations* lesson "Field Day Refreshments" asks students to solve a multistep word problem with a variety of correct solutions. The lesson materials support

teachers' ability to anticipate student responses by including two examples of Sample Student Work. The work is presented in student handwriting as if it were authentic. In this lesson, 8 PSTs (17.0%) attended to this support. 5 PSTs responses were similar to in nature to Amanda's response, "It shows examples of possible student strategies for solving the problem" and coded as descriptive. Three responses were coded as educative. These, like Marilyn's response below, recognized the student work as informing their understanding of what students might do in solving the task and how that would help her in the course of teaching the lesson:

> The lesson also offers copies of two students' strategies for solving the problems, with their actual work displayed. This allows the teacher to know what to expect even before teaching the lesson so that the teacher can be prepared as to what questions to ask during discussion or what strategies may be used in his/her own classroom.

The 500 Hats

In "The 500 Hats", a 3rd grade lesson from *Math Trailblazers*, students are to solve a variety of three-digit addition and subtraction word problems using their own strategies set within the context of "The 500 Hats of Bartholomew Cubbins" by Dr. Suess. The materials remind teachers to "Emphasize partitioning numbers and using mental strategies, such as partitioning and combining, doubling, using compliments of ten and a hundred, or skip counting" (UIC, 2008, p. 29). We viewed the information on what strategies to emphasize as supporting teachers' knowledge of students' thinking as it was a reminder to look for and encourage students' own strategies. Students' use of their own strategies was reinforced in the lesson materials, "Keep in mind, however, that the goal of this activity is to engage students in using their own strategies to add and subtract numbers rather than to teach them formal processes" (UIC, 2008, p. 29).

The "Emphasize strategies" feature was, by far, the most attended to within the five lessons as 34 of 47 (72.3%) of PSTs attended to this lesson feature. PSTs' responses were coded in each of the five categories (see Table 1) with descriptive and educative occurring the most frequently. In Table 2 we present sample responses for each of the five categories of codes for this feature.

Table 1: How PSTs Attended to "Emphasize strategies": The 500 Hats (N = 47)

Feature	**Total**	*Descriptive*	*Interpretive*	*Educative*	*Evaluative*	*Adaptive*
Emphasize strategies	**34**	13	5	11	4	1

Table 2: Examples of How PSTs Attended to "Emphasize strategies": The 500 Hats

Code/PST	Response
Educative Nancy	There was also a small section with strategies to emphasize to students while they are solving problems. I would use this to look for the use of those strategies in students work and ask them to share them with the class.
Descriptive Kara	The lesson also emphasizes which concepts and operations should be applied to the story's mathematical problems (partitioning numbers and using mental strategies such as partitioning and combining, doubling, using complements of ten and one hundred, and skip counting).
Evaluative Mandy	I also like that students are able to use pencil and paper as well as base tens and hundreds to help them organize their numbers. This allows each student to work at their level and needs and leads to many different strategies
Adaptive Kellie	By suggesting various manipulatives that the students can use, like the base-10 blocks, or listing specific strategies to lead students to, the teacher can be more aware in how to phrase questions and responses. With careful planning, the way

	in which the problems are discussed can be directly related to the use of these strategies and get students to think about using them when solving other problems.
Interpretive Cynthia	Students are able to use their mental math skills while solving the addition and subtraction problems of two or three digit numbers. The students are producing their own strategies and figuring out which ones work best for them individually. As a teacher, I would be able to see which types of strategies my students are using and which strategies are working best for them.

Counting One Hundred Seventy-Two

"Counting One Hundred Seventy-Two" is a 1st grade *Math Trailblazers* lesson focused on representing numbers greater than 100 in a variety of ways. We identified four facets of the lesson that could support teachers' knowledge of student mathematical thinking. The lesson begins with the teacher writing several numbers between 101 and 199 on the board and asking students "What does each number mean?" (UIC, 2008, p. 35). The lesson includes four examples of what students might say about the meaning of the number 125. As students work on drawing representations of 172 beans, the lesson informs teachers as to what they may do, "Students may try to draw 172 beans, while others will show groups of beans in containers, organized in some way" (UIC, 2008, p.35). The "Assessment" section of the lesson asks teachers to use students' journal entries to assess their strategy for counting and grouping objects. It explicitly reminds teachers, "Some students may count by ones. Even though counting by ones is an inefficient strategy, it works if done carefully" (p. 37). The lesson also lists appropriate strategies for students to use in completing the Math Facts practice sheet.

Each of these features can support teachers' knowledge of student thinking by: providing sample responses and common student strategies. Further, the lesson materials not only help the teacher anticipate that some students will attempt to draw 172 individual objects, or to count by ones, but also remind them these strategies can work and are acceptable.

As Table 3 demonstrates, 18 PSTs (38.3%) attended to the "Meanings of 125" information within the lesson materials. There were also 18 PSTs (38.3%) who attended to the "Drawing 172" task and the ways in which students might respond. Thirteen PSTs attended to the assessment section of the lesson materials and the student strategies for counting or grouping objects. Finally, 2 PSTs (4.3%) attended to the "Math facts strategies".

Table 3: How PSTs Attended to Educative Supports: Counting 172 (N = 47)

Feature	Total	*Descriptive*	*Interpretive*	*Educative*	*Evaluative*	*Adaptive*
Meanings of 125	**18**	4	5	9	0	0
Drawing 172	**18**	6	10	1	1	0
Assessment	**13**	6	3	1	2	1
Math fact strategies	**2**	1	0	1	0	0

Formula for Area of a Triangle

The 4th grade *Everyday Mathematics* lesson "Formula for Area of a Triangle" is designed to help students generate the formula for the area of a triangle. The main task involves students determining the length of the base and height, and area of a series of given triangles by counting unit squares on a grid. Then students compare the area of the triangles to the area of parallelogram formed by two congruent copies of the given triangles. The lesson begins with a Math Message asking students to "Make a list of everything that you know about triangles"

(UCSMP, 2007, p. 694). In discussing results from this activity, the Math Message Follow Up informs teachers, “The list might include:” and presents a series of six statements regarding triangles. Following the main task of the lesson, students are asked to complete a page of problems. Problem 8 on this page asks students to compare the areas of two shapes, a star and a square. The materials include insight as to how students sometimes respond to this task, “It may surprise students that the star and the square in Problem 8 have the same area. One way to find the area of the star is to think of it as a square with a triangle attached to each of its sides” (UCSMP, 2007, p. 696). The Math Message Follow Up not only provides sample student responses, it simultaneously supports teachers’ content knowledge of triangles, their features and classification. The support for Problem 8 lets teachers know students may struggle with solving the task, as well as making sense of its answer. It also provides content knowledge support by including one way to think about the task.

The “Math Message follow up” was attended to by 5 PSTs (10.6%). Two PSTs’ responses were coded as educative. Cleo wrote, “The lesson also gives specific examples on page 694 of what students might say when asked about the properties of a triangle. That list is a good resource for how you may want to guide the discussion”. Three responses were coded as interpretive, represented here in what Stacey wrote,

> The math message follow up shows me that students will have a chance to develop a list of properties of a triangle. This shows that the students know what a triangle is and some rules or facts associated with triangles. They will then use this knowledge to help them conjure a formula for area. Without this knowledge, they wouldn’t have a starting place to come up with a formula to represent area.

Three PSTs (6.4%) attended to the information on “Problem 8”, with two responses coded as educative and one as descriptive. Allie’s response was coded as educative,

> I also liked the information what might be surprising for students. (Example: the area of a star in comparison to the smaller square on page 696). Receiving information like that is always helpful to put student knowledge in perspective and help the teacher get on the student’s level of knowledge.

Discussion and Implications

Within these five lessons, we identified 10 lesson features that could support teachers’ understanding of students’ mathematical thinking. Across the PSTs’ responses to the five lessons, there were a total of 116 instances of PSTs attending to these lesson facets; resulting in PSTs attending to these particular features roughly 25% of the time (116/470; 24.7%). Of the 116 instances of attending, 36 were coded as descriptive (31.0%), 27 (23.3%) as interpretive, 44 (37.9%) as educative, 7 (6.0%) as evaluative, and 2 (1.7%) as adaptive. Our findings indicate PSTs are not attending to these features at a high rate, although when they do, they are most likely to do so in a descriptive or educative manner.

Moving forward, we are interested in examining what factors influence PSTs’ attention when reading educative materials. Does the manner in which the support is presented or what knowledge is supported influence if and how PSTs attend to it? For example, in the Stickers Lesson, we think the “Students might say:” feature is very clear in its intent, even highlighting the information as possible student responses with photos of students. However, more PSTs attended to the “Students might notice” information, even though to us it seemed to be presented in a less obvious manner. Further, all PSTs who attended to the “Students might say” did so in an educative manner, while some PSTs who attended to “Students might notice” did so in an interpretive manner. We believe structure and organization could be factors influencing how

features are attended to, but in light of these results, we posit the content is equally if not more important. That is to say, although both features were intended to be educative for teachers, perhaps our PSTs did not require as much support in understanding how students count as opposed to understanding if students would be able to recognize the pattern in the tens digit without direct instruction, resulting in more PSTs attending to that feature.

Acknowledgment

This work was supported, in part, by the National Science Foundation under Grant No. 0643497 (C. Drake, PI). Any opinions, findings, conclusions, or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

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HOW PRESERVICE TEACHERS RESPOND TO STUDENT-INVENTED STRATEGIES ON WHOLE NUMBER MULTIPLICATION

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Although students' invented strategies typically prove to be effective in the improvement of students' mathematical understanding, little is known about how preservice teachers interpret and respond to student-invented strategies on whole number multiplication. This study investigated the nature of 25 preservice teachers' interpretations of and responses to students' correct and incorrect strategies for whole number multiplication. Results suggest that the mathematical depth of their responses and their consideration of student thinking differed based on the correctness of the student work. Implications for teacher educators and future researchers are discussed in accordance with the findings of the study.

Keywords: Teacher Knowledge, Teacher Education-Preservice, Number Concepts and Operations

Whole number multiplication is one of the most challenging operations in both calculation and justification in elementary mathematics education (Flowers, Kline, & Rubenstein, 2003). Researchers have documented that not only children experience difficulty in understanding the reasons for the standard algorithm, but also teachers. They also reported that students learn better when they are asked to create their own strategies different from the traditional method (Campbell, Rowan, & Suarez, 1998; Carroll & Porter, 1997; Huinker et al., 2003). NCTM (2000) suggests that teachers should spend a significant amount of time with student-invented methods that arise in a typical mathematics classroom and should think about how to help students build on them before introducing the standard algorithms, because students who invent strategies are involved intimately in the process of making sense of mathematics (e.g., Ball, 1988/1989). Ball (1989) also stresses that teachers need to address the aforementioned student-invented strategies as a window into student understanding and to endeavor to help students understand the conceptual thinking behind the mathematics. However, in a classroom situation where student-invented strategies are encouraged in learning mathematics, it is plausible that students make errors, and little is known about how to prepare preservice teachers (PSTs) for responding to these errors. The purpose of this study is to investigate PSTs' interpretations of and responses to students' correct and incorrect student-invented strategies involving whole number multiplication. Our purpose to uncover PSTs' ideas about teaching multi-digit multiplication and ideas about student-invented strategies is practically significant for teacher preparation programs. The research questions for the study were: (1) How do PSTs interpret correct and incorrect student-invented strategies with whole number multiplication?; (2) How do PSTs respond to correct and incorrect student-invented strategies with whole number multiplication?; (3) Do PSTs' interpretation and responses differ depending on the correctness of the student-invented strategies?

Theoretical Background

Research on Students' Strategies in Whole Number Multiplication

Numerous research studies on students' strategies involving whole number operations have been conducted not only in the US, (e.g., Carpenter et al., 1992; Carroll, 2000) but also in many other countries (e.g., Anghileri et al, 2002; Torbeyns et al., 2006). Table 1 shows some examples of student-invented strategies for multiplication reported from the literature with the example problem (28 x 7 = ?) (Bass, 2003; Carpenter et al., 1992; Carroll, 2000; Carroll & Porter, 1997; Huinker et al., 2003; Selter, 2002).

Table 1: Student-invented Strategies for Whole Number Multiplication

Strategies	**Characteristics**	**Examples (e.g., 28 x 7)**
1. Direct modeling	Use manipulatives or drawings to simulate the problem	
2. Repeated addition	Think multiplication as repeated addition	28 x 7 = 28 groups of 7 =7+7+7+....+7 =196
3. Chunking method	Chunk the addends or successive doubling	28 x 7 = 28 groups of 7 7 10s (70) + 7 10s (70) + 8 7s (56) = 70 + 70 + 56 = 196
3. Compensating (or varying method)	Round one of the factors to a multiple of 10 to make the multiplication easier and compensate by subtracting the extra.	28 x 7 → 30 x 10 = 300 → 300 – 14 = 196
4. Partial Products (or decomposition, or partitioning method)	Use the base-ten structure to break down the factors into partial products and use the distributive property	28 x 7 = (20 + 8) x 7 = (20 x 7) + (8 x 7) = 140 + 56 = 196

Among the student-invented strategies, partial product and compensation strategies were chosen in this study for three reasons—(1) they are frequently used by students, (2) they are mathematically efficient, and (3) they can be difficult to use with large numbers and are thus related to student error. First, these strategies are common strategies students invent on their own when they are asked to solve problems involving multiplication (Carpenter, Fennema, & Franke, 1992). Moreover, Bass (2003) asserts that these can be considered algorithms for whole number multiplication because they involve "a precisely specified sequence of steps" that can be programmed to always produce the correct solution (p. 324). However, these methods would be become cumbersome and difficult to use when applied to a problem involving larger numbers. In particular, without sound understanding of compensation, some students tend to apply these methods incorrectly (Schifter, Bastable, & Russell, 1999). For example, for the problem 28 x 7, a student called Tommy changed 28 x 7 into 30 x 10 and then took away 2 and 3 since he added 2 to the 28 and 3 to the 7. In the use of the compensation strategy, the ability to make an adjustment or compensation is important. In Tommy's case, he increased the 2 groups of 10 and 3 groups of 28. When students make errors, teachers should be able to provide an appropriate intervention. However, there is little research on how PSTs would respond to student-invented strategies, particularly, the compensation strategy on whole number multiplication. Therefore, this incorrect student-invented strategy was chosen for the study.

Research on Teacher Knowledge and Approaches

Researchers have studied the ways teachers understand content knowledge (CK) and/or pedagogical content knowledge (PCK) in several mathematics content areas including whole number operations (e.g., Ball & Bass, 2000; McClain, 2003; Thanheiser, 2009), fraction operations (Author, year 1), and other content areas and the relationship between teachers' CK,

PCK, and their teaching practices (e.g., Ball, 1990, Krass et al., 2008; Prediger, 2010; Author, year 1; Author, year 2). These studies commonly reported that many US teachers hold narrow, procedural understandings of algorithms. In addition, they reported that PCK was highly correlated with CK mastery, thus suggesting that deep knowledge of the subject matter is a critical precondition for PCK (Baumert et al., 2010). Furthermore, they reported that teachers with more PCK display a broader repertoire of teaching strategies for creating cognitively stimulating learning situations (Ma, 1999). Much of this research, however, has focused on *teachers' knowledge* on CK, PCK, and teaching practices on common mathematical concepts focusing on *traditional methods* (or standard algorithms). As a result, little is known about the implications in teacher education programs for developing PSTs' responses to student-generated strategies. This article is intended to address this gap.

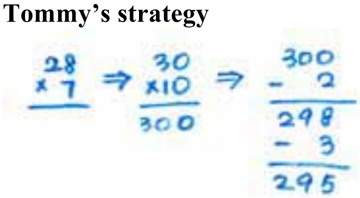
Methods

Twenty-five preservice teachers participated in the study from an elementary mathematics methods class at a large southeast university in the United States which utilizes the Holmes' model of teacher preparation. All participants had completed a required mathematics course equivalent to a 3 credit pre-algebra course either in their freshman or sophomore years. The mathematics methods class, which lasted approximately 14 weeks, was taught by the first author. The course was designed to support PSTs' understanding of approaches that are relevant to the teaching and learning of mathematics, particularly in the elementary grades. During each lesson, PSTs were involved in analyzing children's work through discussion of several samples in small groups and then as a whole class.

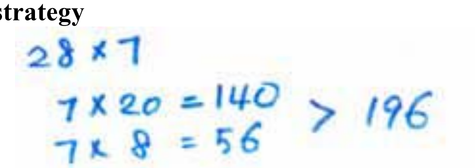
The main task used in this study presents student-invented strategies through a classroom scenario in which two hypothetical students come up with different solution methods to a two-digit whole number multiplication problem. The task consisted of three questions (see Figure 1) that took about 30 minutes to complete. The task was developed based on actual elementary students who appeared on a video from the *Developing Mathematical Ideas* curriculum (Schifter, Bastable, & Russell, 1999). The first student, Tommy, used a student-invented strategy, compensation, but failed to execute the procedure correctly. Dan, the second student, used a different student-invented strategy, decomposition (partial products), and produced the correct answer.

You are teaching whole number multiplication problem 28 x7 to fourth graders. You asked students to solve it. After a few minutes, two students came to the board and explained their methods in a following way

Tommy's strategy



Dan's strategy



1. Explain the logic behind each student's procedure and describe why you think each strategy will or will not work for all whole numbers.
2. How would you respond to Tommy? Describe your intervention using pictorial models / drawings/ numerical expressions. Explain it as much detail as you can.
3. How would you help Dan to develop the multiplication traditional algorithm from his method? Show the connection between drawings and numerical expressions to explain your teaching strategy.

Figure 1: Pedagogical Task: "How Would you Respond to Tommy and Dan?"

The task described above was administered as a survey to the entire class in two methods course sections towards the end of the fall semester of the 2009-2010 academic year. Table 2 shows an overview of the analytical framework associated with each task for the study. Data analysis involved five processes: (1) an initial reading of each PST's response, (2) identifying correctness of the responses, (3) exploring the subcategories under each analytical aspect according to the framework, (4) coding the categories and subcategories, and (5) interpreting the data quantitatively and qualitatively (Creswell, 1988). For example, responses to the first question, after having been identified based on correctness, were analyzed by looking at whether PSTs pointed to the underlying concepts or important properties related to each student's method. Using the conceptual versus procedural distinction guided by the work of Rittle-Johnson and Alibali (1999), a four-point rubric provided a framework for the different depth levels of the preservice teachers' justifications to each student's method.

Task	Item	Analysis aspects	Method
CK	Q1: Student method	• Validity/ Generalizability	Correctness
	Q1: Justification	• Conceptual vs. procedural	Develop a scoring rubric
PCK	Q2-3: Intervention	• Conceptual vs. procedural	Identify teaching category
		• Forms of address—Teacher vs. student-oriented	Develop categories
		• Type of cognitive process—Cognitive status vs. cognitive action	Frequency/Quantiative analysis
		• Type of model and contextual problems(Set, area, concrete, pictorial, etc.)	Frequency/Quantiative analysis

Figure 2: Methods Used for Analyses

The same procedure was used for the analysis of PSTs' responses to the second and third questions. Each code was then documented in Excel according to the task sub-domains, and a data table containing all the categorized responses for each participant was developed. In particular, in order to determine whether and how PSTs respond differently depending on the correctness of the student-generated strategies (research question 3), the Fisher exact value test, which is a non-parametric statistical significance test to determine dependent relationships in contingency tables, specifically for small sample sizes (Ott & Longnecker, 2001), was used for comparisons among 3 variables: (1) the definition of multiplication used in interpreting student logic, (2) depth of reasoning when interpreting student logic, and (3) the use of model-type when responding to students. In addition, we used a Z-test for independent proportions for the analysis of how PSTs' consideration of student thinking might differ based on students' correctness. Furthermore, for the analysis of PSTs' choices to respond with teacher-centered or student-centered responses, a chi-square test was used to determine if there is a dependent relationship between correctness of student work and student-centeredness of the PST's response.

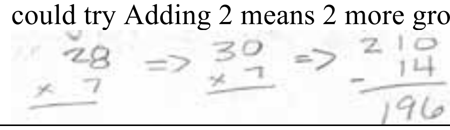
Summary of Results

We found that most of the PSTs correctly recognized the validity and generalizability of Tommy's strategy and Dan's strategy. However, when it came to justifying the reasons behind the procedure and providing good intervention, around half related concept to procedure using meaningful understanding of multiplication. The PSTs mostly relied on the 'show and tell' approaches. In addition, this study provided evidence, with a significance level of 0.05, to

support that the following three variables depended on whether the student work is correct or incorrect: (1) the definition of multiplication used by PSTs, (2) the depth of explanation PSTs gave, (3) how much PSTs talk about student thinking. In this paper, due to the space limit, we answer only the first and third research questions in detail.

Preservice Teachers' Interpretations of Student-invented Strategies Interpretations of Tommy's logic. All 25 participants correctly identified Tommy's strategy as not generalizable to all whole numbers. The PSTs' justifications were analyzed with respect to the following: (1) how to explain the process associated with Tommy's strategy, (2) what is the definition of multiplication used, and (3) what is the depth of justification. With respect to the definition of multiplication, we observed most made use of the equal groups definition of multiplication, while some responses focused on procedure more than on concepts and did not refer to either meaning of multiplication. A majority, 60%, of the PSTs used the equal groups definition of multiplication in their interpretations of Tommy's strategy, while only 2 participants, 8%, used the area definition of multiplication. 32% of the PSTs chose to address neither the area nor the equal groups definition, but chose to explain Tommy's logic referring mostly to the procedures Tommy used.

Table 2: Depth of Interpretations of Tommy's Logic

Rating	Description of response type	Example	Freq. (N = 25)
4	Clear and convincing explanation that includes discussion on the concept of multiplication. Errors are nonexistent.	Tommy tried to make the numbers easier to work with. He added 2 to 28 to make 30. Then added 3 to 7 to make 10. He subtracted the 2 and 3 he added, but when he added 2 to 28, he turned it into 30 groups of 7 and when he added 3 to 7 he added 3 more to each group. Tommy's error is not realizing the value of the numbers he is adding. To make this strategy work he could try Adding 2 means 2 more groups of 7. 	32% (8)
3	Understandable, but less-detailed, explanation that demonstrates some conceptual knowledge of multiplication. Errors are minimal.	This strategy will not work. Tom is thinking that since he added two more to 28, he can just take 2 away and 3 away because he added it to the 10. He does not understand that those 2 and 3 created little pieces contributed towards the area. I would use the area model to explain this problem.	24% (6)
2	Clear explanation, but one that does not give appropriate reasoning why the method is not generalizable.	Tommy has come up with a good strategy but in his rounding up to do this problem when going back to subtract the amount he rounded up he then applies the addition strategy. He obviously has not gained the understanding of groups in multiplication. His strategy works with addition and subtraction of whole numbers but will not work with multiplication of whole numbers.	16% (4)
1	Unclear, incorrect explanation or one that does not address the question.	This strategy will not work unless either the multiplicand or the multiplier is a whole number. What works to make the multiplicand or the multiplier a whole number, then subtract away the average.	28% (7)

To analyze depth of responses, we used a four-point rubric which distinguishes responses of differing depths, ranging from clear and convincing explanations that include discussion of the meaning of multiplication to those that are less-detailed, unclear, or incomplete (see Table 2 above). The ratings for these interpretations have a mean of 2.6 and a standard deviation of 1.2, indicating a moderate measure of spread in the distribution. 32% of the participants' interpretations were ranked at the 4 level, and all of these 4-point interpretations utilized the equal groups definition of multiplication. Also, 24% of the interpretations of Tommy's logic ranked at the 3 level. Among these 6 participants who scored 3 points, 3 used the equal groups definition, 1 used the area definition, and 2 chose to focus on procedure rather than either definition of multiplication. Ranking at the 2 level were 16% of the interpretations, which were clear but did not give appropriate reasoning why Tommy's method was not generalizable. Lastly, 28% participants gave unclear or incorrect interpretations, scoring a 1 on our four-point rubric.

Interpretations of Dan's logic. All participants correctly identified Dan's correct strategy as generalizable to all whole numbers. To analyze definitions of multiplication used to interpret Dan's thinking, we used the same categories observed in the analysis of their interpretations of Tommy's logic--equal groups definition, area definition, and a focus on procedure. We found a different tendency in the use of the definitions. With the interpretations of Tommy's logic, most of the PSTs addressed one of the definitions of multiplication; however, when interpreting Dan's logic, an overwhelming majority chose to only focus on procedure and failed to relate Dan's logic to any definition of multiplication. More specifically, 92% of the PSTs (23 out of 25) failed to connect Dan's logic with any definition of multiplication. Only 8% made a reference to the equal groups definition, such as "Dan's logic is that 28 groups of 7 is equal to 20 groups of 7 plus 8 groups of 7." Also, none mentioned the area definition. The 4-point rubric was used to analyze the depth of the interpretations of Dan's logic (see Table 3). Most of the interpretations were rated as either a 2 or a 3. This indicates that most responses could have been more detailed or could have better explained the generalizability of Dan's method.

Table 3: Depth of Interpretations of Dan's Logic

Rating	Description of Response Type	Example	Frequency (N = 25)
4	Clear and convincing explanation that includes discussion on the concept of multiplication, instead of solely on multiplication procedures. Errors are nonexistent.	Dan's strategy was to multiply the seven by the tens place of the other number and to multiply by the ones place. Then he added those 2 numbers together. This strategy works because he has not changed the problem, but simply isolated the two place values to simplify it.	4% (1)
3	Understandable, but less-detailed, explanation that demonstrates some conceptual knowledge of multiplication of whole numbers. Errors are minimal.	Dan breaks apart 28 into 20 and 8 but still multiplies 20 and 8 by 7, which will work for all whole number as long as all place values are represented.	40% (10)
2	Clear explanation, but one that does not give appropriate reasoning why the method is not generalizable.	Dan is breaking his numbers down so that they are easier to work with. Once he completes his simplified problem he adds those numbers together. His strategy will work for all whole numbers.	52% (13)
1	Unclear, incorrect explanation or one that does not address the	Dan broke 28 into twenty and eight because they are easier numbers to multiply. Then he	4% (1)

	question.	added them together. This strategy does not work for all whole numbers.	

Whether responses differ depending on the correctness of student strategies. In the interest of uncovering how PSTs make their decisions, our third research question asks if there were observable differences in PSTs' interpretations and responses based on the correctness of the student-invented strategies. We performed statistical tests to compare responses to Tommy's incorrect method to responses to Dan's correct method, in particular, with respect to the following five aspects—(1) differences in definition of multiplication used, (2) differences in depth of interpretation, (3) differences in discussion of student thinking, (3) differences in student-centeredness of responses, and (4) differences in type of model used.

First, results from the Fisher Exact Value test suggest a statistically significant difference in PSTs' use of the definition of multiplication with Tommy and Dan ($p = 0.00028$). In particular, we found that the equal groups definition of multiplication was used more often when interpreting incorrect logic. Interestingly, when interpreting correct logic, the PSTs tended to give purely procedural interpretations that were void of any definition of multiplication.

Second, results from the Fisher Exact Value test showed that the depth of explanation given by a PST is dependent on whether interpreting correct or incorrect student strategies ($p = 0.00093$). To discover the nature of this dependence, we compared the frequency on the 4-point rubric shown in Tables 2 and 3. When interpreting an incorrect student-invented strategy, there is more variance in the depth of responses, with more ratings at the highest or lowest end than between. On the other hand, when responding to a correct student-invented strategy, there is less variance, with most of the responses rating in the middle of the rubric and with very few responses at the highest or lowest end. This means that there were more high-quality, in-depth interpretations given of incorrect student strategies than correct student strategies.

Third, results from the z-test reveal that discussion of student thinking is dependent on students' correctness (a p-value of 0.0099, $Z = 2.579$). Although the task only asked how the PSTs would respond to Tommy and Dan, we noticed that many of them chose to use some language about students' thinking within their responses. After noticing that many of the PSTs chose to use this kind of cognitive language, we analyzed their choice of words and phrases. The categories that emerged from this analysis were similar to those of Sfard (1998). We sought vocabulary cues that frame learning either as a cognitive process or as *cognitive status*. For example, if there is think-action taking place, with words like, *recognize*, *see*, *forget*, or *remember*, PSTs' responses were categorized as describing math learning using a cognitive process. On the other hand, a response was categorized as describing math learning using a *cognitive status* if there is acquisitional language, such as "gain understanding," "gaps in knowledge," or "imparting ideas," portray math knowledge as a *cognitive status*, rather than *cognitive action*. We found that sixty eight percent (17 out of 25) were categorized as cognitive action while the rest of the participants showed views of cognitive status when talking about Tommy's learning. However, interestingly, the PSTs gave more neutral responses, not using many words that referred to Dan's thinking. Only two participants used cognitive action wording, which portrayed learning as engagement, and the same number of participants used wording that portrayed knowledge as a substance to be acquired. Results from the z-test suggest that a difference exists in the proportion of PSTs discussing student thinking based on whether the student's solution is correct or incorrect.

Forth, results from a chi-square test suggests that there is no statistical difference in PSTs' responses to Tommy and Dan with respect to student-centeredness ($p = 0.765$, $\chi^2(1) = 0.089$). This suggests that student-centeredness is independent of students' correctness.

Fifth and lastly, we also found that the types of models PSTs used are not dependent on the correctness of the students' work. The Fisher Exact Value test showed no such significant relationship ($p = 0.876$), suggesting that PSTs choose certain types of models to use in their responses, regardless of whether the student to whom they are responding is correct or incorrect in their reasoning.

Discussion and Implications

The findings of this study have implications regarding both CK and PCK for preservice teachers. For example, regarding CK, consistent with the findings from previous studies (e.g., Hill & Ball, 2004; Son & Crespo, 2009), this study stresses the importance of gaining a deeper understanding of the meaning of multiplication requires justifying of how and why an answer works and requires connecting models to computations in responding to students. Teacher education programs should pay more attention to strengthen PSTs' mathematical base and their abilities to provide justification in terms of not only traditional methods for whole number operations but also student-invented strategies. However, different from previous studies, this study investigated CK focusing on correct and incorrect student-invented strategies, and results provides further information about preservice teachers' tendency such as what type of scenario elicits such in-depth justifications and connections. For example, we found that incorrect student work samples elicit more in-depth justifications, whereas choice of model is not influenced by the correctness of the work. These findings suggest teacher educators might use incorrect student-invented strategies to promote the justifications and connections that promote CK.

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MATHEMATICS PROFESSIONAL DEVELOPMENT: EXAMINING FACILITATION AND DISCOURSE

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The authors present an analysis of facilitation across three cohorts of a phase-two professional development project. Four types of discourse are introduced and defined (correcting, eliciting, probing, and responsive). The analysis attends to implementation fidelity and facilitator moves in light of these four types of discourse. Conjectures about the impact of facilitation moves on implementation fidelity and discourse are explored, and implications for conducting professional development are considered.

Keywords: Teacher Education-Inservice/Professional Development, Classroom Discourse, Elementary School Education

Mapping the terrain of research in professional development, Borko (2004) considered three phases of research to examine professional development. Phase-one research focuses on an individual professional development program at a single site and investigates the relation between the program and participating teachers as learners. Phase-two continues to focus on one professional development program, but the program is now offered at multiple sites; such research attends to the relations between the program, the teachers, and the various facilitators of the professional development. Phase-three research focuses on comparing different professional development programs across multiple sites and facilitators, investigating relations among programs, teachers, facilitators and contexts. This papers reports on a phase-two research study that investigated various implementations of one professional development program - Project AIM (All Included in Mathematics).

Borko (2004) stated that professional development programs that could be enacted by multiple facilitators were ripe for the exploration of questions about similarities and differences across sites, with an eye toward generating knowledge about professional development and also refining the program's tasks and materials. She highlighted the importance of developing "well developed and clearly specified" (p. 9) programs that allow the field to move from conducting phase-one research to phase-two research. Such a move creates the necessity of investigating "the balance and tradeoffs between fidelity and adaptations" (p.12), but represents a fundamental step toward offering professional development at scale.

The issue of scale in professional development has received renewed attention with the release of the Common Core State Standards for Mathematics (CCSSM) (Sztajn, Marrongelle, Smith, & Melton, 2012). The national-level need for massive professional development to bring the CCSSM into classrooms represents a change of magnitude in the need for professional development capacity (Ball & Cohen, 1999), requiring knowledge about professional development facilitation that supports productive, rather than fatal, adaptations (Seago, 2007) of professional development programs.

Project AIM, the professional development that serves as the context for the present study, is part of a National Science Foundation-funded, 5-year research and development program. The tasks that comprise the 40-hour program were designed and tested in year 1, implemented in one site in year 2 as a pilot, then revised and scaled to multiple sites in year 3. The present study

focuses on one session of the professional development program as implemented by three different facilitators in year 3. This research attends to whether the materials were implemented as intended. Given that attention to classroom discourse is a central tenet of Project AIM, this study also examines the professional discourse that took place among teachers at the various professional development sites. As we learned, implementing the program with fidelity did not warrant the promotion of teacher discourse as envisioned by the program designers because differences in facilitation moves led to variations in the emergences of professional development discourse.

In what follows, we first discuss the importance of discourse for mathematics learning both at the K-12 and at the professional development level. Based on the literature on discourse in general, and on the work of Hufferd-Ackles, Fuson, and Sherin (2004) in particular, we introduce our definitions of four types of discourse (correcting, eliciting, probing, and responsive), which were used both as a professional development tool for the project and as a lens to analyze the teachers' professional discourse in the context of the project. Next we discuss how issues of fidelity of implementation and integrity of enactment were considered in the study. Then, we define the context of the study and introduce the research participants. We describe the particular professional development session that served as the context for our research, the data collected and the analytical procedures. Finally, we share our findings about the implementation of the professional development program, the professional discourse of teachers in different professional development sites, and the facilitations moves that supported responsive discourse among teachers.

Research Context and Question

The notion that students learn from one another has been posited and investigated by a number of researchers (e.g., Vygotsky, 1978). More recent research in mathematics education confirms the importance of discourse and interaction for student understanding (Kazemi & Stipek, 2001; Yackel, Cobb, & Wood, 1991). However, this notion that knowledge is socially constructed is not limited to the mathematics classroom; it may be applied to mathematics teacher education and professional development as well. McCrory, Putnam, and Jansen (2008) assert that in professional communities "teachers learn through sustained discourse with other teachers, sharing their expertise and learning from the expertise of others, bringing professional development [and university training] closer to the real work of teaching" (pg. 157).

The professional development context of our research was designed to provide practicing elementary teachers opportunities for long-term, collaborative work with ready-to-use strategies that could be implemented in their mathematics classrooms to promote responsive discourse among students. Specifically, participants in Project AIM learn to identify four different types of mathematical discourse in the classroom. Drawing upon Hufferd-Ackles et al.'s (2004) theoretical framework for a "math-talk learning community," the four types of discourse defined in Project AIM classify both mathematical content and responsibility. For the purpose of this study, it is appropriate to share these in light of responsibility classifications. First, correcting discourse is characterized by the Teacher initiate-Student respond-Teacher evaluate pattern of discourse in which the teacher asks questions, a student responds, and the teacher listens to verify whether the answer is right or wrong. In eliciting discourse, the teacher transitions from focusing on correcting students' ideas, to eliciting students' sharing of their ideas and their thinking. This removal of the correcting aspect of the discourse intends to create a safe environment that supports students in communicating their explanations. In probing discourse, the teacher continues to maintain the positive nature of discussions, and purposefully uses

questioning to press students for greater depth in sharing their thinking. Finally, responsive discourse involves a difference in responsibility within discussions. The teacher moves from being the sole authority for the quality of the content and the nature of the discourse, to helping students take increasing responsibility for them.

Recall that this paper reports on a phase-two research study that investigated various implementations of one professional development program. It attends to the central goal of phase-two research, which is "to determine whether a professional development program can be enacted with integrity (LeFevre, 2004) in different settings and by different professional development providers" (Borko, 2004, p. 9). We focused on fidelity of implementation as delineated by O'Donnell (2008) in her review of research as 1) adherence, 2) duration, and 3) participant responsiveness. Analysis of fidelity of implementation allowed us to better understand how facilitators interpreted the professional development materials. In particular, in this study, we address the following overarching research question: *How do elementary grades mathematics professional development facilitators interpret and use with elementary grade teachers a set of professional development materials developed to promote discourse in the classroom?*

Methodology

Participants and Setting

A 40-hour yearlong professional development in Project AIM (*Project Name*) was created with underlying assumptions that 1) students need to learn to participate in productive classroom discourse; and 2) knowledge of mathematics for teaching and how to promote classroom discussions are both necessary, but neither one alone is sufficient. During the first two years of this research and development project, elementary instructional math coaches were recruited from a large school district in the southeast United States. In year one, these coaches were participants of the professional development. The second year, elementary teachers were recruited from the same district and the coaches observed the professional development and served as a focus group for revisions of the materials. During the third year of Project AIM, the time of this current study, eight of the coaches were recruited to be facilitators of the professional development. Facilitators were paired and four new cohorts of second grade teachers were recruited. In all, nearly 80 second grade teachers representing 23 elementary schools in the district were selected based upon their guarantee that at least two teachers from each participating school would commit to the yearlong professional development. The number of teachers in the four cohorts ranged from 15 to 21.

The purpose of Project AIM is to promote mathematics discourse in the elementary school classroom by implementing strategies found in research from the literacy field. The project also includes development of crafted professional development materials along with research that includes documenting teachers' learning and implementation of ideas throughout the school year. A series of three-hour professional development sessions were designed to encourage teachers to analyze classroom video, engage in discourse-promoting activities, and plan for instruction. For the purposes of Project AIM, classroom discourse is defined as patterned ways of using language and other communication tools in the classroom. The professional development began with an intensive 3-day summer institute, during which teachers participated in 6 of the 13 total sessions of Project AIM. Session 3, the focus of this paper, occurred at the beginning of day 2 of the summer institute. To begin answering the above research question, we chose to focus initially on Session 3 because it included the Think Aloud, a strategy often used in literacy, and therefore already familiar to teachers.

Overview of Session 3

In Session 3, participants were introduced to ways to support the implementation of responsive discourse in the classroom. The session included pedagogical discussions about instructional moves that allow students to talk about their mathematical ideas and the teacher to listen in support of learning. Specifically, Session 3 emphasized the launch phase that begins a lesson, and participants had the opportunity to learn about and reflect on one particular launch strategy, called Think Aloud, through the use of classroom video artifacts.

Data Collected

Our initial focus on a single session of the professional development, namely Session 3, also allowed us to look carefully across cohorts. However, facilitators in one of the cohorts shared leader responsibilities within Session 3. Therefore, it was decided to exclude them from the current study so that we could better compare the remaining three cohorts, which only utilized one facilitator as the leader of Session 3.

During the summer institute, when Session 3 occurred, entire sessions were video recorded. Graduate students, not part of Project AIM, served as videographers, one per cohort. Each cohort also had a project liaison in the room during the entire summer institute. One responsibility of the liaison was to provide a detailed observation log for each session. Other written data collected included observation notes from the facilitators who did not lead Session 3 as well as reflections from the facilitators who did lead the session.

Data Analysis

We were interested in reporting traditional implementation fidelity of the professional development materials as well as how facilitators engaged participants with activities and how their moves affected implementation and engagement.

Reporting implementation fidelity. The written plans for individual sessions of the professional development are divided into a few main segments. For each segment, the plans include a rationale/purpose, suggested times, a detailed list of activities, and a list of facilitation notes and main ideas for participants. For some segments, the plans also offer anticipated teacher responses for the facilitators to consider during their planning and preparation or during implementation of the session. Prior to analysis, the number of activities and main ideas for each segment of Session 3 were determined. Videos from each cohort were carefully viewed to determine the time allotted and number of activities implemented for each segment in Session 3. Additionally, the number of main ideas explicitly stated by facilitators during the session was noted. Liaison logs and observer notes were used to confirm such actions.

Discourse in session 3. We selected three focal activities in Session 3 to analyze how teachers interacted with one another and how the facilitator engaged them in whole group discussions. These three activities were identified at the beginning of Project AIM's third year as part of larger research and development goals for Project AIM as potentially rich sources for describing development of participants' mathematical knowledge for teaching and their understanding of responsive discourse during their engagement in the professional development. Three of those focal activities occurred during Session 3. The first was a whole group discussion of a video of a second grade teacher launching a lesson using the Think Aloud strategy with a comparison problem: "Jane and Ernie have some apples. Jane has 6 apples and Ernie has 9 applies. Who has more apples? How many more?" (Fuson, 2009, pg. 222). The second focal activity was a later whole group discussion on the teacher's use of the Think Aloud strategy in her launch. Prior to the second focal activity discussion, participants watched a second video, which included content from the first video as well as what happened in that classroom

immediately following the launch. Finally, the third focal activity in Session 3 was a whole group discussion of participants' reactions to the second grade teacher's written reflection on her use of Think Aloud to launch the mathematics lesson.

Videos of these focal activity discussions were analyzed for both interaction patterns and levels and types of discourse. Two researchers entered a cycle of video analysis and coding modification until agreement was reached. A revised coding scheme, which included four dimensions (correcting, eliciting, probing, and responsive) and two types (questioning and explaining), became the codebook used for video analysis. Reliability was established by the two coders working together until an 85% agreement had been reached, and a third researcher independently agreed with the codes applied to a particular video segment. Then, the two original coders continued coding additional video segments separately. For each cohort, videos for the three focal activity discussions in Session 3 were coded directly using Atlas TI and the quotations formed within the qualitative analysis software were used to look for further themes. Each participant and facilitator turn was coded separately and the time of talk was recorded. Wait times of five seconds or more were also coded. This line-by-line coding provided information about interactions within a cohort. The research team used color codes within an Excel spreadsheet to look for patterns. Each quotation was denoted by a separate row in the spreadsheet. Columns represented the four dimensions of discourse (correcting, eliciting, probing, and responsive). Then, colors were used to distinguish participant from facilitator (yellow and red respectively) and shades of individual colors specified questioning (light color) and explaining (dark color). (See Figure 1 for an example.)

Making conjectures about facilitator moves. Continued video analysis allowed us to make conjectures about why certain discourse types and discussion patterns emerged during whole group discussions. Aligned with the overall goal of Project AIM, which is promoting responsive discourse, it was noted when facilitators explicitly made comments shifting responsibility for discussions to participating teachers. We also attended to facilitators' physical placement in the room during different segments of Session 3, as their physical location within the group might influence the group's sense of who is responsible for generating and shaping discussions. These details, along with graphical representations of discussion patterns and discourse types, permitted us to make conjectures about how facilitators' moves during implementation of Session 3 impacted discourse among participants during the professional development.

Results

Reporting Implementation Fidelity

It was suggested in the lesson plan that 165 minutes be allowed for all activities in Session 3. Overall, we found that facilitators followed the timing recommendation for each segment fairly closely, which resulted in little overall time differences across cohorts for the professional development session. Each cohort went slightly over the suggested total of 165 minutes (between 1.5 and 7 minutes). We also found little difference across cohorts in the number of activities implemented by facilitators. All activities were listed in the detailed plan for Session 3, and facilitators consistently adhered to that plan, implementing the activities in the specified sequence (between 80% and 89%). All facilitators used an accompanying Power Point presentation to help guide their work in leading Session 3. The Power Point file for Session 3 included slides about most of the activities, which may have contributed to the consistent implementation of activities in sequence. Although the accompanying Power Point presentation for Session 3 included, for most segments, the activities in the order they were designed to be

implemented, the slides did not present all of the main ideas. This factor may have had an impact on the number of main ideas that facilitators explicitly stated. For example, during the discussion of the launch video, the facilitator in Cohort 4 stated only those main ideas that were displayed on the slide for participants to read. This decision resulted in a much lower percentage of the main ideas (67%) being stated when compared to what occurred in Cohorts 2 and 3 (91% each).

Despite some differences between facilitators in the number of minutes used in Session 3, the number of activities, and the number of main ideas explicitly stated for participants, the facilitators were using the professional development materials largely as they were intended and were helping participants attend to most of the main ideas within the session.

Discourse in Session 3

Recall that while we reviewed Session 3 in its entirety for implementation fidelity, we focused on only the focal activities within Session 3 for our discourse analysis. Discourse for each of the focal activities was coded for type (correcting, eliciting, probing, or responsive) and dimension (questioning or explaining). Figure 1 below shows discourse coding for one focal activity in two of the cohorts. This example was selected to show how differently one focal

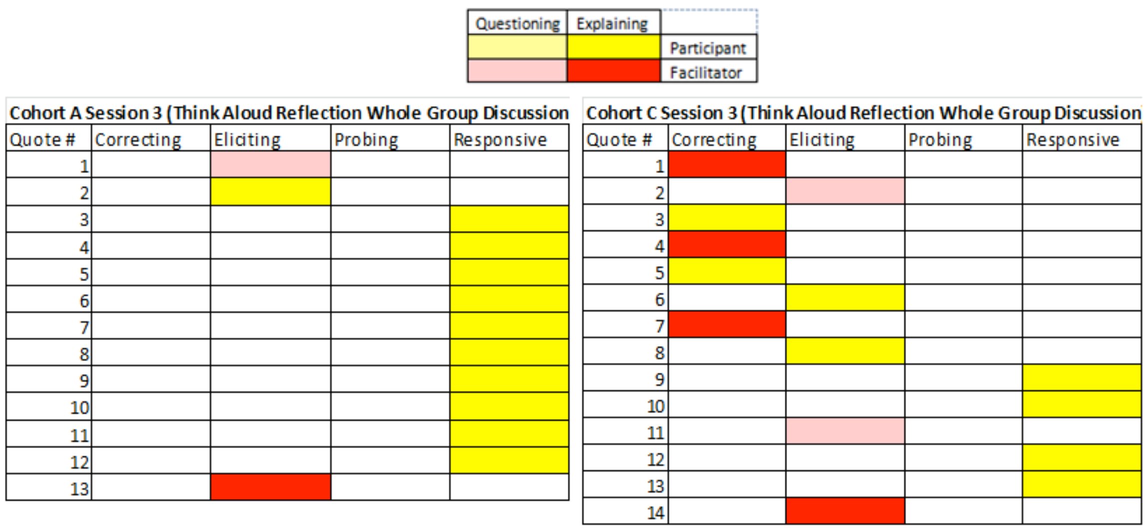

Figure 1. Discourse Coding for One Focal Activity in Cohort A and Cohort C.

activity looked across cohorts, despite having nearly an equal number of distinct talk turns (13 and 14 in Cohorts A and C, respectively). The discussion in Cohort A began with the facilitator asking an eliciting question. This beginning was followed by an eliciting response from a participant and a longer, responsive conversation among the whole group of participants. The facilitator did not enter the dialogue again until the last talk turn when she took stock of the discussion and concluded the focal activity. In Cohort C, the discussion began with the facilitator explaining content and asking an eliciting question. When a participant responded with a short answer, the facilitator's quick reply at the correcting discourse level prompted another participant's correcting-level response. Answering the original question, a participant gave an eliciting remark, and the facilitator once again responded at the correcting level. Responsive discourse was present at the end and maintained for several talk turns, despite an eliciting question from the facilitator.

Apart from the first correcting statement by the facilitator in Cohort C, the two discussions highlighted in Figure 1 began and ended similarly. Both facilitators started the discussion with an eliciting question and ended with an eliciting explanation. What we found interesting were the

differences in what happened during the discussion. The nature of the focal activity discussions described above was not exceptional. Similar discourse patterns were found throughout Session 3. Facilitators in Cohorts A and B were more able to remove themselves from whole group discussions than the facilitator in Cohort C. As a result, participants in Cohorts A and B frequently took responsibility for the conversation. Thus, participants in those groups reached and maintained responsive discourse more often than participants in Cohort C.

Making Conjectures about Facilitator Moves – Considering Implementation Fidelity and Discourse Together

Table 1 provides a summary of adherence to main ideas of the session (percentage of main ideas explicitly shared during the activity), duration (+/- minutes over the suggested time denoted in the professional development materials), and time of facilitator talk (% of total minutes) for

Table 1. Summary of Focal Activities in Session 3.

		Cohort A	Cohort B	Cohort C
Focal Activity 1	Activities	100%	100%	75%
	Main Ideas	92%	92%	42%
	Duration	-4	+7	0
	Facilitator Talk	27%	19%	62%
Focal Activity 2	Activities	100%	100%	100%
	Main Ideas	100%	75%	100%
	Duration	+9	+2	+17
	Facilitator Talk	26%	31%	54%
Focal Activity 3	Activities	100%	100%	75%
	Main Ideas	100%	50%	50%
	Duration	-6	-1	-9
	Facilitator Talk	33%	34%	46%

each of the three focal activities in Session 3. We found that the back-and-forth nature of interactions between the facilitator and participants in Cohort C resulted in more facilitator talk time and less time operating at the responsive discourse level than in Cohorts A and B.

In our analysis of video, we also noted facilitators' physical placement during the session as well as explicit statements about responsive discourse shared with participants. Facilitators in Cohorts A and B tended to remove themselves from the center or front of the room. They would often stand to the side during whole group discussions. Once, the facilitator for Cohort B even sat in a chair outside the semi-circle of participants. We interpreted these physical moves as promoting the expectation that participants should use responsive discourse to share ideas and talk with one another. The facilitators in Cohorts A and B also reinforced this expectation with explicit directions for participants. They directed participants to "look at each other and respond to each other" (facilitator, Cohort A) and to "take control of the conversation" (facilitator, Cohort B). In contrast, the facilitator for Cohort C remained at the front of the room during whole group discussions and did not explicitly remind participants to talk to one another. We hypothesize that the facilitator's physical placement and explicit comments contributed to participants' opportunities for responsive discourse.

Types of discourse used by the facilitators may be another contributing factor for participants' ability to reach and maintain responsive discourse. From our analysis of three focal activities, it appears that responsive discourse may not occur (at least during the early stages of a long-term professional development effort) unless some scaffolding of the discourse occurs.

Changes in responsibility occurred more naturally as the types of discourse also changed (from eliciting to probing to responsive). When correcting discourse occurred, the next talk turn was rarely responsive. In other words, once a discussion moved back to a correcting level, it took some time for responsive discourse to occur again, if at all (e.g., Figure 1).

Conclusion

In the professional development program, Project AIM, practicing elementary teachers learn pedagogical strategies to promote responsive discourse in their mathematics classrooms. Our results from this phase-two research study (Borko, 2004), which investigated implementations of the same session in three different cohorts, revealed that facilitators across the cohorts similarly adhered to professional development materials, implementing most of the activities and sharing most of the main ideas as described in the detailed session plan. However, facilitators' decisions about how to conduct whole group discussions, in particular, affected the opportunities for participants to engage in the responsive discourse in the professional development. Facilitators' physical placement in the group, explicit directions, and use of different types of discourse influenced the degree to which participants were able to take responsibility for conversations in the professional development.

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THE EFFECTS OF FRAMING ON MATHEMATICS STUDENT TEACHER NOTICING

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Teacher education programs have increasingly incorporated activities to support novice teachers in learning to attend to student ideas that surface during mathematics instruction. These efforts have been found to be effective at the end of a particular course, but less is known about their longer-term effects on teaching practice. A related study of the noticing of prospective mathematics teachers who had engaged in such activities found that they demonstrated a focus on students during their own instruction; however, this focus was not always on student ideas. Here, the cases of two of these prospective teachers are used to explore the nuances of their noticing of students, and how the ways in which they framed their instruction—in terms of their knowledge of students and their own expectations of students—affected this noticing. Implications for teacher education programs are discussed.

Keywords: Teacher Education-Preservice, High School Education

Teacher noticing (e.g., Sherin, Jacobs, & Philipp, 2011) has emerged as central to student-centered mathematics instruction (e.g., National Council of Teachers of Mathematics, 2000). Typically defined to include three interrelated skills—identifying important events during instruction, reasoning about them, and making connections between them and broader educational principles (e.g., Sherin & van Es, 2005)—the importance of this construct stems from the fact that teachers cannot intentionally act upon that which they do not notice (Rosaen, Lundeberg, Cooper, Fritzen, & Terpstra, 2008; Sherin & van Es, 2005). In fact, a major difference between expert and novice teachers' practice is their ability to recognize and respond to important instructional events (Berliner, 2001; Hogan, Rabinowitz, & Craven, 2003). Often, novices fail to act upon student-generated instructional instances to which experienced educators intuitively recognize and productively respond (Peterson & Leatham, 2010).

Many teacher education programs have begun to incorporate activities intended to help prospective teachers develop the skills and dispositions necessary to notice important student-generated instances that arise during instruction (e.g. Stockero, 2008; Leatham & Peterson, 2010; Santagata & Guarino, 2011). Some of these activities have been found to be effective in the short term, that is, at the conclusion of a particular teacher education course. Analyses of the outcomes of using video-based activities in mathematics methods courses, for example, have documented that prospective teachers developed a focus on making sense of student thinking (Stockero, 2008), as well as on the teacher moves that help make such thinking visible (Santagata & Guarino, 2011). The latter study also found that prospective teachers began to consider the relationship between teacher moves and student learning. Less is known, however, about the longer-term effects of such activities—what teachers who have engaged in coursework focused on noticing and making sense of student ideas attend to during their own instruction.

This paper focuses on the cases of two novice mathematics teachers who had engaged in such activities during their teacher education coursework. It builds on the results of a prior analysis of the in-the-moment noticing of six teachers during their student teaching experience.

This analysis documented a strong focus on students, with 87.5% of the participants' self-documented instances of noticing having some focus on students (Stockero, 2013). Although this result was encouraging, the analysis also revealed that only 25% of the documented instances were focused on students' thinking, with 15% focused on making sense of an individual student's mathematical thinking—a focus of the student teachers' mathematics methods course. This finding prompted further analysis of the data to better understand the nuances of the participants' student-centered noticing, and what barriers might be preventing them from carefully attending to students' ideas. The cases discussed here highlight how the frame or lens through which the student teachers noticed influenced what they did or did not document as important during the act of teaching. The findings are used to highlight potential implications for teacher education programs.

Theoretical Perspectives

The work is grounded in a vision of teaching where teachers continuously build on student thinking in ways that are responsive to their current mathematical understanding (e.g., NCTM, 2000). This requires that teachers maintain a strong student focus during the act of teaching; specifically, it requires teachers to *notice* (e.g., Sherin et. al, 2011) student mathematical ideas.

Ideas from the psychology literature provide a foundation for the construct of teacher noticing and may help to explain novice teachers' lower level of responsiveness to student ideas. One cause may be *inattentional blindness* (Simons, 2000), defined as a failure to recognize unexpected events. In the context of a mathematics classroom, such blindness may result, for example, in a teacher failing to recognize that an incomplete student idea might be mathematically important because they may not be aware that this could be the case. A lack of responsiveness could also be related to *situational awareness*, defined to go beyond mere awareness of a phenomenon, to include sense-making ability (Endsley, 1995). During a classroom lesson, if a teacher cannot make sense of what a student is saying or how it might relate to the goals of the lesson, they will likely be less inclined to take up the student's idea. Teacher noticing during instruction requires both awareness and sense-making.

This research focuses on understanding a specific type of teacher noticing, *mathematical noticing*. Mathematical noticing focuses on attending to student ideas that surface during instruction that include important mathematics and have the potential to support students' understanding of the mathematical goals for the classroom (e.g., Leatham, Peterson, Stockero, & Van Zoest, 2011). Although a range of instances of student thinking occur during instruction, not all of these instances are equally important to notice in terms of their potential to help achieve the goal of supporting students' mathematical learning. While some instances of student thinking might be important for affective or pedagogical reasons, teachers need to be clear about which instances are important mathematically.

Context and Methodology

The case participants, Adam and Ally, were prospective mathematics teachers completing a semester-long student teaching experience during the final semester of a secondary-school level teacher education program. The teacher education program focused on developing student-centered instruction grounded in inquiry and sense-making. The participants had completed a mathematics methods course with a strong focus on listening to, making sense of, and considering how to use student thinking during instruction. In the course, they had studied Smith and Stein's (2011) *5 Practices for Orchestrating Productive Mathematics Discussions*. They had also analyzed student thinking in written and video cases of instruction, and worked with small

groups of secondary school students on tasks to elicit and build on their thinking to support their mathematical understanding (see Van Zoest & Stockero, 2008 for a description of a course after which this was modeled).

Adam and Ally were typical, or even slightly above average, prospective teachers in the teacher education program. In the mathematics methods course, each had demonstrated an ability to make sense of student thinking in cases of practice and when analyzing their own or peers' work with small groups of students in a local classroom. Both were also able to accurately reflect on their work with students, using classroom-based evidence to highlight ways in which they both helped and hindered student learning.

Each participant was observed and video-recorded teaching three mathematics lessons during their 15-week student teaching experience, early- mid- and late-semester. During each observation, the participant was asked to document important instances that they noticed while teaching by wearing a self-mounted camera that allowed them to capture a 30-second segment of video when they felt an important instance had occurred. The definition of what might constitute an important instance was left open-ended to allow the researcher to understand what participants viewed as important during their teaching. Following each lesson, the participant engaged in an interview focused on discussing self-documented instances, undocumented instances that were deemed to be mathematically important by the researcher/observer, and the participants' general ideas about what might be important to notice during a lesson.

Data included the lesson video recordings, the participants' documentation of important instances they had noticed, and video recordings of the post-lesson interviews. The interview video was segmented into conversations about instances discussed in the classroom video and general ideas about noticing. Building on a framework used in prior work (e.g., Stockero, 2008), each conversation about an instance was coded for agent (Who was the noticing focus?) and topic (What was the noticing focus?). Discussions of undocumented instances were coded for participant-identified barriers to noticing. During the coding process, it became apparent that some participants, including Adam and Ally, seemed to have a particular frame through which they viewed their practice; additional coding was added to document when these frames were apparent. In this work, a frame is defined as a lens through which a participant seemed to make sense of what was important in their classroom.

The Cases of Adam and Ally

Adam and Ally were selected as cases for this study because the way they talked about their self-documented instances of noticing and their general ideas about what is important to notice while teaching highlight two different noticing frames that were evident among the larger group of participants—*knowledge of students* (four participants) and *expectations of students* (five participants). Data from Adam and Ally's post-observation interviews are used to help the reader understand these frames and to highlight how they affected the participants' noticing.

Adam: Knowledge of Students Frame

During the three observed lessons, Adam self-documented a total of 16 instances. Of these, 12 (75%) included some student focus, with individual or groups of students being the primary focus in 7 instances. In the post observation interviews, it became evident that Adam's knowledge of students significantly affected what he deemed important to notice during his instruction. Although Adam's overall noticing shifted across the three observed lessons, he consistently framed the discussion of his documented instances in terms of what he knew about the particular students who were central to the instance.

In the first classroom observation, the instances Adam documented as important mainly focused on his own use of questioning. Even in this early observation, he noted that he felt like he had a good understanding of the needs of individual students in his classroom and was able to tailor his questioning to those needs. It was typical for Adam to frame his noticing both in terms of the 'level' of the class, and his perceptions of the particular student's needs. This framing is evident in the following dialogue, where Adam discussed the importance of an instance in which he scaffolded a student's thinking:

> This class has probably one high achiever and then a lot of pretty okay students and then a couple not-as-good-at-math students, and [Student] is one of those students that is not as good at math and also has like an attention disorder or something. So I try to ask him questions because it keeps him on track and it keeps him working and when I can get him to give me a correct answer, he's actually a lot smarter than he gives himself credit for, so I try to keep working with him until he can give me the correct answer. I keep asking questions and sometimes I have to, I try to start with hard questions because I don't want to give him like dog treats. I don't want to do that. Sometimes I have to work him down that far, most of the time, and actually in this instance he was able to figure out that this [face of a figure] is a rectangle.

During the second observation, Adam shifted to documenting instances in which he facilitated small group interactions, but still framed the importance of each documented instance in terms of the needs of the students in the group. In some cases, he explained how he intervened with a group, while in others he noted how he helped particular students become involved with their group, or why he was able to let students work out difficulties on their own, as in the following:

> This group, in particular, is a very good group because [Student 1] is a very high achiever, [Student 2] and [Student 3] are both hard working students that are okay at math. [Student 4] is, I think, very smart at math but doesn't really try that hard…So I think their group is nicely balanced where they have kind of different levels of motivation as well as achievement. And I think that that discussion is very telling of each of their level of achievement, where [Student 3] wants a lot of, she wants to achieve well, or she wants to do well even though she is not that great at math. So she is always like asking for the clearest clarification she can get or multiple explanations….And then [Student 2] who actually gives the explanation at the end I think has, I don't want to say the lowest understanding, but she's, I mean they're all really pretty good, so of the four she's probably the lowest understanding but her explanations and writing skills are very good, so she gives the explanation to [Student 3] at the end. I don't know, I think that dynamic is very interesting.

After the interviewer pushed Adam to articulate why he had marked the instance as important to notice, he stated that it was the interaction among the students that was important, and concluded by saying, "I didn't do anything good there. I was just listening."

In the third observation, a stronger focus on individual student's thinking started to become apparent in Adam's noticing. In one instance, he described a student as having an attention disorder and then discussed a moment when he was "inspired" by this student's thinking:

> So he is actually very, very bright but he gets confused in the middle of his problems and I think that I found this to be almost like an inspiring moment for me, like [I] let [Student 5] do it. [Student 6] was trying to help him, she actually gave him the answer but we worked to make sure that he knew the process to get to the answer, and he didn't actually give the answer [Student 6] said of 1000…So I don't know, I just thought it was powerful to see

> someone's thoughts happen, like what's happening in their brain happened out loud. He was literally thinking through the problem and I don't know if he knew the answer when I called on him but he definitely, at least I feel like he knew the answer by the time it was done.

Across the three interviews, Adam consistently framed his noticing in terms of his knowledge of students, providing detailed accounts of students' strengths, weaknesses and individual needs to explain why a particular documented instance was important for him to notice. This framing was clearly evident in 11 of his 16 documented instances (4 out of 5 in the first observation, 4 of 7 in the second, and 3 of 4 in the third). Although he showed a strong focus on students, he was not always focused on students' mathematics. Instead, he was often focused on more general needs of the students in his class, such as what kind of support they might need to complete a problem or how they were interacting within a small group. In fact, his knowledge of students caused him not to document student ideas as important on at least two occasions, once because the student was strong mathematically and frequently approached problems differently and the other because he was unsure whether he should put the student on the spot to discuss his idea. In these cases, his perceptions of students caused him not to note the importance of students' mathematical thinking that might have been used to help other students make sense of the mathematics in the lesson.

Ally: Expectations of Students Frame

Ally documented a total of 12 instances as important to notice during the three observed lessons. Her total number of student-focused instances was similar to Adam, with 9 instances (75%) including some focus on students and 7 (58%) with students as the primary focus. Unlike Adam, what Ally considered important to notice was fairly consistent across the three observed lessons and was grounded in students' work on mathematical tasks. A clear frame for noticing also became apparent in Ally's post-observation interviews; she viewed the classroom through a lens of her expectations of students.

In all three of her observed lessons, Ally documented instances that provided evidence of whether students were making the "connections" that she wanted them to make; most often, these were instances when she felt connections were not being made. She often noticed how students were approaching a task or thinking about a problem in comparison to what she expected, using the words "rather than" and "instead of" to contrast the two. An example of this occurred in the first post-observation interview when she discussed an instance she had documented as important while observing the work of a small group of students. They were working on a task related to developing a function to model magazine sales. Here, she explained how the students had used an unsimplified form of a function to answer a question rather than using the simplied form that she had prompted them to find in a previous part of the problem:

> I actually saw in their work, because a lot of them, on number two, when they were putting in 72 individual magazines into their function, they didn't use their simplified function. They went back to [the unsimplified form of the function] and only did .55 times the individual magazines. So they didn't connect with the significance of taking the individual magazines away from them, too…I was trying to prompt them into looking at number one and making the connection, what's the difference, and what function could be [used] for number three.

In this instance, the students' use of the unsimplified function may have eventually resulted in a correct solution, but was not what Ally expected or wanted them to do. Thus, she documented the instance as an example of students not making a connection that she wanted them to make during the lesson.

Another example of an instance in which Ally framed her noticing in terms of her expectations of students occurred in the third interview. She documented this instance as important because an individual student did not respond to her questioning about long division of polynomials with the answer that she expected, which she interpreted as a lack of understanding.

Ally: So, maybe he identified that constant, but that wasn't, and I just swept over it because that wasn't what I was thinking in my head. I was looking for...ahhh!

Interviewer: Well, OK, so you were looking for what?

Ally: I was looking for the $0x^4$, $0x^3$ because [those terms] are not there. So, and then my, the way I was explaining that was not clear at all: “They don't exist but we need them to exist so they probably”, it wasn't clear to them.

Interviewer: So what do you think was important about this [instance]? Do you think that it was the way you explained it or didn't explain it as well as you wanted, or was it [the student] not understanding?

Ally: I think it was important that I didn't really approach it, I approached it looking just for what was in my head, what I wanted the answer to be, instead of actually drawing out of him what his thoughts were, what his thinking was to approach the problem. So doing that doesn't help either of us, it’s just putting an answer on the board. So it is not clearing anything up in his head, cause I am not addressing what he first said, just probably right where his thinking was going. I could have done that in a different way.

In this exchange, Ally again framed her noticing in terms of what she expected the student to say, and interpreted his comment though this frame, initially noticing only that his response did not align with her expectation. It is important to note, however, that there was evidence in this dialogue that Ally was beginning to realize how her expectations were limiting what she noticed as important during her instruction. In fact, at the end of this same interview, Ally discussed this limitation, saying, “I can predict, anticipate what they are going to say, but sometimes, they just, I didn't expect that so much. So then, I don't always have a high level or good question to ask them, so I don't actually focus on it.” She later added, “I think my brain is going another way, and so I'm not, I'm like, OK, this is going to happen, they're going to not realize this, so I'm not thinking about handling or noticing [something else].”

Across the three observed lessons, Ally framed her noticing of important instances in her instruction through the lens of expectations of students in 7 of 12 documented instance (5 out of 7 in the first video, 1 of 2 in the second, 1 of 3 in the third). As seen in the dialogue above, this frame often caused her to dismiss the value of student ideas, rather than viewing them in terms of the important mathematics within them. In both of the above episodes, the students were not incorrect, but rather, were not thinking about the problem in the way Ally expected; her frame allowed her to only notice the inconsistency.

Discussion and Conclusion

The data in this study showed that both Adam and Ally had a strong focus on students in their in-the-moment noticing during instruction. In general, this is an encouraging finding; it indicates that the teacher education activities in which they had engaged may have been effective in moderating the tendency to focus on their own teaching actions, rather than on their students, which is often the case with novice teachers (e.g., Berliner, 2001). The cases of Adam and Ally also highlight, however, that developing an ability to notice and make sense of student thinking in teacher education coursework may provide a foundation for attending to students during instruction, but may not be enough. Although both Adam and Ally noticed students, their focus was not on making sense of student ideas in order to build on them during instruction. Instead,

Adam focused on meeting students' general learning and affective needs, while Ally focused on whether students approached tasks or answered questions as she expected.

The data suggests that the frames through which Adam and Ally viewed their instruction strongly influenced what they noticed as important in their teaching. Because Adam framed his noticing in terms of his knowledge of students, he often only viewed instances as important when they gave him a sense of whether students' specific needs were or were not being met. For example, he noticed the importance of one student's thinking because the student often did not display good mathematical thinking and the instance was a chance to let him do so. In another case, however, he dismissed a student's unique approach because this student often approached problems differently than other students. Ally's frame of her expectations of students often constrained her noticing to student ideas, approaches or answers that did not align with what she wanted to hear. Although this type of noticing has the potential to be productive—if the teacher considers the mathematics behind the student thinking and how that thinking might be built on—Ally's frame caused her to instead interpret these instances as cases where students were not making desired "connections". Thus, Ally's focus in these instances was on how to correct the student's thinking, rather than how to productively use their ideas to develop a better understanding of the mathematics. These results closely align with Levin, Hammer and Coffey's (2009) findings about novice science teachers' noticing. In their work, they concluded that, "whether and how novice teachers attend to student thinking depends significantly on how they frame what is taking place" (p. 146). This was clearly the case for Adam and Ally.

This work suggests that helping novice teachers move beyond noticing students, to noticing the nuances of students' mathematical thinking, during instruction may require rethinking teacher education experiences. Although video cases have effectively developed noticing skills in the context of a particular course, this noticing might be connected to specific analysis frameworks or questions that are used to prompt such noticing. It is not guaranteed that teachers will adopt the same frames when not required to do so. Moreover, the context of a course provides the luxury of time to analyze and reflect on practice that is not available in the act of teaching. One potential solution is to scaffold prospective teacher noticing in coursework by first helping them learn to notice student thinking in a context that is "slowed down", gradually moving toward situations that more closely simulate in-the-moment noticing. It may also be essential to push prospective teachers to frame instructional instances in multiple ways to prevent them from adopting one particular lens. Yet another solution might be to provide different kinds of support and opportunities to engage in structured noticing activities during the student teaching experience; these could focus on both helping novice teachers continue to develop their noticing skills in a new context and pushing them to frame instances of practice in multiple ways. This would require either training classroom teachers to support novice teacher noticing or more substantial engagement between student teachers and university teacher educators. The outcomes of some of these efforts are being investigated in the author's ongoing work.
"[W]hat and how teachers teach depends on the knowledge, skills, and commitments they bring to their teaching and the opportunities they have to continue to learn in and from practice" (Feiman-Nemser, 2001, p. 1013). Given this understanding, the mathematics teacher education community must not only consider ways to develop essential knowledge, skills and dispositions in teacher education coursework, but to study the extent to which the results of this work are evident in teachers' practice—what is taken up, adapted, or dismissed. This study highlights the need to consider what types of ongoing support might be necessary to help prospective teachers build on what was learned in their coursework to notice what is important during instruction—

their students' mathematics. Providing such support has the potential to improve mathematics education by better enabling teachers to implement instruction that is responsive to student thinking even at the start of their teaching career.

Acknowledgements

This material is based upon work supported by the U.S. National Science Foundation under Grant No. 1052958. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the Foundation. The author acknowledges Erin Thomas, Michigan Technological University, and Raymond Gant, Ferris State University, for their contributions to the work.

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TEACHER POSITIONING AND AGENCY TO ACT: TALKING ABOUT "LOW-LEVEL" STUDENTS

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In this paper, we share our dilemma of disrupting teachers' deficit frames of students. The data comes from a professional development program with nine secondary mathematics teachers. In particular, we attended to when teachers talk about "low-level" students. Drawing on Harré and van Langenhove's (1999) idea of positioning, we examined teachers' self positioning and their associated storylines. By noticing and attending to positionings and storylines in professional development, we suggest, mathematics teacher educators can consider how to disrupt these deficit frames by recasting positionings and storylines.

Keywords: Classroom Discourse; Equity and Diversity; Teacher Education-Inservice/Professional Development

Introduction

An enduring dilemma in our work with practicing teachers surfaced recently as we piloted a set of professional development materials we have been designing: how might we disrupt mathematics teachers' talk about students that positions students in deficit frames? This kind of talk often appears in phrases like "my low-level kids." Both anecdotal evidence and published research suggests that we are not alone in grappling with this dilemma. Mathematics education researchers have addressed this problem by focusing on teachers' beliefs (e.g., Fennema, Peterson, Carpenter, & Lubenski, 1990).

Yet, as Parks (2010) pointed out, locating the problem as being about teachers' beliefs is limited and constructs the problem as being something that happens only in people's heads. She proposed, instead, that we understand this deficit view as socially constructed across many different communities and artifacts used in mathematics education. Using a rhetorical perspective, Parks explored the enduring metaphors that occurred across many mathematics education domains: prospective teacher coursework, field experiences, textbooks from methods courses and mathematics classrooms, standards, and university researchers' writings about mathematics students. She argued that the pervasive metaphor of learning mathematics as travelling along a narrow path limits who might be seen as 'successful,' as well as imposes rank ordering of students in many different ways. See the deficit view as socially constructed in this way, she contended, allows us to tackle the problem systemically rather than focusing only on individual teachers and their beliefs.

Recently, other mathematics education researchers have turned to social perspectives to better understand this dilemma. Horn (2007), for example, drew on a sociocultural framework for learning by examining the conceptual resources teachers brought to their work in trying to employ equity-oriented reforms. She took the stance that the broader idea of *conceptions* should be seen as "distributed across individuals and settings" (p. 38) as she explored talk in teacher

communities. She examined the "ways in which conceptions of students, subject, and teaching are embedded in teachers' daily work, particularly as they encounter problems of practice and work to solve these problems in consultation with colleagues" (p. 38). This article begins with a description of teachers' talk about students as being "fast," "slow" or "lazy."

We agree with Horn and Parks that views of students are socially constructed through discourse. We use positioning theory (Harré & van Langenhove, 1999), a slightly different social perspective, to examine teacher talk that positioned students as having low status during a year-long study group on classroom discourse. We seek to better understand the relationships between the ways in which teachers' talk positioned students (third order positioning) and the reciprocal ways in which teachers also positioned themselves (first order positioning). We argue that the teachers' first order positionings and their framings (storylines) matter because they relate how teachers might take action in their contexts.

Theoretical Background: Positioning Theory

We use Harré and van Langenhove's (1999) *positioning theory* as a lens to understand teacher talk in study group setting. Influenced by feminist studies (e.g., Hollway, 1984), positioning theory opposes the static notion of *role* (Harré & Slocum, 2003). Roles are what have been established through consistent interactions that constrain role-holders' moves in the future and are fairly static. Positions, however, are fluid, negotiated in moment-to-moment interactions, and can be accepted or refused. The process of positioning and the acceptance or refusal of a positioning can be explained by attending to *first, second*, and *third* order positioning. The order is determined by who is positioning whom in what context. We draw on an example of small group interaction in a sixth grade science classroom from Ritchie (2002) to further explain orders of positioning:

Nerida: We're smart, aren't we?
Rupert: No, we're [Marcello and I] smart. You aren't the smart ones, we are.
Alicia: We're not doing batteries are we?
Rupert: No, we're doing solar panels [*Marcello and Rupert laugh*].
Nerida: [*Laughing*] This is funny.
Alicia: No it's not.

In this example, Rupert positioned Nerida and Alicia as disengaged or inattentive by pointing out that they were not working on the right science topic. By doing so, he reciprocally positioned Marcello and himself as smart and rejected Nerida's proposed positioning. This direct positioning of oneself and others in an interaction is *first* order positioning. Alicia, however, rejected this positioning by disagreeing with Nerida. She overtly said that what Rupert was doing was not funny. The challenge from Alicia indicated that Rupert's first order positioning was questionable and required negotiation, shifting to a *second* order positioning (i.e., intentional questioning or negotiating a first order positioning). *Third* order positioning happens when positioning occurs outside the original conversation. Rupert might tell this story to a friend after the science lesson that positions Alicia as a lazy and difficult girl. This is a type of third order positioning of Alicia because the conversation is now outside of the original dialogue. By recounting the conversation that happened outside of the immediate space and time, the participants of the initial conversation become subject of third order positioning.

Ritchie later reported on further observations of Nerida and Alicia, who claimed that Rupert positioned them in particular ways because they were *female* group members, locating at least some of the positionings within the storyline of gender—in this case, representative of a larger cultural story that women are less capable than men to do science. *Storyline* is this larger

contextual background of positioning; it “draws on knowledge of cultural structures and the positions that are recognisably allocated to people within those structures” (Ritchie, 2002, p. 27). The storylines that frame positionings restrict, constrain, and shape the moves and positionings that might happen in an interaction. In any given situation, multiple storylines may be at play and different participants may locate positionings in different storylines. Rupert, for example, could have been playing a ‘good student’ storyline, within which he positioned himself as being compliant and Nerida and Alicia as not being compliant.

Although positioning focuses on immediate micro-level interactions, it relates to how a person’s identity develops because identity relates to how one talks about and sees oneself but also how others talk about and see him/her (Esmonde, 2009). By attending to coherency of positioning in a context over time, Anderson (2009) suggested that each person becomes a *kind* of person. The evolution of *kind* requires cultural resources or discursive structures, value of a position in the local context, and what a *kind* of person is allowed to do or is expected to do. Thus, positionings over time can impact one’s identity development. In our work as teacher educators, we have noticed that teachers’ repeated positionings of students sometimes constructs *kinds* of students that position students in deficit ways. Although students are not always privy to these third order positionings, these positionings and their accompanying storylines can constrain the ways in which teachers work with students.

Here we examine mathematics teachers’ third order positionings of students during professional development sessions. In particular, we focus on interactions where they have said things like “my low-tracked student” or “my at-risk group.” We do this in order to better understand the reciprocal first order positionings of the teachers that relate to these third order positionings of students. It is also important to identify the framing storylines of these positionings because, as we stated earlier, storylines can restrict, constrain and shape people’s action, either implicitly or explicitly. We seek to better understand these kinds of comments in order to address these issues in our future work as teacher educators. In particular, we investigate two research questions:

1) What first order teacher positionings seem to surface reflexively with the mathematics teachers’ third order positionings of students as “low-level”?

2) What storylines seem to be in play when the teachers position students as “low-level” and how might they restrict, constrain, or shape teacher action?

Method

Setting, Participants, and Data

Observation data came from the 11 three-hour professional development (PD) sessions in which we piloting a set of case-based materials for secondary mathematics teachers. An overarching goal of these materials is to raise teachers’ awareness of the role of discourse in teaching and learning mathematics. In the materials, participants were asked to analyze and consider different modes of communication (e.g., spoken or written) and were introduced to six focal *teacher discourse moves* as tools that can be used purposefully to encourage classroom discourse that is both productive and powerful for students’ learning. Further, issues of status and mathematical authority are addressed throughout the materials.

Participants in this study group included nine secondary mathematics teachers. The participating teachers included two middle school teachers and seven high school mathematics teachers with 0-17 years of teaching experience. Two teachers worked in urban schools, six teachers in suburban schools, and one teacher worked in a rural school. Three of the teacher participants were men (Xander, Kyle, and Bobby) and six were women (Diedre, Kelly, Bridget,

Donna, Kathy, and Maggie). The facilitation team included two faculty members and four graduate research assistants, all of whom had classroom teaching experience and had worked with prospective and practicing teachers previously. The PD sessions took place outside of the public school sites and occurred during the 2011-12 school year.

Data Analysis

Data analysis was conducted iteratively. We began by parsing the videos based on the particular type of PD activity (e.g., solving math tasks, looking at student work, discussing connecting to practice). As we moved through the video, we transcribed any instances in which participating teachers described students as having low-status or a deficit framing–e.g., using descriptors like “low-level,” “struggling,” or recounting a time when a student “couldn’t do something.” We then each used open coding (Esterberg, 2002) to describe and characterize teachers’ descriptions of these students. Through discussion and comparison, *static* and *dynamic* became particularly salient theme. Static descriptions seemed to frame students relatively stable over time. For example, when teachers described students as part of a “low-level algebra class,” the first order positioning is unlikely to change because the class that students are enrolled in is constant throughout the year. In contrast, dynamic descriptions seemed to indicate that a student’s low-level status is subject to change. For instance, teachers gave examples of how a new content area allowed a low-status student to show their smartness in a previously unrealized way. Although these classifications helped us to understand aspects of the third order positionings, through our discussions, we came to realize that these positionings implicated reciprocal first order positionings for the teachers. We came to consensus that these first order positionings might impact whether a teacher felt as if s/he could do something (e.g., have agency) to disrupt these low-level status positionings. With this in mind, we returned to the transcribed incidences to identify teachers’ first order positionings and to discuss the implications for action on the teacher’s part. As we did this, we also located the third-order positionings within their framing storylines in order to better understand why the teachers may have positioned students as “low” and to understand the potential constraints on the positionings because of these storylines.

Findings and Discussion

We organize our findings based on two of the more pervasive storylines we found: an individual maturation storyline and a tracking storyline. In each sub-section, we provide illustrative examples of the teachers’ reciprocal first order positionings related to the students’ third order low status positionings. We discuss the first order positionings and their framing storylines in relationship to the ways teachers might see themselves as able to act (or not).

Individual Maturation Storyline

One framing storyline that was salient in the teacher’s third order positionings of students related to student’s individual maturation. Oftentimes, students were designated as “low” because they were seen as lacking the maturity necessary to be successful at mathematics:

> *Deidre*: There are low-achieving students. You know. And it's really hard, especially in middle school, to sort out which ones are which. We get tons of kids coming from the middle school to ninth grade that for lack of a better word, grow up. And all a sudden, they're very good mathematically and they're sitting in a class they shouldn't be in.

Here Deidre began her turn by talking about low-achieving students and positioned students within a storyline of developmental maturation. Deidre acknowledged that “low-achieving”

students could be re-positioned as they matured, but that this change might happen "all of a sudden" and often resulted in continuing to "sit in a class they shouldn't be in." This latter point was framed by a storyline of a system of tracking students in mathematics courses (which we say more about in the following sub-section). Because maturation was something that Deidre could not control, there was little room for her to act to change this positioning of the students. Deidre's third order positioning of students was more fluid than static because it saw students as changing over time. Despite her awareness of this fluidity, she did not position herself as responsible for intervening by doing something like arguing in the district that students' mathematical potential is not easily identified in middle school. The reciprocal first order positioning of Deidre as an agent of change was limited; she did not position herself as someone who had agency to disrupt these positionings because the storyline was outside of her influence. Kathy's response to Deidre reinforced this perspective the storyline of individual maturity:

> *Kathy*: Well, some concepts in math are pretty abstract too. And I think that's a brain function, everybody's brain doesn't develop at the same rate. You know, someone who, who understands Calculus in the ninth grade, you know, whereas some kids don't, doesn't mean that they'll never understand it, okay. It might just be later on in life that it starts to click. So, I think it's safe to say that that's, I guess I didn't find people as low-level. I don't know that it's low-level. It's just that they… that's the level they're at.

Kathy explicitly acknowledged the fluidity of low-level students' position, in that it might just take some students a bit longer for things to "start to click." That is, all students can learn, but perhaps this learning takes place at different rates for different students. To Kathy, time and experience in conjunction with brain development were important parts of her storyline. This storyline restricted her from positioning herself as one who has the responsibility or opportunity to provide educative experiences necessary for such change. As such, the corresponding first order positioning of the teacher was one who waits for a student to be biologically or physiologically ready. Positioned in this way limited her agency as one who can provide meaningful mathematics experiences, instead attributing student success to brain development.

Together, these examples illuminated the storyline of developmental and biological maturity that framed the "low-level" status as internal and individualistic. This individual maturation storyline reminded us of Parks' point about the common metaphor of a narrow path. If the students' maturation was the center of attention, the metaphor of a narrow path in which some students must be ahead and others behind limited the range of potential solutions to better support students' learning. Although both examples positioned students in dynamic ways, these examples do not implicate teachers as agents of change. By using the storyline that students' success in mathematics was a function of their maturity, Deidre and Kathy "delimit[ed] a range of reasonable pedagogical responses" (Horn, 2007, p. 74). For example, another viable storyline could relate to calling "school mathematics" into question, moving from a view of mathematics as a "well-defined body of knowledge that is somewhat static and beholden to a particular order of topics" (Horn, 2007, p. 43) in order to reorganize and consider alternative ways in which this mathematics might be taught to make mathematical ideas accessible to more students.

Institutional Tracking Storyline

The other pervasive storyline we found related to institutional tracking, which was briefly mentioned above. Other teachers (in addition to Deidre) also enacted this storyline. For example, Kyle described his surprise when his "low-level" students used more mathematical language than their "high-level" peers on a task that emphasized group work and communicating ideas:

> *Kyle*: I, ah, I got to [video] record two groups. And the surprising part about mine was, I did, ah, I guess what I would call lower level kids and higher level. And the lower level used more terminology than higher level. But the higher level just, they would look at the problem and I think they would just sit there and do it in their head. And just be like, did you get this for an answer. Yep. Ah, okay. Let's move on to the next one. Um...
> *Beth*: So, [Kyle], can you say more about why you thought one lower level? Is it because they're tracked into a lower level class?
> *Kyle*: They're not. They're just like, the one group that I have are, well, they are, I guess next year, they'll be going on to like the ninth grade algebra class. Where like the others kids were just like, don't do as high on the test and they're just going to go into eighth grade math. So, I mean, they're definitely just as capable. They are, they're not like bad students. But, there's a little bit of a difference between the two.

Kyle referred to two groups of seventh grade students with whom he worked and described them as "lower level kids and higher level." In his response to a facilitator's clarifying question about what he meant by "low-level," Kyle responded that the students were not yet tracked. He noted, however, that he already knew that some of his students would go on to ninth grade algebra, but others would "just" be going onto eighth grade math (which is actually at grade level, rather than accelerated). Although students were not yet explicitly tracked, we note that the institutional tracking storyline still framed Kyle's third order positioning of his students. That is, students were positioned as "low-level" because of their likely *future* placement into a particular track. Within this storyline, Kyle had little agency to support all students to be successful in mathematics. To him, losing some students during their mathematics journey was inevitable.

The presence of the institutional tracking storyline seemed strong enough that Kyle's views on how he might take action were limited, even in the face of evidence that may countered his positioning of students. He began by saying that he was surprised about what his low level students had done, highlighting the construction of differential mathematical ability in his classroom. Those students deemed "high-level" have learned that being good at mathematics meant "doing [math problems] in their head[s]" and may not have involved learning to communicate one's ideas to peers. The task, however, was about communicating mathematical ideas using "more terminology." This task provided an opportunity to re-position the students who were positioned as "low-level" because it allowed them to demonstrate that they were capable of doing mathematics. These students stepped up to the challenge and did something that surprised Kyle. Yet, Kyle still attributed a low status positioning to them. An alternative storyline might be to apply a "toolbox" (Parks, 2010) metaphor to describe student engagement with mathematics in order to disrupt hierarchical ways of talking about students. The toolbox metaphor highlighted that "[e]ach students can be seen as having certain tools that he or she can use effectively" (p. 93). With this storyline, the students Kyle previously deemed low-level would be positioned as successful with finding the right tool to be successful at this task: effective and precise communication.

Reframing Storylines

The illustrative examples included thus far show third order positioning of students as having low status. The storylines framing such third order positioning resulted in a reciprocal first order positioning of teachers with little agency to act and to make a difference in the students' learning experiences. These storylines were based on individual maturation and a

system of tracking. We propose that, as teacher educators, we might respond by introducing different storylines in response. Different storylines (e.g., considering alternative views of school mathematics or using a toolbox metaphor) might disrupt teachers' institutional and cultural storylines and at the same time, increase their awareness of how they might act to support students. Below is an example of such a case, although rather than a teacher educator herself disrupting the positioning, it was information the teacher read in an article in the PD. Prior to this excerpt, we read an article by Herbel-Eisenmann (2002) that discussed bridging informal and formal mathematical languages, and how each type of language was valuable to students' meaning making. Although still positioning students as a "low group" (within institutional tracking), Deidre described how she began to use bridging languages with her students, rather than only recognizing when students used official mathematical language:

> *Deidre:* I know with my low group, I'm talking about pre-calc, they are my low group, start to combine like terms, and distribute really simple things. And I found myself on planning this unit, to do a lot with using their terminology or making up, kind of generic terminology and saying the mathematical terms right next to each other over and over. They say, for instance, "terms," we used to call them "clumps." That said, "Where are my clumps?" And they are like, "Where are my like terms? I'm going to combine by like terms. I will combine the same clumps." And I see myself going back and forth with students and when they say weird words, I kind of incorporate them. So that is how I changed my plan. Before I would just give them [inaudible] distributive, commutative, do an example.

Although using the word "low" suggested that the teacher still positioned students in terms of low status, she acknowledged that her "low group" could express mathematical ideas and eventually take up more precise language. Despite the use of the word "low," her reciprocal first order positioning had changed to incorporate how she might better support students to do this. Instead of correcting students' use of informal and contextual language, she incorporated their language as a resource for teaching formal and official language. The original positioning of students as low was shaped within the institutional tracking storyline, but as Deidre continued to speak, she did not question the students' ability to do mathematics as she positioned students as people who can talk about mathematics, shifting the storyline to be about how students' developed language. This suggests that at this moment of time, the storyline of students' language development was stronger than the tracking storyline. As such, she positioned herself as an active teacher with agency who plans her lesson according to the students.

Conclusion

Building on the ideas of Horn (2007) and Parks (2010), we focused on when teachers position students as having low status. Examining the examples through the lenses of positioning and storyline, we came to understand that teachers unintentionally limit their agency by positioning students in particular storylines of individual maturation and institutional tracking. Given the current emphasis on standardized tests and assessment in schools, we understand how these storylines are powerful for teachers' framings of students' positioning. In addition to Parks' (2010) recognition of mathematics education researchers, policymakers, and artifact contributing to the social construction of a deficit view of students, we add aspects inherent to the structuring of schooling: using age to group students and move them through grade levels as if they are all homogeneous and, when they seem not to be the same, creating a set of tracks in which to place them from which they can rarely escape. As teacher educators, we now understand how such

storylines might constrain teachers' own agency in classrooms. The implications of this finding, for us, are to pay closer attention to the storylines that seem to frame teachers' positionings of students and to offer alternative storylines for teachers to consider that might both disrupt these storylines but also recast first order positionings so that teachers see how they might act to better support students' learning.

Acknowledgement

We thank NSF (Award #0918117) and the College of Education at MSU for their support. Any opinions, findings, and conclusions or recommendations expressed in this paper are those of the authors and do not necessarily reflect the views of the granting agencies. We also thank David Wagner for his feedback on this paper and the teachers for working with us to pilot and refine the PD materials.

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USING TOULMIN'S MODEL TO DEVELOP PROSPECTIVE TEACHERS' CONCEPTIONS OF COLLECTIVE ARGUMENTATION

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As creating and critiquing arguments becomes more of a focus in mathematics classes, teachers need to develop their abilities to facilitate productive arguments. Toulmin's (1958/2003) model of argumentation has been useful in analyzing arguments in mathematics education research, raising the question whether it would assist mathematics teachers as well. In this study, we examine how using the model affected prospective secondary teachers' development of conceptions of collective argumentation. Our findings suggest that Toulmin's model facilitated appropriate understandings of what collective argumentation looks like in the mathematics classroom and provided the prospective teachers with a lens for analyzing their observations of practice. This study suggests the use of Toulmin's model in teacher education can be a promising step in helping teachers develop their conceptions of collective argumentation.

Keywords: Classroom Discourse, Teacher Education-Preservice, Instructional activities and practices

Mathematics education researchers and policymakers have called for increased student participation in argumentation in mathematics classrooms (e.g.; Foreman, Larreamendy-Joerns, Stein, & Brown, 1998; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010; Yackel & Cobb, 1996). However, most prospective secondary teachers (PSTs), who continue to learn mathematics through traditional instructional practices (Stigler & Hiebert, 1999), lack the experience of engaging in argumentation in the mathematics classroom as envisioned by those calling for it. Consequently, mathematics teacher educators are tasked with developing PSTs' conceptions of what collective argumentation is and what it should look like.

A complicating factor is that the term *argument* may be interpreted in various ways in our everyday usage of the term (e.g. inquiry, persuasion, negotiation, or disagreement), with the most common interpretation involving conflicting points of view. Contrary to this popular interpretation of the term, collective argumentation in mathematics classrooms is generally characterized by a group of students and the teacher working collaboratively to establish the veracity of a claim. Given the multiple meanings of the term *argument*, it is unlikely that PSTs share a common understanding with each other or with mathematics teacher educators as to what is being asked of them regarding argumentation in the classroom.

Research suggests that facilitating mathematical discussions is difficult for teachers (e.g. Hufferd-Ackles, Fuson, & Sherin, 2004; Stein, Engle, Smith, & Hughes, 2008). Most PSTs have little experience with participating in productive mathematical discussions, and they, like experienced teachers, will likely struggle to master the nuances of facilitating these discussions. This highlights the need for mathematics teacher educators to develop more effective ways to

support the development of PSTs' abilities to facilitate and support collective argumentation and for research to determine how PSTs learn to support and facilitate productive collective arguments. The purpose of this paper is to examine prospective secondary mathematics teachers' developing conceptions of collective argumentation as they worked through an instructional unit focused on facilitating and supporting productive classroom argumentation.

Toulmin's (1958/2003) Model of Argumentation

One model that mathematics education researchers (e.g., Hollebrands, Smith, & Conner, 2010; Krummheuer, 1995; Rasmussen & Stephan, 2008) use to describe and analyze arguments is Toulmin's (1958/2003) model of argumentation. According to Toulmin, an argument consists of claims (a statement, the truth of which is being established), data (the facts from which the claim was based), warrants (the justification for using the data to establish the claim), rebuttals (statements that call the warrant into question), qualifiers (utterances that indicate the certainty of a statement), and backings (supports for the warrant that are usually unstated and related to the academic field). Toulmin conceptualized these components as occurring in a structure as depicted in Figure 1.

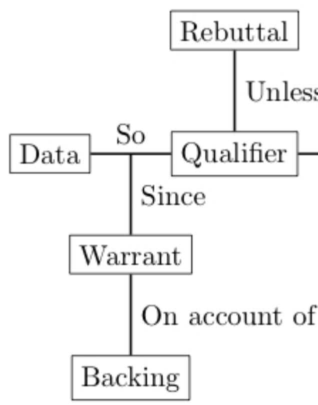

Figure 1. Toulmin's (1958/2003) Model of Argumentation

Krummheuer (1995) adapted Toulmin's (1958/2003) model to account for the social nature of collective argumentation, in which components of an argument are contributed and interpreted by a group of people. Other researchers have further modified the model to capture who (teacher, student, or both) contributed each component of the arguments (e.g. Conner, 2008) and the ways in which the teacher supports each component (e.g., Conner, Gleason, Singletary, Smith, & Wagner, 2011).

We have used these modifications of Toulmin's (1958/2003) model in research to analyze argumentation in mathematics classrooms (Conner et al., 2011). In this work, the model provided useful information about how teachers facilitate and support arguments, and the roles of the teacher and students in these arguments. Our findings would not have been accessible without the use of the model. In light of the recent emphasis on argumentation in the classroom (e.g. National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010), we conjectured that PSTs might find the model useful for analyzing and reflecting on instructional practice. The research questions that guided our study were:

- How do PSTs interpret the components of Toulmin's model of argumentation?

- To which aspects of mathematics classrooms do PSTs attend when using Toulmin's model as a lens for observing?

Methodology

To answer these questions, we conducted a study in a secondary mathematics methods course in which one of the units focused on collective argumentation. In this section, we describe the participants, course, data collection, and analysis methods.

Participants

The participants in this study were 11 PSTs who were enrolled in a 15-week secondary mathematics education methods course taught by one of the authors. The PSTs were in their final semester of coursework prior to a semester-long student teaching experience and had previously or concurrently fulfilled requirements including at least six advanced mathematics courses and two courses focused on high school mathematics content. The methods course was focused on pedagogical topics such as choosing and implementing high cognitive demand tasks, assessment, equity, and questioning. The PSTs were enrolled in a concurrent field experience course, in which they were assigned in pairs to local classrooms to observe and interact with the teachers and students. They were assigned weekly written reflections of their observations of these topics during their field experiences.

Argumentation Unit

The topic of argumentation was scheduled to be taught for one and a half class periods in the methods course and part of a class period in the field experience class (approximately three hours total). Given the potential complexity of collective arguments and the limited amount of time assigned to argumentation in the secondary methods course, we decided to introduce only the core components (data, claim, warrant) of Toulmin's (1958/2003) model of argumentation. We believed that this limited introduction would provide an overview of the most important and common elements of argumentation. We decided to have another author act as a guest instructor for the unit because the guest instructor was more knowledgeable about facilitating argumentation and using Toulmin's model as a means to analyze arguments.

On the first day of the unit, the instructor introduced argumentation by having the students watch a video in which two individuals defined what an argument is and what it is not. After a short discussion about the definition of argument, the instructor introduced the modified Toulmin's (1958/2003) model for argumentation, created a diagram of an argument, and then had the PSTs create diagrams of arguments found in transcripts and videos of mathematics classroom. At the conclusion of class, the instructor gave an assignment in which the PSTs were asked to watch a video clip of a mathematics classroom, to identify and diagram episodes of argumentation, and to bring their diagrams to the next class session.

In the next class meeting, the PSTs presented their argument diagrams and discussed the similarities and differences among them, particularly noting the contents of each component and who contributed them. For their weekly field experience reflection, the PSTs were asked in pairs to identify and diagram at least two classroom arguments, identifying the claims, data, and warrants in each and to describe episodes of argumentation that they believed were effective and ineffective, as well as their reasons for each choice. On the final day of the unit, the PSTs reflected on and shared their observations from their field experience.

Data Collection and Analysis

We videotaped and transcribed the three relevant classroom sessions. We collected the PSTs' notes from each day and the assignment that they completed. We analyzed the data using the constant comparative and grounded theory methods, as outlined by Corbin and Strauss (2008), in

order to capture individual interpretations. Through axial coding, we identified categories, which suggested a thematic framework on the use of Toulmin's (1958/2003) model of argumentation in the PSTs' development of a conception of collective argumentation.

Findings

The PSTs developed conceptions of what an argument is and made certain observations about teachers and students using Toulmin's (1958/2003) model of argumentation. These observations lent themselves to the PSTs conjecturing about characteristics of effective and ineffective arguments. Below we describe their conceptions of argumentation and how they evolved using Toulmin's diagrams. We then describe the PSTs' conceptions of the model components and how attention to these components served as a lens for analyzing collective argumentation. Finally, we describe how this analysis of collective argumentation framed the PSTs' descriptions of effective and ineffective collective argumentation.

Conceptions of Argumentation

Although the initial activity of the unit was focused on dispelling the perception that argumentation implied contradiction, many of the PSTs still perceived collective argumentation as involving disagreements between students. For example, Megan (all names are pseudonyms) posited that collective argumentation happened "when the whole group is actively participating in the argument. So you have different sides and people getting together behind certain, like different ideas." One PST exhibited an interpretation of argumentation that was more aligned with what is usually found in mathematics classrooms: "Maybe [argumentation is] an opinion you could justify" (Lauren). However, the majority of PSTs interpreted collective argumentation as involving controversy, though perhaps undertaken in a respectful manner: "It's not like a personalized kind of thing where you're arguing against the person, but you're just arguing against I guess the viewpoint and you're not being demeaning or just flat out saying they're wrong" (Mitch). It was only after exposure to Toulmin's (1958/2003) model that the PSTs came to an understanding that collective argumentation occurs any time a claim is supported by evidence. For example, at the end of the unit, Mitch and Jared wrote that arguments occur "whenever a claim is made … and [there is] reasoning or warrant behind the claim."

Learning Toulmin's Model of Argumentation

The PSTs' conceptions of argumentation shifted as they learned more about the components of Toulmin's (1958/2003) model and began using the model to create diagrams of collective arguments. Their understandings of the components were evidenced in their explicit statements and the choices they made when producing their diagrams.

Most PSTs treated claims as a *single assertion* or a series of assertions that did not contradict each other, reflecting their emerging understanding that collective argumentation does not require opposing points of view. The majority of the PSTs' diagrams revealed this understanding of claims except for two cases when PSTs diagrammed a *description* of an argument as a claim. In these cases, the claim took the form of a summary. For example, Megan's diagram of an argument from her field placement stated the claim as, "Two students disagree on what the correct answer is."

The PSTs envisioned data as encompassing information that was *explicit* or *implicit* to the student. Explicit data referred to information that was introduced orally or visually and therefore was identifiable by an outside observer. For example, in a transcript that was used during the first day of the unit, the classroom teacher stated that the measure of angle A was 20 degrees. All of the PSTs identified this given information as data. Implicit data included information that the PSTs assumed the students must have held internally in order to participate in the argument. For

example, in an argumentation episode in which students used the properties of vertical and supplementary angles to find angle measures, the PSTs identified "knowledge of the existence of vertical and supplementary angles" as data because, they claimed, without such knowledge the students could not have appealed to their properties.

The PSTs perceived warrants in two different forms. In one form, they saw the warrant as having its foundation in *external mathematical authority*. In these cases, warrants consisted of rules or formulas, calculations, or definitions. For example, Emily described a student's warrant as "the sum of the measure of the angles of a triangle is 180," which was a previously established theorem in the class. The PSTs suggested that warrants of this form could ostensibly be successfully applied even if the student providing the warrant did not fully comprehend the reasoning behind it. In the other form, the PSTs saw the warrant as having its foundation in *sense making*. In these cases, warrants consisted of student reasoning, previous knowledge, or explanations. For example, in describing a warrant, Billy stated, "the student's reasoning was that we want 180 on the inside." The PSTs suggested that warrants of this form reflected and revealed student understanding.

Analyzing Classroom Discussions Through the Lens of Toulmin's Diagrams

The PSTs made a number of observations within the context of diagramming collective arguments that reflect what they noticed when observing the classroom through the lens of Toulmin's (1958/2003) model of argumentation. We categorized their observations as focused on collective argumentation, teacher practice, or students.

Collective argumentation. The PSTs made comments that reflected their developing opinions about collective argumentation itself. One area of noticing was when the students or teacher used *inadequate mathematical terminology*. For example, Sarah observed, "[the students and teacher] should say the measure of the angles, not the angles are." The PSTs also attended to *who contributed* (teacher, student, or both) each of the components, including who contributed to arguments in a general sense (e.g. there was a lot of (or little) student participation). They also noticed *patterns of warrants* that were used throughout their observation period, such as a heavy reliance on rules or formulas for warrants. Finally, the PSTs observed that argumentation is *inherent to instruction*. Mitch and Jared wrote, "teaching is full of argumentation even though teachers and students might not realize it."

Teacher practice. The PSTs remarked on a number of aspects related to teacher practice while using Toulmin's (1958/2003) model. They noticed the *questions* the teacher asked to support the argument, particularly the general nature of the question asked and the results they achieved. The PSTs also noted *patterns* of practice that came to light in their analysis of classroom observations. For example, Emily noted that the teacher in her field placement, "basically gave them (the students) answers for everything."

A large number of PSTs' observations concerned the *shortcomings* of the teacher or *potential improvements* to the teacher's practice. For example, PSTs observed that the teacher did not require justification for claims, gave students little to no opportunity to contribute to collective arguments, or accepted or encouraged inadequate warrants. The PSTs also suggested a number of ways that the teachers they observed could have improved their ability to facilitate arguments. Emily noted, "I was thinking it wouldn't be that hard to make [the argument] effective if you just changed who's giving what answers." Contemplating an improved environment for collective argumentation, Sarah ventured, "Maybe the teacher would ask why instead of telling them why."

Students. PSTs made comments that reflected things they noticed about students when they attended to the argumentation model. Since they had determined that implicit elements act as

data, the PSTs occasionally noted the *prerequisite knowledge* that the students needed in order to participate in or follow the argument. They also identified *weaknesses in components* contributed by students, such as warrants that did not fully justify or were irrelevant to the claim. The PSTs made general comments related to *student thinking* that attention to the argument components revealed. For example, noting the heavy use of drawings in an argument, Harry postulated that students were thinking "visually." Other PSTs used their argument diagrams as a launching point for considering aspects of the mathematical argument that students did not understand. For example, after diagramming an episode of argumentation in their observed field experience, Billy and Megan wrote, "the student did not understand that … the inverse operation of logarithms is exponentiation (sic)." Lastly, the PSTs sometimes noted that *connections* were being formed between topics.

Descriptions of Effective and Ineffective Arguments

The PSTs incorporated what they noticed in collective arguments to inform their opinions about what constitutes effective or ineffective arguments. As part of their field experience reflection, the paired PSTs were asked to identify an effective and an ineffective argument from their observations and explain their choices.

Effective arguments. Mitch and Jared referenced arguments concerning the behavior of absolute value functions and observed that the warrants took the form of analogies of the behavior of quadratic functions. They argued that effective arguments make explicit *connections* between topics in this way. Other PSTs identified effective arguments as those that either contained or lent themselves to *multiple perspectives*. Lauren and Iris argued that a specific argument was effective because it would "allow other students in the class to see different methods [other than their own] for solving the problem." Additionally, Emily and Sarah argued that one of the things that made their identified argument effective was that it was *relevant* to the students in that it was "similar to questions that would appear on the midterm." Another characteristic of effective arguments identified by PSTs was a high level of *student participation*. In particular, many of the PSTs characterized effective arguments as those in which students contributed most of the components, particularly the claims and warrants. Additionally, the PSTs felt *explicit warrants* were a characteristic of effective arguments. During a discussion, Megan offered, "Maybe that's a characteristic of a good argument. Or a clear argument. When the warrant is clarified and really explicit. It's explicitly stated. The other students who aren't really participating in the argument at least see what's going on."

Ineffective arguments. Megan and Billy were dismayed by an argument they observed, in which a student used the product rule of logarithms to warrant her claim that $\log(x^2) = 2\log(x)$, because the teacher had disagreed with the student's warrant. The teacher asserted, "The exponent rule was the rule to use." After noting that the student's warrant could have easily been justified, Megan and Billy characterized the argument as ineffective because the teacher allowed *one method only* and did not allow for alternative perspectives. This characteristic of ineffective arguments closely mirrored Lauren and Iris' depiction of ineffective arguments as relying on *memorized warrants*, that is, warrants that were rules or formulas lacking conceptual bases. Mitch and Jared also focused on the role of the warrant, citing *inadequate warrants* as a property of ineffective arguments, objecting to an argument in which the warrant accepted by the teacher and students did "not give a full understanding as to why that is the answer." Emily and Sarah added the characteristics of *low student participation* and a *rushed atmosphere*: "Students were not given time to look at the questions or try for themselves before the teacher began going over the answer…. [She] did not 'recruit' students to join the argument."

Discussion

Researchers and policymakers have called for increased argumentation in mathematics classrooms, yet few PSTs have experienced collective argumentation in the mathematics classroom in the form envisioned by researchers. Because facilitating classroom discussions is difficult even for experienced teachers, we can expect PSTs to face challenges in developing their abilities to support collective argumentation. A necessary first step is to help PSTs develop an appropriate vision of what collective argumentation is and to provide them the means for analyzing their observations to determine factors that contribute to effective or ineffective arguments.

Using the lens and the language of Toulmin's (1958/2003) model, the PSTs in this study made observations related to students, teacher practice, and collective argumentation, which informed their conceptions of effective and ineffective arguments. In developing their conceptions about what constitutes an effective collective argument, the PSTs gained valuable insights from the observations they made using the lens of the argumentation components (see figure 2). Researchers in mathematics education have yet to clearly describe the elements of effective, or productive, collective argumentation, which suggests the complexity of the endeavor; therefore, it is important for PSTs to begin developing a personal conception that will inform their future practice. By identifying characteristics of discourse they desire in their future classrooms, the PSTs will be better situated to negotiate the classroom and sociomathematical norms (Yackel & Cobb, 1996) that foster their vision. Because practicing teachers also find facilitating classroom discussions difficult, Toulmin's model may similarly impact their conceptions of collective argumentation in ways that will help them improve their practice.

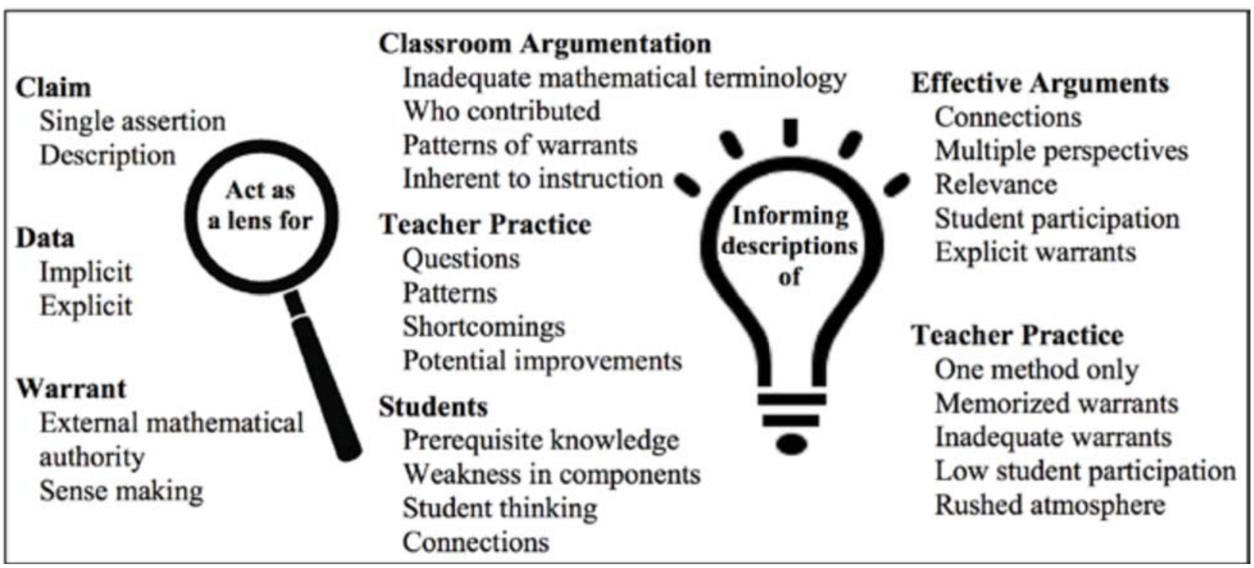

Figure 2. Development of Conceptions of Argumentation

Recent mathematics education research identifies specific ways that PSTs support collective argumentation (Conner, Singletary, Smith, Wagner, & Francisco, 2012), which may help mathematics teacher educators foster instructional practices that lead to effective arguments. Future research can explore the ways in which PSTs use Toulmin's (1958/2003) model of argumentation to develop their skills in facilitating and supporting collective argumentation. It

can also determine whether a more thorough introduction of Toulmin's model, such as including rebuttals, counterclaims, sub-arguments, and qualifiers, contributes to PSTs' understandings of argumentation in a meaningful way. Advances in research in these areas will increase our ability to effectively instruct PSTs in the techniques of collective argumentation. In the meantime, this investigation into the possibilities afforded by introducing PSTs to the basic components of Toulmin's model suggests that it is a promising first step in developing their conceptions of argumentation.

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CONNECTING ABSTRACT ALGEBRA TO HIGH SCHOOL ALGEBRA

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This study examines opportunities to learn in an abstract algebra course for future teachers. Using a case study approach, it addresses the question of the ways in which the course provides learning opportunities to connect abstract algebra to school mathematics. The course focused on proving the fundamental theorem of algebra. Through this novel design, the professor highlighted links to K-12 school math. This provided an opportunity for the future teachers to begin to bridge the "vertical disconnect" between secondary and tertiary mathematics.

Keywords: Post-Secondary Education, Teacher Education-Preservice, Teacher Knowledge

Mathematics content courses have long been considered a key component of math teacher preparation (Liljedahl et al., 2009). Typically, future high school math teachers complete either a major in the field (Graham & Fennell, 2001) or a certain sequence of math courses (Hodge, Gerberry, Moss, & Staples, 2010). Recent recommendations from the Conference Board of the Mathematical Sciences (CBMS) highlight the importance of teachers taking math classes that emphasize a deep understanding of the mathematics they will teach with attention to the development of mathematical practices (CBMS, 2012). However, content courses for future teachers are frequently housed in math departments, while other aspects of teacher preparation are based in the education department (Graham, Li, & Buck, 2000). This implies a potential divide between content learning and other parts of teacher development. Additionally, while content courses may serve to broaden future teachers' perspectives on mathematics and deepen their knowledge of advanced topics, the courses suffer from a "vertical disconnect" (Cuoco, 2001, p. 3). That is, the future teachers are not necessarily able to see how tertiary mathematics connects with the content they will be asked to teach (school mathematics).

Content courses designed specifically for future teachers have a unique opportunity to bridge this disconnect. Some scholars advocate for capstone courses where future secondary math teachers consider high school mathematics from an advanced perspective (e.g. R. Hill & Senk, 2004). Others highlight the value of linking tertiary and secondary level mathematics in more traditional undergraduate math courses (e.g. G. Hill, 2003). However, classes such as these have not yet become commonplace in teacher preparation. Additionally, most of the publications regarding math classes for teachers, including those mentioned above, are self-report, indicating the need for more research on the nature of these courses.

Abstract Algebra Courses for Teachers

At the Joint Mathematics Meetings, Cuoco and Rotman (2013) gave a presentation on the importance of abstract algebra – a course required in a traditional math major – for high school teachers. Typically, abstract algebra focuses on the study of groups, rings, and fields (e.g. Dummit & Foote, 2004). Cuoco and Rotman argue that such topics are not well connected to high school mathematics. Furthermore, university professors are not necessarily aware of all the connections between the content they teach and secondary math (Hodge et al., 2010). An abstract algebra class designed for future teachers could provide stronger connections between tertiary and secondary mathematics. However, researchers have not yet investigated the nature of these connections and the ways they might emerge from such a course. Opportunities to begin to bridge the vertical disconnect requires occasions to encounter links between secondary and tertiary mathematics (Blömeke & Delaney, 2012). This study, therefore, seeks to address the following question: In what ways does an abstract algebra course for future teachers provide opportunities to connect abstract algebra to high school algebra? I investigate this question by drawing on Cuoco and Rotman's (2013) framework for abstract algebra for teachers.

Methods

This study considers the case of an abstract algebra course that makes explicit connections to high school algebra. The purpose of the class, as explained by the professor, was to help future teachers deepen and expand their own content knowledge and connect advanced mathematics with the school curriculum. This class is not necessarily representative of all abstract algebra classes designed for teachers. Instead, it was chosen because it is part of a teacher education program with a deep commitment to enhancing the mathematics education of future teachers and because of the specific focus it has on making connections to the school curriculum.

The abstract algebra class, taught by a mathematician, met for ten weeks in the summer of 2012 in the first term of a yearlong graduate level teacher preparation program. The class met once a week for three hours. I observed all ten classes, and took detailed field notes. Most classes centered on mathematical discussions and problem solving, so I recorded mathematical conversations among the students and with the professor. I supplemented these records of the class sessions by collecting all problem sets (both those completed in class and those assigned for homework) and student work. Additional insight into opportunities to learn came through two interviews with the professor, one at the beginning and one at the end of the course.

To analyze the field notes for evidence of opportunities to connect abstract algebra with school algebra, I divided the field notes into 15-minute blocks. I coded each block for the school-level and college-level math content covered. For example, in one instance participants used multiple representations (algebraic, graphical) to multiply complex numbers. The multiple representations helped connect college-level content (generalizing multiplication in complex numbers) and school-level content (multiplying complex numbers). I looked for patterns in how college-level and school-level mathematics were linked in order to develop propositions for how the course provided opportunities for participants to bridge the vertical disconnect. School-level topics that arose during the course included prime factorization and factoring polynomials, and ranged from arithmetic to complex numbers. I tied the school level math codes to the Common Core State Standards (2010). Finally, I coded the interviews with the professor with a focus on his goals for the course and the learning opportunities he hoped to provide students.

Findings

This abstract algebra course provided opportunities to connect secondary and tertiary mathematics through a novel mathematical approach to abstract algebra: an entire course focused on proving the fundamental theorem of algebra, with connections to school mathematics embedded throughout the course.

A New Approach to Abstract Algebra

Focusing on a single proof of the fundamental theorem of algebra distinguishes this course from typical abstract algebra courses. The professor explained this on the first day:

> The topic of this class is algebra, and one of the challenges is that everyone comes in with a very different background. […] I've selected a topic in algebra, the fundamental theorem of algebra, which is a key topic that impacts high school math and middle school math. We're actually going to prove it this quarter. There are lots of different proofs. I've selected the most difficult and exciting.

He went on to explain that the proof was "very algebraic in flavor" and though it might not be the "easiest way," it would expose the students to "a huge list of topics" in algebra. The class discussed the equivalence of many different statements of the theorem, and eventually proved the form: Every degree n polynomial ($n \geq 1$) with real coefficients has at least one complex root.

Making Secondary-Tertiary Content Connections

Each week class was devoted to unpacking one concept necessary for the proof of the fundamental theorem (e.g. mathematical induction, symmetric polynomials). Each concept provided an opportunity to emphasize links to the high school curriculum. The school content tended to focus largely on topics in

natural numbers and algebra (polynomials in particular), which is unsurprising given the nature of the proof. For example, during week 2, students spent time working to prove that every integer greater than 1 has a prime divisor. The professor's goal was to attend to the nuanced induction proof, since the proof of the fundamental theorem of algebra proceeds by induction. However, the discussion also drew out important school concepts such as prime factorization, the definition of a composite number, and even a brief digression about whether multiplication should be defined as repeated addition. In this case, the school material was addressed from an advanced perspective due to the attention to proof and precision. In other cases, an advanced perspective arose from considering school topics in non-standard settings (e.g. what is a fraction when working mod 7?). Thus, considering school math from an advanced perspective is one way of connecting secondary and tertiary mathematics.

Drawing parallels between integers and polynomial rings is another key way to help bridge the vertical disconnect (Cuoco & Rotman, 2013). In week 3, discussion moved back and forth between factoring and long division in the natural numbers and division of polynomials. The professor motivated this connection by transforming the number they had been analyzing, 1729, into $t^3 + 7t^2 + 2t + 9$. This approach provided future teachers the opportunity to develop greater insight into topics like long division that might seem disconnected from advanced mathematics.

Furthermore, in line with the recommendation of Cuoco and Rotman (2013), the professor made explicit connections between secondary and tertiary content in almost every class session. The exception was the class when the professor connected all of the previous topics by finally presenting the full proof of the fundamental theorem of algebra. Weeks such as those, focusing on polynomials or complex numbers, tended to have more connections, while weeks addressing topics such as induction or the pigeonhole principle tended to have fewer. This might be attributable to the relative attention to these topics in the secondary curriculum. Polynomials, for example, represent a substantial component of high school mathematics, while the pigeonhole principle is not addressed at all in the Common Core Standards (2010). The work of regularly highlighting connections to school math and integrating that discussion into the discussions of tertiary content, represents an additional way to help bridge the vertical disconnect.

The time devoted to discussing school content sometimes came at the expense of covering more traditional tertiary content in depth. For example, field extensions are a necessary part of the proof because they guarantee that one can construct a field containing a root of the polynomial. However, this topic received only minimal attention during the proof itself, and was not unpacked beforehand. Other traditional topics, such as groups and rings, were not discussed at all. This change in emphasis, however, is in line with Cuoco and Rotman's (2013) suggestion for how an abstract algebra course might be designed to better meet the needs of future teachers.

Implications and Directions for Future Research

This case study reveals important new insights into the way an abstract algebra course for teachers can help bridge the vertical disconnect between secondary and tertiary mathematics. Focusing on the fundamental theorem of algebra provided numerous opportunities to link tertiary-level and school-level topics. These opportunities occurred through regular attention to school content, thinking about school mathematics from advanced perspective (through proof and non-standard settings), and drawing parallels between integers and polynomials. The findings support the call for a re-thinking of how abstract algebra can be made relevant for teachers without being "watered down" (Cuoco & Rotman, 2013, slide 10). This new approach to abstract algebra made connections to school mathematics more apparent than in typical abstract algebra courses focused on group theory, following the professor's stated goal for the course. For instance, knowing the statement of the fundamental theorem of algebra is a Common Core standard. In contrast, though some school topics may prepare students to study group theory, the topic is difficult to link to the school curriculum. This case study demonstrates the advantage that math classes for teachers have in making tertiary math relevant to secondary teaching. By focusing explicitly on teachers' needs, the professor provided opportunities for participants to gain a deep understanding of the fundamental theorem of algebra, while simultaneously providing opportunities for them to deepen their understanding of school math.

This study is a first step in unpacking how linking secondary and tertiary mathematics can provide opportunities to learn that enhance math teacher preparation. It is part of a larger study on the ways in which future teachers develop mathematical knowledge for teaching; and my research will next consider the learning that actually occurred in this course. Additionally, future studies must examine how the ability to bridge the vertical disconnect in the university setting translates into changes in teachers' classroom practice. It will also be important to consider additional cases, both of other abstract algebra classes for teachers, as well as other math classes for teachers. This will contribute to better supporting content learning in teacher preparation.

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ONE SOLUTION TO TWO PROBLEMS: TEACHER EDUCATION STUDENTS AS TEACHERS OF UNDERGRADUATE DEVELOPMENTAL MATHEMATICS

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Many post-secondary institutions across the U.S. are confronting two seemingly unconnected problems: first, teaching developmental mathematics successfully to students unprepared for college level mathematics; second, preparing future mathematics teachers to teach diverse, often poorly prepared, students. In this paper, we describe a research program addressing these problems simultaneously. Prospective secondary mathematics teachers (PST-Ms) served as instructors for a developmental mathematics course and implemented high cognitive demand lessons with guidance from teacher education faculty. We discuss preliminary findings from data to measure the effects of this program in supporting learning of the developmental mathematics students as well as the PST-Ms.

Keywords: Post-Secondary Education, Teacher Education – Pre-service, Algebra and Algebraic Thinking

Background

Across the United States, many students enter their first year of college unprepared to take college level mathematics. At public 4-year institutions, 16% of entering freshmen took developmental (remedial) mathematics in 2000 (U.S. DOE, 2003, p. 18). These students are less likely to enter or persist in STEM majors, and even less likely to graduate from college than students who are prepared for, and succeed in, college mathematics in their first year (Adelman, 2006). Most students who drop STEM majors do so during the first two years of college, often because of trouble in their first year mathematics courses (AAU, 2012, p. 4).

In addition, students enrolled in developmental mathematics courses at 4-year institutions tend to be disproportionately students of color. In 2011-12, the undergraduate population for the institution participating in our study was 6.5% African-American and 3.3% Hispanic. In Fall 2011, in the remedial mathematics course at this institution, 29% of students identified as African-American and 7.6% as Hispanic. African-American and Hispanic students are typically underrepresented in STEM majors, thus improving outcomes among these students can support greater diversity in those pursuing STEM careers.

Research on post-secondary education makes clear, however, that current teaching of mathematics is most often focused on algorithms and knowledge of procedures delivered in lecture mode, an approach that is unsuccessful for many students, even those in STEM majors (e.g., Thompson, 2007). An effective method of teaching for mathematical proficiency entails engaging students in problem solving and mathematical discourse, working in small groups and whole class discussions to explain mathematical ideas and justify mathematical solutions (Fuson, et al., 1999; Kilpatrick et al., 2001). This kind of teaching, however, has been difficult to bring to scale because it is hard, time consuming and takes several iterations of enactment and reflection to develop proficiency.

We report preliminary findings from a study of an intervention in a developmental mathematics course that uses prospective secondary mathematics teachers (PST-Ms) to implement research-based teaching methods using innovative materials. In particular, we address

the question: *What is the influence of an intervention involving inquiry-oriented and high cognitive demand instruction, enacted by PST-Ms, in helping students in a developmental mathematics course gain mathematical proficiency?* In the sections that follow, we document the methods of the intervention and research on its effectiveness, followed by the results we have obtained thus far and the implications of those results on the design of the intervention and best practices in teaching developmental mathematics.

Methods

The Intervention

In the pilot project, prospective secondary mathematics teachers (PST-Ms) at a large, Midwestern university were split into two groups of 10 and each group was assigned to work with one section of the face-to-face component (MATH 101E) of a large, online remedial mathematics course (MATH 101) at the same university. MATH 101 is offered entirely online using the ALEKS (undated) web-based tutoring program, and most students in MATH 101 do not participate in any face-to-face component. Students in MATH 101E are specifically identified as needing extra help and are placed in these small, face-to-face sections in addition to the online course.

The work for the PST-Ms involved co-teaching lessons in pairs twice during the semester. The written curriculum for MATH 101E was developed by the course coordinator for the mathematics pedagogy course (Methods I), and consisted of tasks, each designed to be taught in a single two-hour session, to help students develop conceptual understanding, strategic competence, and adaptive reasoning focusing on the topics of rational numbers, proportional reasoning, as well as linear and quadratic functions. The materials are designed to support both small group and whole class discussions, and, as instructors in 101E, the PST-Ms were expected to use methods they were learning about in their Methods I course. We assigned a teacher education graduate student (TA) to supervise each group of 10 PST-Ms enrolled in Methods I. These two TAs attended all sessions of their group's 101E class, provided in-the-moment instructional coaching to the PST-M teaching pair, and attended the debriefing session with the PST-Ms held for an hour each week on the same day as the 101E classes. The TAs also taught the 101E classes for the first three weekly sessions, to model the kind of teaching that was appropriate to the designated curriculum and to give the PST-Ms time to prepare lessons before they assumed responsibility for the 101E course instruction.

Data Collection

We collected data purposefully to investigate the influence of this model on the mathematics achievement and attitudes of the 101E students. In addition to collecting data from participating students in the intervention sections of 101E, we solicited participation from students in two control groups: students enrolled in 101E but not in the intervention sections, and students enrolled in 101 but not also taking a 101E course. The following data were collected (see Table 1 for response rates): 1) scores from a mathematics placement test administered to incoming freshmen; 2) responses to a survey of attitudes and beliefs about learning mathematics based on scales (confidence, effectance, teacher, and anxiety) used by Fennema-Sherman (1976) and Bai (2009); 3) scores on midterm and final exams in the 101 course for intervention and control groups; 4) scores from ALEKS web-based assessments; and 5) final course grades in 101 for intervention and control groups. We are in the process of collecting additional data, such as: (a) responses to a post-survey reexamining the attitudes and beliefs measures along with questions about students' experiences in the course and (b) interviews with a sample of 20-30 students

selected from control and intervention groups (approximately 7-10 students from each of the three groups).

Table 1: Number of Students, Survey Responses, and Response Rates

	Total # students	# responses	Response Rate
101 Only	705	249	35.3%
101E Control	102	30	29.4%
101E Intervention	34	23	67.7%

Data Analysis

Data analysis is underway using data from sources #1, 3 and 4 listed in previous section Data Collection. We are using factor analysis for the attitude and beliefs data (source #2), and currently evaluating fit of a path-analysis model for the pre-data. Once results are in from the post-survey (source #2), we will compare the pre and post-data to assess whether students' attitudes towards mathematics have changed, and how the three groups compare.

Results

Mean scores on all the assessments are shown in Table 2. Scores on the pretest for the MATH 101-only group are higher than those for the 101E students in both groups. This result is not surprising given that the 101E were identified prior to enrolling in MATH 101 as being potentially at-risk for not successfully completing their first-year coursework. ANOVA results (Table 3) show that there is a significant difference among these three, and post-hoc tests (Tukey multiple comparisons for all three possibilities, not shown on the table) indicate significant differences only between the MATH 101-only group and the 101E control group on the placement test ($p < 0.01$). For the ALEKS pre-test, there are no significant differences among the three groups. Thus, the MATH 101E intervention group is statistically indistinguishable from both groups on these pretest measures.

Table 2: Mean and Standard Deviation of Pre and Post Assessments

	MATH 101 only		101E Control		101E Intervention	
	Mean	SD	Mean	SD	Mean	SD
Placement	6.46	2.30	5.85	1.98	5.90	1.58
ALEKS Pre-test	49.21	26.57	45.76	26.92	44.53	25.92
Final Exam	126.10	47.55	127.43	47.89	133.57	24.54
ALEKS Post-test	200.24	42.82	206.44	45.00	215.33	25.53

For both outcome measures – the final exam (the same test for all 101 and 101E students) and the final ALEKS assessment, the intervention group has higher mean scores. The ANOVA shows, however, that on the final exam, the three groups are statistically equivalent. On the ALEKS post-test, there is a significant difference among the groups. Posthoc tests (Tukey multiple comparisons for all three possibilities, not part of the ANOVA table) indicate a significant difference only between the 101-only and the 101E intervention group on the gain scores for the ALEKS tests ($p = 0.07$).

At the end of the semester, scores on the ALEKS web-based post-test and the final exam suggest that the 101E intervention group outperformed the MATH 101-only group. The final

exam scores, although not statistically significant, are in the same direction: in both assessments, the intervention group has the greatest improvement.

Table 3: ANOVA Results For Pre and Post Assessments

	df	SS	MS	F
Placement test	2	31.9	16.0	4.22*
ALEKS Pre-test	2	1528	764.0	1.08
Final Exam	2	1691	845.5	0.38
ALEKS Post-test	2	9178	4589.1	2.53**
ALEKS Gain	2	17841	8920.4	3.93***

*** p=0.020 ** p=0.080 * p=0.015

Discussion

Although the sample is small, we are quite encouraged by these results. Early quantitative analyses suggest that the MATH 101E intervention has made a difference, particularly in students' procedural understanding of mathematics that is assessed by the final exam and ALEKS post-test. This may indicate that face-to-face support courses for online, developmental math courses can be enhanced with more opportunities for group work, discussion, and tasks to develop understanding of why procedures work. The next steps for this research are to complete data collection and analysis of the student surveys and to interview the sample of students from the three conditions. Once all the data is collected, we will develop a model to investigate predictors of the differences in outcomes. Additionally, we will analyze data from the teacher education students (videos of their teaching in 101E; lesson plans; observation notes; and interviews) to understand better whether and how this intervention affected their development as mathematics teachers. Our results from this pilot will inform revisions to the model to be implemented in Fall 2013.

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COACH-MEDIATED OPPORTUNITIES FOR TEACHERS' PROFESSIONAL LEARNING

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In response to the pressures of increased accountability, many schools have turned to teacher collaborative groups as a means for improving student achievement. Many of these groups include instructional coaches to facilitate professional learning in teachers' collaborative time. In this case study, I employ discourse analysis to highlight the ways in which a more expert coach mediates teachers' opportunities to learn using a high quality professional development activity. Despite her greater expertise, the resulting learning was problematic from a view of ambitious teaching. The analysis highlights the need for improved tools and methods for mathematics coaches, if instructional improvement is to have an impact on a larger scale.

Keywords: Teacher Education-Inservice/Professional Development, Instructional Activities and Practices

Introduction

In response to increasing pressure to improve student achievement, many districts and schools have turned to the intentional use of teacher collaborative time (TCT) as a way to facilitate student achievement. In TCT, district and school leaders expect teachers to collaboratively engage in activities that will lead to increased student achievement. However, as Little (2001) notes, these efforts are "more productively seen as a problem of learning than as a problem of implementation" (p. 31) and little is understood about *how* teachers learn or the ways TCT mediates teachers' opportunities to learn. One way that district and school leaders have sought to facilitate learning in teachers' collaborative time is through the use of instructional coaches.

To extend our current understanding of coach-facilitated learning, I draw on literature from the mathematics classroom. Specifically, I draw an analogy between coaches pressing on teachers' understandings in the TCT setting and teachers pressing on students' understanding through the use of the similar interactional moves. Practically, this analysis has implications for both districts that are using instructional coaches as a strategy for instructional improvement and organizations that train instructional coaches.

Conceptual Framework

Opportunities for Teacher's Professional Learning

As noted above, Little reframes the issue of successful reform away from implementation to terms of teacher learning. Due to the collaborative nature of TCT, a situated view, which defines learning as a change in participation in a community of practice (Lave & Wenger, 1991), is a useful lens for analyzing teachers' learning in this setting. In this view, an OTL is an "affordance for changing participation and practice" (Greeno & Gresalfi, 2008, p. 172). A situated framework assumes that opportunities to learn are ubiquitous, although they might not be equally desirable. But what activities and environments might result in OTLs that will support teachers' learning about ambitious practice?

To answer that question, I turn to the professional development (PD) literature. Wilson and Berne (1999), in looking across a wide variety of PD programs, identify three characteristics of effective PD. First, communities of teachers must be involved in redefining their teaching practice. Second, effective PD allows teachers' knowledge to be activated and was not simply delivered. Finally, effective PD privileges teachers' interactions with each other, and includes space for critique of ideas in a collegial and constructive manner. Taken together, these characteristics of effective PD can be thought of as affordances of a setting that allow teachers to change their participation and practice.

Description of the Study

In this paper, I will analyze the opportunities for learning found in math teachers' collaborative time. This research takes place within the larger context of the Middle School Mathematics and the Institutional Setting of Teaching (MIST) project (Cobb, Jackson, Smith, Sorum, & Henrick, 2013). MIST is an investigation of what it takes to support mathematics teachers' development of ambitious and equitable instructional practices on a large scale.

As suggested by the introduction, the focal districts in the MIST study have used TCT as a key improvement strategy. In an effort to understand and describe what effective TCT looks like, we used a "best-case sampling logic" where we asked key informants to nominate certain collaborative groups for close study because they were viewed as particularly "high functioning." The collaborative group in this analysis comes from that sample.

Description of Focal Data

This analysis focuses on the TCT at Lakefront Middle School[1]. The meeting was facilitated by an instructional coach, Jane, who had been at Lakefront for six years. Using MIST's measures of teaching expertise (VHQMI (Munter, 2012) and LMT [Hill, Schilling, & Ball, 2004]), Jane has above-average expertise in teaching mathematics.

In the focal meeting for this analysis, Jane first assigned two mathematical tasks to the teachers and had them work individually to solve those tasks. After the teachers solved the tasks, they discussed their solutions and whether they thought the task was "high" or "low." After this discussion, Jane had the teachers work in small groups to sort a collection of tasks according to whether they are "high" and "low." As the teachers discussed whether certain tasks were of high cognitive demand, Jane created liminal space in which teachers could learn about the cognitive demand of tasks. Thus, my research question is: how does the coach's facilitation of teachers' collaborative talk mediate teachers' opportunities to learn?

Analytic method

To operationalize these constructs about OTL, I use discourse analysis (Gee, 2011). My unit of analysis is an episode of pedagogical reasoning (EPR), which Horn (2005) defines as "units of teacher-to-teacher talk where teachers exhibit their reasoning about an issue in their practice" (p. 215). Within each EPR, I look for evidence that the coach mediated teachers' opportunities to learn in the facilitation of their discourse.

Analysis

Overall, I found that teachers at Lakefront had opportunities to learn about (a) determining the cognitive demand of the task and (b) the use of cognitive demanding tasks in their classroom. The teachers' discourse about cognitive demand initially invoked Bloom's Taxonomy, a familiar way of analyzing task potential. Over the course of the meeting, their talk shifted to the subject-

[1] All school and teacher names are pseudonyms.

specific language of cognitive demand, using terms from Stein's rubric. Note that Bloom's Taxonomy is domain neutral and actually does not attend to the notion of cognitive demand. One could imagine a "knowledge" task that is of high cognitive demand and an "evaluation" task that is highly procedural.

Learning About Cognitive Demand

In EPR 1, the teachers discussed "Martha's Carpeting Task" (M. S. Smith, Stein, Arbaugh, Brown, & Mossgrove, 2010), which asks students to find the number of square feet of carpet required to cover a fifteen-foot by ten-foot rectangular room. Jane asked the teachers to share how they classified the task, and Eve described it as "comprehension level." Jane pressed Eve to explain her decision, and she said that "it's a little more than knowledge." In this EPR, Eve spoke of cognitive demand purely in terms of Bloom's Taxonomy. Jane concluded EPR 1 by revoicing and summarizing Eve's characterization of the task.

However, in EPR 3, there was a shift in conversation. The teachers vigorously debated the cognitive demand of "Task J" (M. S. Smith et al., 2010), which asks students to "mentally compute" several multi-digit multiplication problems and sketch a rectangular area model that "describes your methods for each" multiplication problem. The teachers were evenly split on whether they thought the task is of high or low cognitive demand. In the initial phase of the discussion, there were still lingering references to Bloom's Taxonomy, for example when Eve argued that "it says 'mentally compute' so it's just knowledge based" and later she noted that students are asked to *create* a diagram. Later in the EPR, Rueben and Christopher began to discuss the fact that the task has multiple steps. This recognition of complexity suggests that some of the teachers were butting up against the limitations of understanding cognitive demand purely in terms of Bloom's Taxonomy. However, the critical moment in the debate happened when Jane shared with the group that the task was, according to Stein (1996), a high-level task. After this injection of new information into the group, there was massive reorganization of the teachers' thinking as they tried to make sense of the "right answer." Eve, who had previously suggested it was a low-level task, posited that it was a high-level task because of "the creation of a diagram." Christopher doubted the new information.

But Jane pushed further and read Stein's (1996) entire rubric. The teachers latched on to the phrase "doing mathematics." Mark asked, "So if you're doing any math, that's high level?" and Christopher responded by saying "if you have connections with your math." Jane rebutted them both and said, "Not doing any math . . . procedures with connections" and Eve concluded the EPR with the observation that "that makes a lot of these high level." Eve and the others had begun to reconceptualize what was meant by a cognitively demanding task.

The different responses Jane gave to teachers' ideas mediated teachers' OTL about determining the cognitive demand of tasks. In EPR 1, Jane revoiced and summarized the teachers' ideas in a way that failed to press them towards a more mathematical understanding of cognitive demand. However, in EPR 3, when Jane injected the rubric into the discussion, it created an OTL for teachers to learn about one aspect of ambitious instruction.

Discussion and Conclusion

In these data, we have both a structure (TCT) and an activity (sorting tasks to learn about cognitive demand) consistent with Wilson and Berne's (1999) description of effective PD. In spite of the affordances of the environment, the teachers' opportunities to learn were mediated by

the discourse moves of the content coach who facilitated the meeting. What implications might these data have for both future research and practice?

First, researchers need to investigate and describe a learning trajectory of cognitive demand in mathematical terms. The use of Bloom's Taxonomy in the teachers' talk around cognitive demand was a conception that the teachers brought into the learning activity. Indeed, its ubiquity in their talk indicates that Bloom's is a part of the shared repertoire of their community, and there is no reason to think that the teachers in these data are unrepresentative of the population. The shift in discourse seen in these data is significant for teacher educators. If we want to support teachers to develop a deeper conceptual understanding of the cognitive demand of mathematical tasks, we prefer a taxonomy that is specific to mathematics. Their understanding of cognitive demand in terms of Bloom's Taxonomy interfered with the affordances of the activity that might have otherwise supported the development of a more mathematical view of cognitive demand.

The different ways that Jane pressed on teachers to explain their thinking is analogous to the ways that teachers press on students to explain their thinking in a math classroom. Such a learning trajectory could help coaches assess teachers' knowledge and plan for instruction. Second, these data suggest the need for further professional development for content coaches. The coach in these data has above-average expertise, more experience than the teachers in the collaborative group, and she implemented an activity that research suggests would be highly effective. However, the results are mixed. Teachers did have opportunities to learn about one aspect of ambitious instruction, but those opportunities were mediated by certain discourse moves, even moves that would be difficult to characterize as "bad." Thus, even under the best of circumstances, opportunities for learning may be mediated or blocked entirely. Thus the role of the coach in facilitating teachers OTL is critical, and district and school leaders need to ensure coaches have access to high-quality PD.

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THE MYTH OF PLANNING: TEACHERS' DEVELOPMENT IN SUPPORTING MATHEMATICS DISCOURSE

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When introducing teachers to new practices, many professional development experiences expect mathematics teachers to plan in detailed ways before enacting those new practices. In this study, we examine the ways teachers' planning relates to their learning of new classroom practices, in the context of a study group focused on rich mathematics discourse. Our findings suggest an alternative conception of teachers' engagement in planning related to changing their classroom practice, which involves cycles of enactment and reflection that lead to principled intentionality.

Keywords: Teacher Education-Inservice/Professional Development, Classroom Discourse, Instructional Activities and Practices

A prevailing assumption in the design of teacher preparation and professional development is that planning plays an important role in learning new classroom practices. Planning involves making decisions about the instructional tasks to use, the social structures of the classroom that might organize the work, the pedagogical practices a teacher seeks to implement, the goals of the lesson, and the assessment of progress towards those goals (e.g., Morris, Hiebert, & Spitzer, 2009; Stein, Engle, Smith, & Hughes, 2008; Stein, Smith, Henningsen, & Silver, 2009). When teachers learn a new aspect of practice or seek to refine their teaching in a particular way, one might expect to find evidence of change in the planning process. This notion implicates planning as a site to examine teacher change. In this study, we explore this assumption in the context of a professional development experience for secondary mathematics teachers focused on classroom discourse. Specifically, we examine the ways in which teachers integrated new discourse practices in the work of planning for, enacting, and reflecting on their teaching practice.

Rich mathematical discussions are a critical characteristic of the type of ambitious teaching advocated by mathematics education reform efforts (e.g., NGAC & CCSSO, 2010; NCTM, 1991; 2008). In-depth examinations of teachers explicitly focused on mathematics discourse have identified teacher discourse moves that are visible in practice that support positive student learning outcomes (Chapin, O'Connor, & Anderson, 2003; Herbel-Eisenmann, Steele, & Cirillo, 2013). The findings of these studies suggest few explicit traces of planning for mathematical discourse prior to the enactment of lesson. As such, the ways in which teachers might be supported in planning for rich mathematics discourse in classrooms is an open question. In this report, we study the role that planning played as teachers engaged with professional learning materials designed to support them in changing their classroom discourse practices.

The Mathematics Discourse in Secondary Classrooms (MDISC) professional development curriculum is a set of practice-based, case-based materials. The materials are organized around five constellations of activities (Steele & Hillen, 2012) anchored by a mathematical task and a narrative or video case of a teacher engaging students in work on the task. The materials introduce six Teacher Discourse Moves (TDMs) as tools for teachers in developing their discourse practices (see Herbel-Eisenmann, Steele, & Cirillo, 2013 for more detail).

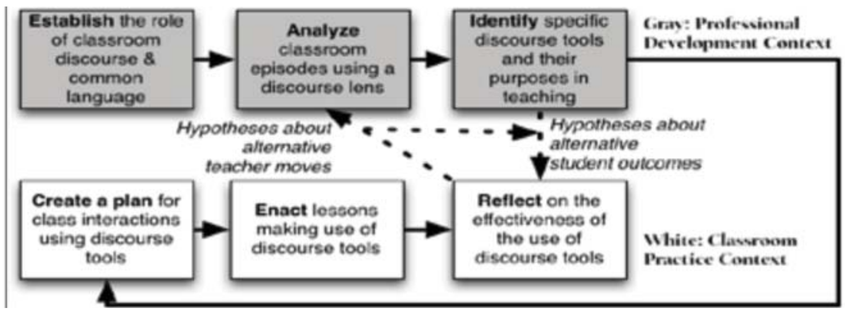

Figure 1: Patterns of Participation Presented in the MDISC Materials

The MDISC materials provide teachers with opportunities to learn about the TDMs and integrate them into their practice. Figure 1 represents the patterns of participation envisioned by the design of the materials and serves as a theoretical framework for understanding teacher learning. We anticipated that teachers would first establish the role of discourse in the classroom, analyze cases of classroom practice with a discourse lens, and then identify tools for supporting rich classroom discourse (the TDMs). Through a set of Connecting to Practice activities, the teacher had opportunities to plan, enact, and reflect on lessons using the TDMs.

Method

Our pilot of the MDISC materials established a community of practice supporting critical and thoughtful discussion of teaching practice. We analyzed teachers' engagement in the practice of planning and related practices from the framework in Figure 1 using data from the study group. Our analysis investigates the video and written records of a yearlong pilot of the MDISC materials with nine local secondary teachers from three districts. Teachers ranged in experience from 0 to 20 years and taught a wide range of content from middle school mathematics through calculus. Professional development sessions lasted 2-3 hours every three weeks throughout the academic year at a nearby university and were videorecorded, with written artifacts collected.

Data Analysis

To examine the ways in which teachers shared information about their planning practices, we identified discussions in the sessions in which teachers reported directly on their classroom practice in whole-group discussions. This allows us to describe the opportunities that all teachers had to share their practices in planning for discourse and consider the practices of others.

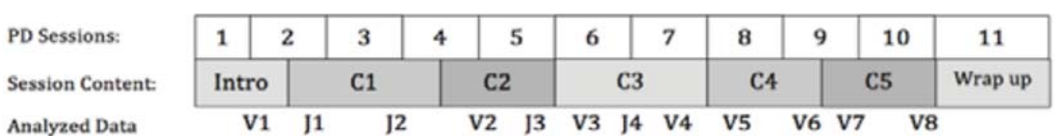

Figure 2: Structure of Professional Development and Data Collection

Figure 2 shows how the constellations (C1-C5) of activities in the materials extended across the 11 sessions. We used a modified grounded theory approach (Strauss & Corbin, 2008) to identify the ways in which teachers engaged in planning as evidenced in the whole group discussions. Using open coding, we identified instances in which teachers talked about planning for classroom discourse, enacting discourse-rich lessons with students, and reflection on that enactment in segments of video records (V1-V8 in Figure 2) and in written journals (J1-J3 in Figure 2). We limited our analysis to statements explicitly related to classroom discourse in

which teachers discussed their actions (enacted or intended), students' actions (enacted or intended), or both. This process developed the coding scheme in Table 1. Through this analysis, we determined the extent to which teachers' participation patterns fit within the framework in Figure 1. Our results compare those patterns to our expectation.

Table 1: Coding Categories

Categories	Description
Enactment: Current discourse teaching practice	Statements that describe current practice related to discourse
Enactment: Impact on students	Statements addressing how students responded to their teaching moves related to discourse.
Planning or Intentionality	Statements explicitly referencing planning for classroom discourse. Such statements may include a focus on discourse tools or expected student outcomes as a result of planning.
Hypothesizing: Desire to change discourse practices	Statements indicating dissonance between current teaching practices related to discourse and desired outcomes.
Hypothesizing: Conjecturing about alternative student outcomes	Conjectures or predictions about how discourse practices may impact students, including thinking about future teaching moves and conjectures about how students might respond.

Results and Discussion

The primary findings from this study indicate that planning played a different role in teachers' patterns of participation than we had initially anticipated. Specifically, planning to use the TDMs was *intertwined with* teachers' engagement in changing their classroom practice. These finding stand in contrast to the typical "plan first, teach afterwards" model. Based on the organization of the materials, we anticipated that planning for class interactions served as the initial bridge between the professional development content and the classroom context. Our analysis suggests that the notion that teachers must first plan to enact new teaching practices may be a myth. Instead, the development of teachers' intentionality in using the TDMs was distributed across their engagement with the materials and their own practice as they engaged in cycles of enactment, reflection and hypothesizing. Figure 3 traces teachers' participation patterns within the study group as was evidenced by their discussions and journal entries. This revised pattern was characterized by teachers' early experimentation with discourse ideas; cycles of enactment, analysis, and reflection; generalizing through hypothesizing about practice across classroom episodes, and ultimately planning as a distributed outcome. This revised model does not indicate a lack of planning, but a revised notion of the meaning and role of planning.

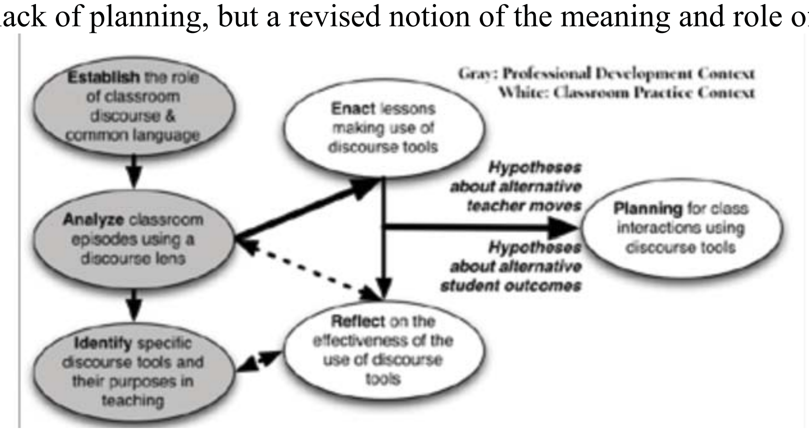

Figure 3: Observed Patterns of Participation

To briefly illustrate the ways in teachers made intentional decisions about their use of the TDMs as an outcome of their exploratory engagement, we present the following quote from Donna, an experienced teacher in the study group. In response to a question that asked how Donna might purposefully consider the issues surrounding classroom discourse when preparing to teach students, she wrote, "I see myself starting the period with some scripted discourse to see how long I can keep it going. When the direction of the hour moves away from the discourse. I need to consider how to revisit the discourse the next day and use it in different contexts," (J3). Donna's conception of planning to use the TDMs resulted in her writing verbatim what she intended to say to students. Her statement was an acknowledgement that she had reached a plateau in her work improvising with the TDMs. Experimenting with the TDMs alone would not allow her to accomplish all that she wanted. Donna's comment is reflective of the teachers' recognition that the process of planning would be helpful in making the changes they were trying to enact in their classrooms.

The teachers in our study group made immediate use of the TDMs in opportunistic ways in their everyday practice. It was through the use of these moves and cycles of enactment and reflection on them that the teachers began to develop ideas that we might have looked for in planning; namely a sense of general principles for using the TDMs and the linking of TDMs to the mathematical and social goals for their lesson. The data reinforce that planning is not necessarily absent in these teachers' practice, but takes place at an unexpected point in the process. As researchers studying the ways in which teachers enact classroom practice, the myth of planning can be replaced by the reality of thoughtful experimenting and focused reflection and analysis.

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EXPLICIT METACOGNITION IN A GEOMETRY CONTENT COURSE FOR PRE-SERVICE ELEMENTARY TEACHERS

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This qualitative case study focused on the actions of the instructor in a sophomore level geometry content course designed specifically for pre-service elementary teachers. Frequency counts of metacognitive episodes from classroom discussions were coded for speaker and content. While this particular course is designed to promote the sharing of thinking by both instructor and students, actual classroom conversations focused primarily on procedural knowledge provided by a "more knowing other." Implications of these findings suggest the need to help teachers understand ways they can encourage in-depth conversations that expose students' thinking and, in addition, to further explore the relationship between non-traditional instructional practices and metacognitive development.

Keywords: Metacognition, Teacher Education-Preservice, Instructional activities and practices

While many formal definitions of metacognition exist, thinking about thinking is an underlying theme across all definitions. Schraw and Dennison (1994) defined metacognition using the components of knowledge of cognition and regulation of cognition. More specifically, knowledge of cognition is comprised of the sub-components declarative, procedural, and conditional knowledge; while regulation of cognition is comprised of planning, information management, monitoring, debugging, and evaluation. Knowledge is thinking about knowing something while regulation is doing something with this knowledge.

Positive correlations between metacognitive abilities and academic success have been well documented in the literature, as well as the effective use of carefully designed instructional activities to improve students' ability to think metacognitively (Lin, 2001; Schraw, 1998; Sternberg, 1998; Young & Fry, 2008). Together, these findings have lead many to suggest the importance for explicit instruction promoting metacognitive development (Wilson & Clarke, 2004)

The literature is saturated with interventions that have been used and tested for their ability to promote metacognitive development, however there are limited studies that bring together the common themes. Lin's (2001) analysis suggested metacognitive activities must include the need to provide students the opportunity to self-assess, help students articulate their thinking, foster a shared understanding of metacognitive goals, and help students develop knowledge of self-as-learner. Schraw (1998) provided a more explicit approach by suggesting that metacognitive instruction should include promoting general awareness, improving the knowledge and regulation of cognition, and fostering an environment that supports metacognition.

Across the literature, an overarching idea is that developing metacognitive thinking through classroom interaction requires opportunities for students to be exposed to the thinking of others, to compare their own thinking to that of another, to ameliorate the differences among the thought structures, and, most importantly, to internalize and adapt the most efficient structures into their own thinking. When thoughts and ideas are shared, it is possible to make comparisons among a variety of strategies thus expanding students' opportunities to improve their metacognition. Since students' metacognitive thinking can be enhanced, it is important to help Pre-service

elementary teachers (PSETs) learn ways to incorporate these strategies into their own future classrooms. The purpose of this study was to describe the learning environment in a student-centered geometry content course developed for PSETs and to identify the types of metacognitive thinking that was shared explicitly during whole class discussions.

Data

This study employed naturalistic inquiry using qualitative data to explore and to identify the metacognitive thinking made explicit during non-participant classroom observations (Patton, 2002). Classroom observations consisted of eight sessions from the fifty-minute MWF section and six sessions from the 75-minute TR section. All 14 observations occurred during weeks 12-14 of the sixteen-week semester and contained all but two of the class sessions for each section between Exam 2 and Exam 3 of the course. All observations were prearranged with the instructor and videotaped. By this point in the semester classroom norms were well established and instructor expectations along with student dialogue were consistent across all sessions. Because of the consistency of format and class norms across the class sessions, 5 of the 14 observations were randomly selected for transcription and in depth analysis. The observations, 3 from MWF and 2 from TR, were transcribed with an emphasis on the classroom dialogue and field notes taken by the researcher during the observations added detail.

Participants

The instructor chosen to participate in this study was purposively selected because she was teaching multiple sections of the student-centered course. Described as a student-centered, this instructor has taught this particular course for several semesters. The course is a sophomore level geometry content course designed specifically for pre-service elementary education majors to provide an opportunity to learn mathematics in a student-centered and problem-centered environment. Based on information provided by the instructor and gleaned from the textbook, students are expected to explore, to discuss, and to explain the mathematics content they encounter in the course. The class sections chosen for this study were selected based on the instructor's and researcher's schedule and for differences that may have potentially existed between the two sections, one section was held during more traditional class hours (MWF morning) while the other section was in a time frame more commonly attended by non-traditional students (TR evenings). However, after close review, the demographics of the students in each section were similar when taking into account gender, ethnicity, marital status, high school graduation date, and number of children at home.

Data Analysis

Because strategies for promoting metacognitive development rely on the sharing of metacognitive thoughts, Schraw and Dennison's (1994) description of the components and sub-components of metacognition was used to explore the types of metacognitive thoughts that were made explicit during the transcribed observations. This framework includes two main components, knowledge of cognition and regulation of cognition. Sub-components of Knowledge of Cognition include Declarative, Procedural, and Conditional Knowledge while the sub-components of Regulation of Cognition include Planning, Information Management, Monitoring, Debugging, and Evaluation. All episodes of metacognition were identified during the first pass through the data. Multiple passes then followed to confirm the inclusion of metacognitive thought in the episode, to label the episode with the appropriate sub-component(s) identifying the type of metacognitive thought that was present, and to identify the provider of the

thought as student, instructor, or both. An episode typically began with a question and ended when the conversation moved to a new topic or problem. A final pass was made through the data to confirm and tally the labels found in each episode.

Findings

Episodes containing explicit knowledge of cognition (77), were more frequent than episodes containing explicit regulation of cognition (51). See Table 1 and Table 2 below for a summary of all data. Further, 57 of the 77 knowledge episodes were procedural in nature, which far surpassed the 3 episodes that were declarative in nature, and the 17 that were conditional in nature. While it does appear that the students provided a great majority of the procedural episodes, these students were typically serving the role of a pseudo-teacher and were providing direct instruction of the procedure another student was questioning. While 34 of the episodes consisted of the student providing the explicit sharing of knowledge of cognition, 31 of the episodes were examples of both the teacher and the students being involved in making the sharing happen. This joint coding of the episode occurred when student struggled to complete their explanation and the teacher scaffold the students' explanation thru a series of questions, thus helping students complete an explanation they would not have been able to complete on their own.

Unlike episodes coded for the knowledge of cognition, two-thirds of regulation of cognition episodes required instructor prompting and scaffolding. Further, about half of the episodes consisted of managing information or debugging ideas that were ill-conceived. These episodes tended to consist of the instructor directing students to reconsider a specific aspect of the geometry problem. Additionally, only 5 of the 51 regulation episodes provided an example of modeling and only 7 of the 51 provided an example of monitoring.

In summary, at the end of the semester, students explicitly shared very few metacognitive thoughts during their class discussions. The metacognitive thoughts that were shared tended to be procedural in nature. Further, there were few opportunities to discuss how one knows they know something, when or why to use different strategies, or how to regulate one's thinking through the problem solving process.

Table 1: Frequency of Shared Knowledge of Cognition

	Declarative			Procedural			Conditional			Totals		
	S*	B	T	S	B	T	S	B	T	S	B	T
Frequency	3	-	-	31	18	8	-	13	4	34	31	12
Totals	3			57			17			77		

*S: student only; T: teacher only thought; B: integrated student and teacher thought

Table 2: Frequency of Shared Regulation of Cognition

	Planning			Info. Mgmt.			Monitoring			Debugging			Evaluation			Totals		
	S*	B	T	S	B	T	S	B	T	S	B	T	S	B	T	S	B	T
Frequency	1	1	3	4	8	3	2	4	1	1	11	1	-	11	-	8	35	8
Totals	5			15			7			13			11			51		

*S: student only thought; B: integrated student and teacher; T: teacher only thought

Discussion

The goals of this study were to describe the metacognitive thinking explicitly shared in this student-centered classroom and to better understand the metacognitive development that may occur in this instructional setting. Unfortunately, the thinking shared by students was primarily procedural thinking and thus does not meet the process outlined by Schraw (1998) for promoting metacognition. Metacognitive thoughts are rarely made explicit and the importance of metacognition in the overall learning process is not discussed. While there was prompting by the instructor to promote regulation thoughts, this had not become a class norm and, thus, students left the content course lacking the internalized process of knowing when, why, and how to regulate their own thinking. Lastly, the focus in class became deferring to a "knowing other" for an explanation of the procedure rather than on students explaining their own thought processes as they worked through the problem.

When teacher education programs seek to develop teachers who will promote thinking and reasoning ability in their future students, modeling metacognitive thinking and promoting the sharing of metacognitive thinking should also occur in the content coursework as well. This instructor for the course was selected for this study because she was committed to student-centered instruction. However, there is a heavy reliance on the teacher or a pseudo-teacher to provide the knowledge and endorsement rather than having students rely on, share, and evaluate their own thinking processes. Future research is needed to explore why this mismatch exists and it's impact on metacognitive development. The overall lack of metacognitive thoughts shared in the classroom also suggests the need for further exploration of the relationship between student-centered classrooms and the development of students' metacognitive thinking processes as well as exploring approaches to develop PSETs that will in turn develop metacognitive thinking in their future K-12 classrooms.

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HARMONIZATION PERSPECTIVE ON TRANSITION ISSUES

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The transition between secondary and post-secondary levels is mainly looked at in terms of differentiations: different ways of thinking (e.g. Tall, 1992); different natures of mathematics (Robert, 1998); different mathematical organizations (e.g. Bosch et al., 2004); different mathematical cultures (Artigue, 2004); etc. The idea of a possible harmonization between the levels is rarely studied. This paper presents a new way to conduct research on transition issues, introducing a harmonization perspective in which dialogue between teachers of both levels facilitates new ways of thinking and doing mathematics at their specific levels.

Keywords: Teacher Knowledge, Secondary Education, Post-Secondary Education

The transition from secondary to post-secondary is a challenging period in education. Difficulties are encountered by students (Praslon, 2000) and by teachers as well (Corriveau, 2007). Research in mathematics education about secondary/post-secondary transition has mainly studied transition issues through differences (e.g. Bosch & al, 2004). Few researchers have studied the relationship between the two levels. Studies by Bloch (2000) and Praslon (2000) focus on the link between the teaching and learning of calculus at the secondary and post-secondary level; yet, in these cases, researchers, and researchers alone, have supported the issue of secondary/post-secondary articulation, by implementing *ad hoc* teaching devices. In our research, we want to consider the views of teachers of both levels. Their teaching experience can certainly be put to good use in the exploration of what can be done at their respective levels to overcome transition problems. We also aim to initiate a dialogue between them in order to approach transitional issues. These teachers do not usually work together and, as such, live in different *worlds* and mathematical *cultures* (Artigue, 2004). These considerations allow, on the one hand, the exploration of what is done at both levels but, on the other hand, go further on the possible meaning of a harmonization perspective when working around transition issues. If any harmonization occurs, how does it emerge?

Collaborative Research

A group of six participants (three secondary teachers and three post-secondary teachers) were invited to join a collaborative research project, in which the researcher worked with practitioners on an issue related to their practice (Desgagné et al., 2001). It is often said that collaborative research means researching "with" teachers rather than "on" teachers. Since we work with teachers, the precise object of the study has to be related to their practice. As we said, Artigue (2004) addresses transitional issues by characterizing the culture (Hall, 1959) of secondary mathematics based on an analysis of its curriculum (the explicit part of a culture). However, according to Hall, it is the implicit "ways of doing" that lead to the greatest cultural differences. In the classroom, teachers do mathematics in a certain way and make their students do mathematics. So, we decided to work on transitional issues through teachers' "ways of doing mathematics" (WDM) at each level. The project ran from November 2010 to November 2011. There were regular meetings: one meeting to explain the project and six days (9 a.m. to 4 p.m.) of dialogue meetings. All these meetings were recorded and transcribed.

Mathematical Ethnomethods

Ethnomethodology (Garfinkel, 1967) provides relevant elements to better understand, on a theoretical level, familiar "ways of doing" to a group sharing a practice (see Corriveau, 2012). It studies everyday methods people use to accomplish and to constitute a socially organized activity, such as teaching mathematics at a specific level. We have developed the concept "mathematical ethnomethods" (ME), which are WDM mobilized by teachers in their daily working lives. The ME include action: WDM *shared* by teachers of a specific level, indexed to a *rationality* and *circumstances* of this action. They also include actor considerations; the teachers are assumed to be knowledgeable, meaning that "doing mathematics" requires an interpretation from the teacher to give meaning to what they do, and to recognize the circumstances surrounding their actions. This theory is likely to be useful in understanding familiar "ways of doing" mathematics and new "ways of doing" likely to be developed through the joint exploration of transition by teachers and researchers.

The Example of Function

Although WDM was the object of the research, the participants focused on a specific topic: functions (an important object at both levels). The dialogue revolved around the teachers' usual activities: the researcher submitted tasks developed around the everyday actions teachers face in their practice. The first part of our analytical work highlights a territory of ME with functions at each level. Table 1 presents a part of the ME territory at each level.

Table 1: ME Territory Around Functions at Each Level

Secondary level	College level
WDM. The teachers involved in the research associate a table of values, a graph or an algebraic expression to a function (to be studied). They sketch the graph of a function from a table of values or an algebraic expression (using the parameters). They use basic symbolization to represent basic function (e.g f(x)=x^2) and they gradually introduce parameters to show all the possible functions (e.g. f(x)=$a(bx-h)^2+k$). **Rationality.** The different representations are seen as a way to give meaning to each group of functions to be studied (e.g. a quadratic function has this kind of shape on a graph, this kind of variation in the table of value, is recognizable by this kind of algebraic expression).	**WDM.** College level teachers taking part of the research retrace the behaviour of any function from its algebraic expression. They mobilize tools (e.g. table of variations) to retrace the behaviour of any function. They use condensed symbolization (e.g. $f(x)=x^2$ represents the whole group of quadratic functions). **Rationality.** They want to prepare college student to study science at university.

As part of the second session, we began working on harmonization.

Reconstruction of a Harmonization Trajectory for Functions

From the analysis, we were able to reconstruct three related steps of a harmonization trajectory: a first step in which new ideas emerge; a second step in which the group works on different representations of a function (used to enrich the ideas in part 1); and a third step in which we develop specific tasks. In this presentation, we will present the first part, which is divided into three parts and is based on data from a discussion between a secondary teacher and two post-secondary teachers. The discussion took place in the second session.

Part 1: The emergence of a first key element linking both levels. The first part of the trajectory comes from an informal discussion between teachers (while the researcher was preparing the

material). The analysis of these informal exchanges gradually revealed the importance of the graphic register for teachers and, eventually, the reasons for such a choice. Scott (secondary) is talking about a system of two equations: “Algebraically what you get is special [he refers to a resolution that would be e.g. 0 = 7]. It is as if students had been mistaken. Graphically, some realize why: this is because it’s parallel; there's no meeting point.” Thus, Scott says he works with the graphic representation when solving a system of equations—a “clear” illustration that is meaningful to students and gives meaning to the algebraic solution. "So graphically, there’s something 'clearer' to students than the algebraic representation?” asks the researcher. Colleen (college) goes on to describe an instance where students had to find the domain of a function: “I was surprised to find that the students who had used a graph were all successful.” She also confirms that the graphic mode is a "clear" representation for students. As seen in Table 1, we thus find that the graphical representation allows students to *make sense* of a resolution at the secondary level and allows them to find *mathematical characteristics* of a function at the post-secondary level. At this point, there is a link between the two levels, but each teacher remains within his or her own territory.

Part 2: The creation of a contrast and a void to fill. Based on the informal discussion and the work started during the first meeting, the researcher felt the need to explicitly identify important elements at each level (see Table 1). The researcher wrote on the board what had emerged globally, up to that point, for the secondary and post-secondary levels, making sure that the teachers were in agreement. The researcher used Figure 1 as a basis for discussion focused on harmonization. The following question was therefore put forward: What can be done to bridge the gap between the two levels?

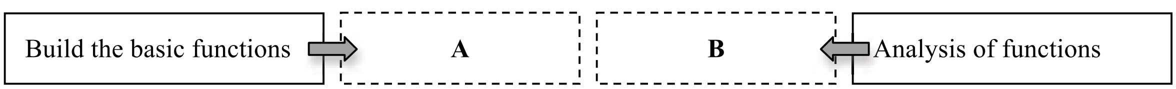

Figure 1: Figure Used as a Basis for Harmonization

Several ways to schematize could have been put forward by the researcher. The layout of this figure, as a basis for thinking about possible harmonization, implies that links are built at each level (a certain way of conceiving harmonization was implicitly put forward by the researcher).

Part 3: Possible bridges at each level to bring the two closer together. Scott noted that “Ultimately, even if students have to deal with a ‘combination’ of many basic functions (referring to the work ahead in post-secondary), they must be able to tell themselves ‘When I see x square by the square root of x [$x^2\sqrt{x}$], for example, it’s all right ‘cause I know what $\sqrt{x}$ is, and I also know what x^2 is...’ In reality, at the secondary level, we don’t really use this approach." What Scott was saying is that operations with functions are not seen as new functions. The researcher, having in mind the graphic register as an important element at both levels, asks, "If it is a mode of representation “clear” to students, is it possible to graphically observe the effect of operation on functions?” She is attempting to establish the first bridge, suggesting that they “combine’ functions and see it as a function as a result, referring to Scott, in the graph. Scott agrees and the researcher writes it on the board (rectangle A, Figure 1). Then, Colette (college) makes a suggestion to fill the last empty rectangle (B). She writes $1/(x^2+1)$ in the empty rectangle and $x^3/(1-x^2)$ in the last one, saying she sees a progression. She says that post-secondary teachers could review the layout of basic functions with parameters (as they are used at secondary level), and introduce function where it is no longer possible to work in the same way as at the secondary level

(using parameters to draw its graph). In other words, she is revisiting her own territory, with a new horizon—secondary level. Colette says she had this idea when describing an experience with her students (they don't have the intuition to sketch the graph of functions). Although apparently the function in rectangle B ($1/(x^2+1)$) looks like something that could be studied in secondary school, it is not part of their territory. Yet Colette mentions that it is not found in the territory of post-secondary schools either. It is therefore, according to her, an "in between." How should these functions be graphed, then? It is not studied at the secondary level and, in the first calculus course at the post-secondary level, students already use the derivative. New work around function emerges from this, which is to develop a different way to "analyze" or retrace the behaviour of a function by sketching the graph (intuitively). Moreover, we see that all the work is based on a particular type of representation: the graph, which serves as a "bridge."

Conclusion

What Brings This Reconstruction to the Harmonization Rrajectory?

The first part of the harmonization trajectory, aimed at the emergence of ideas, is developed around the identification of links (the graph representation), the creation of a contrast from discussion about WDM around functions (a gap to fill) and finally, building bridges. This trajectory is characterized by the contribution network between the teachers from both levels and the researcher—a reorganization of the ME territory at each level. The researcher identifies issues of transition, common elements and contrasts, and as well, she establishes a scheme as a basis for harmonization, and teachers from both levels bring ideas from their respective levels to fill the void created and establish a plausible reconciliation. Teachers constitute the links by revisiting their territory with a new horizon: the territory of the other.

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MEASURING MATHEMATICAL KNOWLEDGE FOR TEACHING STUDENTS TO PROVE

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Proof and proving are essential parts of mathematics and being a mathematician. A curriculum lacking in proof has the potential to portray a false image of mathematics and what it means to do mathematics. Unfortunately many teachers' conceptions of proof are limited and even those with robust conceptions of proof often view it as having a limited pedagogical role (Knuth, 1999). In-service and pre-service need different experiences with proof to better prepare them to teach proof. To create more fruitful experiences teacher educators need to assess the knowledge of proof that pre-service teachers hold to help them prepare more meaningful lessons. This paper reports preliminary results from an instrument designed to measure mathematical knowledge for teaching students to prove.

Keywords: Reasoning and Proof, Mathematical Knowledge for Teaching

Introduction

A great deal of research has been done in the area of proof and several researchers have categorized the different modes of argumentation used by people when proving in mathematical situations (Balachef, 1988; de Villers, 2003; Harel & Sowder, 1998). In the same time frame, mathematical knowledge for teaching (MKT) has blossomed into a fruitful area of research in mathematics education (Hill, Ball, & Schilling, 2008). This research study intends to capitalize on both areas by utilizing the categories of arguments to inform the development of a measure to investigate MKT needed to teach students to prove (MKTP) in order to inform the preparation and professional development of teachers.

Purpose

Proving is an essential part of mathematics and being a mathematician. However, studies conducted across a variety of countries have shown proof to be a skill that is largely underdeveloped in students (Healy & Hoyles, 2000; Recio & Godino, 2001; Martin & Harel, 1989). Although proof might be thought of as too challenging for younger students, Lester (1975) showed that students in upper elementary grades (4 – 6) were as capable of hypothetico-deductive reasoning as high school students. Similarly, de Villers (2003) cited several dissertation studies that found logical reasoning used by kindergarten and younger students. Finally, researchers have also had success teaching proof to early and upper elementary students using a variety of teaching strategies (Stylianou, Blanton, & Knuth, 2009). Unfortunately, many teachers' conception of the pedagogical role of proof is often limited (Knuth, 1999). Knuth's work (1999) work suggests that teachers need more experience or different kinds of experiences with proof to be able to effectively teach their students to prove.

This paper reports preliminary findings for an instrument designed to gauge teachers' mathematical knowledge needed to effectively teach students to prove. This instrument was designed for middle and high school teachers. The measure of MKTP was based on ideas from the literature that are considered essential for teachers to know. From a review of this literature, I suggest that creating a measure of MKTP requires an emphasis on four domains; (1) evaluating

proofs that are accurate, inaccurate, and that utilize various modes of argument from naïve to formal, (2) utilizing proof as a method of explanation or instruction, (3) constructing proofs, and (4) creating counter examples for false conjectures. This paper focuses on the first domain.

The organization of this paper is as follows. First, I define the terms proving, proof, and teaching proof. Second, I describe the factors of teaching proof that researchers suggest lead to effective teaching and learning to prove. Then I describe how I used these factors to design a portion of an instrument for measuring MKTP. Finally, I present preliminary findings from a study to validate the instrument.

In this paper, proving implies a process that an individual or community uses to remove doubts about the truth of a statement (Harel & Sowder, 1998). Furthermore, proof is meant to be an argument that an individual or a community considers to remove the doubt about the truth of a statement (Harel & Sowder, 1998). This interpretation of proving and proof is used because students consider various types of arguments to be proofs. For instance, numerous studies (de Villers, 2003; Harel & Sowder, 1998) have investigated the modes of argumentation used by various age levels and populations, and Harel & Sowder (1998) have synthesized those into three classes of proof schemes that are subdivided into numerous subclasses. Only two of the classes were considered for this study, empirical and analytical. An empirical proof scheme uses one or more examples to establish the truth of a conjecture, whereas an analytic proof scheme implies arguments where each statement is assumed to be true or follows from earlier statements. Finally, the goal of teaching proof ought to be to help guide students from empirical proof schemes toward formal, deductive arguments.

Reid (2011) offered five key aspects of fifth grade teacher Vicki Zack's teaching that he suggests contributed to her effective teaching of proof. (1) She utilized tasks that focused on problem solving while (2) encouraging conjecturing, and (3) giving students sufficient time to explore the problem. While solving the problems, the teacher held (4) high expectations concerning the nature of students' communication and (5) utilized listening closely to seize opportunities to provoke student discussions (Ball & Stylianides 2008). Capitalizing on student discussions to provoke productive conversations about proof requires at least three key aspects of MKTP. First, the teacher needs to anticipate students' proofs, as well as recognize and ideally refute inaccurate statements through the use of counterexamples. Second, when anticipating students' proofs, the teacher needs to recognize the various types of proof schemes students might hold in order to help them productively transition their schemes toward analytic proof schemes. Third, the teacher has to have an analytic proof scheme herself.

Methods

Only a portion of the instrument will be described in this paper. The task is based on the statement that the sum of the interior angles of a triangle is 180 degrees. The participants were asked if they thought the statement was true or false and how confident they were with their choice on a scale from one to five. There are four arguments to verify the statement. Three arguments are empirical and the fourth argument (A4) is the traditional deductive proof of the statement that utilizes creating an auxiliary line parallel to one of the sides. For each argument the participants are asked if it is a proof and to explain their response. In the first argument (A1) a triangle is made and the angles are torn off and put together. Since they make a straight line it is concluded that they add to 180 degrees. The second argument (A2) utilizes knowledge that a square has four right angles that add to 360 degrees. Since a square can be made into two triangles the angles must add to half of 360 degrees. There are five triangles in the third

argument (A3), one acute, right, obtuse, isosceles, and equilateral and all of the sums of their interior angles are measured and found to add to 180 degrees. Finally participants were asked to consider a situation in which they planned to teach about the sum of the interior angles but also teach about proof. They were asked to select a grade band (6 – 8 or 9 – 12) and describe which of the four arguments they would use. They were allowed to select more than one method, but if they did they were asked to specify the order for them and to provide a rationale for their organization. If they choose none of the arguments, they were asked to provide their own.

There are several reasons for this format. First, individuals take different actions depending on their view of the validity of a statement hence asking for their initial opinion. Second, different actions are taken depending on how confident in the validity of a statement an individual is. Empirical arguments can be a step in the process of creating a deductive proof and therefore to categorize someone as only having an empirical proof scheme from that one instance would be wrong. Their confidence also offers insights into participants' conception of proof. If an individual is convinced the statement is true, then it must be true for all cases. Individuals with an analytic proof scheme who are confident the statement is true ought to seek deductive arguments. Finally, the three empirical arguments although similar are different. A1 might visually help students understand a deductive proof and be useful for teaching the content, but the triangle could also be wrongly attributed as arbitrary and therefore A1 would be a proof. A2 is the least sophisticated because not all triangles can be put together to make a square, but A2 uses properties and might appear deductive. Finally A3 includes examples from the different classes of triangles and might be considered by participants as being a proof because they might think each instance generalizes for the whole class of triangles.

Twelve pre-service secondary mathematics teachers (PST) were the sample. They were in the second semester of their third year in their undergraduate program. Most students had finished the majority of their mathematics requirements, they were taking their first mathematics methods course, and none had student taught. The PSTs proof scheme was based on their answers to whether or not the arguments were a proof or not. Their reasons for why the arguments were or were not proofs were analyzed using constant comparison method of grounded theory to establish what the PSTs felt did and did not make an argument a proof (Corbin & Strauss, 2008).

Results

Based on their responses on the entire instrument, the PSTs fell into three categories: those with an analytic proof scheme, those with an empirical proof scheme and those that showed signs of both. However only evidence from the task described above will be presented. All of the PSTs stated the conjecture was true and were highly confident with the statements veracity. Only one PST indicated all the justifications were proof and her reasons gave no indication of having an analytic proof scheme. Only 3 of the PSTs accurately categorized all four arguments. Two of the three gave reasons that suggested they held an analytic proof scheme by stating the inadequacies of the empirical arguments.

The remaining eight PSTs expressed mixed proof schemes. To better understand the implications of this, the reasons provided by the PST who correctly classified the arguments are provided. Although her assessment of the arguments would categorize her as having an analytic proof scheme, her reasons for accepting highlight key findings of the PSTs conceptions of proof. For A1 she stated "there is a lot of human error that could be involved" and for A2, she felt that it was missing reasons; specifically it needed to state the diagonal bisects the angles. Her responses were void of the lack of proving for all cases and suggest she would have accepted

better designed empirical proofs. Finally, in her response to A3 she stated that a list of examples could not prove a statement. Similarly, the PSTs with mixed proof scheme accepted empirical arguments but rejected others strictly because they were empirical. These responses suggest the PSTs were aware that empirical arguments are not proof but they still relied on empirical proofs.

Two key aspects that constituted a proof for the PSTs that were of note were their attention to arbitrary and the use of properties. For instance, A1 was accepted by many because the triangle was assumed to have been generated arbitrarily and therefore proved all cases. Similarly for A3 many suggested that an arbitrary triangle needed to be used to make it a proof or accepted it because it had shown it was true for an arbitrary triangle for each type of triangle. Secondly, the use of properties or previously known concepts appeared to signal proof for the PSTs. A2 was accepted by many because not only were they aware that a square had four right angles, but that most middle grade students would also be aware of that fact. That fact then could be used to build on to create more facts. This aspect I feel is tied to the conception of arbitrary because A2 only verifies for isosceles right triangles.

Finally, to teach the lesson the PSTs with analytic proof schemes selected one of the empirical proofs to convince the students of the veracity of the conjecture followed by the deductive proof. The remaining PSTs felt the deductive proof was too challenging or did not show why the statement was true and should not be used. Although based on a small sample this corroborates that few teachers hold an analytic proof scheme (Martin & Harel, 1998) but more importantly provides evidence that not holding an analytic proof scheme hinders one's ability to adequately teach students to prove. Finally, the common acceptance of property based empirical arguments by the PSTs might suggest a need for axiomatic proof schemes.

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IDENTIFYING CURRICULUM SPACES FOR CONNECTING TO CHILDREN'S MULTIPLE MATHEMATICAL KNOWELDGE BASES IN ELEMENTARY MATHEMATICS

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Elementary mathematics curriculum materials can serve as a lever for instructional change. In this paper we promote a particular kind of instructional change: integrating children's multiple mathematical knowledge bases (MMKB), including children's mathematical thinking and children's home and community-based mathematical funds of knowledge, in instruction while using curriculum materials. We focus on existing and potential spaces in written curriculum materials, or curriculum spaces, so as to better support PSTs in enacting curriculum that opens spaces for connecting to children's MMKB.

Keywords: Curriculum, Equity and Diversity, Teacher Education - Preservice

Elementary mathematics curriculum materials can serve as a lever for instructional change, given their ubiquitous presence in teachers' classrooms and potential for incorporating features that would be "educative" for teachers and students (Ball & Cohen, 1996). Our work is focused on promoting a particular kind of instructional change: integrating children's multiple mathematical knowledge bases (MMKB), including children's mathematical thinking and children's home and community-based mathematical funds of knowledge, in instruction (Turner et al., 2012). We conjecture that a powerful means of supporting PSTs in integrating children's MMKB in instruction is to scaffold PSTs' noticing of potential spaces in elementary mathematics curriculum materials for connecting to children's MMKB and then developing practices for leveraging these spaces during instruction. In this study we analyzed a wide range of elementary mathematics curriculum materials, asking: a) To what extent do spaces for connecting to children's MMKB exist across the range of elementary mathematics curriculum materials? b) If such spaces exist, what is the nature of these spaces?

Our focus on curriculum *spaces* is informed by the work of Moje and colleagues (2004), in which *third space* is denoted as a place where individuals integrate multiple funds of knowledge. Moje and colleagues discuss multiple perspectives on third space, including third space as "a way to build bridges from knowledges and Discourses often marginalized in school settings to the learning of conventional academic knowledges and Discourses" (Moje et al., 2004, p. 43). We consider *curriculum spaces* as opportunities for students and teachers to use students' MMKB to support the learning of school mathematics. In this study, we closely analyzed the features of elementary mathematics curriculum materials that PSTs might notice and use in

instruction that connects to children's MMKB. The written curriculum provides a starting point for the enacted curriculum, necessitating an understanding of the written curriculum so as to better support PSTs in enacting curriculum that opens spaces for connecting to children's MMKB.

Methods

We examined sets of three lessons from each of eight curriculum series to identify existing and potential spaces in written curriculum materials, or *curriculum spaces,* as a start for understanding how to support PSTs in enacting curriculum that leverage or open spaces for connecting to children's MMKB. The three lessons within each lesson set focused on mathematical content prominent in elementary mathematics curriculum: a) multi-digit addition, b) single-digit multiplication, and c) introductory fractions. Through an emergent coding process, three types of spaces were identified: a) spaces for real-world contexts, b) spaces for student strategies and c) spaces for student explanations. Spaces for real-world connections are places that refer to real-world contexts to support students in learning mathematics. Spaces for student strategies are places where students are to develop or choose a strategy to make sense of mathematics and solve a task. Spaces for student explanations are instances where students are asked to describe and/or explain a strategy for solving a problem. We then developed and refined secondary codes with several passes through the data set, leading to sub-codes within each of the primary codes.

The first three authors coded all 24 lessons using this coding scheme. Each lesson was coded by at least two researchers for reliability purposes. Researchers then met in pairs to discuss coding, and differences were resolved through discussion and consensus. Finally, we analyzed *how often* each type of space was occurring; *where*, within the lesson, the space was located; and *what* kind of space, open or closed, was provided. For the latter, we noted secondary codes and analyzed the design features that seemed to further open or close a space.

Results

Our coding scheme helped us see that spaces ranged not only in number and type from lesson to lesson, but also in the specific features of the spaces, the different ways in which spaces were combined within lessons, the locations of spaces within lessons, and the types of supports provided to teachers that might lead to spaces becoming more or less open during enactment. Specifically, we identified three major findings: a) almost all of the lessons attempted to make real-word connections; however, the connections were rarely presented in ways that would help students use the connections to leverage their MMKB to make sense of the mathematics; b) there were particular design features that seemed to open and close the opportunities within curriculum spaces for connecting to MMKB; and c) significant differences existed among the curriculum spaces found in the main lessons from those found in the peripheries of lessons (e.g., in the teaching notes, differentiation activities).

Real-World Connections

Of the 24 lessons, all but one of the lessons made connections to real-world contexts.

Real-world objects as replacement manipulatives. In seven lessons, real-world connections consisted of real-world objects as replacement manipulatives. In *Arrays and Multiplication* (Altieri et al., 2009), for example, a teacher is to introduce the lesson by arranging counters into an array of 4 rows of 6 counters, asking students to identify the number of rows, counters in each row, and the total number of counters. Next, the teacher is to do the same with 3

rows of 5 counters, which matches a picture of 3 rows of 5 cups on the students' textbook page. The cups are simply a real-world object put in place of the mathematical counters.

Real-world contexts as application: strategy- or model-focused lessons. Five lessons had contextual real-world connections within the lesson, presented *after* a strategy or model was introduced to students. The use of a particular strategy was the focus of the lesson, and the context(s) provided a place to apply the strategy. In *Two-Digit Addition* (Altieri et al., 2009) students are shown how to use base-ten blocks for 28 + 7. Next, two problems labeled as "real-world examples" (e.g., "Gaspar has 8 game tokens. His brother has 24 tokens. How many tokens do they have in all?") are provided and then solved for students. While students *may* leverage their MMKB related to the provided contexts to access the mathematics, it would not be necessary and is not encouraged.

Context-focused lessons. In contrast, seven of the lessons *began* by making a connection to a real-world context. For example, one lesson began with the problem: "Ari has 63 baseball cards. His mother gave him her collection of 26 cards. How many cards does he have now?" (TERC, 2008). After some discussion of the context to establish student understanding (e.g., How many baseball cards did Ari have to start?) students are asked to solve the problem using the method of their choice. Because of the openness of this space, students have opportunities to leverage their MMKB, including their knowledge of and experiences with collecting and counting baseball cards or other objects and their knowledge of strategies for multi-digit addition, to make sense of and solve the problem.

Spaces opened in beginning of lesson. Finally, only two lesson plans had spaces sufficiently open for students to make their *own* real-world connections in the beginning of the lesson. These lessons aimed to support students in eliciting personal experiences or connecting to their own real-world experiences, presumably leveraging their MMKB when engaging in the mathematics of the lesson.

Design Features that Open and Close Curriculum Spaces

Here, we report the design features, most prominent within the two codes related to students' strategies and discussion of their own solution strategies, that seem to open and close spaces for students to draw on their MMKB.

Opening spaces for student exploration before strategies are presented. In five lessons, spaces for student development of strategies were provided *before* any solution strategy was presented by the textbook. Similarly, with respect to spaces for students to *explain* their mathematical work, the openness of the space was primarily influenced by when the students were asked to explain strategies (i.e., before or after a teacher/textbook strategy). In *Sharing Among Friends* (TERC, 2012), students are asked to solve problems that involve three and then four friends sharing objects. The curriculum materials did not suggest a strategy for students to use and students are asked several times throughout the lesson to explain their strategies.

Closing spaces for student exploration after strategies are presented. In contrast, in many lessons, a contextualized word problem was given *after* the textbook presented a strategy. For instance, in *Fractions of Whole Numbers* (Fuson, 2009), students were asked to solve the following: "Carlos saw 35 fish at the aquarium. 1/5 of them were clownfish. How many fish were clownfish?" This problem and others like it came after the textbook presented a "fraction times a whole number formula." Seemingly, the message in these cases is that students should use the strategy presented by the materials. However, a directive was later given to have students explain their strategies. Conflicting messages within single lessons about students' opportunities

to develop and explain their own strategies versus using, practicing, and explaining the strategy provided by the textbook can further complicate making connections to students' MMKB.

Opening, then closing, spaces for student exploration. Several lessons also started by asking students to solve a problem, thus providing an open space to generate a solution. However, the very next part of the lesson presented a strategy that students were expected to use for the remaining problems. In other words, an initially open space was closed almost immediately.

Peripheral Curriculum Spaces

Finally, we found that differences existed among the curriculum spaces in the main lesson and lesson peripherals (e.g., opening routines/ messages, ideas in the margins, and homework). This finding is important, because if most of the curriculum spaces – including many of the most open spaces - exist in the peripherals, then there is less of a chance that teachers will elicit and build on children's MMKB when teaching the lesson. When comparing where the spaces existed (main lesson vs. peripheral), we found three ways in which curriculum spaces were distributed: a) curriculum spaces existed primarily in the opening routine, but not in the main lesson; b) curriculum spaces existed primarily in differentiation activities, but not in main lesson; and c) curriculum spaces were evenly distributed across main and peripheral components.

Discussion and Implications

These findings can support PSTs in learning to *read* curriculum materials in order to *identify* spaces for children's MMKB, to *adapt* curriculum materials in ways that expand these spaces, and to *enact* curriculum materials in ways that use the spaces to elicit and build on children's MMKB. In addition to the implications for mathematics teacher education, implications for curriculum designers include considerations for a) how to provide more open spaces, b) where to locate the open curriculum spaces within a particular lesson, and c) how to design curriculum materials that avoid conflicting messages about how students are to develop strategies. This analysis shows that open curriculum spaces exist in curriculum materials, but not frequently enough (particularly with respect to connections to children's mathematical funds of knowledge) or in the most useful places. In closing, we return to Ball and Cohen's (1996) argument that curriculum materials could serve as a lever for instructional change. Because it is common for school districts to mandate a particular set of curriculum materials, we contend that opening and leveraging third spaces that engage children's MMKB within curriculum materials is essential.

Acknowledgments

This material is based upon work supported by the National Science Foundation under Grant Nos. 0736964 and 1228034. Any opinions, findings, conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of NSF.

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TOWARD DEVELOPING AN INSTRUMENT ASSESSING THE EFFECTS OF STUDENT-GENERATED WORKED EXAMPLES IN CALCULUS

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Student success in first year calculus is instrumental for future success in the areas of science, technology, engineering and mathematics. Students participating in an intervention program for first year calculus students not only encounter more worked examples than non-participants, but also actively engage, through group collaboration, in the production of these examples. Do students in this program develop a higher ability to determine the effectiveness of solutions to calculus problems? We report on the development of an initial instrument requiring students to critique, rate and review proposed solutions based on sample student work to exam items.

Keywords: Assessment and Evaluation, Curriculum, Research Methods

The national push within the United States to produce more students with degrees in science, technology, engineering and mathematics (STEM) has been widely publicized, as have views that calculus often serves as a stumbling block for students pursuing degrees in these areas (Fullilove & Treisman, 1990; Oehrtman, Carlson, & Thompson, 2008; President's Council of Advisors on Science and Technology, 2012; Treisman, 1992). This work aims to gain insight into how to assess whether students can determine valid solutions to calculus problems after significant experiences with worked examples.

We identified a need for an instrument to investigate the question: Do students participating in a structured intervention program requiring collaborative work and peer-review develop a higher ability than non-participants to determine the completeness of solutions to calculus tasks? Although grades provide information about ability to correctly solve an exam item, our interest rests in how students view the work of others and how this may correlate with their performance.

At a large (33,000 students) university in the southwestern United States with 25,000 undergraduates, the Arlington Undergraduate Research-based Achievement for STEM (AURAS) project supports an intervention program for STEM students enrolled in a first-semester calculus course. While working in groups of 3-4 students, participants actively collaborate constructing solutions to challenging calculus tasks and defend their personal beliefs about mathematics as they contribute to group discussions. No studies exist, however, which examine a student's ability to judge the quality of work present in a solution proposed to a calculus exam item.

Worked Examples As Teaching Tools

Students who encounter worked examples catalog them within a personal identification system much like a library; they access their personal library as they work to formulate solutions to new tasks (Schwarz & Hershkowitz, 1999; Sinclair, Watson, Zazkis, & Mason, 2011).

Sweller (2006) refers to schema, or actions that are repeatable and generalizable. When presented with new problems, students possessing an expansive pool of worked examples may access their schema as they interpret their own work or the work of their peers. The reviewer of peer work, with regard to mathematical exercises, becomes an active participant in judging mathematical arguments and processes (Zerr & Zerr, 2011). By the end of the peer review process, students come to an understanding of the acceptability of a proposed solution.

The Emerging Scholars Program Model

Many adaptations of the Emerging Scholars Program (ESP) model exist (Asera, 2001; Duncan & Dick, 2000; Epperson, 1999; Treisman, 1992). This adaptation, called A-ESP, targets first-time, first-semester freshmen majoring in physics, mathematics, chemistry or engineering. Students commit to meeting twice weekly for two-hour workshops led by a graduate student (TA). The TA provides feedback on several tasks each week to encourage and promote exposition in mathematics. Challenging, non-routine tasks drive group interactions.

Students participating in A-ESP are a subset of a larger section of calculus; all students meet for the same fifty-minute lecture three times weekly, attend one fifty-minute TA-led recitation session, and one fifty-minute problem-solving session (lab) co-led by a TA and the instructor. The A-ESP problem sets allow students not only to generate a greater number of worked examples than their peers not in A-ESP but in the same larger section (non-A-ESP group) but also to actively engage in the creation of the solutions for these examples.

Development Of The Instrument

To construct the assessment instrument, we collected sample student work from items on a common midterm exam. At this university, all instructors of Calculus I contribute to both the writing and grading of each common exam in Calculus I. A course coordinator facilitates construction of the exams that consist of both multiple choice (MC) and free response (FR) items with FR items contributing to about 50% of the total exam weight.

Problem Selection

Each instructor writes several MC items and two FR items based upon topics assigned by the course coordinator. The final version of the exam results from a collaborative feedback process. Each instructor grades the same FR item on every exam using a rubric vetted by all instructors.

We reviewed the finalized first midterm (M1) exam FR section and selected three of the five items for inclusion on our instrument. A factor beyond our control for this iteration was that all five FR questions relied heavily on student's algebraic and analytic skills; visualization or other reasoning methods were minimal to non-existent. We then constructed the initial instrument using sample student work collected after M1 exam administration.

Selection and Analysis of Sample Student Work

After item selection, instructors assigned to grade each item assisted us in identifying sample student work by selecting at least three student papers fitting each of the descriptions in Table 1. We only considered student work from those in course sections not included in this study.

Table 1: Descriptions of Student Work Classifications

Classification	Description
Exemplary	Fully complete and robust; may exhibit a novel approach; highly organized; asserted solution is fully supported; may exhibit minor algebraic errors
Mediocre	Disorganized and may present gaps in logic.
Poor	Fail to show sufficient work to support asserted answer; blanks or failure to follow instructions not considered.

We reviewed each solution identified by the instructors and selected combinations of exemplary, mediocre, and poor solutions for each question (see Table 2). In Table 2, Q1PS1 refers to the first item (Q1), first proposed solution (PS1) on the instrument.

Table 2: Items Selected For The Instrument

	Item	Classification of Proposed Solutions
Q1	Let $f(x) = \begin{cases} \frac{\sqrt{x}-1}{x-1} & \text{if } x \neq 1 \\ c & \text{if } x = 1 \end{cases}$. Find the value of *c* that will make *f* continuous at $x = 1$. Justify your result using the definition of continuity.	Q1PS1: Exemplary Q1PS2: Poor Q1PS3: Exemplary
Q2	<u>Use the limit definition</u> of the derivative to find $f'(2)$ if $f(x) = \frac{2}{x}$.	Q2PS1: Poor Q2PS2: Poor Q2PS3: Mediocre
Q3	Write an equation for the tangent line to the graph of $f(x) = \frac{(x-2)e^x}{x^2}$ at the point $(1, -e)$.	Q3PS1: Mediocre Q3PS2: Poor Q3PS3: Mediocre

Student solution Q1PS1 applies the property of the difference of two squares to factor the denominator of $\frac{\sqrt{x}-1}{x-1}$ into $(\sqrt{x}+1)(\sqrt{x}-1)$. This demonstrates fluent algebraic understanding. In Q1PS1, the student states the definition of continuity and fully justifies their conclusion.

In Q1PS2 the student fails to state or use the definition of continuity. The student tests values near $x = 1$ and asserts that $c = 0.5$. The formal definition of continuity is not used. Although a correct answer is proposed, the work shown does not support the claim.

For Q2, stated conclusions are not fully supported by student work. We chose these to gain insight into how students view justification for presented solutions. For example, Q2PS1 is difficult to follow and algebraic steps provide no justification for the correct conclusion. The student uses the definition of limit but the remaining work is nonsensical. Plausibly, the student used another method (the power rule) to find the derivative. This solution is the only one in the group for Q2 that contains a correct conclusion, but shows the least amount of supporting work.

Q2PS3 begins with the proper statement of the limit definition of derivative and shows more detail with regard to algebraic simplification. As the result of distribution error, the student conclusion, $f'(2) = \frac{1}{2}$, is incorrect. The student fails to use equal signs until the final statement.

In Q3PS1 the student work explicitly indicates the need to employ both the product and quotient rules. The student correctly executes the product rule as well as the quotient rule. Almost as an aside, the student shows how they simplified $(xe^x - 2e^x)(2)$; an error is made and the result is given as $-(2xe^x + 4e^x)$. However, the value of the derivative at $x = 1$ is correctly stated as $2e$. Another algebraic error is exhibited when attempting to simplify the equation of the tangent line.

The final proposed solution for Q3 is four lines in length. The student fails to properly employ the product rule and thus concludes that the derivative value at $x = 1$ is $3e$. The student uses this value to find the corresponding equation for the tangent line correctly. This solution exhibits knowledge regarding how to find the equation of a line tangent to the graph of a function at a given point, though errors were made within the solution. This solution was chosen to gain insight into how deeply the different student groups might look in an attempt to understand the student's approach.

Production and Administration

Once the nine proposed solutions were chosen and transcribed, the researchers wrote comprehensive instructions for administration of the instrument. Sixty-eight students completed the instrument; of those, 48 were in A-ESP targeted majors. Only these were analyzed for comparison of the A-ESP and non-A-ESP groups. Students were asked to spend at least 30 minutes working individually on the assignment. They were directed to provide feedback and

comments to the “student” on each solution presented and then rate the effectiveness of each solution as: not effective, partially effective, or fully effective. Students were also asked to consider the group of proposed solutions for each question and indicate which of the three was the best presented solution and which solution was the worst.

Future Modifications and Academic Uses

After the first administration of the instrument and analysis of the data, refinements were made for future comparisons. For example, on the first document students rate and rank the solutions with no explanation requested other than markings made on the proposed solution itself. Many students made rudimentary marks (including simply writing question marks in lieu of any explanation) and in these cases, it was difficult to determine specific reasons for the ranking or rating. To correct this, future instruments included a sheet asking students to explain their ranking and rating of proposed solutions.

This instrument provides researchers a way to gather information regarding student analysis of proposed calculus solutions, and is also a tool for instructors. The instrument was distributed, upon request, to instructors at the culmination of the subsequent semester. Instructors used the instrument as a tool for exhibiting how calculus problems are graded and how acceptable solutions might look. Instructors were asked to offer students time to first review each of the proposed solutions and consider their effectiveness. One instructor provided feedback to the researchers that her students found the process insightful and the instrument provided them the opportunity to consider how difficult it is to determine what a student is attempting to communicate when writing mathematics.

Acknowledgements

This material is based upon work partially supported by the National Science Foundation under grant no (DUE #0856796). Lynn Peterson serves as Lead PI and James Epperson, Ramon Lopez, Kevin Schug and Carter Tiernan serve as Co-PIs of AURAS.

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TEACHER REPORTED IMPACTS ON PRACTICE: USE OF A MODELS AND MODELING PERSPECTIVE IN MATHEMATICS TEACHER EDUCATION

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Researchers have incorporated a Models and Modeling perspective to examine teacher thinking about mathematical topics in teacher education settings. In addition to calls for additional research, questions remain about how this perspective might apply to domains beyond mathematical topics. How can a Models and Modeling perspective advance teacher thinking about what it means to teach a mathematical idea, and how that idea manifests in the classroom? In this paper the researcher will discuss findings from a study that used a Models and Modeling perspective in a teacher education setting to examine teachers' development of quantitative reasoning with a focus on the impacts in the teachers' classroom. The implications of this work contribute to researchers and teacher educators using a Models and Modeling perspective in teacher education.

Keywords: Modeling, Teacher Education-Inservice/Professional Development

Connecting teacher education to practice has long been a challenge for mathematics educators, and one approach to addressing this gap is through research examining aspects of teacher learning and systems of interpretation (Doerr & Lesh, 2003; Zawojewski et al., 2008; Potari, 2012). In response to this challenge, Lesh and colleagues have described how a Models and Modeling approach to in-service teacher education can challenge teachers to develop ways of thinking to help their students while simultaneously documenting their development for research purposes. This approach uses Model Eliciting Activities (MEAs), which are tasks that engage teachers in thinking about realistic and complex problems embedded in their practice in order to foster ways of thinking that can be used to communicate and make sense of these situations (Doerr & Lesh, 2003; Lesh & Zawojewski, 2007). MEAs have contributed to teacher development because these activities make teachers engage in applicable mathematics, consider student reasoning more deeply, and reflect on beliefs about problem solving (Chamberlin, Farmer, & Novak, 2008; Schorr & Koellner-Clark, 2003; Schorr & Lesh, 2003).

While studies have documented how this can be done, the merit of this approach for teacher education is still needed. There is also a need for additional research given the recent demands that education reform place on teacher education programs, such as the Common Core State Standards (Garfunkel, Reys, Fey, Robinson, & Mark, 2011). Teacher educators subsequently need to ensure teachers, both pre- and in-service, are pushed in productive ways to think about the mathematics and its connection to the classroom (Potari, 2012). The purpose of this study was to contribute to the body of work by focusing on the latter of these objectives, as this approach can broaden the perspective of the traditional impact of a Models and Modeling approach on teacher education. In this study I examined the development of in-service teacher models of quantitative reasoning in relation to teachers' classroom practice. The research question was: how did teacher models of how quantitative reasoning develop through Models and Modeling approach in a teacher education course, with specific focus on how teachers communicated their teaching and assessment strategies in their own classroom.

Methods

This study was grounded in a Models and Modeling Perspective, as described by Lesh and colleagues. In addition to being a powerful lens for examining teacher education, a Models and Modeling Perspective also provided guidelines for methods that can be used to provide significant findings given this study's research question. With these methods, a Models and Modeling Perspective offered a framework for understanding teachers' ways of thinking, their development, and provides a mechanism for analyzing and piecing together findings (Koellner-Clark & Lesh, 2003; Sriraman & English, 2010).

The setting for this study was within a master's program in mathematics, where teachers took a combination of mathematics and mathematics education courses over two years. I focused the study on a newly developed mathematics education course in the program, called Quantitative Reasoning in Secondary Mathematics, which was offered in the summer. This course was taught by an instructor familiar with the Models and Modeling perspective. The course content was framed to follow a models and modeling perspective, as the instructor incorporated an MEA aligned with this perspective as 50% of the course grade. In the MEA, 21 teachers worked in six groups to receive feedback about their group's task during the summer from the instructor, each other, and undergraduate students who completed the task. The final, optional, iteration occurred during the fall as teachers implemented their task in his or her classroom. Each feedback cycle prompted an updated iteration of the task and supporting documents that captured how the teachers' ways of thinking developed.

For the purpose of this report, I focus on the four teachers (Table 1) who implemented the task their group created in their own classroom. This allowed me to document the influence the course had on their view of quantitative reasoning with particular focus on how teacher models related to the classroom. Data collection consisted of the iterations of MEA documents generated by the individual teacher, who was sometimes working within a larger group. Using content analysis (Patton, 2002), I adapted Hjalmarson's (2008) analytical tool to identify themes on teachers' communication quantitative reasoning.

Table 1: Participant Description

Participant:	Joyce	Tiffany	Allie	Penny
Group	A	B	B	B
Years teaching	4	3	3	18
Grade bands	9-10	9-10	9-11	10-12
Subjects	Geometry	Algebra	Algebra	Pre-calculus

Findings

The MEA documented how the teacher models of quantitative reasoning developed, and here I summarize four themes relating to how these teachers communicated how quantitative reasoning was framed, taught, and assessed in the his or her classroom. The first theme evident in all teachers' development was an increased value placed on student thinking in the quantitative reasoning tasks. Teachers acknowledged how their thinking about quantitative reasoning developed through receiving student feedback on their task. For example, Joyce's group commented on the student feedback, saying, "did they show us enough work? How do we elicit that out of them? How do we change our questions so we get a little more verification that they know what they're doing rather than just doing it?" The other group also made comments about needing to adapt the questions in their task to better elicit student thinking. These changes were subsequently observed in how teachers in both groups communicated the task should be

framed and implemented in the classroom.

A second theme was that all teachers communicated increased attention towards promoting productive ways of student thinking about quantitative reasoning. Initially, the teachers of both groups were focused on constructing questions that examined students' quantitative reasoning. Through the MEA iterations, teachers began scaffolding questions to elicit answers, and ways of thinking, that the teachers desired. For example, group A split questions into subparts and group B added a guiding questions list for the facilitator. These changes were driven again by the student feedback as student misconceptions were identified through the task. In some cases, questions that were too difficult for students were abandoned; group A's questions on relationships between functions and group B's question about slope defined as a relationship between quantities were such cases. The answers the teachers had listed for expectations were not met, and while they identified student's lack of success, they did not address how to overcome these difficulties.

A third theme was generalization becoming an essential component in their model of quantitative reasoning. This element was manifested in two ways: the first in promoting students to generalize the mathematical ideas being explored in the tasks. For instance, Alice's final reflection said:

> My understanding of quantitative reasoning truly developed during the revision process of our MEA. By being able to take the feedback and discuss it with my group, I was able to really grasp how to incorporate the reasoning component into a given task. It was very neat to be able to take a relatively simple task, such as the plumber scenario, and turn it into an activity that reveals how students are thinking about the problem and also be able to help the students generalize what they learned from our MEA to any function representation.

The second place this theme revealed was the task revision. Group A was especially explicit about how they were generalizing successful strategies, such as pattern recognition, to incorporate in updated versions of the task in order to promote productive student thinking.

The final theme was the difficulty teachers had in acknowledging and accepting multiple answers to quantitative reasoning questions. While both Tiffany and Allie indicated quantitative reasoning and their task should incorporate multiple ways to think about and solve a problem, the actual task and assessment of student work did not reflect these goals. Group A identified this issue and altered one question in the final version in an attempt to incorporate multiple ways of thinking from students, but task as a whole did not reflect these goals. Group B did not comment on the issue, though overall this challenge was prevalent in both groups throughout the MEA.

Implications

The aim of this study was to describe how teacher models of quantitative reasoning were developed in relation to their classroom practice through a models and modeling perspective in teacher education. Common themes from the participant indicated (1) quantitative reasoning was presented more conceptually to students in comparison to teachers' original models; (2) productive student thinking was promoted through targeted decisions in the design and implementation of the task in ways that were not present originally; (3) teachers began seeing quantitative reasoning as a generalized process that appeared in the classroom as more of a habit of mind rather than tied to specific tasks; and (4) quantitative reasoning challenged teachers to consider multiple ways of thinking, one of which being its connection to their classroom.

These themes indicated teacher education that follows a models and modeling perspective could have positive impacts on classroom practice. Teachers were put in positions that challenged their prior models of quantitative reasoning and promoted development in ways that

align with teachers' practice, a characteristic of successful teacher education (Potari, 2012). Identifying areas of difficulty, such as accepting only single ways of thinking, was also an important finding, setting the stage for future work to be done investigating why these challenges occur and how to support teachers in more advanced ways of thinking about the problem. Using this information, teacher educators are better suited to structure teacher education in ways that advance teacher thinking and change teacher practice (English, 2003; Lesh, Middleton, Caylor, & Gupta, 2008).

The documentation of development reported in this study can support both teacher educators and researchers of teacher education to continue considering what it means for teachers to think productively about quantitative reasoning and other mathematical ideas. Using a Models and Modeling approach encouraged teachers to decide what productive ways of thinking are in terms of their students and classrooms. Identifying patterns of development can guide future teacher education efforts taking this perspective and support improvement to address the issues occurring in this study. Finally, this work helps set the stage for the theoretical groundwork for thinking about teacher development using a Models and Modeling perspective.

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EXAMINING PRE-SERVICE TEACHERS' EMERGING ABILITIES TO READ STANDARDS-BASED CURRICULUM MATERIALS

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Remillard (1999) writes that reading is the process most closely connected to how teachers use curriculum materials. In this study we investigated how PSTs read Standards-based curriculum materials by identifying which features PSTs attended to and how they attended to them. Our results indicate that PSTs attended to educative and non-educative features of curriculum materials in educative and non-educative manners. This study has implications for curriculum developers, researchers, and mathematics teacher educators in terms of the design of curriculum materials and supporting PSTs in reading curriculum materials in educative ways.

Keywords: Teacher Education-Preservice, Curriculum, and Elementary School Education

Introduction

There are three key ways that teachers interact with curriculum materials – by reading, by evaluating, and by adapting (Sherin & Drake, 2009). Though each of these modes of interaction is important and interrelated, we posit that without effective *reading* of curriculum materials, teachers are less likely to effectively evaluate and adapt the materials. “Reading is the intellectual process most closely connected to teachers' curriculum decisions. Thus, change in the enacted curriculum must occur through change in how or what teachers read” (Remillard, 1999, p. 338). Little is known however, about how pre-service teachers (PSTs) read curriculum materials. In this study, we investigate *what* features of curriculum materials PSTs attended to while reading *Standards*-based curriculum materials and *how* they attended to those features.

Theoretical Frame

For this study, we draw primarily from two studies conducted by Remillard (1999, 2000). In both studies, Remillard investigated the ways in which two teachers (Catherine and Jackie) interacted with curriculum materials. Framing the Remillard (1999) study was the idea that reading is central to teachers’ curriculum development. Remillard (1999) defined reading in a way that extended beyond its conventional usage:

> Scholars of reading describe it as a dynamic and constructive process that involves interaction between the reader and the text, situated within a particular context (Pearson and Stephens 1994; Rosenblatt 1994)… In this process readers "use their existing knowledge and a range of cues from the text and the situational context in which the reading occurs" to make meaning. (Dole et al. 1991, p. 241, 318)

Remillard points out that reading is not an objective activity conducted in isolation. Instead, it is an interactive process between text and reader where the reader interprets ideas through “prior knowledge, experiences, beliefs, and attitudes.”

In investigating how PSTs read curriculum materials, we were most interested in the educative features of the curriculum materials – those that were meant to promote teacher learning (Davis & Krajcik, 2005). Davis and Krajcik (2005) set forth a set of design heuristics for the ways in which materials could be designed to support teacher learning: 1) “help teachers learn how to anticipate and interpret what learners may think about or do in response to

instructional activities" (p. 5); 2) support teachers' learning of subject matter knowledge including facts, concepts, and disciplinary practices; 3) could help teachers consider ways to relate units during the year (p. 5); 4) "make visible the developers' pedagogical judgments" (p. 5); 5) and promote pedagogical design capacity (the ability to perceive and mobilize existing resources to achieve instructional goals (Brown & Edelson, 2003) (p. 5)). Were PSTs attending to the materials' educative features, and if so, how were they attending to those features?

Methods

Data were collected from 47 PSTs during elementary mathematics methods courses and consisted of written responses to the following prompt - *How does this lesson plan help you to teach this particular concept? In other words, what information does it give you that is most important or helpful? Please be specific.* – after reading five different *Standards*-based curriculum lessons over the course of one semester. Each response was first unitized into differing ideas. Next, we analyzed each unit of data in two ways. First, to determine *what* PSTs attended to in their reading of curriculum materials, we identified the feature (e.g., teaching note, sample student work) they were referring to. The first two authors analyzed the data separately and had 90% reliability. All disagreements were discussed until consensus was reached. Second, to determine *how* PSTs attended to the various features, the first two authors began the coding process by making a first pass through responses to one lesson. In doing this, we considered the literature related to teachers' use of curriculum materials and developed an initial set of five codes – *descriptive, evaluative, educative, interpretative,* and *adaptive*. After several passes through the data, codes were refined and assigned definitions. All data units were assigned a code through a process of focused coding (Charmaz, 2004) by the first two authors. Reliability was 87% and all disagreements were discussed until consensus was reached.

Because we were interested in the educative features, each lesson was analyzed to identify the educative features. The first two authors jointly identified the features (educative and non-educative) within each lesson from the curriculum materials and individually coded the educative features with one of Davis and Krajcik's five high-level guidelines (2005).

Results

To frame our results section, we provide the following table (Table 2). The columns depict the type of feature (educative or not) while the rows depict the way the feature was read (educatively or not). Thus, there are four different combinations of curriculum features versus reading. We next provide examples from the data to illustrate each of the four quadrants.

Table 1

Educative Feature Read Educatively	Non-educative Feature Read Educatively
Educative Feature Read Non-Educatively	Non-educative Feature Read Non-Educatively

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Educative Features Read Educatively

The first quadrant indicates that there were educative features in the materials that were read educatively by PSTs as in the following example (all names are pseudonyms).

<u>*Mandy's response:*</u> One part that was helpful to me was the whole class activity where students are coming up with their formula for area of a triangle. It is so easy for teachers to

just tell students what a formula is, but allowing students to attempt to put it together is more meaningful to their learning because they are the ones putting their experiences and ideas out there…

Mandy attended to a section of text titled "Developing a Formula for the Area of a Triangle" in a 4th-grade *Everyday Mathematics* lesson (UCSMP, 2007). We considered this section of text educative because the curriculum developer's pedagogical judgments were made visible. Mandy's response indicated that she learned a particular pedagogy around how to facilitate a lesson with this content. Instead of telling students the formula, teachers can have students "put it together" instead. By putting it together, it seemed that Mandy was referring to an aspect of the task that entailed taping two triangles, with the same area, together to form a parallelogram.

Non-educative Features Read Educatively

Non-educative features of the materials serve a function (e.g., describing a task) in supporting teachers, but their primary role is not teacher learning. In a few instances, however, these non-educative features were read educatively as in the following example,

Allie's response: The lesson also did a good job of not only making the problems concrete and visual for the students, but also for the teachers, such as the breakdown of the problems on page 30.

Allie's response indicated that she attended to the task educatively. The task is meant to be educative for students, but Allie found the representations of two-digit numbers useful for her own content knowledge understanding.

Educative Features Read Non-educatively

In many cases, the educative features of the lessons were not read educatively (i.e., in ways that supported teacher learning). We provide examples for all these response types below.

Edith – Description: As a teacher, this lesson helps to teach the relationship between the area of a triangle and a parallelogram by giving step-by-step instructions with corresponding worksheets.

In the response above, Edith described the text as "step-by-step" directions, which means that she read the text as telling the teacher what to do. She identified the ways in which this feature could help the teacher support student learning, but did not describe any learning of her own.

In some cases, PSTs made evaluative comments about an educative feature:

Megan – Evaluation: The transparency where it has boxes labeled Picture, Strips of 10, Singles, # Stickers, and Equation; was really great.

Megan attended to an important feature of the *Stickers: A Base-Ten Model* lesson from *Investigations* (TERC, 2008), but simply indicated that this feature was great.

In several instances, the educative features were interpreted by PSTs. That is, they made an interpretation of why a particular feature was included in the materials or why it was useful:

Cleo – Interpretation: An aspect of this lesson that I found to be very positive is the fact that it stated, "Most of you probably remember Sticker Station from second grade." That is an important piece of information to note, because that means that right off the bat, the students will understand the context of the problem and be able to focus on the actual concept trying to be taught rather than worrying about the type of problem.

Cleo referred to a teaching note in the Stickers lesson (TERC, 2008), which provides a means for teachers to relate units across grade levels. Cleo interprets this information to mean that students will understand the context of the problem allowing them to focus on the mathematical concept.

PSTs also identified educative features that supported lesson adaptation:

James – Adaptation: The lesson plan gives many suggestions after each section. Teachers who want to use a certain material sometimes have difficulty coming up with a relevant or integrated activity. By giving examples, the teacher is able to use these exactly, or better, use the examples to come up with their own to better fit their unique class.

James attended to the example problems given in *500 Hats* (UIC, 2008) and discussed the difficulty teachers may have adapting an activity that is relevant for their students.

Non-educative Features Read Non-educatively

Last, there were non-educative features read non-educatively. Most interesting in these cases were the interpretations.

Cynthia-Interpretation: The beans make the problem more visual and hands on for students. They can actually see how counting by larger amounts of beans is faster than counting by ones. The lesson plan tells me whether or not the students are ready for multiplication and division.

Cynthia attended to the jar of 172 jellybeans in the *Counting 172* lesson (UIC, 2008) and made an interpretation as to whether or not students would be ready for multiplication and division based on their representation of 172.

Discussion

Ball and Cohen (1996), Davis and Krajcik (2005), and others have suggested that curriculum materials might be designed in ways that are educative for teachers as well as students. However, an assumption underlying these suggestions has been that teachers will read features that are designed to be educative in educative ways. In other words, they will understand that the information provided in curriculum materials is intended for their own learning, as well as their students' learning, and they will read the features from that perspective. In this study, we were interested in whether this assumption held true for PSTs. The results were mixed. In some cases, PSTs did read educative features in educative ways, but, in many other cases, they read educative features in non-educative ways. The same could be said for non-educative features.

Acknowledgement

This work was supported, in part, by the National Science Foundation under Grant No. 0643497 (Corey Drake, PI). Any opinions, findings, conclusions, or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

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UTILIZING COGNITION-BASED THEORY TO IMPACT PRE-SERVICE ELEMENTARY TEACHERS' MATHEMATICS CONTENT KNOWLEDGE AND BELIEFS ABOUT TEACHING

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The question of how best to prepare future mathematics teachers has no clear answer. This research project investigated the impact of integrating learning progressions of children's mathematical thinking into a mathematics content course for pre-service elementary teachers. This paper reports on the finding that pre-service elementary teachers demonstrated a substantial shift towards constructivist beliefs about mathematics teaching and learning, as well the development of more conceptual and accurate analyses of children's mathematical thinking.

Keywords: Teacher Beliefs, Teacher Education-Preservice, Teacher Knowledge

Introduction and Literature Review

There is a significant body of research investigating the knowledge and beliefs that can help facilitate effective mathematics instruction. Carpenter, Fennema, Peterson, Chiang, and Loef (1989) found research-based knowledge about student thinking to have a significant impact on in-service teachers' beliefs about mathematics teaching and learning as well as student achievement. While much is conceptualized about the form and type of mathematical knowledge and beliefs necessary for effective teaching (Ball, Thames, & Phelps, 2008), it is important to investigate methods designed to impact pre-service teacher growth and development.

For this research project, pre-service elementary teachers' (PSTs') mathematical knowledge and beliefs about teaching were explored in the context of an undergraduate mathematics content course intended for future teachers. While the focus of the course was mathematics content typically taught in the Kindergarten through third grade curriculum, the approach to teaching mathematics content was unique. PSTs tend to approach undergraduate mathematics content courses designed to address knowledge of elementary mathematical content with a disinterested attitude. Appealing to PSTs' interest in young children while simultaneously reaching the teacher educator's goal of increasing their understanding and desire to teach elementary mathematics content has been found to positively impact their beliefs about teaching, learning, and mathematics as well as improve their mathematical knowledge (Philipp et al., 2007).

In consideration of the above results with in-service and pre-service teachers as well as the emphasis on learning progressions in the design of the Common Core State Standards for Mathematics (CCSSM), we chose to integrate learning progression materials from the Cognition-Based Assessment (CBA) research project into a required mathematics course for PSTs. The CCSSM were designed in part based on learning progressions of children's thinking, or the typical paths that children follow in their development of a mathematical concept. The CBA project, in particular, designed learning progressions materials (Battista, 2012) to help teachers 1) Assess how their students understand mathematics, 2) Understand the different strategies that students use to solve mathematics problems, 3) Understand the typical progression children's thinking follows, and 4) Design instruction based on their students' current understanding of mathematics. CBA materials have been found to be an effective means by which to promote teachers' development of mathematical knowledge for teaching (Harrison, 2012).

Methodology

Considering the complex construct of teacher knowledge and teacher beliefs, a variety of data sources, both qualitative and quantitative, were utilized.

Design of the Study

Participants included 117 PSTs, all enrolled in their first, of three, mathematics for teachers course at a mid-sized Midwestern University. Content objectives in the course focused on problem solving, number theory, place value, and the operations of addition, subtraction, multiplication, and division of whole numbers. In order to discuss and address specific areas of content such as place value and the previously mentioned operations, PSTs were asked to consider written student work and assess the strategy as well as what understanding was demonstrated. Much of the student work used in class came from the CBA materials. PSTs were also asked to assess the CBA level of understanding demonstrated by the student work as described in the CBA materials. Additionally, PSTs completed a series of two interviews with a K-2 student where they posed mathematical problems for the student to solve and assessed the student's understanding using CBA.

Instruments and Data Collection

All participants completed the 18-item Likert-scale Mathematics Beliefs Scale (MBS, Capraro, 2001) survey at the beginning and end of the 15-week course. The MBS scale consists of three subscale factors: (1) beliefs about how children learn, (2) the role of the teacher in sequencing of teaching both computational and application skills, and (3) relationships between teaching computational skills and problem solving skills. Responses to each item ranged from Strongly Agree (6) to Strongly Disagree (1). As a result, the instrument did not allow the PSTs to be "neutral" in their beliefs. Additionally, participants provided qualitative responses to prompts asking them to describe and assess understanding for written student work on each of the four operations (Appendix). These responses were collected at the beginning and end of the 15-week semester as well.

Results

Understanding mathematics content beyond basic comprehension assists teachers in their ability to interpret and utilize written and verbal student work to inform instructional moves. It is also important for teachers to hold the belief that children's thinking is valuable for informing their understanding of what children know and understand. We investigated PSTs' changes in beliefs about mathematics teaching and learning as measured by the MBS, as well as their interpretations of children's mathematical thinking on a sample of student work related to addition, subtraction, multiplication and division strategies.

Looking globally at the quantitative MBS data, the PSTs' beliefs about mathematics teaching and learning changed from an average total of 63.61 on the pre-test measurement to an average total of 78.9 on the post-test measurement. To contextualize this numeric shift, a 63.61 on the pre-test corresponds to an average response of 3.53 per item, or an overall "neutral" average.

The average score of 78.9 on the post-test corresponds to an average response of 4.38 per item. This represents an average shift of .85 per question. Students responding in this range on the MBS were characterized as being "constructivist" in their beliefs (Capraro, 2001). These numbers indicate a substantial global shift from a neutral perspective to a more constructivist way of thinking for the PSTs.

Looking at the main factors associated with the MBS, an average shift of responses on Factor 1 from a 3.12 to 4.04 per question, or a .92 average change was observed. For responses on Factor 2 we observed a shift from a 3.37 to a 4.33 per question, or a .96 average change. Finally, responses on Factor 3 shifted from a 4.17 to a 4.73 per question, or a .56 change. Ultimately, the average PST moved from neutral, to the "constructivist" range in all three factors.

The qualitative data on the PSTs' interpretations of student work was analyzed using a grounded approach for common themes and codes. The PSTs were given samples of student work (Appendix)

relating to addition, subtraction, multiplication and division. Results from responses to each will be discussed separately.

The student work for addition was analyzed for PSTs' discussion of place value. To operationalize this code, it was determined that it was insufficient for PSTs to simply describe the steps involved in the student's work to be considered a 'place value' code. Rather, PSTs needed to explicitly mention how the student handled the 'tens' and 'ones' places, and the consequences of his/her error on the resulting place values in the answer. On the pre-test, 30 PSTs (25.6%) met this threshold, while 81 (69.2%) did so on the post-test.

The student work for subtraction was analyzed for correctness, as well as conceptual description of the student's strategy. On the pre-test 25 PSTs (21.4%) believed the student to be incorrect in his/her thinking, while 71 (60.7%) believed the student to be correct. On the post-test only 3 PSTs (2.5%) believing the student was incorrect and 105 PSTs (89.7%) believing the student was correct. In order to be considered a conceptual description of the student thinking, PSTs needed to describe where the numbers in the student's algorithm came from, and why they were used appropriately. Only 55 PSTs (47%) were able to meet this threshold on the pre-test, while 99 PSTs (84.6%) were able to conceptually describe the student thinking on the post-test.

The student work for multiplication was analyzed for correctness, as well as conceptually accurate descriptions of the partial products approach to multiplication. It was not necessary that the PSTs use technical language such as 'partial products' or 'expanded algorithm' to be included in this category, but descriptions of the student's algorithm needed to include the conceptual characteristics (e.g. - implicitly mentioning distributive property). 31 PSTs (26.5%) admitted that they did not understand the student work on the pre-test, while only 1 (0.8%) admitted so on the post-test. The number of PSTs believing the student's work was correct increased from 79 PSTs (67.5%) on the pre-test to 102 (87.2%) on the post-test. Finally, the number of PSTs mentioning conceptual elements of the strategy increased from 82 PSTs (70.1%) on the pre-test to 100 PSTs (85.5%) on the post-test. Interestingly, more PSTs were able to describe the strategy reasonably well on the pre-test than the number of PSTs who stated the student was correct.

The results for the student work for division were almost identical to those just described for multiplication. 29 PSTs (24.7%) on the pre-test admitted to not understanding the student work at all, while only 3 PSTs (2.6%) did so on the post-test. 76 PSTs (65.0%) believed the student to be correct on the pre-test, while 94 PSTs (80.3%) did on the post-test. The additional theme for division related to whether or not the PSTs effectively connected the answer given by the student to the process of repeated subtraction. 72 PSTs (61.5%) were able to connect the student's answer to the strategy on the pre-test, while 102 PSTs (87.2%) were able to on the post-test. Again, there were more PSTs, this time on the post-test, which accurately described the strategy of the student than stated the student was correct.

Discussion

The results on the MBS indicate a shift in thinking for many of the PSTs, moving towards a "constructivist" belief about the teaching and learning of mathematics. While the beliefs instrument measured 'espoused' beliefs outside of the context of actual teaching scenarios, the changes in PST responses on the pre and post survey of children's thinking indicated that many of the PSTs developed the ability to more deeply analyze mathematics.

The PSTs demonstrated clear improvement in evaluating student work not simply for correctness, but for conceptual elements such as place value understanding. What was especially encouraging is that while many of the PSTs readily admitted on the pre-test that they did not understand several of the children's strategies, virtually none felt the need to on the post-test. Whether or not this can be directly attributed to exposure to children's strategies in the context of the CBA learning progression materials is unknowable, but it is encouraging nonetheless to know that PSTs can develop more

constructivist beliefs and demonstrate improvement in their ability to analyze mathematical work within the context of a mathematics for teachers content course. This finding supports Philipp et al.'s (2007) hypothesis that building off of PSTs' inherent caring about children, by making CBA learning progressions a component of the mathematics for teachers content course, can lead to profound gains in constructivist beliefs about teaching as well as understanding of children's mathematical thinking.

Acknowledgment

Much of the work described in this paper was supported by the Center for Excellence in Teaching and Learning Scholarship of Teaching and Learning summer grant program funding at the University of Wisconsin-Eau Claire.

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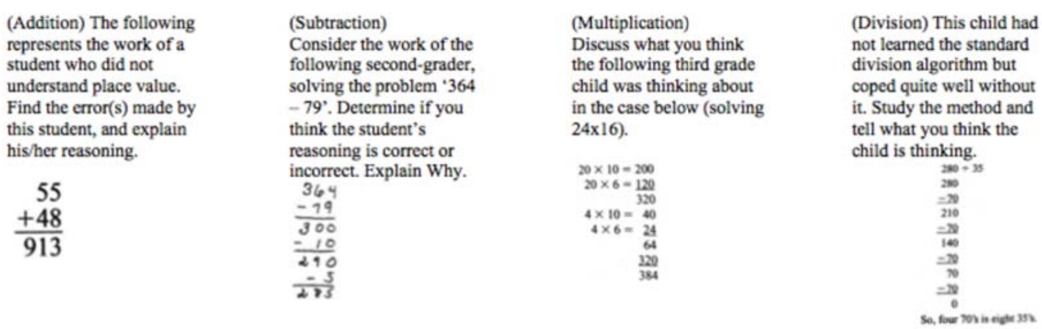

STATE OF THE RESEARCH ON ELEMENTARY PROSPECTIVE TEACHERS IN UNIVERSITY MATHEMATICS CONTENT COURSES: 1990-2012

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This paper describes an investigation of the state of the research focusing on elementary prospective teachers completing university mathematics content courses during the time period of 1990-2012. The researchers searched two general sources for relevant empirical articles: electronic databases and targeted journals with high scholarly regard in the fields of mathematics education and teacher education. The publications were analyzed for predominant research methods employed. This research is the first step in a multi-step project, with additional evaluation focusing on who conducts the research, interventions and outcomes investigated, and major findings, among other factors.

Keywords: Elementary School Education, Teacher Education-Preservice, Teacher Knowledge

Historical Perspective and Background

During the 1970s, National Assessment of Educational Progress (NAEP) published its first two reports (1972, 1975) revealing disappointing results for student achievement in mathematics in the U.S. Subsequently, the National Council of Teachers of Mathematics (NCTM), with support from the National Science Foundation (NSF), initiated the Priorities in School Mathematics (PRISM) project to collect research-based evidence illuminating differences between enacted and desired practices in K-14 mathematics education. An outcome of this project included recommendations published in *An Agenda for Action: Recommendations for School Mathematics in the 1980s* (NCTM, 1980). Recommendation #1 was that *problem solving be the focus of school mathematics*. This call for change in the focus of mathematics curriculum also raised awareness of the need for teachers who were better prepared to teach mathematics. While the *Agenda* recommended stringent standards for teaching mathematics (Recommendation #3), little was offered about how to reach these goals for teachers.

Following publication of *An Agenda for Action* mathematics education research experienced a significant paradigm shift. Prior to that time Kilpatrick (1992) claims "mimicking of research designs in the natural sciences, seemed to lead [*mathematics education*] researchers to conduct studies that were methodologically impeccable but conceptually barren" (p. 31). The shift in research toward increased use of qualitative methods allowed for new avenues of research, producing emergent theories and inquiry on important features of teacher effectiveness (e.g., learning mathematics, affect related to mathematics, and beliefs about mathematics teaching and learning). Mathematics educators began to consider teacher change and teacher development in response to recommendations grounded in this research. Effective implementation of reformed curriculum would necessitate changes in teachers' attitudes, beliefs, and knowledge.

While the areas of beliefs, affect, and learning received significant attention during the 80s, it was not until the end of that decade that teacher content knowledge in mathematics began to gain attention. This was not a totally new area of concern; in fact, in Begle's (1979) review of the empirical literature on research in mathematics education through 1976, he identified teacher knowledge of mathematics as one of two significant teacher variables impacting student learning, with the second being teacher affective characteristics. In this early interpretation, content knowledge was almost exclusively evidenced by completion of mathematics courses or content knowledge tests. Begle concluded that at a certain point completion of more mathematics courses by teachers "contributes nothing to student achievement" (p. 51).

Nonetheless, in the 1990s a few researchers (e.g., Ball, 1993) continued to explore the content knowledge needed by elementary mathematics teachers; and when Ma (1999) introduced the idea of a profound understanding of fundamental mathematics, a fuller realization of the complexity of content knowledge needed for elementary teaching gained momentum in the U.S. Policy documents developed by panels supported by the U.S. Department of Education also highlighted the critical role of the teacher and the need for modifications in teacher preparation (for example, Knowing and Learning Mathematics for Teaching, 2001).

The new millennium brought fresh insights into understanding teachers' mathematical content knowledge, specifically the *specialized content knowledge* (SCK) needed for effective teaching of mathematics (Ball & Hill, 2009). However, 20-plus years after NCTM's publication on reformed mathematics education, *Curriculum and Evaluation Standards for School Mathematics (1989),* most elementary teachers continue to be prepared as generalists (Tatto, Lerman, & Novotna, 2010), with little attention to their mathematics content preparation as compared to their secondary education counterparts.

With adoption of the Common Core State Standards, there is an even greater need for appropriate content preparation in mathematics for elementary teachers. Moreover, given the substantial amount of time that has passed since the NCTM first proposed changes, it is disturbing there has not been more emphasis on mathematical content preparation.

Purpose

The purpose of this investigation was to review the state of the research on the mathematical content knowledge preparation of prospective elementary teachers since publication of the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989). Specifically, the investigation focused on articles published from 1990 to 2012 focusing on prospective elementary teachers (i.e., undergraduate elementary education majors), either completing university mathematics content courses or "mathematics for teachers" content courses. As described, the need for the study is self-evident: too many elementary teachers continue to lack profound understanding of fundamental mathematics and the Common Core State Standards require a high level of mathematical competence. Given the historical context, a pause to review the body of research on this issue is appropriate.

Methods

A research team comprised of five researchers engaged in a comprehensive assessment of the literature focusing on elementary prospective teachers completing mathematics content courses (to include "mathematics for teachers" content courses) since publication of the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989). The time frame for searching for relevant articles was from 1990-2012. We limited our search to publications on mathematics

content preparation in university courses. We did not include mathematics teaching methods or pedagogy courses. We also limited our search to initial teacher preparation rather than studies of practicing teachers. We conducted both an electronic and manual search.

After two searches with limited results, we expanded our electronic search to include ERIC, PSYCH Info, Academic Search Complete, Professional Development Collection, and Psychology and Behavioural Sciences Collection databases using all combinations of the following key words (elementary education, elementary teachers, mathematics, undergraduate, mathematics courses, pre-service teachers, prospective teachers, content courses, and early childhood education). This yielded a total of 53 publications for the time period 1990 – 2012. Results were as follows: Australian Journal of Early Childhood (1), Early Childhood Education Journal (1), Educational Research for Policy and Practice (1), Educational Studies in Mathematics, (1) International Commission on Mathematics Instruction, (1) International Handbook of Mathematics Teach Education (1), International Journal of Mathematics Education in Science and Technology (2), Issues in Teacher Education (1), Issues in Undergraduate Mathematics Preparation of School Teachers (14), Journal of Research in Mathematics Education (7), Journal of Engineering Education (1), Journal of Mathematical Behaviour (1), Journal of Mathematics Teacher Education (6), Knowledge Management and Dissemination (1), Mathematics Teacher Education and Development (3), National Council on Teacher Quality (1), Notices of the American Mathematical Society (1), PRIMUS (2), School Science and Mathematics (6), Statistics Education Research Journal (1), and The Institution of Engineering and Technology (1).

Two researchers conducted a manual search of 11 journals considered to have high scholarly regard in the fields of mathematics education and teacher education. The journals included in this search were: American Educational Research Journal (3), Cognition and Instruction (0), Educational Studies in Mathematics (4), Elementary School Journal (1), Journal of Mathematical Behaviour (2), Journal of Mathematics Teacher Education (8), Journal for Research in Mathematics Education (8), Journal of Teacher Education (5), School Science and Mathematics (9), Teachers College Record (0), Teacher and Teaching Education(0) for a total of 40 articles.

As expected, there was some overlap of articles between the electronic database and manual searches. After eliminating duplicate articles, there were a total of 80 articles in the pool. Since the original search cast a very wide net, the next analysis sorted the articles as research or as *not* research (e.g., theoretical, positional, etc.). Of the 80, 38 were empirical and included in the final pool. (See Table 1.)

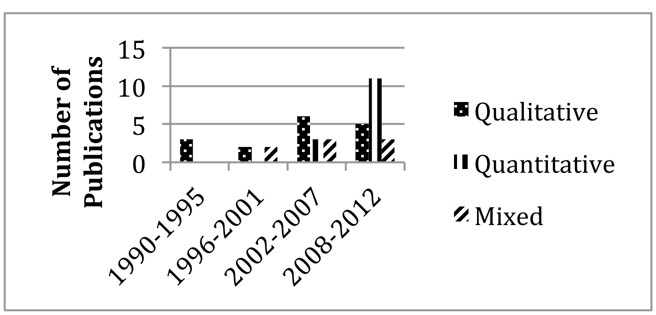

Figure 1: Research Methods by Time Periods.

Discussion and Next Steps

The investigation reported here is only the first product in a multi-step project to describe the state of the research with elementary prospective teachers as participants completing mathematics content courses. As Bekdemir (2010) points out, many elementary teachers today still have high levels of mathematics anxiety. Disconcertedly, many do not feel prepared to meet the challenges of teaching the rigor and conceptual depth of the mathematics in the Common Core State Standards. Since NCTM's first publication of the *Standards*, examination of university courses that build elementary prospective teachers' content knowledge remains woefully inadequate. More research is clearly needed in order to understand the issues, challenges, and possibilities associated with developing elementary teachers' content knowledge.

For the next pool of articles, the following questions will be addressed: Who conducts the research? What interventions and outcomes are investigated? What are the major findings? What claims of impact are made? It is anticipated at the conclusion of the analyses, we will be able to discuss contributions and limitations of the entire body of research, as well as implications and suggestions for future research.

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THE DEVELOPMENT OF ATTITUDES ABOUT MATHEMATICS DURING PRESERVICE TEACHER EDUCATION

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This paper reports on the development of attitudes about mathematics among preservice elementary teachers in relation to their experiences as K-12 learners of mathematics. Using a combination of the Rasch Rating Scale Model and traditional parametric analyses, results indicate that significant changes in attitudes occurred over the duration of mathematics methods coursework among those who entered the course with "unfavorable" K-12 mathematics experiences. Further, these changes can be explained, in part, by particular experiences in mathematics methods coursework.

Keywords: Affect, Emotion, Beliefs, and Attitudes; Teacher Beliefs; Teacher Education-Preservice

Objectives of the Study

This paper reports findings from two administrations of the Mathematics Experiences and Conceptions Surveys ([MECS] Jong, Hodges, & Welder, 2012) during the fall 2012 semester. Preservice elementary teachers (PSTs) enrolled in mathematics methods coursework first completed MECS-M1, a Likert item survey administered at the beginning of mathematics methods coursework measuring, in part, attitudes about mathematics and K-12 experiences in mathematics. Second, PSTs completed MECS-M2, a Likert item survey administered at the end of a mathematics methods course, which contains the same set of attitudes items, but a different set of experiences items focused on mathematics methods coursework and corresponding field experiences (see Table 1).

Table 1: Sample MECS Items

Item	*Construct*	*Instrument*
I like mathematics.	Attitudes	MECS-M1
I think mathematics is boring	Attitudes	
I enjoy solving mathematics problems	Attitudes	
I struggled with mathematics as a K-5 student	K-12 Experiences	
My experiences learning mathematics were generally positive as a K-5 student.	K-12 Experiences	
My mathematics methods course emphasized selecting cognitively demanding mathematics tasks.	Methods Experiences	MECS-M2
My mathematics methods course emphasized assessing student understanding of mathematics.	Methods Experiences	
My field experiences this semester generally reinforced what I learned in my mathematics methods course	Field Experiences	

The overall aims of this research were to (a) determine if MECS could capture the link between K-12 experiences and entering attitudes; (b) investigate what, if any, changes occur in attitudes, particularly among those with relatively negative K-12 experiences and; (c) explore whether particular types of experiences in mathematics methods coursework help explain changes in attitudes. While a number of surveys exist that measure attitudes, the inclusion of experiences items contextualized to specific points in teacher education provides a unique way of understanding the development of PSTs' attitudes. Consequently, our focus adds to the knowledge base in mathematics teacher education by observing the relationship between K-12 experiences and attitudes, alongside factors within teacher education that influence the advancement of attitudes about mathematics.

Perspectives

The attitudes construct in MECS is defined as judgments made about places, events, people, or objects. To a large extent, attitudes are either positive or negative (Breckler & Wiggins, 1992). Attitudes change more quickly and are less cognitive than beliefs (Philipp, 2007), thus evidence of change can be seen in shorter time increments than constructs such as beliefs, which tend to be more entrenched. Attitudes about mathematics among preservice elementary teachers are strongly linked to their experiences as students of mathematics in K-12 classrooms. These attitudes become a part of the *apprenticeship of observation* (Lortie, 1975), framing how PSTs make meaning of their mathematical experiences in teacher education. Attending to attitudes is significant since many students develop negative attitudes towards mathematics during K-12 experiences (Ignacio, Blanco Nieto, & Barona, 2006). Teacher educators often underestimate the effects past experiences have on PSTs' attitudes, which can overshadow the role teacher education has in developing positive attitudes (Feiman-Nemser, 1983).

Mathematics methods courses aligned with reform recommendations have shown promise in positively influencing PSTs' attitudes towards mathematics (Hodges & Jong, 2012; Conner, Edenfield, Gleason, & Ersoz, 2011). Many studies of PSTs' development of attitudes in mathematics during methods course experiences have focused on examinations of mathematics autobiographies (Drake, 2006; Ellsworth & Buss, 2000). These studies suggest that making explicit PSTs' entering attitudes provides opportunities to develop positive attitudes towards mathematics. However, gaps remain in detailing the connection between changes in attitudes and the types of mathematics experiences occurring in methods courses. Consequently, this study investigated the relationship between K-12 experiences and entering attitudes, changes in attitudes among those with relatively negative K-12 experiences, and the extent to which changes in attitudes can be explained by particular types of methods course experiences.

Methods

The data presented here include seventy-seven PSTs enrolled in mathematics methods courses at two universities in the United States during fall 2012. MECS-M1 was administered during the first week of class in each of the mathematics methods courses while MECS-M2 was administered during the final week of each of the same courses. The Rasch Rating Scale Model ([RRSM] Andrich, 1978) was selected to create a common metric and common rating scale, as the RRSM is well-suited for polytomous data that contain the same number of possible response options. Through an iterative process of estimation, items were calibrated first, followed by person estimates. MECS-M1 was selected for anchoring. Since attitudes estimates from MECS-M1 were anchored, the rating scale thresholds from MECS-M1 were anchored using additional coding in a Winsteps (Linacre, 2012) control file. The attitudes construct from MECS-M2 data was then reanalyzed with the item and rating scale in place. This resulted in new person calibrations for each individual in the data set. A new series of quality control checks to ensure the attitudes construct was stable, fit statistics acceptable, and reliability and separation values comparable took place to ensure the anchored MECS-M2 analyses were valid. Once these interval scales were created, traditional parametric analyses were conducted on the logit values.

To determine whether MECS instrumentation captured the relationship between K-12 experiences and entering attitudes about mathematics, a Pearson's r was computed. Of particular interest are those PSTs which report negative K-12 experiences, as these individuals are likely to have more negative attitudes towards mathematics. Twenty-eight PSTs (36%) had logit values less than zero and were selected for additional analyses. A t-test was used to determine whether there was a statistically significant change in attitudes over the duration of mathematics methods coursework. Further, multiple regression models were used to determine whether particular experiences during the mathematics methods courses helped to explain any changes in attitudes.

Results

To determine whether MECS-M1 captured the relationship between K-12 experiences and entering attitudes, a Pearson's r was computed using the logit values produced from the RRSM. The variable

K12_exp had a strong positive correlation with the variable *attitudes_pre*, $r(75) = .605$, $p < .01$. A cutoff *K12_exp* logit score of zero was selected to identify cases with relatively "unfavorable" K-12 experiences in mathematics. This cutoff score resulted in the selection of 28 of 77 cases (36%) with *K12_exp* scores less than zero. A single sample *t*-test was calculated using the variable *attitudes_chng*, representing the difference in logit values (*attitudes_post* – *attitudes_pre*). There was a statistically significant gain (M = 0.781, SD = 0.252) in attitudes among those with unfavorable K-12 experiences, $t(27) = 3.098$, $p < .01$.

A backward elimination regression model technique was used to explain changes in attitudes. The model was loaded with the constructs of mathematics methods pedagogy (*MM_ped*), materials (*MM_mat*), environment (*MM_env*), and students (*MM_stu*) – constructs used to identify foci in methods course experiences. Additionally, a field experience (*FE*) construct was included. When *attitudes_chng* was predicted it was found that *MM_stu* (β = -0.595, p =.089) and *MM_ped* (β = 1.440, p = 0.022) were the best predictors. The overall model fit was $r^2 = .449$. While this regression model was limited, which may have been due, in part, to the small sample size, it identified two potentially influential factors. Methods courses that focus on mathematics pedagogy and strategies to meet the needs of a variety of students seemed to have a positive change in PSTs attitudes.

Discussion

The first aim of this research was to determine if MECS could capture the link between K-12 experiences and entering attitudes. Existing research linking PSTs' K-12 experiences to attitudes about mathematics has primarily included the use of qualitative data to make such claims (Ball, 1990; Ellsworth & Buss, 2000). Results of the Pearson *r* computation indicated that MECS instrumentation was capable of measuring this link via Likert scale items. This is significant, since the nature of MECS instrumentation allows for its use in large-scale research projects across a variety of teacher education contexts to observe both attitudes and K-12 experiences.

Mathematics experiences in teacher education are of particular importance to those with negative entering attitudes towards mathematics. These negative attitudes influence PSTs' perception of teacher education experiences, the knowledge and skills developed, and ultimately the instructional practices used with students. As a practical matter, those with negative attitudes also represent PSTs with the greatest potential for change during methods course experiences. The lowest 36% of participants were selected as cases representing relative negative attitudes. The single sample *t*-test demonstrated significant gains in attitudes among those with negative attitudes. This is a promising finding as the short duration of a single mathematics methods course provides little time to demonstrate changes in attitudes. Critically important is the extent to which these changes can be built upon in subsequent teacher education experiences. Further, given the link between beliefs and instructional practices, it remains to be seen if changes in attitudes serve as a precursor to changes in beliefs.

The regression model provided limited, yet important insights into methods course experiences which influence changes in attitudes. While the overall model fit was relatively weak, methods courses which focus on pedagogical strategies appear to provide opportunities for PSTs to change attitudes. The pedagogy subscale includes items focused on the use of multiple representations of mathematics, developmentally appropriate and varied instructional strategies, and assessing students' understanding of mathematics. These foci likely provide opportunities for PSTs to engage in mathematics content at the grade levels they are expected to teach, with emphasis on conceptual understandings. Other subscales, such as instructional materials and learning environments, while likely critical, do not appear to have the same influence on changes in attitudes among those with entering negative attitudes.

While surveys are limited by, among other things, reliance on self-reported data, it was promising to see links to K-12 experiences and changes during mathematics methods coursework. Explaining changes in attitudes in relation to the nature of methods course experiences remains a challenge. However, the connections made here provide a glimpse into how PSTs who demonstrate significant change in attitudes characterize their experiences in methods courses. An important next step in this line of inquiry is to include more longitudinal data to observe whether changes in attitudes are sustainable, lead to changes in beliefs, and can be furthered through additional teacher education experiences.

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EXAMINING PRESERVICE TEACHERS' CONCEPTIONS ABOUT TEACHING MATHEMATICS FOR SOCIAL JUSTICE

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The purpose of this study was to examine elementary preservice teachers' conceptions of teaching mathematics for social justice before and after completing a mathematics methods course. Preservice elementary teachers (n = 230) enrolled in mathematics methods coursework at three universities across the United States described what it meant to teach mathematics for social justice as an open-ended question on the Mathematics Experiences and Conceptions Surveys (MECS). We used open coding to examine responses to discover patterns and themes that emerged from the data. Findings suggested that preservice teachers: 1) were aware of a the range of themes consistent with conceptualizations of teaching mathematics for social justice in the literature, and 2) enter mathematics methods courses with promising ideas and are able to make important connections between teaching mathematics and social justice

Keywords: Teacher Beliefs; Teacher Education-Preservice; Equity and Diversity

Theory and Objectives

Teaching for social justice is a critical pedagogy that is used to empower students to be social agents in the world they live. Several educators and researchers have advocated the importance of preparing teachers to teach for social justice in teacher preparation programs to enrich the learning of all students by challenging inequities (Cochran-Smith, 2010). Gates and Jorgensen (2009) presented a framework with three levels of understanding social justice beginning with fairness and equity, to recognizing structural inequalities, to taking a radical approach to address structural inequalities. We claim that teaching mathematics for social justice provides opportunities for all students "to learn rigorous mathematics in culturally specific, meaningful ways that seek to improve the economic and social conditions of marginalized individuals and groups, and that work toward[s] reduc[ing] deficit-oriented beliefs about who is or is not 'good' at mathematics" (Leonard & Evans, 2012, p. 100).

Although mathematics educators have made considerable efforts to understand social justice within the contexts of mathematics teaching and learning (de Freitas & Zolkower, 2009;Wager & Stinson, 2012), these efforts have not examined preservice teachers' conceptions of what it means to teach mathematics for social justice. The research questions underlying this study are:

(1) What are elementary preservice teachers' conceptions about teaching mathematics for social justice?

(2) To what extent do elementary preservice teachers' conceptions of teaching mathematics for social justice change after completing an elementary mathematics methods course?

Methods and Data Sources

To examine preservice teachers' (PSTs) conceptions about teaching mathematics for social justice, we examined open-ended responses from the Mathematics Experiences and Conceptions Surveys ([MECS], Jong, Hodges, & Welder, 2012). MECS are a set of instruments designed to examine teachers' conceptions about mathematics teaching and learning over time. The instruments consist primarily of Likert-scale items and four open-ended questions. For this study,

we analyzed pre and post open-ended responses to the following question: "What does teaching mathematics for social justice mean to you?" MECS was administered in fall 2011 and spring 2012 at three universities in the Eastern United States. The three teacher education programs in this study were all initial certification in elementary education. Data consisted of pre and post responses written by preservice teachers at the three participating universities (n=230).

In order to analyze our data, we used an inductive content analysis approach (Grbich, 2007). We initially organized raw data into an excel spreadsheet, read all of the responses, and created codes based on the raw data. Then, we coded the data in the excel spreadsheet by including a "1" when a particular code was present within a response. This allowed us to compute frequencies. Finally, we categorized the codes into themes and created a clear description for each theme.

To establish consistency between the raters, we computed a percentage agreement as a measure of interrater reliability (Huck, 2012). Percent agreement for a set of 40 responses was calculated by dividing the number of agreements by the number of agreements and disagreements and multiplying by 100. We completed two rounds of coding until an interrater reliability of 90.5% was achieved. After each round of coding, we discussed our rationale for coding responses, clarified definition of codes, and revised existing codes. Then we divided the responses in half, individually coded, and calculated frequency counts for each code. Altogether, we coded 460 pre and post responses, then we examined aggregate frequency counts.

Results

To answer our first research question about PSTs' conceptions, we created 36 codes based on the 460 pre and post responses and categorized the codes into seven themes, as listed in Table 1. Below are two examples of the type of responses we examined.

> *Teaching mathematics for social justice means ensuring that all students, regardless of race, class, or socioeconomic status, are given the opportunity and support to succeed in mathematics. Because mathematics is fundamental to success in society, helping students to succeed in the subject ensures more equality in society.*
>
> *I think it means making sure each student has an equal opportunity to learn. You have high expectations for every student and expect them to meet those standards. You teach things multiple ways to make sure all of your students understand.*

We realize many of the ideas mentioned in the responses were concise and not fully fleshed out. It is also possible that PSTs who wrote statements such as "empowering students through mathematics" or "closing the achievement gap" may not have fully understood what these ideas meant beyond the surface level. However, we thought it was valuable that PSTs appeared to have an awareness of a range of topics related to social justice that are often discussed within the mathematics education literature (Wager and Stinson, 2012).

Two major findings resulted from our analyses, suggesting that 1) the range of themes present were consistent with the various conceptualizations of teaching mathematics for social justice in the literature, and 2) preservice teachers enter into mathematics methods courses with promising ideas about teaching mathematics for social justice (pre-responses) and are able to make important connections between teaching mathematics and social justice (pre and post-responses).

Table 1. Themes, Codes, and Descriptions about to Teaching Mathematics for Social Justice

Theme	Code	Description
Unsure; Limited	Not sure	Not knowing what it meant, not seeing a

	No connection Teaching math the same way to all	connection, or having a limited idea such as teaching all students the same way
Mathematics Content	Math is neutral Integrate subjects Math for Understanding	Mentioning ideas about mathematics content ranging from mathematics as a universal or neutral subject to including critical thinking
Access and Opportunity	Opportunity to learn Everyone can learn Math is important/society Achieving academic success Achievement gap	Emphasizing the importance of mathematics, being able to access it, providing students with an opportunity to learn to achieve academic success, and addressing the achievement gap
Demographics	Different backgrounds/awareness Race, ethnicity, nationality Gender, sex Ability Levels (including SPED) ELLs SES, class Religion	Having an awareness of students from different backgrounds, meeting the needs of students regardless of their race, age, gender, religion, language, or ability
Learning Environment	Unbiased, No discrimination Treating students fairly; Respect Inclusive classroom environment Adequate resources Equal opportunity, quality teaching Math attitude/interest Accountability	Creating an inclusive learning environment where students are respected and treated fairly, providing adequate resources, and helping students develop an interest in and positive attitude toward mathematics
Instructional Strategies	Differentiation Embed cultures Connect math to real world Connect math to students' lives High expectations Help students overcome challenges	Differentiating instruction to meet the varied needs of students, making mathematics meaningful to students by connecting it to their lives, and having high expectations
Critical	Empower Learn about social issues with math	Empowering students through mathematics; teaching students about social issues and the world via math

To address research question 2, which focused on the changes in PSTs conceptions, the data in Table 2 shows the percentages of pre and post responses by themes. We were surprised to find that the percentages were actually quite similar from pre to post. For example, 30% of pre responses mentioned access and opportunity, which stayed consistent in the post response. It was assuring to see that there was a major decrease from 36% to 19% in preservice teachers stating that they did not know what it meant to teach mathematics for social justice or did not see a connection between teaching mathematics and social justice. The responses below are representative on how several PSTs' understanding of teaching mathematics for social justice changed from pre to post.

> Student Pre: *I'm honestly not sure what that means or how the two relate but I am looking forward to learning about how teaching math for social justice works.*
> Student Post: *I want to teach mathematics to my students so that they all feel included and interested in my classroom instruction. I want for all students no matter what their class, race or learning level might be, to feel comfortable and secure in my teaching*

environment. I want for my room to be socially just for all children, and for all children to benefit from my teaching.

Two themes that increased in responses from pre to post were *Learning Environment* and *Instructional Strategies*. This was likely due to the fact that the mathematics methods courses focused on both themes, and PSTs were able to specify approaches they would use to teach mathematics effectively, which was perceived as teaching mathematics for social justice.

Table 2. Overall Percentages of Responses by Themes

Theme	Pre	Post
Unsure; Limited	36%	19%
Mathematics Content	13%	11%
Access and Opportunity	30%	31%
Demographics	17%	22%
Learning Environment	27%	32%
Instructional Strategies	17%	29%
Critical	17%	13%

Scholarly Significance

While there exists variation within the framework on the forms of social justice, it does not include a spectrum of understandings needed for PSTs in relation to teaching mathematics for social justice. We contend the themes that arose from this study are foundational in developing this needed continuum of conceptions about teaching mathematics for social justice. The themes can be categorized under Gates' and Jorgenson's (2009) first two levels of understanding social justice. The majority of our themes fit into the *fairness and equity* level while a few mentioned the second level of *recognizing structural inequalities*. However, none of our themes, or individual responses, mentioned topics related to level three—*taking a radical approach to address structural inequalities*. We, as mathematics teacher educators, must be cognizant of preservice teachers' conceptions in our mathematics methods courses. It is with this awareness that we can incorporate meaningful experiences and assignments that will further develop and refine preservice teachers' conceptions about teaching mathematics for social justice.

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PEDAGOGICAL KNOWLEDGE FOR ALGEBRA ON THE RUNWAY: TURNING RESEARCH FROM THE ICME-12 INTO PRACTICE

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This study addresses comparable concerns regarding pedagogical content knowledge (PCK) from different countries, experiences of teachers around the world regarding knowledge for teaching algebra, and the use of such experiences for teaching pre-service teachers. Proceedings from the most recent International Congress on Mathematical Education were analyzed focusing on the importance of PCK and specialized knowledge for teaching algebra. Analysis of multiple international studies provided a holistic approach to bridge research into practice. In this paper, I present the findings of this analysis which were subsequently used to develop contents for teaching algebra in a methods course for pre-service secondary mathematics teachers.

Keywords: Mathematical Knowledge for Teaching, Algebra and Algebraic Thinking, Teacher Education-Preservice, Instructional activities and practices

Introduction

One of the concerns regarding international studies is that the culture of each country is so different that it is not realistic to adapt teaching ideas from one country to another. While I was participating in the 12th International Congress on Mathematical Education (ICME-12), I saw similar ideas regarding teaching mathematics, especially regarding pedagogical content knowledge (PCK). In 2012, ICME was held in Seoul, South Korea, including approximately 3000 participants from about 100 different countries (ICME-12, 2012). After experiencing this conference, I realized that I could learn more by analyzing the proceedings presented on the website (http://icme12.org/). Specifically I aimed to answer the following research questions: What are the concerns regarding PCK from different countries described in the proceedings?; what are the experiences of teachers around the world regarding knowledge for teaching algebra?; and how can a method instructor use these experiences of teachers in her classroom?

Perspectives and Conceptual Framework

PCK addresses teachers' understanding of how to present a specific concept and knowledge of strategies that make student learning of the topic easy or difficult (Shurman, 1986). Hill, Ball, and Schilling (2008) describe PCK regarding mathematics, called mathematical knowledge for teaching (MKT). This study focuses on some of the MKT components including Knowledge of Content and Teaching (KCT), and Specialized Content Knowledge (SCK). KCT includes knowledge of designing instruction, such as choosing representations. SCK has been considered as a construct to focus teachers' learning of teaching content (Hill, Ball, & Schilling, 2008). Silverman and Thompson (2008) discuss another conception of MKT that supports pre-service teachers (PSTs)' ability to continually develop key mathematical ideas and reflectively develop their MKT with a transformative model, adopted from Gess-Newsome (1999)'s study, rather than positioning PSTs to develop a particular MKT component. I focus on both frameworks related to MKT, especially providing teachers with opportunities to connect their content and pedagogical knowledge to develop a new knowledge related to teaching algebra.

Algebra has been discussed in many countries. Especially in the U.S., learning algebra has served as a gatekeeper for entrance to college course work (Reck & Harrington, 2000) and as a civil right for students (Moses & Cobb, 2001). McCrory, Floden, Ferrini-Mundy, Reckase, & Senk (2012) introduce three kinds of knowledge of algebra for teaching: decompressing, trimming, and bridging. Decompressing addresses "simplifying expressions and…moving among representations" (p. 18); trimming includes "making mathematics accessible to students while retaining the integrity of the mathematical ideas" (p.23); and bridging connects "mathematics across topics, courses, concepts, and goals" (p.23). McCrory et al.,'s framework is related to KCT and SCK (e.g., using representations to express algebraic concepts for teaching), which I adopted for this study, along with both frameworks described earlier.

Methods

Proceedings from the ICME-12 website were used for this study. The second column in Table 1 refers to the number of proceedings included in each session of the conference. For instance, Topic Study Group includes 37 topics (e.g., mathematical knowledge for teaching at secondary level, problem solving in mathematics education, teaching and learning of algebra), while the total number of proceedings was 812. Poster sessions were not included in this study because their proceedings were brief while the other proceedings include 10-12 page single-spaced papers. I read all the titles of the 964 papers to see if they were related to PCK or algebra. After 97 papers were sorted by reading titles, I read all the abstracts for these papers to see if they were related to the purpose of this study. For example, if a paper addressed PCK without specifically focusing on algebra, the paper was ruled out after the abstract review. Finally 38 studies remained, which I documented for specific examples of significance of PCK and specialized knowledge for teaching algebra. I especially took detailed notes on studies that included examples which were applicable for my methods course where 18 secondary mathematics PSTs could develop their PCK regarding algebra.

Table 1: Number of proceedings for each session

Types of sessions in ICME	On the website	First sorting by title	Second sorting by abstract
Plenary Lectures	8	2	0
Survey Teams	5	2	0
Regular Lectures	76	23	8
National Presentations	5	3	0
Topic Study Group	812	63	29
Discussion Groups	17	2	0
Workshop & Sharing Group	41	2	1
Total	964	97	38

Results

Comparable concerns regarding PCK have been discussed in different countries. This section includes some of these issues and descriptions of specialized knowledge for teaching algebra studied in various countries. Some countries are concerned about the lack of focus on PCK in their practice and research. For example, there has been little focus on mathematical knowledge required for teaching in India (Takker & Subramaniam, 2012). Also O'Meara and O'Donoghue (2012) mention that in Ireland, there have been very few studies regarding teachers' subject knowledge for teaching. Li, Huang, and Shin (2008) explain that in China, teacher preparation

programs tend to focus less on pedagogical training than content knowledge training for pre-service secondary mathematics teachers. Similarly, Gronmo, Kaarstein, and Ernest (2012) state the importance of students' PCK, saying student teachers in Norway seem to lack knowledge in teaching mathematics, especially in algebra, which needs to be seriously addressed.

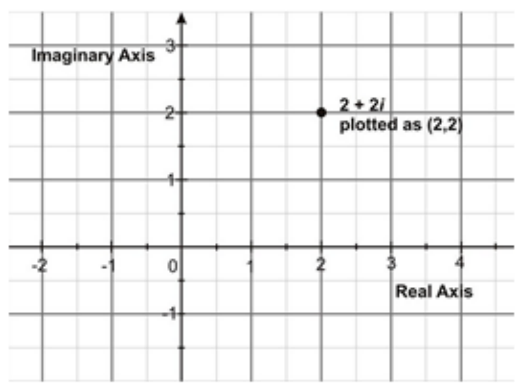
International studies focused on specific topics, such as assessing or developing teachers' knowledge and knowledge that teachers can take into their K-12 classrooms, were integrated into a method course that I taught. Below I describe some of the activities on which my PSTs gave positive feedback. A study from China focused on evaluating teachers' knowledge for teaching algebra (Huang, 2012). The researchers developed open-ended items assessing PSTs' knowledge for teaching the concept of functions in order to determine what types of PCK for algebra that PSTs are confident or struggle with. The result demonstrated that PSTs in China were confident with using different representations, but struggled with using geometric representation to express algebraic concepts (Huang, 2012). I wondered how my PSTs might solve the same problem by utilizing the concepts in the items for their teaching. I introduced the following modified problem from Huang's study (Figure 1 left). PSTs were intrigued by the fact that these items were for evaluating teacher knowledge in China and tried to show others that they could easily solve them. It took a while for some of the PSTs to figure out, but most PSTs shared their geometric representations with others to justify their reasoning. At the end of the lesson, some of them mentioned in the exit slip that they would introduce these geometric representations when they teach the distributive property and the expansion of the square of a binomial.

Figure 1: Examples of activities

Developing open-ended items focusing on secondary pre-service mathematics teachers' knowledge for teaching the concept of function (Huang, China)

Which of the following can be represented by areas of rectangles?

1) The distributive property of multiplication over addition: For all real numbers a, b, and c, we have a(b+c)=ab+ac
2) The expansion of the square of a binomial: $(a+b)^2 = a^2 + 2ab + b^2$



Another study from Turkey showed a way of teaching complex numbers in their content course (Aslan-Tutak, 2012). PSTs were provided a geometrical understanding of the imaginary numbers. This way of teaching shifted PSTs' view towards complex numbers from algebraic to graphic representation (Aslan-Tutak, 2012). I assumed that most PSTs might find it difficult to teach the concept of imaginary numbers due to their lack of MKT. To address this concern, Aslan-Tutak's study would be helpful because it addressed McCrory, et al.'s knowledge of algebra for teaching. I asked PSTs to make of a list of ten mathematical topics that they considered most difficult to teach. As I assumed, all the groups included imaginary numbers in their lists. I shared the following standard from *Common Core State Standards for Mathematics (CCSSM)* (National Governor's Association Center for Best Practices & Council of Chief State School Officers, 2010): "Represent complex numbers and their operations on the complex plane" (p. 57). Then I introduced one way of teaching the concept from Aslan-Tutak's study by asking how to plot (2, 2) on a coordinate plane, and then asked them how to plot (2, 2i) on a new plane (Figure 1 right). PSTs answered that they needed a new axis to express 2i on a new plane when I introduced the formal names for these new axis and new plane with a historical background.

After discussing how they apply this knowledge for teaching in their classroom, including their own way of teaching the concept, they chose one of the difficult topics from their lists in order to plan, implement, and evaluate their own mini lessons.

Conclusion and Discussion

Researchers from several countries discussed how little PCK is a focus in their countries. They also emphasized the importance of PCK including knowledge for teaching algebra. Teachers need knowledge of using geometric representations to express symbolic algebra expressions, which are referred to as decompressing and bridging in the study by McCrory et al. (2012). Similarly, teaching a difficult algebraic concept by connecting it with prior knowledge of a coordinate system involves knowledge of making algebra accessible to students (trimming) and connecting algebra across topics and concepts (bridging). By asking students to collaboratively develop a lesson on a difficult topic and teach the lesson, teachers can develop KCT which addresses knowledge of designing instruction, selecting representations, and implementing lesson plans (Hill, 2010). When PSTs participated in some of the activities introduced in the international studies, they connected their content knowledge with pedagogical knowledge and created their own knowledge that they attempted to use in the future, which was recommended by the Silverman and Thompson's study (2008). Admittedly, these findings are limited due to the fact that my first search was limited to the short titles of the documents. This could perhaps have excluded relevant papers that did not include reference to PCK or Algebra in the title. However, analyzing studies from international proceedings; such as one that I have described above; provides a holistic approach to move from research into practice.

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SETTING UP AND ENACTING APPROXIMATIONS OF MATHEMATICAL TASKS OF TEACHING IN A PRACTICE-FOCUSED CURRICULUM

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This paper argues that it is important to understand what is involved in teaching tasks of teaching via approximations of practice. Specifically, it identifies elements of setting up and enacting activities for improving teachers' abilities to meet the demands of teaching mathematics. Observations of a methods course for secondary mathematics teachers and regular interviews of the teacher educator following each class were analyzed. From the analysis, we propose seven elements of such teaching and illustrate these elements with episodes from the observations and interviews. The paper concludes with a discussion of implications for mathematics teacher education.

Keywords: Teacher Education-Preservice, Mathematical Knowledge for Teaching

Introduction

Recent decades have seen increased emphasis in mathematics teacher education on the notion of content knowledge for teaching (Ball, Thames, & Phelps, 2008; Shulman, 1986) and the core tasks that teachers execute to help students learn mathematics (Ball & Forzani, 2009).Yet it is still difficult to help teachers "understand instruction as the "complex weave of interactions and interpretations" (Ball, 2012, p. 352). Furthermore, work on the mission of teacher education currently depends on each individual teacher educator's efforts; there is scant common knowledge about the pedagogy of preparing mathematics teachers (Ball, Sleep, Boerst, & Bass, 2009).

One contributor to this problem is the lack of a knowledge base that supports systemic development among teacher educators (Morris & Hiebert, 2009). In the proposed paper, we seek to contribute to the development of such a knowledge base by proposing a potential framework that could be used to refine the pedagogy of approximations of practice (Grossman et al., 2009) in methods courses. We see this framework not as static but as a work in progress shaped by how it functions in use.

The context for this research is the instruction of a mathematics methods course for prospective secondary teachers. This course aimed to engage prospective teachers in approximations of practice, termed by the teacher educator as "rehearsals." The teacher educator intended the rehearsals to approximate tasks of teaching for which skillful performance would require use of mathematical knowledge for teaching (Ball et al., 2008). We refer to such tasks as *mathematical tasks of teaching*. We examined the instruction of these rehearsals, asking: What elements compose the instructional work of setting up and enacting approximations of mathematical tasks of teaching?

Conceptual Framework

Instruction in teacher education is at least as complex as the practice of teaching mathematics. The content of instruction not just mathematics or students; it includes practice and its demands as well. Cohen, Raudenbush, and Ball (2003) identified teachers, students, content, and environments as the main constituents of instruction, and emphasized interactions between

these elements in teaching and learning. Ball (2012) extended the idea of this instructional paradigm to teacher education, emphasizing teaching practice as content of teacher education, as shown in Figure 1.

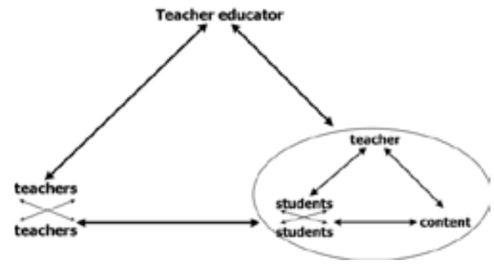

Figure 1: Instructional Triangle in Teacher Education (Ball, 2012)

This framework shows that teacher educators need a multifocal approach, with an eye on the mathematics, on students, on teachers, and on their learning and interactions (Ball, 2012, p.351). Teacher educators in this triangle are both designers and instructors. They need to consider the content as well as dynamics in the instruction of teacher education. Teacher educators have the fundamental challenge of building bridge between teachers and the practice of mathematics teaching. Teacher educators interpret what teachers as learners say and do in terms of mathematics and practice. Planning a lesson in teacher education requires substantial knowledge and skill, and deep understanding of the nature of instruction, or teaching and learning, and enacting the plan asks managing the dynamics of instruction and of teachers' professional learning of and from practice (Ball, 2012). The research reported here uses this framework to see teacher education and analyze the data.

Data and Method

This study collected data from methods course for preservice secondary mathematics teachers, taught by a first-time teacher educator who designed the course to be practice-focused, and whose goals included the teaching of mathematical tasks of teaching. The experience of being a first-time instructor is likely to make the entailments of instruction more visible (Heaton, 2000). The course enrolled twenty prospective teachers comprising thirteen undergraduate students and seven master's degree certification students, all of whom were concurrently enrolled in field placements.

To investigate the research question, the authors used a grounded approach to the data (Strauss & Corbin, 2008), identifying aspects of interview statements and actions in the classrooms that could be construed to be informed by setting up and enacting activities for the lesson. The analyses of the teacher educator's interview statements and classroom actions were conducted in parallel. The authors also analyzed the course syllabus and slides for each class. Using these resources strengthened the data interpretation and helped identify possible factors influencing the teacher educator's setting up and enacting of activities. Iterative analysis of audio clips, field notes, and interviews were used to identify the eventual proposed elements of instruction across classes.

Results

The purpose of the analysis was to identify what is entailed in setting up and enacting approximations of mathematical tasks of teaching. We looked at this in terms of what was intended and actually performed in and for the instruction of these mathematical tasks of

teaching characterized their features. From the analysis of the data, we identified seven prominent elements in setting up and implementing mathematical tasks of teaching in a practice-focused curriculum. The elements are distinct yet can be mutually informing.

Purposes of Teacher Educator

Teaching is purposeful (Sleep, 2012). That is, instructional goals shape what instruction is designed, how the instruction is steered, and what teachers as learners can learn in teacher education. For instance, one rehearsal approximated the task of going over a mathematics problem with multiple representations of a function. The instructional goal was for teachers to practice making explicit and sustained mathematical connections between representations by planning and performing a coherent explanation of a solution that tied together three hypothetical students' work, each using different representations. The statement of the instructional goal was used as a resource to design supplementary documents, including for formative assessment.

Pedagogical Setting

Classes of teacher education are very particular places because they are asked to create two different layers of situations: a context to see and discuss mathematics for teaching and practice of teaching in general, and a hypothetical setting that is a class of K-9 mathematics classroom. We call the latter as pedagogical setting. Such settings in teacher education with a focus on practice and its disciplinary demands offer sites that teachers as learners practice tasks of teaching that demand content knowledge for teaching. Four elements are noticeable as elements.

Topic. This element includes a topical name of a course or unit (e.g., geometry), a more focused topic (e.g., similar triangles), or a specific issue (e.g., establishing correspondence). When the teacher educator chose to focus on a specific topic, she considered its importance in school mathematics and its relationship to other mathematical objects in disciplinary and applied mathematics. Moreover, she considered the way that mathematical problems in the topic area might present challenges embedded in mathematical tasks of teaching.

Teaching purpose. This element is about the goals and outcomes for teachers to consider within the setting of the rehearsal. Note that this purpose differs from the previously discussed purposes of teacher educator. The teaching purpose is often a direct statement about a pedagogical goal or an implicit concern, orientation, or awareness that bears on teaching. In the class, the teaching purpose was stated explicitly to the teachers when introducing a rehearsal as a way to focus the teachers' planning for their rehearsals.

Instructional materials. This element includes excerpts from textbooks, example problems, student work, board work by either a teacher or students, manipulatives or instructional representations, student explanations, classroom dialogue, or other records of practice. These are objects or talks that are potentially used inside of instructional interactions.

Student background. This element identifies information about students' prior knowledge, performance, disposition, or other characteristics. For instance, this element could be used to determine what constitutes mathematically and pedagogically appropriate explanations.

Assessment

Assessment guides and enhances learners' learning and provides useful information to instructors and learners. Assessment is in the context to see and discuss mathematics for teaching and practice of teaching in general rather than in the pedagogical setting that we discussed previously.

Assessment by a teacher educator. The purposes of this course involved mathematics and teaching practice. The teacher educator assessed in terms of what kinds of performance were valued and what knowledge was emphasized. When the teacher educator recognized potential

areas for the preservice teachers to improve facility at a task of teaching, this appraisal shaped the subsequent instruction, including the design of the homework assigned following the rehearsal. The assessment is driven by the instructional purposes of the teacher educator.

Assessment by teachers. The second way in which assessment is used by a teacher educator is the opportunities for the preservice teachers to assess the rehearsal of the volunteer. Following the rehearsal, the teacher educator asked the teachers to assess the rehearsal in small group discussions. This type of assessment approximated the task of analyzing instruction for improvement.

Discussion

Support of teacher educators' work has received little emphasis despite the challenging responsibility of teacher educators to design and develop teachers' learning experiences (Zaslavsky & Leikin, 2004). Through the analysis, the research found seven elements that compose the setting up and enacting of approximations of practice that entailed in a practice-focused curriculum. The findings may contribute to identification of what is salient in instructional decision making by teacher educators in a practice-focused curriculum. Moreover, the identification and illustration of these components support the possibility for local teacher education systems to enact the features identified by Morris and Hiebert (2009) by laying out components that can be modified in bounded ways. These seven components encompass shareable instructional goals for teacher educators, allow for the rehearsal to be conceived of as a product that can be seen and changed, and provide a way to make and test changes to the rehearsal in instructionally relevant ways.

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CONVICTION AND VALIDITY: MIDDLE SCHOOL MATHEMATICS TEACHERS' PROOF EVALUATIONS

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This paper examines 55 middle school mathematics teachers' proof evaluations and their perceived criteria for what constitutes convincing arguments and valid mathematical proofs. The results suggest that the majority of teachers find: (1) a number-based argument to be neither convincing nor a proof, and (2) an algebraic argument and a visual argument to be both convincing and a proof. The results also suggest that many teachers perceive generality as the major criterion in order for arguments to be considered as convincing and a proof.

Keywords: Reasoning and Proof, Middle School Education, Convincing Arguments

Introduction

Both the Common Core State Standards for Mathematics (Common Core State Standards Initiative [CCSSI], 2010) and the Principles and Standards for School Mathematics (National Council of Teachers of Mathematics [NCTM], 2000) suggest that students at all grade levels should be able to evaluate and to understand mathematical arguments. Successfully responding to this call requires that teachers have adequate understandings of mathematical proof. Research on mathematics teachers' competencies in understanding and evaluating proofs, however, has documented that teachers have difficulty recognizing given arguments as mathematical proofs (e.g., Knuth, 2002; Martin & Harel, 1989; Morris, 2002). We suggest that examining teachers' criteria for what constitutes convincing arguments and mathematical proofs may help teachers develop a better understanding of valid proof structures (e.g., proof by induction). Yet, to date, little is known about what arguments middle school mathematics teachers find convincing and accept as mathematical proofs. To address this research gap, this study investigates middle school mathematics teachers' reasons for deciding whether or not a given argument is convincing and whether it is a valid mathematical proof.

Theoretical Background

To understand middle school mathematics teachers' criteria for what constitutes a convincing argument and a valid mathematical proof, we conceptualize characteristics of convincing arguments and mathematical proofs using Knuth's (2002) taxonomies that he used to characterize experienced high school mathematics teachers' views on proof. Knuth (2002) found with in-service high school mathematics teachers the following criteria for evaluating whether an argument is convincing: concrete features (i.e., the use of numbers or diagrams), familiarity, sufficient level of detail, generality, shows why, and valid method (i.e., proof by induction). He also found that valid methods, mathematically sound (i.e., the argument showed the underlying mathematics), sufficient detail (i.e., the argument included all of the steps), and knowledge dependent (i.e., the teacher's understanding of the mathematics presented in the argument) are four major criteria that teachers used to consider an argument to be a valid proof. Given that personal criteria have the potential to influence a teacher's evaluation of whether an argument is convincing and/or a valid proof, we examine the types of arguments that teachers found convincing and acceptable along with the criteria they used to justify their evaluation.

Methods

A total of 55 teachers from nine public school districts in the Northeast region of the United States completed an online survey. The survey presented teachers with one conjecture and three arguments for the same conjecture—a number-based argument, an algebraic argument, and a visual argument (see Figure 1). Teachers were asked to: (1) indicate how convincing they found the given

argument using a “very convincing,” “convincing,” “somewhat convincing,” or “not at all convincing” scale; (2) decide if each given argument was a mathematical proof using a “yes” or “no” response; and (3) describe what criteria they used to make their decisions. This paper focuses on teachers’ proof conviction and validation. A response was coded as convincing if a teacher indicated the argument was “very convincing” or “convincing” and was coded as not convincing if a teacher indicated the argument was “somewhat convincing” or “not at all convincing.” The teachers’ criteria for convincing arguments and mathematical proofs were analyzed based on the theoretical background described previously. As the data were examined, different types of criteria emerged in participants’ written responses. The characteristics for convincing arguments and proofs with a representative example are listed in Table 1.

Rob comes up with the following conjecture: The sum of any two even integers is an even integer.

Jodi’s answer

2+4=6
4+6=10
1002+856=1858
20,147,938+3,144,028=23,291,966
Thus, Jodi says Rob’s conjecture is true.

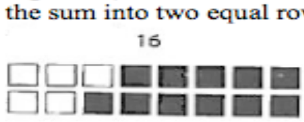
Matt’s answer

If a and b are even integers, then a and b can be written a=2m and b=2n, where m and n are other integers.
So $a+b=2m+2n$. We can factor out a 2 and then get $a+b=2(m+n)$. Since $(m+n)$ is an integer, $a+b$ must be even.
Thus, Matt says Rob’s conjecture is true. (Adapted from Smith & Stein, 2011, p. 46)

Teresa’s answer

If I take the numbers 6 and 10 and organize the counters as shown, you can see the pattern.

You can see that each number can be made into two equal rows. When you put the sets together (add the numbers), the blocks are still two equal rows. So, since you can arrange the sum into two equal rows, it must be an even number.



Thus, Teresa says Rob’s conjecture is true. (Adapted from Smith & Stein, 2011, p. 46)

Figure 1: The Conjecture with Three Arguments

Results and Discussion

Which Arguments are Convincing

Thirty-nine of the 55 participants did not find Jodi’s answer (the number-based argument) to be convincing, while the other 16 teachers judged it as convincing. This result is not in alignment with prior research showing that the majority of teachers are convinced by arguments based on specific examples (e.g., Knuth, 2002). Thirty-five teachers identified Matt’s answer (the algebraic argument) and Teresa’s answer (the visual argument) to be convincing, which are consistent with Knuth’s (2002) findings that some in-service high school mathematics teachers were convinced by arguments with algebraic or visual features. The most prevalent criterion for convincing arguments was the use of generality (41 cases out of 176), as many teachers indicated a belief that a convincing argument should have generalization for all cases.

It is interesting to note that there were 24 cases in which the teachers were convinced by arguments based on Table 1: Characteristics of Convincing Arguments and Mathematical Proofs to be true. We also found that a few teachers thought that Matt should test specific numbers to support the argument he had made using symbols, which suggests that they did not seem to understand that a generalized argument provides certainty for all cases. However, the findings only represented 22 cases that the teachers used for convincing arguments. In addition, there were 18 cases in which the teachers considered if arguments

were easy to understand to be convincing. This finding suggests that teachers should have certain knowledge of mathematics in order to understand arguments.

Table 1: Characteristics of Convincing Arguments and Mathematical Proofs

Category	Definition	Representative Example of Convicting Arguments	Representative Example of Proofs
Use of Generality	An argument has/has not shown the certainty for all cases.	Only 3 specific examples, out of an infinite number of possibilities. There was no generalization for all cases.	It (Matt's answer) proves the statement for the general case.
Use of Numerical Representations	An argument has/has not provided a variety of numbers.	Jodi shows several various examples that prove the statement.	I tried plugging in Matt's proof using random #s and couldn't get it to prove.
Use of Counterexamples	An argument has/has not included a counterexample to disprove the statement.	Jodi only used whole positive numbers. She needed to look for a counterexample.	She (Jodi) did not eliminate the chance of missing an exception - it only takes one "wrong" solution to invalidate the postulate.
Use of Visual Representations	An argument has/has not used visual diagram.	I gave Teresa very convincing because she used a diagram to explain her reasoning. I think using pictures to organize her thinking is a very effective and clear way to convince someone she is correct.	I read through her thinking and she used pictures as a proof.
Use of Clarity	An argument is/is not easy to follow.	It (Teresa's answer) is a clear and understandable argument.	I couldn't follow it (Matt's answer) so I thought no, but I realize that it is not a valid answer
Use of Mathematical Facts	An argument has/has not included known definitions or theorems.	(Matt) [u]sed mathematical evidence such as the fact that an even number is divisible by two and the distributive property	If it (Jodi's answer) showed logical reasoning backed up with known (proven) mathematical concepts, it did not.
Use of Logical Structures	An argument has/has not (1) stared with the givens or (2) included all steps that follow logically from previous assertions.	He (Matt) used logical steps that are beyond dispute to reach a conclusion.	She (Jodi) did not use a series of logical steps that are beyond dispute to reach a conclusion.
Use of Symbolic Manipulations	An argument has/has not (1) presented algebraic symbols or formulas or (2) manipulated symbols correctly.	(Matt's answer) [u]s[ed] algebraic variables instead of specific numbers proves the argument in more general terms.	She (Jodi) needs to state some kind of rule(s) and then base her statement off of that.
Use of Explanations	An argument has/has not provided a clear explanation.	Matt is adding variables, but not explaining why he can write that a=2b and so on. He is multiplying by two and does not explain why that makes numbers even.	Teresa is clearly showing two even numbers will always equal an even number. She proves this by showing us why.
Use of Similarity	An argument has/has not used the similar way as the teacher teachers/produces a proof.	Jodi's argument is similar to my own in that she used real math facts to prove she was correct.	We do a similar activity in our math class using tiles to create rectangles. Teresa demonstrated this using the same strategy we teach our students.
Other	A teacher's response is not clear.	She needed more proof!	I don't know.

Which Arguments are Mathematical Proofs

Thirty-nine of the 55 participants did not accept Jodi's answer using specific numbers as an acceptable proof, while the remaining teachers had a different belief. This finding is in contrast to previous results suggesting that pre-service elementary and secondary mathematics teachers tend to view number-based examples as valid proofs (e.g., Martin & Harel, 1989; Goetting, 1995). Thirty-seven teachers found Matt's answer using algebra to be a proof, and 27 teachers judged Teresa's answer using a visual representation to be a proof. The former result is consistent with Raman's (2002) findings that many undergraduate students believe that arguments including mathematical symbols are proofs. The latter finding is reminiscent of Weber's (2010) suggestion that diagrammatic arguments are acceptable forms of proofs because they provide insight into why statements are true. We also found that there were 31 cases in which the teachers judged the argument to be a mathematical proof because it had or had not used algebraic rules or mathematical symbols. These results are not entirely surprising considering the literature on undergraduate students' competencies in evaluating proofs suggesting that they are inclined to accept arguments with mathematical symbols as proofs (e.g., Raman, 2002). In order to explore middle school mathematics teachers' views on what constitutes proof, mathematics teacher education programs might consider providing more opportunities for teachers to evaluate various types of arguments.

Acknowledgment

This research is supported by the National Science Foundation (NSF) under grant MSP- 0962863. The opinions expressed herein are those of the authors and do not necessarily reflect views of the NSF.

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MATH AND SCIENCE MASTER'S COHORT PROGRAM: IMPACT ON TEACHERS AND STUDENTS

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This study examines the influence of an elementary math and science education master's cohort program on teachers' identities and student performance in mathematics. A mixed methods approach was used to study how being part of a cohort impacted the teachers and their students. Qualitative data was collected on the teachers' perceptions of how being part of a cohort impacted their roles as teachers. The findings reveal that teachers developed an "affinity identity" as a cohort (Gee, 2001). The quantitative data indicated an increase in student learning. Implications for developing cohort programs are discussed.

Keywords: Mathematical Knowledge for Teaching, Teacher Education-Inservice/Professional Development

Theoretical Framework

Improving teachers not only involves enhancing the qualifications of practicing teachers, but also teaching them how to improve their instructional practices in the classroom (Hiebert & Morris, 2012). This process can be accomplished by designing specialized master's programs for practicing teachers. However, very little research exists on how to design such programs for teachers. Elementary teachers are generalists, and therefore, there is a need for elementary teachers to develop more in-depth knowledge in content areas such as math and science.

Teaching is a complex process that involves making professional judgments to support student learning. Hiebert and Morris (2012) assert that effective teachers are able to pose problems, provide explanations, respond to student thinking and lead class discussions. Therefore, teachers require feedback, opportunity for reflection, and discussion with peers (Guskey, 2002; Secada, 1989). Ideally, in order for a professional teaching community to form, sustained professional development must occur so that teachers have each other's support in improving their teaching (Dean, 2005).

Designing a master's program in math and science can take into account the development of a professional teaching community—teachers should engage in sustained professional development as a group. This can lead to the creation of a "joint enterprise" and identity in which people organize around a particular area of knowledge (Lave, 1988; Wenger, 1998). Thus, teachers must interact with each other over time, which can lead to the development of mutual engagement and a shared repertoire. Therefore, we can draw from Lave (1988) and Wenger's (1998) definition of a "community of practice". According to Lave (1988) and Wenger (1998), a community of practice is formed when joint enterprise, mutual engagement, and shared repertoire exist. A joint enterprise is a shared goal that guides how the community functions. Mutual engagement involves the relationships that people develop with each other that become part of the group norms. A shared repertoire encompasses the kinds of tools, discourse, actions, and artifacts that the group produces and on which they interact. Additionally, Gee (2000-2001) defined "affinity identity" as "a group that share allegiance to, access to, and participation in specific practices" (p.105). In these groups, members must choose to embody the group's norms in order to become part of the affinity identity.

This paper reports on a math and science cohort program that was developed by a land grant institution in a western state. The program was designed with input by district leaders to develop teacher leaders to support mathematics and science instruction in the district. Moreover, the program was designed for the working teacher. Initially, the program focused on increasing teacher content and pedagogical content knowledge and supporting shifts in teacher practice. At the conclusion of the program, a course was specifically designed to provide teachers with experiences in leadership to prepare them to eventually be teacher leaders in the district. In order to be accepted into the master's program, the teacher needed to be a part of a cohort of teachers from each school. In other words, at minimum there needed to be two teachers from each school. Additionally, other criteria such as GPA, letters of recommendation, an essay indicating interest in math and science disciplines, and evidence of leadership potential was also considered. The math and science courses were designed to complement each other so that teachers developed standards based approaches and all teachers took the same sequence of courses. Fourteen in-service teachers participated in this study. The questions investigated in this study are as follows: What impact did the cohort program have on teachers and what impact did it have on their students' learning?

Methods

Data was collected throughout the cohort program of 2010-2012. Qualitative data was collected through teacher reflections, teaching observations, and one-on-one interviews (Creswell, 2002). Quantitative data sources included student MAP test scores in mathematics from Fall 2010, Winter 2010, Spring 2011, Fall 2011, Winter 2012, and Spring 2012.
A mixed methods methodology was used to analyze data over two academic years. According to Tashakkori and Creswell (2007), mixed methods research involves collecting and analyzing data, integrating the findings, and drawing inferences using both qualitative and quantitative methods. This study is a convergent parallel design (Creswell & Plano Clark, 2011) because we used concurrent timing to implement the quantitative and qualitative strands of the research. Qualitative data was analyzed using coding to organize it into broader themes (Maxwell, 2005). All text was systematically searched and organized during the ongoing study by emergent themes in an iterative pattern over the research period, following the approach of qualitative data analysis outlined by Bogdan and Biklen (2003). Quantitative data was analyzed using a one-way analysis of variance (ANOVA) to compare means of MAP test scores in mathematics.

Results

Qualitative Findings

The following themes emerged from the teacher interview data on how they perceived each other in their cohort experience.

A shared sense of passion and motivation developed. For instance, a cohort member remarked, "I have created bonds with like-minded teachers in the district who I can share ideas with and ask questions of." Another teacher stated: "The ability to work with other teachers in the cohort who have a passion for teaching and learning has been something I can't put a price tag on. It has pushed me to step up my game with teaching and continue to not only challenge my students, but myself as well."

Provided an environment to connect research and practice. For example, a teacher commented, "The cohort has impacted my teaching by giving me the research to support teaching through inquiry, discussions, and writing."

Teachers began to view themselves as experts and enjoyed the experience of impacting other teachers at their schools. "I have created multiple professional development sessions for my school in math and also lead my 6th grade PLC. It has been a great experience to become a leader at my school."

Teachers began to notice that others at their schools began to view them as experts. Teachers reported that they had an increased level of confidence in their ability to teach math and wanted to support other teachers at their school. Teachers at their school started to see the cohort as "experts" to bounce ideas back and forth and learn new ideas. For example, a teacher made the following comment: "Teachers (at my school) have been enjoying the new ways of teaching and are constantly coming to me for new ideas and reflecting on math lessons they have taught."

Teachers reported that being in a cohort at the same school gave them more confidence to work with other teachers and the principal at their school.

Quantitative Findings

The MAP Testing Group Scores revealed significant growth in student learning at all grade levels. Mean numbers of correct responses on the MAP test were calculated. We can conclude that the scores rose and became steadily higher from Fall 2010 to Winter 2011 to Spring 2011 and from Fall 2011 to Winter 2012 to Spring 2012. Figure 1 shows the increase in mean scores on the MAP in Mathematics Test.

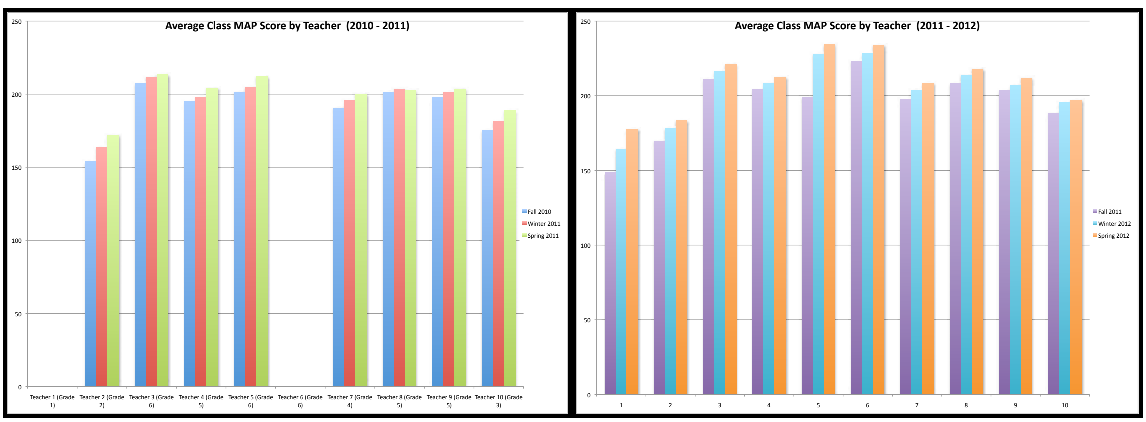

Figure 1: Average Class MAP Score by Teacher for 2010 to 2011 and 2011 to 2012

Discussion

The results of this study support the theory that developing a master's cohort program, while simultaneously taking into account the development of a community of practice (Lave, 1988; Wenger, 1998), can positively impact student learning. Moreover, teachers developed an affinity identity (Gee, 2001) as a group as they became a community of practice. The teachers built relationships that allowed them to learn from each other and the cohort was a support system for the participating teachers, as well as for teachers at each school. The cohort has encouraged teachers to become leaders, not only in their schools, but also in their districts. Many of the cohort members have become leaders in their schools by creating professional development sessions, leading Professional Learning Community (PLC) sessions, and being the lead Response to Intervention (RTI) representatives for their grade levels. Members of the cohort became "adaptive experts" (Hammerness, Darling-Hammond, & Bransford, 2005) and enabled dialogue about teaching and student learning at the school and district levels. The cohort experience enabled teachers to organize around common interests and shared goals.

Furthermore, the cohort master's program connected teachers with other teachers by building a community, ultimately, defined by the members of the cohort. We conjecture that carefully selecting the talented teachers into the program made a difference in the level of enthusiasm, the development of relationships, and even in our ability to positively influence teacher practice so that it impacted student learning.

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EXAMINING PRE-SERVICE TEACHERS' EMERGING ABILITIES TO READ *STANDARDS*-BASED CURRICULUM MATERIALS

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Remillard (1999) writes that reading is the process most closely connected to how teachers use curriculum materials. In this study we investigated how PSTs read Standards-based curriculum materials by identifying which features PSTs attended to and how they attended to them. Our results indicate that PSTs attended to educative and non-educative features of curriculum materials in educative and non-educative manners. This study has implications for curriculum developers, researchers, and mathematics teacher educators in terms of the design of curriculum materials and supporting PSTs in reading curriculum materials in educative ways.

Keywords: Teacher Education-Preservice, Curriculum, Elementary School Education

Introduction

There are three key ways that teachers interact with curriculum materials – by reading, by evaluating, and by adapting (Sherin & Drake, 2009). Though each of these modes of interaction is important and interrelated, we posit that without effective *reading* of curriculum materials, teachers are less likely to effectively evaluate and adapt the materials. "Reading is the intellectual process most closely connected to teachers' curriculum decisions. Thus, change in the enacted curriculum must occur through change in how or what teachers read" (Remillard, 1999, p. 338). Little is known, however, about how pre-service teachers (PSTs) read curriculum materials. In this study, we investigate *what* features of curriculum materials PSTs attended to while reading *Standards*-based curriculum materials and *how* they attended to those features.

Theoretical Frame

For this study, we draw primarily from two studies conducted by Remillard (1999, 2000). In both studies, Remillard investigated the ways in which two teachers (Catherine and Jackie) interacted with curriculum materials. Framing the Remillard (1999) study was the idea that reading is central to teachers' curriculum development. Remillard (1999) defined reading in a way that extended beyond its conventional usage:

> Scholars of reading describe it as a dynamic and constructive process that involves interaction between the reader and the text, situated within a particular context (Pearson and Stephens 1994; Rosenblatt 1994) […] In this process readers "use their existing knowledge and a range of cues from the text and the situational context in which the reading occurs" to make meaning (Dole et al. 1991, p. 241). (p. 318).

Remillard points out that reading is not an objective activity conducted in isolation. Instead, it is an interactive process between text and reader where the reader interprets ideas through "prior knowledge, experiences, beliefs, and attitudes."

In investigating how PSTs read curriculum materials, we were most interested in the educative features of the curriculum materials – those that were meant to promote teacher learning (Davis & Krajcik, 2005). Davis and Krajcik (2005) set forth a set of design heuristics for the ways in which materials could be designed to support teacher learning: 1) "help teachers learn how to anticipate and interpret what learners may think about or do in response to instructional activities" (p. 5); 2) support teachers' learning of subject matter knowledge, including facts, concepts, and disciplinary practices; 3) could help teachers consider ways to relate units during the year (p. 5); 4) "make visible the developers' pedagogical judgments" (p. 5); 5) and promote pedagogical design capacity (the ability to perceive and mobilize existing resources to achieve instructional goals (Brown & Edelson, 2003, p. 5)). Were PSTs attending to

the materials' educative features, and if so, how were they attending to those features?

Methods

Data were collected from 47 PSTs during elementary mathematics methods courses and consisted of written responses to the following prompt—*How does this lesson plan help you to teach this particular concept? In other words, what information does it give you that is most important or helpful? Please be specific.*—after reading five different *Standards*-based curriculum lessons over the course of one semester. Each response was first unitized into differing ideas. Next, we analyzed each unit of data in two ways. First, to determine *what* PSTs attended to in their reading of curriculum materials, we identified the feature (e.g., teaching note, sample student work) they were referring to. The first two authors analyzed the data separately and had 90% reliability. All disagreements were discussed until consensus was reached. Second, to determine *how* PSTs attended to the various features, the first two authors began the coding process by making a first pass through responses to one lesson. In doing this, we considered the literature related to teachers' use of curriculum materials and developed an initial set of five codes—*descriptive, evaluative, educative, interpretative,* and *adaptive*. After several passes through the data, codes were refined and assigned definitions. All data units were assigned a code through a process of focused coding (Charmaz, 2004) by the first two authors. Reliability was 87% and all disagreements were discussed until consensus was reached.

Because we were interested in the educative features, each lesson was analyzed to identify the educative features. The first two authors jointly identified the features (educative and non-educative) within each lesson from the curriculum materials and individually coded the educative features with one of Davis and Krajcik's (2005) five high-level guidelines.

Results

To frame our results section, we provide the following table (Table 2). The columns depict the type of feature (educative or not), while the rows depict the way the feature was read (educatively or not). Thus, there are four different combinations of curriculum features versus reading. We next provide examples from the data to illustrate each of the four quadrants.

Table 1

Educative Feature Read Educatively	Non-educative Feature Read Educatively
Educative Feature Read Non-Educatively	Non-educative Feature Read Non-Educatively

Educative Features Read Educatively

The first quadrant indicates that there were educative features in the materials that were read educatively by PSTs as in the following example (all names are pseudonyms):

> <u>Mandy's Response</u>
> One part that was helpful to me was the whole class activity where students are coming up with their formula for area of a triangle. It is so easy for teachers to just tell students what a formula is, but allowing students to attempt to put it together is more meaningful to their learning because they are the ones putting their experiences and ideas out there…

Mandy attended to a section of text titled "Developing a Formula for the Area of a Triangle" in a 4th-grade *Everyday Mathematics* lesson (UCSMP, 2007). We considered this section of text educative because the curriculum developer's pedagogical judgments were made visible. Mandy's response indicated that she learned a particular pedagogy around how to facilitate a lesson with this content. Instead of telling students the formula, teachers can have students "put it together" instead. By putting it together, it seemed that Mandy was referring to an aspect of the task that entailed taping two triangles, with the same area, together to form a parallelogram.

Non-Educative Features Read Educatively

Non-educative features of the materials serve a function (e.g., describing a task) in supporting teachers, but their primary role is not teacher learning. In a few instances, however, these non-educative features were read educatively as in the following example:

> Allie's Response
>
> The lesson also did a good job of not only making the problems concrete and visual for the students, but also for the teachers, such as the breakdown of the problems on page 30.

Allie's response indicated that she attended to the task educatively. The task is meant to be educative for students, but Allie found the representations of two-digit numbers useful for her own content knowledge understanding.

Educative Features Read Non-Educatively

In many cases, the educative features of the lessons were not read educatively (i.e., in ways that supported teacher learning). We provide examples for all these response types below.

> Edith - Description
>
> As a teacher, this lesson helps to teach the relationship between the area of a triangle and a parallelogram by giving step-by-step instructions with corresponding worksheets.

In the response above, Edith described the text as "step-by-step" directions, which means that she read the text as telling the teacher what to do. She identified the ways in which this feature could help the teacher support student learning, but did not describe any learning of her own.

In some cases, PSTs made evaluative comments about an educative feature:

> Megan – Evaluation
>
> The transparency where it has boxes labeled Picture, Strips of 10, Singles, # Stickers, and Equation; was really great.

Megan attended to an important feature of the *Stickers: A Base-Ten Model* lesson from *Investigations* (TERC, 2008), but simply indicated that this feature was great.

In several instances, the educative features were interpreted by PSTs. That is, they made an interpretation of why a particular feature was included in the materials or why it was useful.

> Cleo - Interpretation
>
> An aspect of this lesson that I found to be very positive is the fact that it stated, "Most of you probably remember Sticker Station from second grade." That is an important piece of information to note, because that means that right off the bat, the students will understand the context of the problem and be able to focus on the actual concept trying to be taught rather than worrying about the type of problem.

Cleo referred to a teaching note in the Stickers lesson (TERC, 2008), which provides a means for teachers to relate units across grade levels. Cleo interprets this information to mean that students will understand the context of the problem allowing them to focus on the mathematical concept.

PSTs also identified educative features that supported lesson adaptation:

> James – Adaptation
>
> The lesson plan gives many suggestions after each section. Teachers who want to use a certain material sometimes have difficulty coming up with a relevant or integrated activity. By giving examples, the teacher is able to use these exactly, or better, use the examples to come up with their own to better fit their unique class.

James attended to the example problems given in *500 Hats* (UIC, 2008) and discussed the difficulty teachers may have adapting an activity that is relevant for their students.

Non-Educative Features Read Non-Educatively

Last, there were non-educative features read non-educatively. Most interesting in these cases were the interpretations.

Cynthia-Interpretation

The beans make the problem more visual and hands on for students. They can actually see how counting by larger amounts of beans is faster than counting by ones. The lesson plan tells me whether or not the students are ready for multiplication and division.

Cynthia attended to the jar of 172 jellybeans in the *Counting 172* lesson (UIC, 2008) and made an interpretation as to whether or not students would be ready for multiplication and division based on their representation of 172.

Discussion

Ball and Cohen (1996), Davis and Krajcik (2005), and others have suggested that curriculum materials might be designed in ways that are educative for teachers as well as students. However, an assumption underlying these suggestions has been that teachers will read features that are designed to be educative in educative ways. In other words, they will understand that the information provided in curriculum materials is intended for their own learning, as well as their students' learning, and they will read the features from that perspective. In this study, we were interested in whether this assumption held true for PSTs. The results were mixed. In some cases, PSTs did read educative features in educative ways, but, in many other cases, they read educative features in non-educative ways. The same could be said for non-educative features.

Acknowledgement

This work was supported, in part, by the National Science Foundation under Grant No. 0643497 (Corey Drake, PI). Any opinions, findings, conclusions, or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

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A SNAPSHOT OF PRE-SERVICE TEACHERS' CAPACITY FOR DESIGNING CURRICULUM

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The study examined pre-service teachers' capacity for designing curriculum. First, field supervisors (n =19) used a chart of standards for pre-service teachers and provided their perceptions of typical abilities of using and designing curriculum demonstrated by top performing pre-service teachers. Second, pre-service teachers (n =126) wrote lesson plans by three different modes of production. These modes include synthesizing, creating, and modifying. The pre-service teachers (n =47) responded to questions on self-efficacy of curriculum design. The findings indicated that high performing pre-service teachers were perceived to have yet to develop skills in designing lesson plans. Additionally, pre-service teachers demonstrated high self-efficacy for the process of implementing prescribed lesson plans, while revising them with student feedback and assessment data.

Keywords: Teacher Education-Preservice, Standards, Curriculum

Introduction

There is a conventional assumption that beginning teachers should know how to design an effective curriculum. One recent study reported that a majority of mathematics methods professors spend a bulk of their instructional time in teaching how to write effective lesson plans as well as unit plans (Kastberg, Edenfield, Sanchez, & Tyminski, 2012). However, this could occur at the expense of addressing other important knowledge and skills essential and necessary for the first year teachers. With limited instructional time, mathematics education professors are under pressure to prioritize knowledge and skills to cover in their methods courses.

In reality, mathematics educators in teacher preparation programs claim that most of their teacher candidates and first year teachers struggle to see the key math ideas within excellent tasks, let alone create such lessons (Fernandez, 1997; Ma, 1999; Menon, 2008). Experienced mathematics educators also suggest that designing lessons should not be on the list of qualities beginning teachers are expected to achieve, and that teacher education focus more on implementing excellent curricula rather than on having beginning teachers learn to develop curricula (R. Rubenstein, personal communication, May 21, 2012).

A Theory of Skill Acquisition and Needs for Research

There is an increased awareness that pre-service teachers' capacity for lesson planning might be developmental, and the art of writing a lesson plan is something to perfect over a longer period of time (Feiman-Nemser, Schwille, Carver, & Yusko, 1999; Korthagen, 2001). The Dreyfus model of skill acquisition shows that people go through a pattern of five stages—novice, advanced beginner, competence, proficiency, and expertise to acquire various skills (Dreyfus & Dreyfus, 1980). Although it is commendable for a novice to aim to master the knowledge and skills of experts, the essence of the theory is to support progress in the development of skills and to provide learners with opportunities and wait time to develop. Research also suggests that using students' feedback and assessment data should be key ways to shape the curriculum design

(Fink, 2003). It follows then that the task of designing curriculum might be more appropriate for experienced teachers who have years of experience with children and have access to student data.

If research provides a progressive development of learning to teach, people might begin to recognize teaching as a developmental progress just like learning. There exist, however, few studies to answer questions, such as: What is reasonable and important for teachers to know and be able to do as pre-service teachers, as early career teachers, as developing teachers, as master teachers, as curriculum developers, and as teacher educators.

Purpose of the Study

The study's intent was to examine the capacity for curriculum design demonstrated by pre-service teachers. There are different ways to understand constructs like curriculum, curriculum use and curriculum design, and they have been framed variously across studies (Remillard, 2005). In this study, curriculum design is defined to be an activity related to producing lesson plans and unit plans beyond the level of selecting, revising, and implementing curriculum. As the study presents a snapshot of pre-service teachers' capacity regarding curriculum design, it is hoped that the findings can inform the literature as to conceptualizing curriculum design and the developmental aspect of it, if any. In turn, mathematics educators and professional organizations of mathematics teacher preparation can develop a consensus on clarifying and revising current standards for beginning teachers. The study asked the following questions: (1) To what extent, can teacher candidates perform tasks related to curriculum use/design? and (2) Which areas in curriculum use/design, do teacher candidates demonstrate competency?

Methods

In order to answer the first research question, a group of experienced field supervisors (n = 19) participated in a survey in which (a) they were asked to choose the top five high performing teacher candidates in the last two years of their field supervision in middle grades mathematics classrooms and (b) they were to rate the pre-service teachers' capacity for curriculum use/design against state and national standards via a competency chart. The chart included two state standard documents and two national standard documents on curriculum use/design with fifteen items relating to instructional or curricular use/design activities. Each item includes the action verbs, such as create, modify/revise, synthesize, research, identify, select, organize, justify, enact/deliver, construct, sequence, and evaluate, which are typically associated with the words, use and design. Four sample items are provided as follows: (1) The teacher candidate can create developmentally appropriate and challenging learning opportunities grounded in mathematics education research. (2) The teacher candidate can use existing lesson/unit plans and revise them in ways that improve student learning. (3) The teacher candidate can select age-appropriate effective activities that relate to the linguistic, cultural, and socioeconomic background of students. (4) The teacher candidate can organize learning activities and teacher-led discussions that are sequentially appropriate and effective.

In order to answer the second research question, middle grades mathematics pre-service teachers (n =126) in five cohorts at two teacher preparation programs submitted lesson plans using a specified method (see below). Volunteer experts (n = 8) graded the submitted lesson plans using a rubric developed by a state department of education. The experts were master teachers and two doctoral candidates in mathematics education with ten or more years of mathematics teaching experience. Two of the experts had 4-5 years of teaching experience.

Table 1: Three Production Modes of Lesson Plan

	Specified Method	Competency Domain
Group A (n= 42)	To combine three short lesson plans to create one for a 90-min class with appropriate sequencing and select activities.	Synthesis Sequencing Selecting
Group B (n= 39)	Create an original lesson plan for a 90-min class.	Planning Creating
Group C (n= 45)	Given a prescribed lesson plan for a 90-min class, revise the lesson plan and provide teacher related notes/comments	Modifying Evaluating

The study then asked volunteer teacher candidates (n = 47) to use their lesson plans and actually teach a lesson. Afterwards, the study asked them to respond to the following questions: (1) How useful was your lesson plan in guiding instruction? (2) Rate the level of your satisfaction with the planning process.

Preliminary Findings

Preliminary data analysis indicated that those who were identified successful by the field supervisors were most competent in the teaching task of using, revising, or implementing curriculum and were least competent in the teaching task of selecting, sequencing, and creating. Table 2 shows four levels of competency and the related action verbs.

Table 2: Four Levels of Competency and the Related Actions

Rating	Extremely Low	Low	Average	High
Action verbs of standards	create, construct, sequence	organize, justify, evaluate, research, synthesize	identify, select	enact/deliver, modify/revise

Regarding the lesson plan task, in which experts evaluated lesson plans grouped by the modes of production, Group A displayed inconsistent qualities; Group B displayed low quality work; and Group C displayed high quality work. It was also found that teacher candidates in Group A demonstrated a moderate level of overall satisfaction about the process of writing a lesson plan and enacting the plan; Group B demonstrated the lowest level of satisfaction; and Group C demonstrated the highest satisfaction. Concerning the effectiveness of the sequencing of their lesson plans, Group C was the most satisfied followed by Group B and Group A. Unlike Group A and B, it was found that pre-service teachers in Group C examined assessment data and used student feedback to inform their revision process.

Discussion

The initial findings suggest pre-service teachers have yet to develop skills in sequencing lessons, but most were proficient in identifying appropriate learning activities. Pre-service teachers may benefit from organizing activities sequentially that lead to understanding a body of interconnected mathematical ideas. Also, pre-service teachers have yet to mature in designing lessons but are comfortable with revising existing lesson plans. This supports the notion that the efficacy of first year teaching is measurable, more appropriately, in the area of enacting the

curriculum, such as presenting information, interacting with students, or managing learning environments, than in the area of designing curriculum. It is then plausible that pre-service teachers need to implore effective skills for delivering curriculum and actually might be able to improve with appropriate learning opportunities in mathematics methods or mathematics education courses.

The survey found that pre-service teachers displayed the highest self-efficacy for revising, followed by synthesizing and creating. It is then conceivable that pre-service teachers might be able to build skills for synthesizing curriculum after they teach with a prescribed curriculum, refine the skills of presenting and interacting with students, and revise the curriculum with feedback and assessment data. Pre-service teachers could benefit from a prescribed curriculum with autonomy for revision in their field experience. Overtime, such experiences can prepare them then for organizing related activities together and eventually creating curriculum.

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STUDENT TEACHING EXPERIENCES AND EARLY CHILDHOOD MATHEMATICS PEDAGOGICAL BELIEFS: IDENTIFYING BARRIERS TO SUCCESS

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Much of the research on teacher pedagogical beliefs in mathematics focuses on teachers at the elementary level or above. In this study, a mixed methods sequential explanatory design was used to examine changes in preservice teacher beliefs related to early childhood mathematics during their student teaching experience. Findings indicate positive shifts in pedagogical beliefs occurred following student teaching, however follow up interviews identified barriers impeding increased shifts that were apparent across participants.

Keywords: Early Childhood Education, Teacher Beliefs, Teacher Education-Preservice

Purpose

Visions calling for shifts in beliefs about the nature of teaching and learning mathematics have been prominent in the literature over the past decade (Wilkins, 2008). However, few studies focus on early childhood teacher beliefs relating to mathematics. Results from international assessments like the Third International Mathematics and Science Study (TIMSS) indicate that teachers are still using instructional practices that do not align with the goals of the reform movement (Givvin, Jacobs, Hollingsworth, & Hiebert, 2009). These results indicate the need to examine teacher beliefs regarding mathematics teaching and learning prior to entering the classroom. This study examines changes in preservice teacher beliefs related to early childhood mathematics during their student teaching experience.

Perspectives

Ambrose (2004), in her study of initiating change in mathematics teaching, stated, “providing prospective teachers with intense *experiences* (emphasis added) that involve them intimately with children poses a promising avenue for belief change” (p. 117). Several researchers (Rethlefsen & Park, 2011; Burton & Pace, 2009), including Ambrose (2004), examined the change in mathematical beliefs of preservice teachers after completing a brief number of hours in a field experience completed as part of a course requirement. Positive changes in beliefs, such as an increase in self-reported efficacy in teaching mathematics to students with disabilities (Burton & Pace, 2009), were concluded in these studies.

Positive results were also found in studies when preservice teachers’ were completing their student teaching experience. In a longitudinal study by Swars, Hart, Smith, Marvin, and Talor (2007), the researchers examined the math pedagogical beliefs, teaching efficacy beliefs, and specialized content knowledge of 103 elementary pre-service teachers from the onset of a two course mathematics methods sequence to the conclusion of their student teaching experience. Interestingly, while student teaching, the participants’ pedagogical beliefs remained stable, but personal teaching efficacy increased. Another study that involved a twelve week student teaching experience (Nyaumwe, 2004), concluded that changes in learning mathematics were negligible, but there were many instances of change from teaching mathematics with a subject centered approach to a student centered approach.

Much of the research on teacher pedagogical beliefs in mathematics examines changes as the result of tasks implemented in a mathematics methods course, short field experiences included as a component of a methods course, or a specific professional development intervention in mathematics and does not extend the research to determine if belief change is sustainable. In addition, the overwhelming majority of this research focuses on teachers at the elementary level or above. The present study examines how a student teaching experience, immediately following an intensive course on early childhood methods for mathematics, can influence belief change related to mathematics pedagogy for early childhood preservice teachers (pre k-third grade). Arguably, mathematics pedagogical beliefs for these teachers should be at their highest immediately following a methods course emphasizing best practices in early childhood mathematics teaching. The research team in this study sought out to determine how a sixteen week student teaching experience could alter these beliefs and what factors were influential in producing change.

Methods

The central research question guiding this study was: How do preservice early childhood teacher beliefs related to mathematics pedagogy shift following a sixteen week student teaching experience? This question was examined through a mixed methods explanatory sequential design (Creswell & Plano Clark, 2011). Participants were the senior level cohort of the early childhood program at public university in southeastern United States. These participants (n=32) completed a methods course relating to early childhood mathematics in the fall of 2010 and began student teaching in the spring of 2011.

The quantitative portion of this study utilized a pre/post design where an instrument assessing beliefs relating to mathematical pedagogy was administered to participants prior to and following student teaching. The instrument, Integrating Mathematics and Pedagogy (IMAP), uses a case-based survey design to test teacher beliefs (Ambrose, Philipp, Chauvot, & Clement, 2003). Data analysis for the IMAP survey included examining individual responses and then coding them along seventeen different beliefs rubrics. Once this first round of coding was complete and agreement between coders was established, a Rubric of Rubrics (Ambrose, et al., 2003) was used to determine overall belief levels along seven constructs that center on three areas: Beliefs about mathematics; Beliefs about learning or knowing mathematics, and Beliefs about children's (students') learning and doing mathematics (Ambrose, et al, 2003). In phase two, the research team utilized a phenomenological approach to determine how or why shifts in beliefs about mathematics pedagogy occurred. Data were collected from 25 of the original 32 participants in the form of semi-structured interviews. Interviews lasted approximately 45 minutes to an hour and were conducted at each student teaching site during the last week of the spring semester in 2011. Qualitative data analysis followed steps outlined by Moustakas (1994) for a transcendental phenomenological design.

Findings

In phase one, descriptive statistics were used to analyze individual beliefs to determine if shifts in thinking occurred between pre and post administrations of the survey. The most dramatic positive shifts occurred in results for beliefs two (Pre: 1.0, Post: 2.3) and seven (Pre .8, Post 2.1). Belief two focuses on the notion that procedural understanding is connected to conceptual understanding, meaning that if the student could use an algorithm to solve the problem, then conceptual understanding is inherent. Belief seven indicates that the teacher's role

should allow for more student ownership over lessons. Although these two ideas were a central point of discussion throughout the methods course prior to this study, these results indicate that students did not agree prior to student teaching. However, shifts in these beliefs occurred for participants in the post test implementation.

Scores for beliefs three (Pre: 2.2, Post: 2.5) and four (Pre: 2.0, Post: 2.3) remained virtually unchanged. These scores measure beliefs regarding the importance of conceptual understanding in the mathematics classroom. Participants responded to these sections in the pre-test with an emphasis on building conceptual understanding before encouraging the development of procedural knowledge during lessons. The post-test results indicate that whatever shifts occurred for participants as a result of a course on early childhood mathematics methods immediately prior to this study were sustained throughout the student teaching experience.

It was unclear why shifts in beliefs occurred overall and what barriers impacted participants' ability to move towards strong levels of evidence for the seven belief areas. In phase two, participants were interviewed to determine what barriers, if any, existed during their student teaching experience. Four themes emerged following data analysis. These themes are described briefly below.

Surface Level Pedagogy

Participants described lessons as being hands-on and using manipulatives when asked to describe what they remembered from their methods course and how they taught math during student teaching. Few participants discussed the use of questioning, the role of the teacher or student, or any deeper notions about teaching mathematics.

Lack of support

Participants described a lack of feedback from cooperating teachers and university supervisors when teaching mathematics lessons. In addition, they were given very little support in lesson planning and many were discouraged from adapting a traditional textbook based lesson.

Mathematical Encounters

Participants remembered discussing the importance of group work and building conceptual understanding during their methods course, but found that it wasn't until they saw students struggling during lessons that they realized the importance of collaboration and the need to focus on concepts in addition to procedures.

Increased Self-efficacy

All participants described math as one of their favorite subjects to teach and that they felt confident with the subject matter and with developing effective lessons.

Significance

Generally, student teaching is the most meaningful part of a preservice teachers' preparatory program. If cooperating teachers and university supervisors are not encouraging preservice teachers' use of best practices and developing a deeper sense of understanding of early childhood mathematics, then these teachers will not be as well prepared to enter the classroom and work with young children. In addition, it is clear that these students did not have a deep understanding of what it means to teach young children mathematics, even though results on the IMAP showed positive shifts in these beliefs. Future research must be conducted to determine how much experience in teaching mathematics is necessary before students can move from a cursory to a deeper level of understanding about mathematics pedagogy. Further, future research must be conducted to further examine classroom practice during student teaching and the longitudinal impact on teacher beliefs and instructional practice in early childhood mathematics. Many

students described teaching lessons that seemed teacher directed in nature even though when asked to describe characteristics of effective lessons, they spoke of lessons being hands-on, engaging, and student centered. It is unclear if their perceptions of mathematics pedagogy translated to practice during student teaching experiences and if these practices will be sustained as they move towards teaching in their own classroom.

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MATHEMATICAL UNDERSTANDING IN A SOCIAL JUSTICE CONTEXT

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We discuss prospective teachers' engagement with, and reflections on, a task which situates mathematics in issues of social justice. Responses were analysed with respect to participants' abilities to notice and connect relevant and familiar mathematics to a new context. The intent of this exploratory study is to shed light on how knowledge of mathematics can develop and "thicken" through its contextualisation in important world issues.

Keywords: Teacher Education-Preservice, Teacher Knowledge

We present initial findings from a larger study which investigates individuals' understanding of mathematics when embedded in issues of social justice. Current research has emphasized the potential of mathematics in helping learners make sense of important world issues (e.g. Bartell, 2011; Gutstein, 2006), however, as of yet, there are no studies which focus on teachers' capacity to recognize and use relevant mathematics when addressing such issues. We see this as an important area for research, and one directly related to how teachers may (learn to) teach mathematics for social justice. As such, this paper explores the question: *What mathematics are prospective teachers able to notice and mobilize when engaging with issues of social justice?*

To address this, we presented prospective teachers with word problems that embedded secondary school mathematics concepts within the context of important world issues, and analysed their engagement with, and reflections on, the problem solving experience. In this paper we focus on one task, *Issues in Fair Trade*, and use elements of the Pirie-Kieren Theory for the Growth of Mathematical Understanding to analyse the nature of participants' understanding and use of mathematics. In particular, we consider whether participants observed the relevance of mathematics, in general, to the task, and whether they could identify and use specific, appropriate mathematical concepts.

Background and Context

A critical area of research in mathematics education continues to be that of developing pedagogy that will raise levels of achievement and engagement through the use of 'context problems', such as ones that frame mathematics within issues of social justice (e.g. Bartell, 2011, 2013; Gutstein, 2006). Such problems are touted as offering students "instruction that includes the mathematics deemed necessary for success in the current [school] system while simultaneously providing students an opportunity to use mathematics to expose and confront obstacles [in society]" (Bartell, 2011, p.1). Notwithstanding popular advocacy, "enthusiasm for context problems appears to be in advance of the evidence for their effectiveness" in enhancing understanding of mathematics (Beswick, 2011, p.387).

Addressing the concerns of effective use of context problems in teaching requires, in our view, initial attention paid toward the teacher's capacity to interpret, reason with, and resolve, such problems themselves. Research (e.g. Watson, 2008) has identified a need for teachers to accrue more, and different, personal experiences with mathematics as a means to enhance their pedagogy. Rowland, Thwaites, and Huckstep (2003) note a relationship between teachers'

mathematical knowledge and their choices of examples and problems, while Bartell (2011, 2013) reports on challenges of lesson planning with context problems related to issues of social justice. Recognizing that teaching mathematics for (and through) social justice requires the negotiation of learning goals aimed at developing awareness of social issues and developing mathematical fluency and academic success (Gutstein, 2006), we see an important need for research that invites teachers to accrue personal experiences reasoning mathematically about issues of social justice and to explore how these experiences may shape understanding of both the mathematics and mathematics teaching. In this paper, we focus our attention on the former.

Theoretical Framework

The Pirie-Kieren Theory for the Dynamical Growth of Mathematical Understanding (e.g., Pirie & Kieren, 1994) offers a way to characterise understanding as an emergent and ever-changing phenomenon. Central to the theory is the notion of "images" – the mental representations that learners make and hold for mathematical concepts and ideas. Through the notion of "folding back" the Theory emphasises the non-linear nature of the growth of understanding and that while learners may make and have specific images at a particular moment these are not static or fixed. Instead, as learners meet new contexts or problems for which their existing images are inadequate or insufficient, the Theory suggests they will fold-back in order to modify and re-make these to cope with the new situation (the metaphor of folding back employs the notion of "thickening"). In this study, folding back offers a way to talk about the way in which prospective teachers, who already have understandings and images for particular mathematical concepts, are prompted to fold back and thicken these from the social justice context. The intent of the problems posed was to invite participants to select appropriate mathematical concepts and then apply them in their solutions.

Methodology

We presented a selection of tasks to 30 prospective mathematics teachers enrolled in a secondary education pre-service program, and invited them to choose one and work on it over the span of two months. Group meetings were organized one to two times per month, where participants were able to collaborate on the problems, pose questions to the task designers, and to discuss some of the embedded issues. Participation was voluntary, and after the initial tasks were completed, participants were asked to respond to a follow up questionnaire designed to help "unpack" their solving strategies and explore their pedagogical beliefs in connection to social justice context problems. For each task, participants were given an introduction to the issue via readings, online resources, and group discussion, and were subsequently asked to respond to word problems, as depicted in the task presented in Figure 1.

> You've recently been hired as a photo-journalist to produce a documentary investigating fair trade. You've decided to focus your story on issues in chocolate trade from, and to, various countries. You'll be given full funding for the project for 30 days provided you submit a proposal that includes an itinerary with a justification for your route.
>
> 1. What social justice issues, which are specific to chocolate trade, inform current and prospective international policies on fair trade? Include a mathematical and/or statistical justification for why these specific issues are of concern to policy makers.
> 2. Create a detailed travel plan for your 30 days. Please include travel times and modes of transportation between and within countries, and explain what considerations you took into account in determining your route. Include a brief statement to justify that you've made the best choice.

Figure 1: Social Justice Context Problem: *Issues in Fair Trade*

We analyse the responses of two participants, Aster and Cody, whose responses exemplified the broader trends we observed in the data. Our analysis sought evidence of folding back by considering instances where participants were invited (explicitly or implicitly) to make a connection to known mathematical concepts, and thus "thicken" their understanding by modifying their images of the concepts via links to the new (for them) context of social justice issues in fair trade. These instances were cross-analysed with responses to follow-up questions which invited participants to reflect on the mathematics they could have used to solve the task.

Results

Although there is not space here to offer a full discussion of our results, we offer a few key findings. In short we observed that while participants saw a need to use mathematics to respond to the problem, they had difficulty identifying what mathematical concepts were relevant and how to utilize them. For instance, in response to item 1 both Aster and Cody were able to identify pertinent social issues related to chocolate fair trade, such as child labour, uneven distribution of profits, and inequitable working conditions, but neither provided mathematical or statistical evidence to support their claims – despite the explicit invitation to do so. This was particularly noticeable for issues such as profit distribution where a connection to numerical data would be a natural way to illustrate the extent of the imbalance. It was also surprising, as both Aster and Cody had, in previous group discussions, identified statistics, currency, budgeting, charts, and diagrams, as relevant mathematical concepts to the problems.

In response to item 2, where the request for mathematical reasoning was implicit in the request for an itinerary with justification of the "best choice" in route, we again found that participants' attention was strongly focused on the social issues, and avoided using mathematical representations, organizations, or specific content, in their responses. For example, Aster organized her itinerary by day and country (e.g. Days 1-5, Cameroon, Days 6-10, Nigeria), and while she included (sometimes very) specific times estimated via Google maps (c), these figures were often included in parentheses as an aside, rather than as organizers for time management in the itinerary. Similarly, Cody's justification for the "best" route focused on issues of photography (e.g. what could be "easily captured by camera") and social justice (e.g. "to provide the opportunity to be part of [a reform] initiative"). While both participants included reasoning that used mathematics such as data management and optimization (e.g. choosing hotels based on proximity to locations of interest, taking a helicopter to save time) these considerations were sparse, avoided precise mathematical terminology, and tended to be secondary to the social issues. Thus, while we could identify instances of mathematics in their responses, we interpret their lack of explicit attention toward these concepts as missed opportunities to fold back and create new contextual links. The follow-up questionnaire aimed to unpack the mathematics with the prompt: "Some individuals identified as key mathematical content for the task: optimization, data management, and statistics. Do you agree?" Aster and Cody responded as follows:

> *Aster:* "To a certain extent, I do agree with this mathematical content being present… however I am unable to identify where the mathematical content is present in question two. We spent several hours working on question two, creating a detailed, logical, interesting travel plan, and we could not develop ideas on how to incorporate math."
>
> *Cody:* "I agree… for example, optimization could have been useful to determine the best route for our travel plan… For data management, this concept would have been useful to organize the researched material and create a set up to answer the problem… Looking at the statistics from the countries we were planning on visiting would have allowed us to access

facts about the social issues in these countries."

Hence, although participants noticed a need to use mathematics in general, and raised the question "*what* mathematics would I need to understand and examine this issue?" (Bartell, 2013, p.160, emphasis added) on their own, they were not able to answer it, even though the concepts were within their repertoire, as evidenced by Cody's reflections. We suggest participants recognised the need to fold back and draw their existing mathematical understandings into the new context throughout their engagement with the task, but could not "develop ideas on how to incorporate math." Cody's reflection suggests that when his attention was specifically directed to the relevant mathematics he realised ways in which the concepts could illuminate the social justice issues, however his remarks remained fairly general, and more research is needed to determine the strength of these connections.

Concluding Remarks

It is beyond the scope of this paper to offer a full explanation of why the participants struggled to fold-back and engage with the appropriate mathematical concepts in the way the task intended. However we would note that to simply pose problems may be insufficient. Even when confident that people possess the mathematical understandings necessary for the task it is the connection of these to the new context that remains a challenge. In our research the course instructor deliberately did not intervene to help the students locate and draw on specific mathematical ideas (e.g. optimization) and this kind of explicit, focused invocative intervention (XXXX) may be necessary for effective folding back to occur.

Both Aster and Cody were enthusiastic about exploring mathematics in a social justice context problem and expressed a desire to incorporate similar problems into their own teaching. Given this enthusiasm, and the current popularity of teaching through context problems, our findings suggest a need for further research. Namely, if we expect (or hope for) teachers to teach mathematics through issues in social justice, then our research shows a need for them to accrue appropriate experiences that will help them "see" the relevant connections that may be made to the context or by their students. Building on our initial findings, our on-going research explores what interventions are necessary or sufficient to provoke effective folding back in connection to a social justice context. In particular we wonder: *How can we facilitate robust, contextualised, images that connect relevant mathematics concepts to issues of social justice? And, how can these experiences translate to teachers' own lesson planning and pedagogical choices?*

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PROVIDING HIGH-QUALITY PROFESSIONAL DEVELOPMENT FOR ALL

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In this paper we describe the preliminary results of the distance education professional development program designed to meet the needs of thirty-seven secondary mathematics teachers from the southeastern part of the United States. During this study, researchers measured: (1) gains in teacher content knowledge, (2) impact on student achievement, (3) progress towards meeting the assessed needs of participating school divisions and private schools. The researchers developed and administered rigorous measures of teachers' knowledge, analyzed assessment data of student achievement, and conducted surveys and classroom observations. Preliminary results indicate a significant growth in teachers' subject matter knowledge, their students' achievement on standardized tests, change in teachers' professionally situated knowledge, and substantial progress towards meeting needs of the local educational agencies.

Keywords: High School Education, Teacher Education – Inservice/Professional Development

In this paper, we report preliminary results of a professional development program designed to meet the needs of 37 secondary mathematics teachers from 33 school divisions in the southeastern part of the United States. Participating teachers previously had limited access to quality professional development opportunities and were from different parts of the state. They participated in the synchronous distance learning/on-line professional development program of six to ten credit hours of graduate coursework. Researchers' goal was to increase teachers' mathematical content knowledge and mathematical knowledge for teaching in order to change classroom instruction (Author, 2006; Author 2011; Ball, Bass, Goffney, & Sleep, 2006; Hill, Ball, & Schilling; 2008; Simon, 1997). During this study, researchers measured: (1) gains in teacher content knowledge, (2) impact on student achievement, and (3) progress towards meeting the assessed needs of participating school divisions and private schools. The researchers developed and administered rigorous measures of teachers' knowledge, analyzed assessment data of student achievement, and conducted surveys and classroom observations during this project. Preliminary results indicate a significant growth in teachers' subject matter knowledge, their students' achievement on the standardized tests, change in teachers' professionally situated knowledge, and substantial progress towards meeting needs of the local educational agencies.

Data Collection

The data collection was designed to measure any change in teacher content knowledge, impact on student achievement, and progress towards meeting the assessed needs of participating schools and divisions. All 37 teachers provided data required in this study.

Measures of Gains in Teachers Content Knowledge

Pre-test and post-test for participating teachers. One of the main goals for this project was to increase participating teachers' mathematical content knowledge of Algebra, Geometry, and Data Analysis. Teachers completed pre- and post-assessments for each mathematics content course and mathematics education course. These assessments consisted of a range of problem-solving activities which were analyzed for the initial and final content proficiency. These pre- and post-assessments will allowed the researchers to analyze and observe any changes in teachers' content knowledge of Algebra, Geometry, and Data Analysis as well as gains in mathematical knowledge for teaching this content. The researchers relied on assessments produced by researchers at: 1) Michigan State

University, Knowing Mathematics for Teaching Algebra (KAT) Project (NSF REC- 0337595; 2006); 2) University of Michigan, Content Knowledge for Teaching Mathematics Measures (CKTM) Project (NSF REC-9979873, REC- 0207649, HER- 0233456 & HER-0335411); and 3) [Author, 2011] to inform choices as they created assessment instruments for each course offered in this professional development program.

Classroom observations and analysis. In addition to quantitative data, researchers collected qualitative data through multiple classroom observations of a representative selected group of five participating teachers. The purpose of these observations was to gather additional data on whether changes that might have occurred in teachers' content and pedagogical content knowledge were also influencing students' mathematical experiences in the classroom. Through this data, researchers assessed the quality of teacher-student interaction and teachers' level of mathematical content and pedagogical content knowledge presented in a mathematics classroom. Instructional Quality Assessment (IQA) rubrics were used to evaluate mathematics teachers' content knowledge and mathematical knowledge for teaching through the quality of students' classroom experiences in Algebra 1; Algebra 2; Algebra, Functions and Data Analysis and Geometry (Matsumura et al., 2006).

Teachers' survey. In addition to summative feedback in the form of participant pre- post assessments and classroom observations and analysis, the researchers implemented teacher surveys. These assessments consisted of specific prompts/questions which addressed either content or pedagogical content knowledge. These surveys provided the researchers with valuable, real-time feedback about participants' understanding about mathematics content, beliefs about mathematics teaching and learning, their perceived abilities to design quality assessments in their mathematics courses.

Measurement of Gains in Student Achievement

Pre-test and post-test for students in participating schools. One of the major aims of this project was to increase student's understanding and knowledge of mathematics, specifically Algebra 1, Algebra 2, Geometry, and Data Analysis. The researchers used standardized test items provided by the state. Researchers collected and scored students' pre-test and post-test scores at the beginning and the end of each course.

Classroom observations and analysis. Observation and analysis of lessons of five participating teachers were used by the project directors to assess the quality of students' experiences and the level of their interactions with peers and teachers in the classroom. Instructional Quality Assessment (IQA) rubrics were used to measure the students' understanding of the mathematics discussed. (Matsumura et al., 2006).

Measurement of progress in meeting needs of LEAs. Researchers used assessment results to develop indicators when tracking the progress towards meeting assessed schools' and divisions' needs at each stage of the project.

Surveys of participating school administration. The surveys of school administrators were conducted twice during the project. Participating school administrators were be able to share their insights on how well this project progresses towards meeting their schools needs including but not limited to the teachers professional development in the areas of Algebra, Geometry, and Data Analysis.

Surveys with participating teachers. Anonymous on-line surveys gave participating teachers opportunities to discuss strength of and give suggestions for this program in an open, relaxed and non-threatening environment.

Course fidelity assessments. Course fidelity assessments were conducted to gather data on the courses included in the project. They examined if the courses were designed and taught in ways that corresponded with needs identified by local schools during the initial assessment of needs process.

Analysis &Preliminary Results

The data collection and analysis has not been completed for this project. We are reporting preliminary data and are discussing the portion that has been collected and analyzed.

Impact on Increasing Teacher's Knowledge

Pre and post-tests were developed by instructors and implemented at the beginning of the course and again at the end. Tests were scored based upon percent correct. Results were analyzed via paired sample t-tests to determine if any significant improvements were made in teachers' knowledge after participating in these courses. Thirty-four teachers completed assessments.

A paired samples t-test found a statistically significant improvement between pretest (M=28.42, SD=17.7) and post-test scores (M=94.68, SD=11.20) for the first graduate professional development course with a content focus on Linear Algebra for Teachers. Improvements in test scores ranged from 21% to 90% with 84% of teacher participants improving their score by 50% or greater.

A paired samples t-test found a statistically significant improvement between pretest M=40.00, SD=18.42) and posttest scores (M=91, SD=10.92) for second graduate professional development course with the content focus on Number Systems. Improvements in test scores ranged from 25% to 77% with 60% of teacher participants improving their score by 50% or greater.

A paired samples t-test found a statistically significant improvement between pretest (M= 33.53, SD=18.69) and posttest scores (M=93.06, SD= 11.07) when combining both courses' test scores. Improvements in test scores ranged from 21% to 90% with 73% of teacher participants improving their score by 50% or greater.

Superintendents Reports

A 5-item on-line survey was developed and superintendents that had teachers enrolled in the professional development project were asked to complete the survey during late fall 2012. Six superintendents completed on-line survey. Based on the survey results, superintendents were satisfied with the professional development program as demonstrated through their responses to an on-line anonymous survey.

Teachers' Feedback on the Professional Development Program

An 18-item on-line quantitative survey (with 1 additional open-ended question) was developed to get feedback from teachers in the professional development program during late fall 2012. Fourteen students who began participation in the program in the summer or fall of 2011 completed this on-line survey.

Table 1: Student Feedback on the Professional Development Program.

	Strongly Agree	Agree	Disagree	Strongly Disagree
The program provided me with the professional development opportunities I needed.	64%	36%	0	0
I had access to all the resources needed to be successful in the program.	71%	29%	0	0
Knowledge (mathematical and pedagogical) I gained in the program will help e to teach high school level mathematics more effectively.	46%	46%	8%	0
I was able to share my new professional knowledge gained from the program with other teachers in the field.	46%	46%	8%	0
The program gave me opportunities to meet and network with new people.	62%	38%	0	0
Meeting and networking with new people in the	38%	54%	8%	0

program will help me be a more effective high school teacher.				
The distance education format for the program worked well for me.	38%	62%	0	0
The information I learned in the program will improve the way I interact with my students.	46%	46%	8%	0
My participation in the program has increased my own students' ability to learn.	33%	58%	8%	0
I would recommend the program to my colleagues.	54%	46%	0	0

Teachers' scores significantly increased between the beginning and the end of the courses demonstrating that the goal of having teachers' knowledge increase was met. In addition, based on the self-reported data, teachers reported positive experiences with the distance education professional development program.

The data collection and analysis on this project is on-going and the complete report will be provided at the PMEMA presentation along with copies of all surveys and selected assessment instruments developed in the project.

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PRE-SERVICE ELEMENTARY TEACHERS' PROCEDURAL KNOWLEDGE OF THE GREATEST COMMON FACTOR AND THE LEAST COMMON MULTIPLE

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Pre-service elementary teachers demonstrated on a survey either superficial or deep procedural knowledge of the greatest common factor (GCF) and least common multiple (LCM). Those exhibiting deep procedural knowledge varied the processes used to determine the GCF and LCM based on the numbers' representation. Those exhibiting superficial procedural knowledge followed the same procedure regardless of the numbers' representation. Statistically significant evidence suggested that pre-service elementary teachers who define the GCF (and LCM) through the relationships between two or more numbers and their GCF (and LCM) also demonstrate deep procedural knowledge of the GCF (LCM).

Keywords: Teacher Knowledge, Teacher Education-Preservice

Introduction

Research has shown that teachers sometimes lack the content knowledge that is needed to teach mathematics in the elementary school (Ma, 1999). Without possessing this mathematics content knowledge "teachers lack resources necessary for solving central problems in their work—for instance, using curriculum materials judiciously, choosing and using representations and tools, skillfully interpreting and responding to their students' work, and designing useful homework assignments" (Ball, Lubienski & Mewborn, 2001, p. 433). A first step in discovering what mathematics content knowledge is needed to teach elementary mathematics is determining what mathematics knowledge teachers currently possess. A critique of the research in this area by Mewborn (2001) notes that only a narrow range of content areas have been investigated. Teachers' content knowledge in areas such as division, place value, rational numbers, and geometry have been researched by many studies, while contemporary areas such as number theory, probability, and data analysis have been investigated by fewer studies. Studies in a variety of mathematics content areas, such as number sense and division, have shown that in-service and pre-service teachers can perform standard computations, but do not necessarily possess the knowledge needed to explain the meaning of the procedures (Mewborn, 2001).

Zazkis and Campbell (1996) studied pre-service elementary teachers' understanding of divisibility and its relation to division, multiplication, prime and composite numbers, factorization, divisibility rules, and prime decomposition. They found that their respondents showed a disposition towards procedural thinking, even when they displayed a conceptual understanding of the topic. Hiebert and Lefevre (1986) characterize procedural knowledge as memorization of facts and algorithms used to solve mathematical tasks. Star (2005) later argued for a reconceptualization of procedural knowledge and asserted that a distinction must be made between those learners with superficial and deep procedural knowledge. "There are subtle interactions among the problem's characteristics, one's knowledge of procedures, and one's problem-solving goals that might lead a solver to implement a particular series of procedural actions" (p. 409). If a solver possesses a superficial knowledge of the procedures s/he may fall back on the known standard procedure to solve the problem, regardless if it is the most efficient process. If the solver instead possesses deep procedural knowledge, s/he may use various

techniques to produce a solution that best matches the form of the problem.

Research Questions

The research questions driving this study are: (a) What knowledge do pre-service elementary teachers' possess of the GCF and the LCM, and how would this knowledge be classified through the lens of superficial and deep procedural knowledge? (b) How do pre-service elementary teachers define the GCF and LCM, and is there an association between their definitions of these concepts and the forms of knowledge that they exhibit?

Methodology

In this study, I gathered data from a survey instrument in order to draw generalizations regarding the associations between displayed forms of knowledge in the population of pre-service elementary teachers.

Participants

The participants for this study were 48 (45 female, 3 male) students enrolled in an elementary teacher preparation program at a large, private university in the northeastern United States. The students were enrolled in a mathematics content course designed for pre-service elementary school teachers. They spent one day of coursework and activities discussing the GCF and LCM, less than one month before completing the survey.

Survey Data Collection

The respondents completed a survey instrument during the first 20 minutes of their regular class time. Two items asked the participants to define the GCF and the LCM, and the remaining items asked them to determine the GCF and LCM of two numbers, where varying representations of these numbers were given to assist them. I constructed the survey in this manner in order to investigate if the participants would apply the same procedure to find the GCF or LCM in each of these situations, or if the change in the numbers' representations would allow them to find the solutions in different and possibly more efficient ways. The survey tasks asking to determine either the GCF or LCM are listed below:

1. Find the greatest common factor of 650 and 275. Explain your reasoning.
2. Find the greatest common factor of the numbers $M_1 = 2^2 \cdot 3 \cdot 7 \cdot 11$ and $N_1 = 2 \cdot 3^2 \cdot 5 \cdot 11$ Explain your reasoning.
3. Find the greatest common factor of the numbers 884 and 510. The factors of each number are given below. Explain your reasoning.
   Factors of 884: {1, 2, 4, 13, 17, 26, 34, 52, 68, 221, 442, 884}
   Factors of 510: {1, 2, 3, 5, 6, 10, 15, 17, 30, 34, 51, 85, 102, 170, 255, 510}
4. Find the least common multiple of 48 and 56. Explain your reasoning.
5. Find the least common multiple of $M_2 = 2^3 \cdot 3 \cdot 13$ and $N_2 = 2^2 \cdot 3^2 \cdot 7$. Explain your reasoning.
6. Find the least common multiple of 112 and 252. The first ten multiples of each number are given below. Explain your reasoning.
   Multiples of 112: {112, 224, 336, 448, 560, 672, 784, 896, 1008, 1120, ...}
   Multiples of 252: {252, 504, 756, 1008, 1260, 1512, 1764, 2016, 2268, 2520...}

Findings

The respondents demonstrated what I categorized as either superficial procedural knowledge or deep procedural knowledge. Those demonstrating superficial procedural knowledge of the GCF and LCM applied the same procedure to each survey item regardless of the numbers' representation. Those exhibiting deep procedural knowledge varied the processes used to determine the GCF and LCM and applied more efficient methods based on the numbers' representation. One survey item asked respondents to find the GCF of two numbers which were represented by a list containing all of their factors. Those demonstrating superficial procedural knowledge did not utilize this list, found the prime factorization for each number, and then used these prime factorizations to determine the GCF. Those demonstrating deep procedural knowledge exhibited an understanding that this representation of the numbers is transparent with respect to their GCF, and determined the GCF by finding the largest factor shared on each list of factors.

The survey responses defining the GCF and LCM fell into two categories: (1) those describing the relationship between two or more numbers and their GCF and LCM; and (2) those detailing a process that can determine the GCF and LCM for two or more whole numbers involving the numbers' prime factorizations. Table 1 shows typical correct and incorrect survey responses for the definitions of the GCF and LCM. The percentages of the participants who fell into each category are also listed.

Table 1: Typical Participants' Definitions of the GCF and LCM

		Definitions Describing Processes		Definitions Describing Relationships	
GCF	Correct	"You find the prime factors of the whole numbers and multiply the lowest powers of the common factors"	23%	"The highest number that can be divided evenly in two or more whole numbers"	38%
GCF	Incorrect	"Do prime factorization, then find the largest number in both prime factorizations"	13%	"The biggest number that is divisible by all of the numbers involved"	21%
LCM	Correct	You multiply the highest powers of common factors and the not common factors"	17%	"It is the smallest number that two numbers can go into"	42%
LCM	Incorrect	"The least common multiple is multiplying all the factor numbers together, not just the shared ones"	15%	"The smallest number that divides into both numbers evenly"	23%

Note: The remaining 6% (2% incorrect, 4% correct) of participants' responses for GCF definitions gave a definition that combined both types of definitions. The remaining 4% of responses for LCM definitions left the item blank.

GCF Definitions and Use of the Factor List

Of the 48 participants, 28 used the list of factors for each number in item 3 to correctly determine the GCF. In these responses, the participants observed that 34 was the largest factor shared by each list of factors for the numbers, and recognized it to be the GCF without performing the procedure that requires determining each number's prime factorization. Of this group of 28 participants, 19 defined the GCF through relationships, with 15 of those defining it

correctly. Only nine of the total participants choose to use the prime factorization procedure to correctly solve for the GCF in item 3 rather than use a different procedure that took advantage of the given list of factors. Of these nine participants, four had defined the GCF though relationships, with only one of those defining it correctly. I compared the number of participants who defined the GCF correctly through relationships, out of those who used the list of factors to solve for the GCF in item 3, with the number of participants who correctly defined the GCF though relationships, out of those who performed the prime factorization procedure to determine the GCF in item 3. I used Fisher's Exact Test and the data revealed statistically significant evidence ($p=0.03$) suggesting that pre-service elementary teachers who use the list of factors to correctly determine the GCF on this survey item are more likely to correctly define the GCF through relationships than those who found it correctly using the prime factorization procedure.

LCM Definitions and Use of the Multiples List

Of the 48 participants, 35 correctly used the list of multiples in item 6 to determine the LCM. In these responses, the participants observed that 1008 was the smallest multiple shared by the list of multiples for each number, and recognized it to be the LCM without performing the procedure that requires determining each number's prime factorization. Of this group of 35 participants, 26 had defined the LCM through relationships, with 18 of those defining it correctly. Only 12 of the total participants choose to use the prime factorization procedure to correctly solve for the LCM in item 6 and did not use the given list of multiples. Of these 12 participants, five had defined the LCM through relationships, with two of those defining it correctly. I compared the number of participants who defined the LCM correctly through relationships, out of those who had used the list of multiples to correctly solve for the LCM in item 6, to the number of participants who had defined the LCM correctly through relationships, out of those who had used the prime factorization procedure to correctly solve for the LCM in item 6. I again used Fisher's Exact Test and the data revealed statistically significant evidence ($p=0.04$) suggesting that pre-service elementary teachers who use the list of multiples to correctly solve for the LCM on this survey item are more likely to define the LCM correctly through relationships than those who found it correctly using the prime factorization procedure.

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USING SCHOOL-UNIVERSITY PARTNERSHIPS TO CONNECT THEORY TO PRACTICE: A MODEL FOR ELEMENTARY MATHEMATICS TEACHER PREPARATION

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This study explored a model for teacher preparation that closely links theory taught in elementary mathematics methods courses with classroom practice. In this study, four elementary student teachers and six practicing teachers participated in professional development that focused on the teaching and learning of fractions.

Keywords: Teacher Education-Preservice, Teacher Education-Inservice/Professional Development, Elementary School Education

Introduction

The preparation of elementary teachers is often inadequate for producing the types of knowledge needed to enact practice where all children learn meaningful and significant mathematics (National Research Council, 2001). We argue that this is, in part, a result of the current structure of traditional teacher preparation programs, where foundational knowledge is separated from methods courses and coursework is disconnected from practice (Grossman, Hammerness, & McDonald, 2009). If teacher education programs are not currently organized to best support teacher learning about practice, to what extent do alternative models of preparation have the potential to do so? Specifically, we explore the extent to which a Book Study supports teacher learning.

Review of the Literature

The existing structure of teacher education programs are problematic for several reasons. Theory is often separated from the practical work that teachers do in the classroom, making practice a nonessential aspect of preparation (Grossman et al., 2009). Prospective teachers rarely use theory taught in teacher education programs, leaving them unprepared to solve problematic situations they encounter as novice teachers (Korthagen & Wubbels, 2001). Additionally, prospective teachers need to understand teaching differently than what they experienced as learners in traditional classrooms (Hammerness, Darling-Hammond, Bransford, Berliner, Cochran-Smith, McDonald, & Zeichner, 2005). Lortie (1975) first described this dilemma as the *apprenticeship of observation*, where prospective teachers' beliefs about teaching are based on the ways in which they were taught.

Book Study Model

The literature suggests that an effective model of teacher preparation should emphasize a connection between theory and classroom practice that provides opportunities for enacting clinical experiences. Thus, we propose a model for teacher preparation that closely links theory taught in methods courses with classroom practice. In our model, teams of student teachers (STs) and their cooperating teachers (CTs) participated in a CGI Book Study, focusing on theory in relation to children's thinking about fractions and how to utilize this knowledge in practice (Empson & Levi, 2011). An important component of this model involved ST/CT teams

implementing CGI problems, selecting student work, and reflecting on their interpretations of students' understanding in their discussions with other STs and CTs.

Methods

Context and Participants

This year-long study took place in a rural elementary school in the southeastern United States. Four ST/CT pairs participated in this study, two 3rd Grade teams and two 5th Grade teams. In addition, a 3rd Grade and 4th Grade teacher at the school also participated. Teachers' years of teaching experience ranged from 1 year to 16 years (M_{years} = 6.5 years, s = 4.9805). CTs with STs years of teaching experience ranged from 5 years to 16 years (M_{years} = 8.75 years, s = 5.1881).

Data Collection and Data Analysis

Sources of data for this study include interviews and video recordings of Book Study sessions. STs and CTs were asked to participate in interviews. Three of the 4 STs agreed to be interviewed, and all of the CTs participated. Semi-structured interview protocols were used. In the analysis process, a codebook was created through an iterative process to develop data-driven codes (Decuir-Gunby, Marshall, & McCulloch, 2011). Open coding and axial coding were used to analyze the interview data (Corbin & Strauss, 2008). The Powell, Francisco, and Maher (2003) model was used to analyze video recordings of the Book Study sessions.

Results

All participants indicated that one of the most valuable experiences in the Book Study was learning about the progression of students' thinking in relation to fractions across the grades. For example, Wendy a 5th Grade ST explained that "it was really beneficial to look at the work of the other grade levels." Further, she stated that it was beneficial to look "across the spectrum of K through 5th. So, having those work samples to see in 3rd Grade they're learning this, and by 4th Grade they should be at this step. And, by 5th Grade this is where they should be." Phoebe, a 5th Grade CT, explained that "for the first time we had a vertical PLC, essentially. I mean it was three-five, and we don't do that often. We just don't have the time. So, to have all of us there represented and watch the total trajectory." Another 5th Grade teacher, Leigh, stressed the advantages of working with her 3rd and 4th Grade colleagues:

> I really find that it's refreshing to get together with colleagues. It's hard for us to have that vertical alignment from 3rd Grade to 5th Grade to be able to talk about what we're teaching and how we're teaching and what's working in our classes and what's not working. And, by using the Fraction Study as a vehicle, that gave us a lot of opportunity to share and listen to one another and also just that refreshing new input from the student teachers, as well. So, that was I think one of the biggest strengths, as well, that we were just able to take that time and spend some time talking about a subject that was important to us and seeing it from different perspectives and that 3rd through 5th Grade alignment of what's going on with our students.

While all teachers benefited from learning about how children's mathematical thinking progresses, some findings of this study relate specifically to the ST/CT pairs and STs.

ST/CT pairs. Two of the three ST/CT pairs, the 3rd Grade teacher pairs, indicated that collaborating together in the Book Study was a catalyst for building common understanding about the teaching and learning of fractions between the STs and CTs. Specifically, the Book Study supported the 3rd Grade teacher pairs in developing a common language, or common

reference point, to identify children's strategies. Similarly, studying the Empson & Levi (2011) book together provided these ST/CT pairs with a common language for discussing their interpretations of children's understanding of fractions from analyzing children's strategies. These teacher pairs also indicated that collaborating in the Book Study provided occasions for designing common assessments and lesson planning. For example, Penny, a 3rd Grade CT, explained how her past STs had difficulty with content and pedagogical content knowledge in relation to fractions and contrasted this with her current ST:

> They have difficulty with the background, just understanding the background of fractions and understanding where our kids come from and moving them and progressing them. And, so I thought the Book Study was really good because, um, it gave us an opportunity to work with the student teachers, and it gave them the background knowledge they needed before they started teaching and before they planned the lessons. And, so, when they were actually watching the kids do the work they had a lot more background knowledge to pull from about what they were thinking and how they were thinking, and, then, it just allowed us to be on the exact same page so when we communicated on plan lessons. We were using the same language. We were talking about the same thing. We were talking about the CGI stuff, and it wasn't like we had two different beliefs in terms of teaching. We both had the exact same beliefs, and we all were going, working towards the same goal.

Penny's statement highlights another important aspect of building common understanding, sharing beliefs and goals. The 3rd Grade ST/CT pairs shared a vision for teaching mathematics that focused on: a) posing problems to children (without showing them how to solve the problems), b) encouraging children to make sense of these problems using their own strategies and representations, c) listening to children's explanations of their strategies, and d) providing opportunities for children to explain and discuss their reasoning with their peers.

STs. All three of the STs learned to focus on eliciting children's strategies and explanations rather than simply looking for "right" or "wrong" answers. They encouraged children to use pictures, diagrams, and other representations to model their mathematical ideas. Wendy stated:

> I feel like this Book Study, an important thing that I have learned is to always have that explanation time at the end of the time of where the students are solving the problem, to always come back and, um, have the students share their ideas. And, not necessarily view control or, um, put words in their mouth as they're presenting, but kinda lead them on and what types of questions to ask when they are presenting problems and that type of thing. So, I feel during MATH 307, we didn't really have that opportunity to do that. It was more of us being, we were the ones being, um, provoked and you know asked questions to put us in their shoes almost. And, now it was our turn, in this Fraction Book Study, to say, "Ok, this is what you need to be asking to lead these students into deeper thinking."

Additionally, all three STs learned the importance of making instructional decisions based on students' thinking, which they attributed to the Book Study and working with their CTs. Part of this focus on student thinking involved regularly using student work from CGI problems to assess children's understanding. Kristy, a 3rd Grade CT explains:

> I had a really great model that uses [student thinking] to plan everything because I learned

> really quickly that you can make plans for the whole week, but if that's not where the kids are going, than that's not where you are gonna end up going. Almost everyday, we would have some type of work sample from kids. We would choose to group them, plan the next activity, and maybe stay on the topic or keep going. It was we like we re-evaluated work samples everyday.

For all three of the STs, the Book Study connected the theory that they had learned in their methods courses and mathematics classes with the knowledge they acquired from their interactions in their classrooms. Wendy summarizes the sentiments of the STs:

> The Book Study gave me a widespread view of how students can interpret and solve these equal sharing problems, CGI problems. Like, truly, I just feel like the Fraction Book Study kinda brought together all the student work, which we had talked about in MATH 307, you know, the ways different, the ways they could solve it. Um, but we actually saw it in the Book Study, that made sense from our actual students.

Conclusions

The results of this study indicate that alternative learning experiences, such as Book Studies, have the potential to be a powerful mechanism for connecting theory about children's learning to the practical experiences of teaching in ways that increase professional knowledge. In our study, while all participants benefited from focusing on the progression of children's thinking from 3rd through 5th grades, the 3rd Grade teams of STs and CTs reported additional benefits. It was clear that these teams of teachers, as well as the university researchers/teacher educators, shared the same beliefs and philosophies about teaching children mathematics. In this case, an *apprenticeship of observation* (Lortie, 1975) reinforced a student-centered perspective of teaching. This would imply that CT selection in this model is crucial. The participants saw themselves as a community of learners, who shared similar beliefs and knowledge. This model provides a way to connect theory and practice.

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THE PRIMACY OF MATHEMATICAL CONVENTIONS IN STUDENT MEANINGS

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We present of clinical interviews with PSTs indicated that, in some cases, particular mathematical conventions were inherent to the PSTs' ways of operating, while in other cases, PSTs' ways of thinking were not restricted to canonical situations. We provide examples of the varying degrees by which the PSTs' meanings were tied to particular mathematical conventions and draw implications for teacher education that stem from these findings.

Keywords: Teacher Education-Preservice; Algebra and Algebraic Thinking

In an attempt to gain insights into in situations where pre-service teachers (PSTs) mathematical conventions appeared to be inherent to their ways of operating and had the potential for inhibiting their ability to make sense of non-canonical mathematical situations, we conducted clinical interviews exploring ten PSTs' thinking on tasks that posed internally consistent (from our perspective) but non-canonical situations. We report the results of these interviews and present an emerging theoretical framework exploring the relationship between mathematical conventions and mathematical meanings. Specifically, we observed that in many cases mathematical conventions (e.g., the output of a function is placed on the vertical axis) were essential components of PSTs' meanings. In such situations, the PSTs had difficulty making sense of problems and solutions in which particular mathematical conventions were not followed. Collectively, we provide examples of the varying degrees by which the PSTs' meanings were tied to particular mathematical conventions and draw implications for teacher education that stem from these findings.

Theoretical Framework

Our work with PSTs explores the interplay between common mathematical practices (conventions) and the disciplinary knowledge of mathematics and how those lines may blur for PSTs and mathematical learners more generally. We ascribe to the radical constructivist perspective (Glasersfeld, 1995), wherein individuals construct their own internal system of meanings through processes of assimilation and accommodation and develop an internal system of meanings that have proved viable for him. We seek to better understand and characterize PSTs' internal systems of meanings as a way of explaining conundrums similar to those described in the previous section. Specifically, we employ Glaserfeld's (1995) conceptual analysis, an analytic method designed to better understand the conceptual structures and mental operations that enable individuals to interact with and organize their world in particular ways. Conceptual analysis provides a tool to formulate and test conjectures concerning the particular meanings by which individuals organize their experiences and, in the case of this work, the way PSTs negotiate non-canonical representations and situations.

Methodology and Setting

In order to develop models of PSTs meanings with respect to graphs, functions, and quantitative relationships, we conducted clinical interviews with ten undergraduates at a large state institution in the Southeast United States. We selected the participants on a voluntary basis from a pre-service secondary mathematics education content course consisting of third year (in credits) undergraduate students, all of whom had completed at least two courses past a typical university calculus sequence. Each interview lasted 90-120 minutes, during which students completed the same set of interview tasks. All interviews were videotaped and transcribed. The data was then analyzed using conceptual analysis techniques (Glasersfeld, 1995), where we sought to build viable models of PST thinking by collectively developing conjectures about their thinking, seeking contradictory and corroboratory evidence for our models, and continually revising and developing conjectures.

We designed the interview protocol to draw insights into the PSTs' meanings for graphs, functions, and quantitative relationships, particularly in non-canonical situations similar to the pair of graphs that spawned the classroom event motivating the study. In designing the tasks, we leveraged a quantitative reasoning perspective, which provides us a lens to characterize ways of thinking that may help students conceive relationships between quantities and graphs in meaningful and flexible ways (Smith III & Thompson, 2008; Thompson, 2011).

As an example, consider the relationship $y = 3x$ graphed such that y is on the horizontal axis and x is on the vertical axis. We presented this graph to the PSTs as a hypothetical student solution to the task of graphing $y = 3x$. By providing this graph without the axes labeled, we were interested in determining how the PSTs would make sense of a quantitatively correct, yet non-canonical solution. For instance, we conjectured that the PSTs might restrict y and x to their conventional axes (e.g., vertical and horizontal, respectively) when making sense of the first graph. Based on this conjecture, we also designed the task to include a second graph that had the axes labeled in order to further explore their thinking to a non-canonical but more clearly defined solution. We provided the PSTs the second graph after their explanation for the first graph.

Results

Three primary characterizations emerged from our attempts to develop viable models of the PSTs' activity. Due to space constraints, only one will be discussed in length; illustrative examples of all three will be included in the presentation.

Conventions as Inherent to Meanings

Over the course of the study, several PSTs' actions indicated that particular mathematical conventions had become intrinsically tied to their meanings in such a way that they did not notice the non-canonical nature of the solutions. We also identified a subset of PSTs who did recognize the non-canonical nature of the tasks but who encountered much difficulty in making sense of these situations. One such example occurred when a PST, Maria (all names are pseudonyms), encountered the student work for the graph of $y = 3x$. Prompted with the picture described above, Maria hypothesized that the student misinterpreted "slope" and went "over three" and "up one" rather than "up three" and "over one." When asked to grade such a solution out of ten, Maria decided on a score of five out of ten. Upon being presented with the second solution (with the axes labeled), Maria turned the paper clockwise 90 degrees so that the x-axis is horizontal and stated that the equation of the graph is $y = -3x$. Her interpretation stemmed from reasoning that a left to right downward sloping line had a negative "slope" (we note that in her orientation the negative x-values were right oriented). Unsettled by the student's solution, Maria

rotated the paper from clockwise 90 degrees and back to its original orientation numerous times. When the interviewer asked her to assign a grade out of ten the following conversation ensued.

Maria: (*with paper in normal orientation*) Welllll I would think that... they don't know the difference between the *x* and *y* axes because if they, if you're looking at it, they labeled them wrong, but... if they had, if they had labeled them like this (*rotating the paper 90 degrees clockwise*), then these numbers were like ten (*indicating changing the orientation of the numbers on the x-axis*)... well no, then it'd still be wrong if they flipped it this way (*turning the graph 180 degrees so it is now 90 degrees counterclockwise from its original orientation so the x-values are oriented negative to the left and positive to the right*) unless they flipped it this way. Okay well if these were like in this direction, if they had written these in in this direction (*indicating changing the orientation of the numbers on the axes itself*) then I would think they interpreted it correctly, the axes correctly. Like okay, did the student draw this whole graph?

Int: Yeah, that's all their work.

Maria: Okay, and so like... if these (*indicating the numbers on the axes*) weren't written up and down (*indicating changing the orientation of the numbers on the axes*), if these were written sideways like this then I would think okay they thought this was *x* one two three four five (*motioning in the positive x direction*), negative one, negative two, et cetera (*motioning in the negative x direction*), um... no but then that's wrong (*indicating the values on the y-axis*), so I have no idea.

After continuing to attempt to make sense of the student work, Maria said the student deserves a three because "they have something there, but it's, a lot of it is wrong." Based on Maria's actions, it appears that her unease with the student's solution stemmed from the orientation of the axes. Throughout the entirety of the interaction, Maria attempted to orient the proposed graph such that the *x* and *y* axes had a conventional orientation (both in direction and in sign). Because she was unable to achieve this orientation, she could only make broad statements about why she thought the graph was incorrect, most of which centered on the axes orientation. Rather than working with the solution as presented to consider the graphed relationship, Maria needed the graph to be conventionally oriented and drew conclusions based on this convention. This is just one of many situations in which mathematical conventions appeared to be essential to the PSTs assimilation of situations. As another example, but one in which the PSTs were not aware of the non-canonical nature of the solution, several PSTs interpreted the statement "*x* as a function of *y*" as no different from "*y* as a function of *x*", even when both phrases were repeated several times and made explicit objects of discussion; the interviewer may have *said*, "Is *x* a function of *y*?", but the PSTs *heard* (assimilated), "Is *y* a function of *x*?". Like Maria, the PSTs appeared to hold meanings restricted to particular mathematical conventions (e.g., *y* is the output variable and *x* is the input variable).

Additional Characterizations

Conventions as creating conflicts with meanings. A second characterization involved situations where conflict emerged as PSTs attempted to negotiate the conundrum that existed at the intersection of the conventions and the relevant mathematical meanings. Megan's activity was one example of many that illustrated conventions creating conflict. For example, some PSTs concluded that a pair of graphs (one with the traditional axes orientation, the other with the axes reversed: the *x*-axis oriented vertically and the *y*-axis oriented horizontally) represented the same underlying quantitative relationship. But, when asked if the graphs represent functions, they encountered a perturbation because their meanings were tied to considering only the horizontal

axis as the input quantity (which supports the vertical line test). This conflict–how two graphs can convey the same relationship but yet not both represent functions–led the students to question the input axis convention. For instance, a PST claimed, “Well it’s the same graph, [but] then ones a function and ones not. It just seems like it wouldn't matter!” The PSTs were unable to reconcile such conflict in the interviews, ultimately relying on the conventions. In cases such as these, we found the PSTs were able to conceive the situations in terms of quantitative relationships, but they remained perturbed by the non-canonical nature of the situation. These students were particularly aware of the situations departure from conventions and often questioned the importance of the convention in these situations.

Conventions as secondary to meanings. The final category we propose involves PSTs who reasoned explicitly about relationships among quantities and were, for the most part, unperturbed by non-canonical situations. For instance, while Megan became conflicted by a solution straying from convention, another PST, Jacob had a different reaction to solutions that deviated from conventions. When prompted with a student’s idea of x as a function of y, and in contrast to nearly all other PSTs that we interviewed, Jacob was comfortable assimilating this into his current understandings. He was able to make sense of the middle school student’s rationale and evaluate it based on the relationship between the quantities rather than only focusing on the common mathematical practice the student did not follow (e.g., x as a designated input quantity). Jacob provides one example of a PST whose mathematical meanings foregrounded relationships between quantities as opposed to common mathematical practices.

Discussion and Implications

While the observations and continuum discussed above have implications for all mathematical learners, the implications for PSTs and mathematics teachers are significant. As states and districts integrate the *Common Core State Standards* (2010) within their curricula, teachers are often left to their own devices to implement non-curricular aspects such as the Standards of Mathematical Practice. We argue that PSTs and teachers who are able to reason about relationships between quantities, with conventions as secondary to mathematical meanings, are better prepared to teach in ways that exhibit Standards for Mathematical Practice in their classroom (specifically, reasoning abstractly and quantitatively, constructing arguments and critiquing the arguments of others, attending to precision, and looking for and making use of structure). As the above examples show, when PSTs’ mathematical meanings are closely tied to conventions, their ability to conceive mathematical correctness in students’ non-canonical solutions is limited. As a result, instead of instruction being connected with and building on students’ current conceptions and understandings (e.g., one should not expect students to follow common mathematical practices when first learning the topics within which these practices exist), it is likely that the instruction will be focused on assessment and remediation.

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EXPLORING CONNECTIONS BETWEEN CONTENT KNOWLEDGE, PEDAGOGICAL CONTENT KNOWLEDGE, AND CONTENT COURSES THROUGH TEDS-M DATA

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Using an international data set from TEDS-M, this work explores current assumptions underlying the relationship between content knowledge, pedagogical content knowledge, and the number of mathematics topics taken in teacher preparation programs. Our analysis of the TEDS-M data indicates generally weak relationships between these three constructs for most participating countries. Germany was somewhat of an exception, in that the number of mathematics topics taken was moderately associated with gains in future teachers' content knowledge and pedagogical content knowledge.

Keywords: Teacher Education-Preservice; Teacher Knowledge

Introduction

Content knowledge (CK) and pedagogical content knowledge (PCK) are often considered necessary components of an effective teacher's knowledge base. These assumptions have taken on new meaning with the recently released report, *The Mathematical Education of Teachers* II (MET II) (Conference Board of the Mathematical Sciences, 2010), and with the emergence of a lively discussion between mathematicians and mathematics educators about how best to prepare future mathematics teachers. While the literature suggests that CK and PCK exist and are related, the nature of this relationship is poorly defined (e.g., Hill, Sleep, Lewis, & Ball, 2007). Furthermore, questions about how best to foster CK and PCK among teachers have resulted in a slew of suggestions, many of which are at odds with one another. Some argue that CK and PCK can be developed in content courses offered by teacher preparation programs (U.S. Department of Education, 2010). Others contend that CK and PCK are best developed in the mathematics classroom as teachers are actively engaged in the act of teaching (Seymour & Lehrer, 2006).

This paper explores many of the assumptions underlying the literature above using the Teacher Education Study in Mathematics (TEDS-M) data. TEDS-M is a comparative study of teacher education with a focus on the preparation of mathematics teachers at the primary and secondary levels that offers a unique and powerful lens for exploring the above constructs.

Literature Review

Content knowledge (CK) is widely accepted as a necessary construct within any framework of teacher knowledge. CK refers to a teacher's knowledge of a given subject and the subject's organizing structure. Pedagogical content knowledge (PCK), introduced by Shulman (1986) in an attempt to elaborate on knowledge that is relevant in training teachers and in research on effective teaching, refers to the knowledge of how to effectively teach content to others. More

recently, researchers have offered three components of PCK specific to mathematics: knowledge of students' errors and strategies, representations and explanations of problems, and multiple ways to solve a problem (Baumert et al., 2010; Kleickmann et al., 2013).

It is widely accepted that CK can develop as future teachers take more advanced content courses in their preparation programs (Kleickmann et al., 2013; Schmidt et al., 2007). Factors believed to support the development of PCK are more numerous. For example, Baumert and colleagues (2010) contend that that teachers need to develop an advanced understanding of math content, or CK, to have higher PCK. However, the MET II report (2010) did not find strong evidence in support of this contention because many studies that appeared to show a correlation between CK and PCK were small and had inconsistent findings. Therefore, questions remain about the relationship between, and development of CK and PCK. In fact, some scholars question the existence of PCK as knowledge distinct from CK, due to the lack of past empirical evidence (Kleickmann et al., 2013). It is also noteworthy that much of the research on CK and PCK happens in the United States and Germany, raising further questions about the existence and utility of these constructs in other countries. With these outstanding questions in the literature, we sought to answer the following questions:

1. For future primary and secondary teachers, what correlations exist between CK, PCK, and mathematics topics taken?
2. Do any of these differ within and between countries?

Method

To answer these questions, we used a large-scale international data set from TEDS-M. TEDS-M investigated mathematics teacher preparation and differences in teacher education programs across 17 countries. The current study focused on data from future primary (N = 15,163) and secondary (N = 9,389) teachers. This data set provides comprehensive data on future teachers, including their CK, PCK, and opportunities to learn about mathematics. For additional details about the study, see Tatto and Senk (2011).

Instruments

Assessment. CK items measured future teachers' knowledge of number, geometry, algebra, and data. PCK items measured future teachers' ability to identify and explain students' errors and thinking. Future teachers' CK and PCK scale scores were estimated using Item Response Theory. The scale scores were calculated on a logit scale and standardized to have a mean of 500 and standard deviation of 100 for equally weighted countries (see Appendix B of Tatto (2012) for details).

Survey. As part of the opportunities to learn survey conducted during TEDS-M, future teachers were asked whether or not they had studied–within their preparation program–particular topics at the university, primary, or secondary school level.

Data Analysis

To answer our research questions, we used the IEA International Database Analyzer software (Tatto, 2012). We calculated means and correlations using the appropriate estimation weights. Estimation weights were needed to obtain design-unbiased estimates of population features (Tatto, 2012). Because of the very large sample size, all the correlations reported here are statistically significant ($p < .01$). As a result, instead of relying upon statistical significance as an indicator of the importance of a correlation, we report noteworthy relationships between constructs.

Results

The purpose of the analyses was to explore correlations between CK, PCK, and number of mathematics topics studied by future primary and secondary mathematics teachers.

CK and PCK

Before reporting correlations, it is useful to note that the mean CK and PCK assessment scores at the country level for both future primary and secondary teachers were comparable, within one-half standard deviation of each other. However, the standard deviations for CK and PCK for the sample as a whole were quite large. Therefore, we looked at the relationship between CK and PCK at the individual teacher level. A high correlation indicated that, for most teachers, high CK was associated with high PCK. A low correlation indicated that many teachers had high CK and low PCK (and vice versa).

The highest correlations between CK and PCK scores at the primary level occurred in Germany ($r = 0.62$), Poland ($r = 0.68$), and the Russian Federation ($r = 0.58$). The highest correlations at the secondary level occurred in Germany ($r = 0.70$), Poland ($r = 0.67$), the Russian Federation ($r = 0.68$), Singapore ($r = 0.55$), and the United States ($r = 0.64$). In these cases, the countries did not necessarily have small differences between overall CK and PCK scores, so the fact that their correlations are higher illustrates that there was a smaller range of values at the individual teacher level.

Mathematics Topics Taken

Overall, correlations between number of topics taken and PCK or CK were low, with some moderate correlations in specific countries. For *primary* teachers, the highest correlations for the total number of courses taken and CK occurred in Germany ($r = 0.45$) and Poland ($r = 0.57$). This was consistent for PCK, where Germany ($r = 0.36$) and Poland ($r = 0.42$) again showed the highest correlations, which were substantially higher than the correlations for any other country in this particular analysis. Looking at future *secondary* teachers, the highest correlations occurred in Germany ($r = 0.44$) and the United States ($r = 0.54$). For PCK, the only country to have a correlation over 0.25 was the United States ($r = 0.39$). These results suggest that increased opportunities to learn mathematics was generally not associated with higher levels of CK or PCK, except perhaps in a subset of countries.

Discussion, Conclusions, Implications

Our analysis of the TEDS-M data suggests that future teachers' CK and PCK assessment scores are not strongly correlated, nor are they strongly correlated to the number of mathematics topics taken in a teacher preparation program. These results appear to run counter to claims in the literature (e.g., Ball, Thames, & Phelps, 2008), that CK and PCK should be, and are, strongly related. Yet for Germany and, to some extent, Poland and the United States, moderate correlations point to a somewhat stronger relationship between CK and PCK.

There are two possible explanations for these results. First, it may be the case that CK, PCK, and topics taken are all closely related to each other, and that the findings from countries other than Germany, Poland, and the United States are somehow flawed. In particular, TEDS-M was a very large, complex study, with possible methodological or sampling anomalies accounting for results that generally do not align with established claims. However, our reading of published technical reports of TEDS-M (e.g., Tatto, 2012) indicate that TEDS-M had national and regional partners to help alleviate difficulties, tested and piloted all measures extensively, and used well-established analysis techniques to deal with complexities inherent in this kind of study.

Alternatively, perhaps Germany, Poland and the United States have innovative teacher preparation programs and/or content courses such that the relationships hypothesized in the literature between mathematics taken, CK, and PCK are found to exist. Within the United States, teacher preparation programs have long sought to target the development of CK and PCK. Teaching methods courses have been reconceived as places where teachers learn general pedagogical strategies in addition to developing critical CK and PCK competencies (Steele & Hillen, 2012). Furthermore, many mathematics departments are developing new courses that can build specific content knowledge for prospective mathematics teachers that connect secondary and tertiary mathematics content in meaningful ways (Murray & Star, in press). These recent developments may help explain the findings in the United States, but further exploration of Germany and Poland's teacher preparation programs is warranted.

As the TEDS-M data indicate, our understanding of how teachers develop the knowledge that is critical for effective math teaching is complex. Future work is needed to better understand what is most important for teacher efficacy and how we can foster among teachers the development of the necessary knowledge for teaching mathematics.

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DEVELOPING RELATIONAL THINKING IN PRESERVICE TEACHERS

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In this brief research report, we present preliminary data on the effects of a mental computation unit on the relational thinking of elementary preservice teachers. Relational thinking is an important component of algebraic reasoning and plays an important role in the preparation of future teachers. The unit was implemented in an elementary mathematics methods course, and consisted of mental computation activities, discussions about a variety of strategies, and paper-and-pencil practice problems. The findings showed a significant increase on the preservice teachers' relational thinking skills, even on problems that were particularly challenging for them.

Keywords: Algebra and Algebraic Thinking, Teacher Education-Preservice

Our research concerns the nature and development of elementary preservice teachers' relational thinking, identified as a major component of algebraic reasoning (Empson et al., 2011; Sfard, 1991). Recent calls for integrating algebra and arithmetic at the elementary level (e.g., Kaput, et al., 2007) have drawn researchers to study the role of relational thinking in the preparation of mathematics teachers. In the present study, we investigated the effects of a mental computation unit in a mathematics methods course on preservice teachers' relational thinking. We present preliminary results on the effects of the unit and briefly describe the nature of the observed changes in the relational thinking of the students in the course. We argue that relational thinking has an important place in the professional development of future teachers; aside from strengthening their content knowledge in algebra and arithmetic (Van Dooren et al., 2002), it can provide preservice teachers with a window on children's thinking and can assist them to "unlearn to teach mathematics" in pedagogically useful ways (Ball, 1988; Stephens, 2006).

Literature Review and Theoretical Framework

We adopt Empson et al.'s (2011) conceptualization of relational thinking as the theoretical framework for our study. Serving as the foundation for early algebra, relational thinking involves reasoning in a coordinated fashion about quantities and the mathematical properties that govern them. In this view, relational thinking occurs when the focus is on the relationships between the quantities in a problem rather than on existing procedures. As such, it requires exploiting the specific numerical features of a problem (Smith, 1995), and, when coordinated simultaneously with prior knowledge about operations and associated mathematical properties (such as commutativity, for example), results in the creation of strategies that are, by their very nature, non-algorithmic. Because relational thinking is characterized by the examination of mathematical representations from a structural (i.e., relational) perspective as opposed to a procedural one (Molina & Ambrose, 2008), it permits one to simplify computations, and, depending on the problem, may even render them unnecessary (Carpenter et al., 2005). Moreover, relational thinking relies on accurate and flexible understandings of the equal sign, in part because of the recurrent use of substitution—the replacement of one expression with another that is numerically equivalent (Jones et al., 2012). For instance, in calculating 8 x 3/8, one can substitute 3 x 1/8 for 3/8, because 3 x 1/8 *is equivalent to* 3/8 (Empson et al., 2011).

The nature of preservice teachers' relational thinking and ways to foster it in the context of professional development has been little researched. We propose that a viable route toward relational

thinking is through mental computation, which appears to share several features with relational thinking. Threlfall (2002), for example, claimed that a hallmark of mental computation is strategy flexibility: when computing mentally, the individual notices specific numbers in the problem and, together with his or her own prior knowledge of how they work, creates a strategy that is invented "in the moment" (p. 44). Similar to Empson et al.'s (2011) treatment of relational thinking, mental computation can be fostered through repeated opportunities to represent and reason with quantitative relationships; Beishuizen (2001) underscores the importance of encouraging students to share their strategies, and Threlfall (2002) adds that the role of the teacher, in providing appropriate scaffolds in the context of such dialogue, is critical.

Method

In this paper, we report preliminary data on the effects of a four-week mental computation unit in an elementary mathematics methods course. The specific research questions were as follows: (a) Will the preservice teachers' relational thinking improve after engaging in mental computation activities?, and if so, (b) what is the nature of the changes?

Participants and Mental Computation Unit

The sample consisted of 33 preservice teachers (referred to here as "students") enrolled in the first of three required mathematics methods courses in an elementary teacher training program at a large, urban university in Canada. The unit consisted of three 45-minute class sessions devoted to mental computation and three in-class paper-and-pencil activities. The entire unit was conducted during 6 class periods over a span of 3 weeks.

During the in-class activities, we provided each student with a dry erase whiteboard and marker. The course instructor (first author) verbally read out problems for the students to solve mentally. After each problem was read out, the instructor gave the students 20 seconds to arrive at an answer. When time was called, the students wrote their answers on their boards, which they lifted up for the instructor to see. They were not permitted to use any tools (e.g., paper and pencil, fingers) during the activity, and they were specifically instructed not to use algorithms to solve the problems. In the first session, the problems involved addition and multiplication (e.g., 57 + 199, 37 x 5), and in the second session, subtraction and division (e.g., 741-75, 184 ÷ 8). In the third session, the problems were more complex (e.g., "Is 9675 ÷ 43 closest to 2, 20, or 200?").

After each problem, the instructor called on the students to describe their strategies at the board. Her role was to make sure that a variety of strategies were shared, that the students explained their reasoning to each other, and that key mathematical concepts and principles (e.g., mathematical properties, meanings of operations) were highlighted within and across strategies.

After each mental mathematics session, students were given worksheets, each with five problems. The students were required to solve the problems mentally in more than one way and to explain their strategies. The problems were similar to those covered during the class sessions. At no point during the mental computation unit did the instructor discuss relational thinking, the equal sign, or algebra. Additionally, the instructor was careful not to expose the students to any non-standard symbolic expressions involving the equal sign (e.g., 120 ÷ 8 = 60 ÷ 4), which has been shown to foster a relational understanding of the equal sign symbol (McNeil et al., 2006).

Measures and Coding

Measure. A paper-and-pencil pretest was administered two weeks before the start of the unit and an isomorphically similar posttest was given one week after the unit was completed. The test consisted of six arithmetical expressions involving whole numbers. The students were required to "indicate whether the following number sentence is true or false" and to justify their responses. (i.e., true/false sentences, Carpenter et al., 2005). The items are listed in Table 1.

Table 1: Items on Pretest and Posttest

Item	Pretest	Posttest
1	347 + 108 = 108 + 347	228 + 319 = 319 + 228
2	82 x 3 = 82 x 2 + 83	156 x 4 = 156 x 3 + 157
3	10000 ÷ 4 = 5000 ÷ 2	5000 ÷ 4 = 2500 ÷ 2
4	117 + 56 – 117 = 58	4508 + 124 – 4508 = 124
5	25000 – 7899 + 1 + 7899 = 25001	24780 – 893 + 1 + 893 = 24781
6	8607 x 56 = (60 - 4) x 8607	1350 x 48 = (50 - 2) x 1350

Coding. We created a rubric based on the frameworks of relational thinking outlined by Empson et al. (2011) and Carpenter et al. (2005) to code the students' justifications on each item of the pre- and posttest. The rubric consisted of three major categories: Relational thinking (RT); Computational thinking (C); and Other (OT). There were two types of relational thinking: (a) justifications in which mathematical properties, and no computation, were used to determine whether the expression was true or false; and (b) justifications in which relationships among the numbers were identified and computations were used only as a way to explain or illustrate the relationships. Computational thinking was characterized by computing the amount of each side of the equal sign and comparing to see if the amounts were the same. Justifications in the Other category were those that were either irrelevant, incomplete, or that demonstrated incorrect use of mathematical properties or concepts.

Results

Scoring involved assigning 2 points to justifications assigned an RT code, 1 point for those with a C code, and 0 points for justifications assigned an OT code. A 2 (time: pre, post) x 6 (item) repeated measures ANOVA was conducted with time and item as within subjects factors. We found a significant main effect of time, F (1, 32) = 18.46, $p < .001$, $\eta^2 = .37$, indicating that the participants improved on their relational thinking scores from pretest (M = 1.63, SD = .05) to posttest (M = 1.86, SD = .03). The analysis also revealed a main effect of item, F (5, 32) = 10.91, $p < .001$, $\eta^2 = .25$. Pairwise comparisons revealed that the lowest mean relational thinking scores were seen on items 2 and 3 across pretest and posttest (ps < .05). Finally, no significant time x item interaction was found. These results indicate that the students improved significantly on their relational thinking, but that some items were more conducive to relational thinking than others. In particular, significantly lower means scores were observed on the division item (item 3) and on item 2, in which two operations were presented (i.e., addition and multiplication).

A McNemar test, using item as the unit of analysis (N = 198), indicated that the proportion of justifications that contained relational thinking across all six items was significantly larger on the posttest than on the pretest (88.4% vs. 68.2%, respectively, $p < .001$). The differences in relational thinking from pretest to posttest differed by item, however. McNemar tests conducted by item (with a corrected alpha of .05/6) revealed that the proportion of relational thinking codes increased on item 2 ($p < .008$) and item 4 ($p < .008$). The proportions are presented in Figure 1.

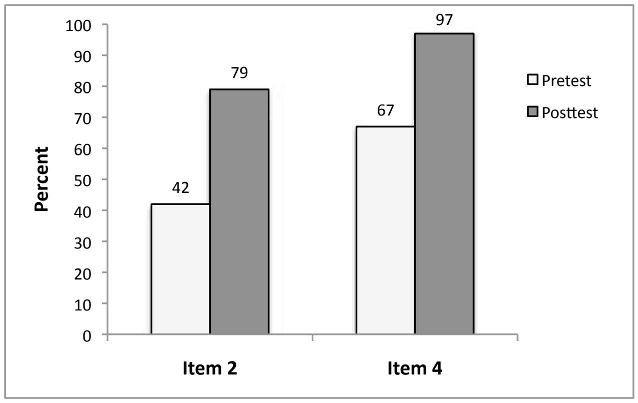

Figure 1: Proportion of Relational Thinking Codes from Pre- to Posttest on Items 2 and 4

Discussion

We are currently in the process of collecting data from a comparison group to strengthen our claims about the effects of mental computation, but for now, the preliminary results show that mental computation is a promising approach to enhancing the relational thinking of preservice teachers. Repeated opportunities to think flexibly about quantities and their relationships appear to be effective in this regard, even on tasks that are particularly challenging. The findings also suggest that some tasks may be more conducive to relational thinking than others. This is in line with previous research on strategy flexibility, shown to be, at least in part, dependent on the task itself (for a review, see Verschaffel et al., 2011). Taken together, the present study shows possible theoretical relationships between mental computation, strategy flexibility, and relational thinking, and thus contributes to the literature on algebraic thinking more generally.

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THE EVOLUTION OF MATHEMATICS TEACHERS' CONCEPTIONS OF KNOWING STUDENTS IN A CULTURALLY RESPONSIVE PEDAGOGY CLASS

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Teachers need to understand their students in order to engage in culturally responsive pedagogy. In this study, we analyzed how teachers' conceptions of knowing students evolved from the teachers' participation in a course called Culture in the Math Classroom, which had the goal of supporting in-service secondary mathematics teachers in developing the knowledge, skills, and motivation to implement culturally responsive pedagogy in their classrooms. We report how teachers' conceptions changed as a result of engaging in four course projects and position these changes in a framework of caring with awareness (Bartell, 2011). Teachers' conceptions of knowing students broadened to include some aspects of caring with awareness that relate to understanding students' cultures and communities, but did not evolve to include critical elements, such as helping students to develop positive racial, cultural, and political identities.

Keywords: Equity and Diversity, Teacher Education-Inservice/Professional Development

Excellence in mathematics education rests on equity—high expectations, respect, understanding, and strong support for all students. (NCTM, 2008)

This quote from NCTM's position statement on equity indicates the importance of equity in mathematics education. Yet, research advances in teaching, learning, curriculum, and assessment have not changed the continued underperformance of African American, Native American, and Latino students (Martin, 2009). Martin suggests this signals that the needs of these students are not being addressed. Broadening our perspectives on mathematics teaching and learning to better understand the students we teach, especially those from minority populations, is therefore a promising avenue for achieving equity in mathematics education. In this paper, we describe research on the impacts of a graduate course called *Culture in the Math Classroom* (CIMC) on teachers' perceptions of knowing students. The findings from this research contribute to what is known about how to support teachers in knowing their students in ways that improve teachers' ability to develop mathematics instructional practices that 'rest on equity.' Specifically, the objective of our work was to examine the research question: What are teachers' perceptions of what they learned in the CIMC course about i) what it means to know their students and ii) the value of knowing their students.

Theoretical Framework

Our work is informed by research that suggests that addressing gaps in students' opportunities to learn requires teachers to "move beyond a narrow focus on measureable performance…to attend to students' interests, cultural backgrounds, and concerns: it requires getting to know students well enough to engage them in learning and relating to students across cultural, racial, and socioeconomic lines" (Bartell, 2011, p. 50-51). Understanding students' cultural backgrounds requires teachers to be aware of the needs and interests of students' parents, families, and communities (Irvine, 2003). Bartell (2011) uses the expression 'caring with

awareness,' to capture when teachers develop "caring teacher-student relationships that explicitly attend to issues of race, culture, and power" (p. 69).

Method

The CIMC Course

The overarching goal of the CIMC course is to support in-service secondary mathematics teachers in developing the knowledge, skills, and motivation to implement culturally responsive pedagogy in their mathematics classrooms to help students become internally motivated and, ultimately, become more successful math learners. The course was conducted online with synchronous classes held once a week using Blackboard's Collaborate software. Weekly readings were required before each class and for a majority of the classes, the teachers engaged in an asynchronous discussion board on class topics. The first part of the course delved into theoretical foundations such as: what is culture, how students' home culture can influence their experiences at school, whether mathematics is culture free, and the tenets of culturally relevant pedagogy (Ladson-Billing, 1995). The second part of the course explored issues involving culture and student learning including: student motivation, status and small group work, language in the classroom, the purpose of mathematics education, and teaching math for social justice.

Four projects formed the core of the teacher work for the course. The Does Culture Matter Project involved groups of teachers reading a set of vignettes designed to support teachers in exploring the role of culture in mathematics teaching and learning. Each group presented to the rest of the class the nature of the vignettes and what the group discussed and learned. The CIP project follows the cultural inquiry process (Jacob et al., 1996) to help teachers maximize student success through action research about cultural influences on students' mathematics participation and learning. The Community Engagement project (adapted from Bartell et al., 2010) was designed to have teachers gain culturally grounded knowledge of their students by engaging with their students' communities. The Motivation project engaged teachers in surveying their students using a survey based on Wlodkowski and Ginsberg's Motivational Framework (Ginsberg, 2005), which describes four classroom conditions (inclusion, attitude, meaning, and competence) that support students' intrinsic motivation. The survey included Likert-type items for students to indicate the degree to which they agreed that these conditions existed in class. Using the survey results, the teachers designed instructional changes, implemented those changes, and administered a post-survey to evaluate the effectiveness of the instructional changes.

Participants

This research is based on the Fall 2010 class, which had 13 teacher participants consisting of 6 men and 7 women, all of whom were White. Their teaching experience ranged from 2 to 22 years, with an average of 10 years. The teachers taught in suburban or rural schools that typically had 20-25% minority students. Hispanic students were the largest minority group in most schools.

Data Collection and Analysis

Data for this study included the text from the asynchronous discussions and the teacher work from the four course projects. For each project, the teachers wrote a report on what they did for the project and a reflection on what they felt they learned from doing the project. Each project had the potential to contribute to teachers' conceptions around knowing students from a unique lens, so we used the projects as a framework for examining how teachers' perceptions of knowing students were influenced by the course. For data analysis, we drew on Wolcott's (1994) protocol of description, analysis, and interpretation for transforming qualitative data. For the

descriptive phase, we identified text in the discussions and projects that related to teachers' perspectives on knowing students and the value of knowing students. In the analysis phase, we identified themes in the teachers' comments and used these to characterize teachers' perspectives at the beginning of the course and as a result of doing the projects. For the interpretive phase we compared the teachers' perspectives on knowing students with Bartell's (2011) framework for 'caring with awareness.'

Results

We analyzed each project individually and then looked for ideas about knowing students that teachers seemed to garner from all the projects. We found four such ideas. First, that culture does matter: a student's culture does influence their engagement at school in general, and in the mathematics classroom in particular. Second, students are not necessarily what they appear to be. That is, teachers realized they were sometimes prone to making assumptions and falling into stereotyping that led them to create false understandings of students. Third, understanding students' culture sometimes required that the teachers make particular efforts to better know students, such as interacting with students outside school, purposefully reaching out to students in school, and asking students how they feel. Fourth, teachers found that having a richer cultural perspective on students could lead to helping students be more successful in school. Prior to the projects, the teachers did not explicitly express the value of knowing students in terms of meeting student needs and supporting student success. Instead, they spoke of knowing students as a means for improving their teaching. Thus, they had a shift in perspective from valuing knowledge of students as a means to improve teaching to valuing knowledge of students to support student learning.

Discussion

To put our findings into perspective, we compare them to a description of what it means for a teacher to 'care with awareness' (Bartell, 2011). To develop this framework, Bartell drew on theories of care (including theories of culturally responsive pedagogy) to lay out a theoretical map on what an effective, caring teacher-student relationship that supports student learning might "look like." The essential activities of teachers who engage in caring with awareness are:

- Understand students' cultures and communities
- Use knowledge to support students' academic success
- Make connections with students cultures and communities
- Reflect critically on personal assumptions about students' cultures and communities (esp. regarding deficit and colorblind perspectives)
- Help students develop positive racial, cultural, and political identities
- Labor to neutralize status differences

We summarize the degree to which the teachers who took the CIMC course moved towards developing a mindset of caring with awareness. Our findings suggest the CIMC teachers did learn strategies for better understanding students' cultures and communities and typically found that such knowledge was valuable for supporting students' academic success. While the process of learning more about students' cultures and communities provided insights to the teachers, there wasn't evidence to suggest that the teachers made strong connections with the cultures of students that were different from the teachers' cultures. A couple teachers made comments that suggested they hoped to increase their interactions in students' communities, but this was not common among the teachers. Rather than making connections with students' communities, the teachers more often talked about making connections with students.

The CIMC teachers certainly engaged in reflecting critically on their personal assumptions about students' cultures and communities and this seemed to be one of the more powerful activities for the teachers. However, the teachers did not discuss how their assumptions might reflect a deficit attitude towards students and their communities. This was probably a function of their not learning about this during the course.

An aspect of caring with awareness that the CIMC teachers did not seem to develop was the critical perspective: helping students develop positive racial, cultural, and political identities, and laboring to neutralize status differences. This may be a reflection of the discomfort many of the teachers exhibited when introduced to the idea that teaching mathematics can have a critical element. The teachers tended to believe that teaching mathematics had nothing to do with, nor should have anything to do with, politics, social inequalities, or other "controversial" issues. While for the most part they recognized teaching and learning as cultural activities, they typically did not connect this with schools being a reflection of the social norms of society at large and the option of schools being a place to empower students to understand and address social inequities.

Given where teachers began the course, developing this critical perspective was probably not likely. It might be that the way the teachers' perspectives about knowing students developed in the course represents a progression that would be common to White middle-class teachers and that this initial movement is necessary before these teachers could be open to understanding and adopting critical thinking. It would be interesting to see how these teachers would develop through a second course that focused more on these ideas.

Acknowledgement

This material is based upon work supported by the National Science Foundation under Grant No. DUE0832026. Any opinions, findings and conclusions or recommendations expressed in this manuscript are those of the authors and do not necessarily reflect the views of the National Science Foundation.

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A COMPARISON OF INSTRUCTION THAT DEVELOPS PRESERVICE TEACHERS' SPECIALIZED CONTENT KNOWLEDGE OF DIVISION WITH FRACTIONS

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Our study centered on the differential effect of two types of instruction on preservice teachers' (N = 8) specialized content knowledge of division with fractions (SCK-DF). One approach emphasized one type of mathematical model (i.e., conceptual models) and one type of representation (i.e., pictorial) to support fraction division and to justify the standard algorithm. The other approach made use of multiple types of models (i.e., conceptual and formal algebraic models; Wu, 2001) and multiple representations (i.e., pictorial and symbolic notation) to support the same learning objectives. We also explored whether components of SCK not addressed during instruction would be promoted, namely analysis of common errors and the effective use of follow-up examples. The multiple models/representations group experienced greater gains in SCK-DF, a difference attributed to participants' more strategic use of follow-up problems.

Keywords: Teacher Knowledge; Teacher Education-Preservice; Rational Numbers

Research has demonstrated that mathematics knowledge for teaching is a key predictor of student success (e.g., Hill et al., 2005). Moreover, the types of instruction advocated by the reform movement (e.g., NCTM, 2000) require greater proficiency in such knowledge than do traditional approaches to mathematics teaching (Turner & Rowland, 2011). For these reasons, some teacher educators (e.g., Osana & Rayner, 2012) stress the importance of studying the development of different aspects of preservice teachers' mathematical knowledge for teaching. One aspect of mathematical knowledge for teaching that has received a great deal of attention in recent years, and is the focus of the present study, is *specialized content knowledge* (SCK): mathematics knowledge that is specialized in that it is not needed or used in settings other than mathematics teaching (Ball et al., 2008).

Theoretical Framework and Objectives

The objective of the present study was to examine changes in preservice teachers' specialized content knowledge of division with fractions (SCK-DF) as a function of two types of instruction. The two types of instruction differed in terms of the mathematical models used to explain the concepts and procedures associated with fraction division and the types of representations (i.e., pictorial and symbolic) used in those explanations. More specifically, one approach emphasized one type of mathematical model (i.e., *conceptual models*) and one type of representation (i.e., *pictorial*) to support fractions division and to justify the standard algorithm (i.e., "invert and multiply"). The other approach made use of multiple types of models (i.e., *conceptual and formal algebraic models*) and multiple types of representations (i.e., *pictorial and symbolic notation*) to address the same learning objectives. A second objective of the study was to explore whether developments in the components of SCK addressed during instruction would extend to other components of SCK, namely the analysis of common student errors and the generation of follow-up problems in response to such errors.

Our decision to compare instructional approaches that differ in terms of the number of types of models stems from the research of Star and Seifert (2006), who demonstrated that learning how to solve problems using multiple procedures supports procedural flexibility and the ability to "invent" new strategies to solve similar problems. In addition, including models and representations that emphasize the generality of mathematics (such as formal algebraic models, which focus on statements that are true for all numbers), abstraction (i.e., the process of focusing on a given mathematical property), and use of symbolic notation follows Wu's (2001) recommendation for the teaching and learning of fractions. That said, previous research has characterized preservice teachers' knowledge of fractions as being primarily

dependent on procedures (i.e., algorithms) and limited in conceptual understanding (Ball, 1990). Taken together, we were interested in comparing instruction that integrates the results from Star and Siefert and recommendations from Wu to instruction that places a greater emphasis on deepening knowledge of the concepts integral to division with fractions.

Method

Participants

We used a mixed-method two-group pretest-posttest design. A sub-sample of the preservice teachers (N = 8) enrolled in an elementary teacher education program at an urban Canadian university participated on a voluntary basis. Group assignment was based on a random assignment and matching.

Instruction on Division with Fractions

Each group received one instruction session per week for three weeks. Each session lasted 75 minutes: 55 minutes were dedicated to instruction and 20 minutes during which the participants completed a paper-and-pencil assessment of their understanding of the content presented during the session.

Conceptual Models Group (CMG). In Session 1, the *measurement model* and *pictorial representations* were used to explain why one inverts the divisor, why the dividend and inverted divisor are multiplied, and how to interpret remainders. Session 2 involved solving division with fraction problems using the *partitive model* and *pictorial representations*. In Session 3, the relationship between multiplication and division was highlighted using a *product and factors* word problem.

Conceptual and Formal Algebraic Models Group (CFAMG). Session 1 in the CFAMG was identical to CMGs Session 1. In Session 2, the standard algorithm was justified using *symbolic notation* and a *formal algebraic model* based on the relationship between division and multiplication. In Session 3, two alternate algorithms (i.e., Common Denominator and Complex Fractions; Tirosh, 2000) were justified using *formal algebraic models* and *symbolic notation*.

Measures and Data Collection

Pre- and post-interviews were conducted on an individual basis with the first author. Each interview lasted 30 to 45 minutes and was audio-recorded. The interview consisted of three sets of items, all presented in the context of hypothetical classroom scenarios, and each assessing a different aspect of SCK (see Ball et al., 2008): (a) justifying the standard algorithm, (b) conducting error analyses, and (c) creating follow-up problems with “pedagogically strategic intent.”

Common student misconceptions, and ways of correcting them, were not addressed during the instructional sessions or during the mathematics methods course. Thus, items on error analyses and the creation of follow-up problems were designed to determine whether any instructional effects would extend to other components of SCK not addressed during instruction. Prior to the pre-interviews, the Knowledge of Fractions Assessment (Saxe et al., 2001) was used to match the participants’ conceptual and procedural knowledge of fractions across the two groups.

Scoring and Coding

For both the pre- and post-interview, participants’ responses to each item were scored on a scale of 0 to 2. In this paper we report group means calculated for: (a) the error analysis (EA) items (maximum M = 10), and (b) the follow-up problems (FUP) items (maximum M = 10).

Interview responses and previous research (i.e., Tirosh, 2000) guided our development of a coding rubric designed to capture, in qualitative terms, differences from pretest to posttest as a function of instruction. In this paper, we report our analysis of the EA and FUP items. Coding for the EA items specified the student error identified by the preservice teachers. Codes for the FUP items characterized the follow-up problems they generated, and in particular, the type of representation used in the problem (i.e., manipulatives, symbols, algorithms, word problems).

Results

Mann-Whitney U indicated no significant difference between groups at pretest, U = 7.00 (corrected for ties U = 7.50), p = .886. Posttest results revealed a significant difference between groups, U = 1.00

(corrected for ties $U = 1.00$), $p = .05$, one-tailed.

Error Analysis Items

A comparison of the group means for the EA items revealed that the CFAMG's mean improved from pretest, $M = 6.25$, to posttest, $M = 8.25$. The mean of the CMG declined from pretest to posttest ($M = 7.75$ to $M = 7.50$, respectively). These findings should be tempered by the pretest scores, however, which were not equivalent to begin with.

In the CFAMG, one participant identified the Algorithmic Error (i.e., inverting the dividend and dividing the inverted dividend by the divisor) and one participant identified the Measurement Error (i.e., confuse dividing by ½ with dividing by 2) on the pretest. Compared to the other EA items, the CFAMG preservice teachers improved the most on these two items after instruction. The preservice teachers in the CMG also improved on identifying target errors, in particular, the Algorithmic Error and the Intuitive Error. It is difficult to determine whether these changes can be attributed to the instruction emphasizing Measurement and Partitive Models, however, because all four participants identified the Measurement Error and three identified the Partitive Error (i.e., confusing division with multiplication) on the pretest. Nevertheless, we propose that the observed improvements in identifying the Intuitive Error in the CMG could be a result of the pictorial representations used during the instruction. To illustrate, one item described a hypothetical elementary student indicating that one cannot compute ¼ ÷ ⅗ because the quotient cannot be larger than either the dividend or divisor (the Intuitive Error). The preservice teacher was asked to describe the student's understanding of fraction division. On the pre-interview, Stephanie (pseudonym), from the CMG stated,

> "[It's] just that bigger numbers represent bigger fractions. So again, we would have to back track and [determine] what is really bigger in these two contexts. Not even the division problem, that is not even the problem in this case."

On the post-interview, Stephanie explained, "When you are working with fractions, and even whole numbers, you could always keep [partitioning] things and distributing with the amount of people even [if] they are small numbers." This and other similar observations lead us to suggest that using pictorial representations over numerical manipulation to illustrate a partitive division model helped the CMG participants understand, or "see," how a fraction can be equally distributed among any number of groups, even when the number of groups is less than a whole.

Follow-up Problems

On the posttest, the CFAMG showed substantial improvement (i.e., $M = 5.25$ on the pretest and $M = 9.00$ on the posttest), while the group mean for the preservice teachers in the CMG increased only slightly (pretest was $M = 5.50$ and posttest was $M = 5.75$). In general, we found that on the post-interview, the types of representations used in the generated follow-up problems appeared to be specific to the instruction received. Within the CFAMG, for example, a comparison of FUP codes revealed a substantial decrease in the frequency of Pictures (i.e., from 44.44% to 19.23%) and an increase in the frequency of Algorithms (i.e., from 2.78% to 13.46%) following instruction. For the CMG, there was a relatively large increase in the frequency of Word Problems (i.e., from 23.33% to 47.50%), perhaps highlighting the effects of the conceptual emphasis of fraction division. In some instances, CMG participants generated word problems even if the task itself contained only symbols (i.e., no word problems), demonstrating flexibility in applying what was learned.

Conclusions

In sum, our results revealed that the CFAMG experienced greater gains in SCK-DF. Analysis of pre and post-interviews showed that this improvement could be primarily attributed to the types of follow-up problems this group of preservice teachers produced in response to scenarios of teaching children. Our analysis also indicated that the specific changes in the preservice teachers' thinking about division with fractions were dependent on the instruction they received. Similar to what was reported in Star and Seifert (2006), instruction that is centered on multiple models and representations of division with fractions can

support preservice teachers' ability to justify and "re-justify" the standard algorithm. We argue that the approach combining conceptual and formal algebraic approaches equipped the preservice teachers with a form of SCK that comprised a variety of ways to understand and represent division with fractions, thereby influencing their skills in creating pedagogically useful follow-up problems.

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CHANGE IN TEACHERS' MATHEMATICS BELIEFS AFTER PARTICIPATING IN PROFESSIONAL DEVELOPMENT

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This report presents the findings of a study that was designed to investigate the effectiveness of a professional development (PD) program to change teachers' mathematical beliefs. The results reveal K-3 teachers had stronger student-centered teaching beliefs and weaker teacher-centered teaching beliefs after participating in the PD program.

Keywords: Teacher Beliefs, Teacher Education-Inservice/Professional Development, Measurement

There has been increasing awareness of the importance of improving the quality of mathematics taught to young children through developing and implementing professional development (Varol et al., 2012). Professional development can contribute to teacher change, including "change in beliefs, teaching styles, and materials, which can come about only through a process of personal development in a social context" (Fullan, 2001, p. 124). Change in beliefs is an important component of teacher change that has been difficult to study due to the lack of appropriate validated and reliable instruments.

A focus on teacher beliefs toward mathematics teaching and learning cannot be divorced from teaching practices. Teachers' beliefs about mathematics, teaching, learning and students are associated with their teaching practices (e.g., McLeod & McLeod, 2002). In mathematics education, there has been a shift from valuing teacher-centered teaching to valuing student-centered teaching. Teachers who embrace student-centered teaching often use various strategies and tools to engage students in learning (Şahin & Yılmaz, 2011). They hold the belief that students construct their own knowledge through active investigation and meaningful exploration (Capraro, 2001). Improvements of teaching in the classroom are likely to occur when teachers embrace student-centered teaching and exhibit corresponding teaching practices (Reid & Zack, 2010). However, change in beliefs is often a slow process that requires substantial resources, external support, time and effort (Pajares, 1992; Raymond, 1997). Also, such changes are not possible if teachers do not have a deep understanding of mathematics (Ma, 1999).

Some reform-oriented professional development programs have demonstrated that it is possible to change teachers' beliefs in ways that are more consistent with the aims of reform and for them to develop reform-oriented practices (Fennema et al., 1996). However, to our knowledge, no sound instrument has been developed and used widely to measure teachers' beliefs toward mathematics teaching and learning (Chamberlin, 2010; Jong et al., 2012).

The present study has three specific aims. First, we describe the courses from a professional development program that are related to teacher change in mathematics beliefs. Next, we validate the short form *Mathematics Beliefs Scales* (MBS, Capraro, 2001; Fennema, Carpenter, & Loef, 1990) to ensure it is psychometrically sound. Finally, we examine the impact of the professional development program on teachers' beliefs among K-3 mathematics teachers.

The Study

The data for this study were obtained as a part of NebraskaMATH, an NSF-funded Math Science Partnership examining the effectiveness of a K-3 mathematics specialist program, Primarily Math. This initiative aims to strengthen mathematical knowledge for teaching, to the extent that it creates measurable gains in mathematics achievement of K-3 students. Primarily Math is a six-course (18 credit hours), 13-month program. It is framed on the premise that deepening K-3 teachers' mathematical knowledge for teaching and pedagogical skills, and teaching them more about child development will improve mathematics teaching and learning.

Of the six courses, three are focused on increasing teachers' mathematical knowledge for teaching. These focus on number and operations, geometry, and algebraic thinking. While the majority of the mathematics is focused on more deeply understanding K-3 mathematics, the third course especially extends to upper elementary mathematics. The three mathematics courses are designed to develop teachers' mathematical habits of mind, and are taught in ways that model student-centered teaching. The other three courses focus on increasing teachers' knowledge of pedagogy and child development. The overarching goal of these three courses is to help teachers become increasingly *intentional*, *planful*, *observant*, and *reflective* practitioners. We believe these courses together contribute to teacher change in mathematics beliefs. We expect teachers to embrace student-centered teaching more and teacher-centered teaching less after they know more about child development and have more pedagogy knowledge, along with the deeper mathematical knowledge necessary to support such teaching and learning.

Methods

Measure

The *Mathematics Beliefs Scales* (MBS, Fennema, Carpenter, & Loef, 1990) were designed to measure teachers' mathematics beliefs. Capraro (2001) developed an 18-item short-form MBS using exploratory factor analysis. Three saturation factors were identified: *Student Learning*, *Stages of Learning*, and *Teacher Practices*.

Participants

A total of 116 K-3 teachers across the state of Nebraska completed the survey in the first year of the study, having applied to be part of Primarily Math, or agreeing to serve in a matched (school-level) control group. Confirmatory factor analysis was conducted to assess the reliability and dimensionality of the short-form MBS using all these 116 teachers' baseline data.

Results

Confirmatory Factor Analysis

Confirmatory factor analyses (CFA) were conducted to test the psychometric qualities of the short-form MBS using Mplus v. 6.11. The three-factor structure proposed by Capraro (2001) was tested first. The obtained model Chi-Square (χ^2), the Comparative Fit Index (CFI), the Root Mean Square Error of Approximation (RMSEA), and the Standardized Root Mean Square Residual (SRMR) were used to evaluate the model fit.

CFA results showed that the *Student Learning* factor fit well; the *Stages of Learning* factor did not fit at all; and the *Teacher Practice* factor fit only if we deleted two items. A new two-factor model was examined next since the three-factor model did not fit well. An expert in mathematics education grouped the 18 items conceptually into two categories: *Student-centered* (6 items) and *Teacher-centered* statements (12 items).

We conducted reliability analyses first. Two *Teacher-centered* items were deleted due to low item-total correlation. Next, CFA were performed separately for each factor. The *Student-centered Teaching* factor obtained a good model fit, and all factor loadings were all significant. The *Teacher-centered Teaching* factor did not fit well because three items should have been more correlated than the model suggested. The three items seemed to be repeated since they all describe whether students should be shown exact procedures to solve simple word problems, and thus, we decided to keep only one of them. We kept the item that resulted in the best model fit. The final model obtained a good fit, and each item had a significant factor loading.

Teacher Change in Mathematics Beliefs

We believe the professional development program described earlier helped change teachers' beliefs toward mathematics due to the courses deigned to increase teachers' mathematical knowledge for teaching and knowledge about child development and pedagogy. Our database now includes data from 42 teachers who have completed the program. These teachers completed the short-form MBS across three

years: baseline, post-treatment, and follow-up (a year after completion of the program). Twenty-eight non-participating teachers have also completed the survey across three years. We hypothesize that after participating in professional development, teachers would embrace student-centered teaching more and teacher-centered teaching less.

Mean *Student-centered Teaching* and *Teacher-centered Teaching* scores were calculated. We fit a separate repeated measures model for each of the two scores, in which scores on the two subscales were included as dependent variables, and treatment group (treatment vs. control) and time (baseline, post-treatment and follow-up), as well as their interaction, were predictors.

For *Student-centered Teaching* scores, there was a significant interaction between treatment and time the survey was taken ($p = .002$). Follow-up analysis revealed that for the control group, there was no difference between teachers' baseline and post-treatment scores ($p = .106$). However, for the treatment group, teachers' post-treatment *Student-centered* scores were significantly higher than their baseline scores ($p < .001$). Additionally, the treatment group's post-treatment and follow-up scores did not change significantly ($p = .382$). See Figure 1.

For *Teacher-centered Teaching* scores, there was also a significant interaction between treatment and time the survey was taken ($p = .002$). While there was no significant mean difference between the control group's pre-treatment and post-treatment scores ($p = .694$), there was a significant mean difference between the treatment group's pre-treatment and post-treatment scores ($p < .001$). Neither the control group's ($p = .804$) nor the treatment group's ($p = .115$) post-treatment and follow-up scores changed significantly.

These results suggest that teachers had a stronger belief that students construct their own knowledge through active investigation and meaningful exploration after participating in professional development. Moreover, this change is fundamental, instead of a temporary artifact.

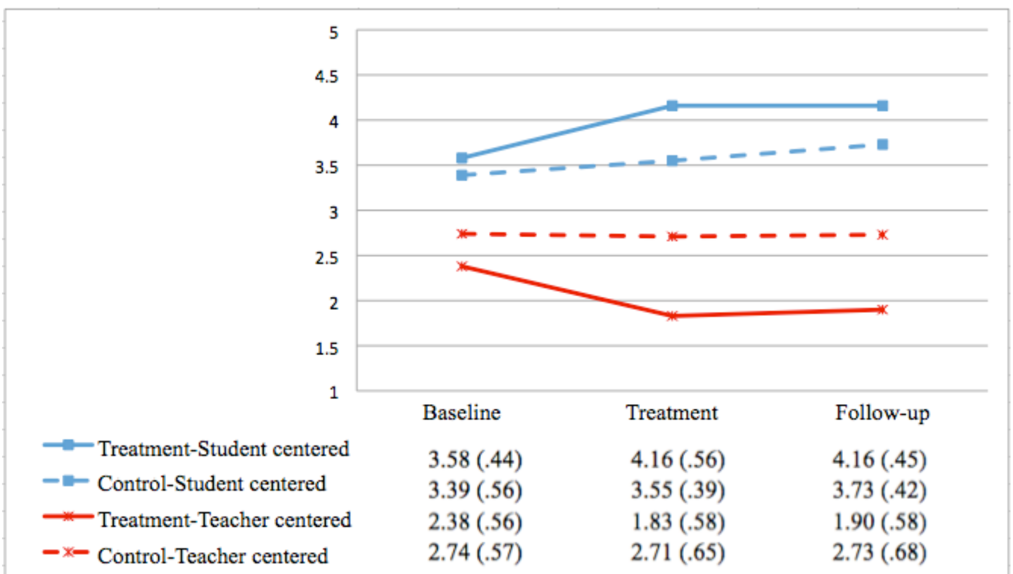

Figure 1. Teacher Change in Mathematics Beliefs (Standard deviation in parentheses)

Discussion and Conclusions

The Primarily Math program, particularly the mathematics and education courses, significantly impacted teachers' beliefs. The short-form MBS results suggest that after participating in the program, teachers had a stronger belief in student-centered teaching and weaker belief in teacher-centered teaching, compared to a non-participating control group. Moreover, the change was retained even a year after completing the program, which indicated that the program had a fundamental influence on teachers' mathematics beliefs.

The current study adds knowledge to the research of teachers' beliefs toward mathematics, especially research to study whether professional developmental programs have a direct impact on teachers' beliefs toward mathematics.

Acknowledgement

The authors acknowledge the support of the National Science Foundation, DUE-0831835, in doing this research. Ideas expressed in this paper are our own and do not reflect the views of the funding agency.

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RESEARCH ON ALGEBRAIC THINKING AND ITS ROLE IN ACCELERATING EARLY CAREER TEACHERS' EFFECTIVENESS IN CLASSROOMS

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Veteran teachers develop an understanding of students' algebraic thinking and misconceptions through the experience of interacting with them over years of instruction. This knowledge of content and students, a component of Mathematics Knowledge for Teaching, is one of the primary differences between early career teachers and veterans. With statistics indicating the dire circumstances with failure in algebra, teachers need to increase their effectiveness in the early years of their instruction. The Algebraic Thinking Project synthesized over 800 articles on students' algebraic thinking into three primary resources: an Encyclopedia of Algebraic Thinking, a Formative Assessment Database, and virtual manipulatives to address challenging concepts in algebra. Using those resources in restructured Mathematics Methods courses, the project aims to accelerate early career teachers' effectiveness in the classroom.

Keywords: Algebra and Algebraic Thinking, Mathematical Knowledge for Teaching, Teacher Education-Preservice, Teacher Knowledge

Introduction

It is well documented that we lose approximately one third of new teachers in their first three years of teaching and up to one half of new teachers by the fifth year of teaching (Darling-Hammond & Sykes, 2003; OECD, 2005). During those early years, new teachers gradually improve their practice (Hanushek, Kain, O'Brien, & Rivkin, 2005). Accordingly, schools have a revolving door of teachers developing their craft that are not at their peak effectiveness. Meanwhile, we know that algebra is the gatekeeper to higher education and future employment in well-paid careers (Ladson-Billings, 1998; Moses & Cobb, 2001). Yet, our failure rate is alarming as districts such as Los Angeles Unified report that 44% didn't pass and 17% received D's, triggering increases in drop outs (Helfand, 2006). This collision between students' algebra struggles and new teachers' slowly developing expertise is resulting in a disaster that is compounded by states increasing graduation requirements in mathematics and new expectations with the Common Core State Standards. If we are to meet the algebra challenge, early career teachers' effectiveness must be accelerated.

Veteran teachers develop extensive knowledge of how students engage with concepts—their misconceptions, ways of thinking, and when and how they are challenged to understand. Veteran teachers learn to evaluate whether an incorrect response is a simple error or the symptom of a faulty or naïve understanding of a concept. Preservice teachers, on the other hand, struggle to make sense of what students say in the classroom and determine whether the response is useful or can further discussion (Peterson & Leatham, 2009). They often assume students' understanding and don't perceive when a student's thinking is problematic. Early career teachers lack confidence in monitoring and reporting on student progress (Ewing & Manuel, 2005).

Learning to use students' thinking during instruction is complex, especially for early career teachers. (Feiman-Nemser, 2001; Sherin, 2002). However, the Cognitively Guided Instruction (CGI)(Carpenter, Fennema, Loef-Franke, Levi, & Empson, 2000) and Integrating Mathematics

Assessment (IMA)(Gearhart et al., 1999) projects found that teachers who learned how students' thinking develops made fundamental changes in their beliefs and practice that ultimately resulted in higher students' achievement (Carpenter et al., 2000; Gearhart & Saxe, 2004). Ball, Thames, & Phelps (2008) define this domain of Mathematical Knowledge for Teaching as *knowledge of content and students.* In the CGI project's study, teachers with higher amounts of this knowledge facilitated increased levels of achievement with their students than teachers who had less knowledge of their students' thinking (Peterson, Fennema, Carpenter, & Loef, 1989) and these effects were long lasting for students and teachers (Fennema, et al., 1996).

For decades researchers have worked to define students' struggles, misconceptions, and ways of thinking about algebra. Much of the knowledge of content and students that early career teachers need is embodied in over 800 articles written about students' algebraic thinking. In lieu of veteran teachers' experience developing understanding of students' thinking, the Algebraic Thinking Project (ATP) is using this research to accelerate early career teachers' knowledge of content and students. Supported by a Fund for the Improvement of Post-Secondary Education (FIPSE) grant for three years, the ATP synthesized these articles into multiple research-based resources. The project's premise is that early career teachers can use these resources to anticipate students' struggles and ways of thinking as they prepare their lessons, make instructional decisions, facilitate student learning, and debrief their instruction—as veteran teachers do based on their experience.

The ATP has created the Center for Algebraic Thinking (www.algebraicthinking.org) to house three primary resources to meet early career teachers' needs for knowledge of content and students. First, the Encyclopedia of Algebraic Thinking consists of over 70 entries that articulate students' misconceptions and struggles to understand algebra. Each entry's use of the research is guided by five questions:

1. What Common Core State Standard(s) does this research address?
2. What is the symbolic representation of thinking with the idea? (What does it look like? How does students' written work indicate how they are thinking about the idea?)
3. How do students think about the algebraic idea? (What does it sound like? What do students say when they discuss the idea?)
4. What are the underlying mathematical issues involved? (What prerequisite knowledge and concepts are essential, and where does the concept fit in a trajectory of mathematics?)
5. What research-based strategies/tools could a teacher use to help students understand?

One key feature of this literature is well developed formative assessment problems that are designed and empirically tested to elicit students' range of algebraic thinking. These problems form the second resource from the project, the Formative Assessment Database, that can be searched by CCSS or keyword and delivered through a tablet based tool that instantly provides teachers with easy to consume evidence of the range of students' thinking with that concept.

Lastly, the ATP has used research as a basis to develop eleven virtual manipulatives that address topics that students typically struggle with in algebra. One example of our manipulatives addresses research by Monk (1992) that students tend to draw graphs that imitate reality, such as a hill, regardless of the labels of the axes ('iconic translation'). The "Action Grapher" app shows a bike climbing various hills while simultaneously three separate graphs of height, distance, and speed versus time appear alongside. The student draws what she thinks each graph will look like, then animates the bike and compares her hypotheses against the actual graphs that unfold. These are apps that take little time to use in class and focus on a challenging concept or misconception.

Method

Four public and private universities in Oregon formed a consortium for the purpose of implementing the Algebraic Thinking Project. Mathematics Methods courses at each of the institutions were chosen as the delivery vehicle for utilizing the developed resources with preservice teachers. Fifteen modules were designed by the project to orient preservice teachers towards considering and using students' algebraic thinking in their preparation, instruction, and debriefing of their lessons. Each module incorporated at least one entry in the Encyclopedia of Algebraic Thinking, formative assessments, video of middle and high school students describing their thinking while engaging in the formative assessments, problem based activities that engaged preservice teachers in developing their knowledge of students and content, and discussion of implications for instruction. Methods faculty had the option of using as many of the modules as appropriate for their course. 35 preservice teachers were enrolled in the Mathematics Methods courses at the four campuses. 24 were seeking Basic Mathematics endorsements (to teach up to and including Algebra I) and 11 were seeking Advanced Mathematics endorsements (to teach through Calculus). Each preservice teacher was enrolled in either one or two terms of methods courses depending on the campus requirements. Preservice teachers ranged in ages from 22-53 and were placed in public school environments for student teaching in middle and high school during the spring semester. 16 were female and 19 were male.

The Algebraic Thinking Project had two primary goals. First, to accelerate preservice teachers' knowledge of algebraic content and students thinking in algebra. Using the modules in the Mathematics Methods courses, each institution worked to develop preservice teachers' understanding of students' misconceptions and ways of understanding algebra. However, it is impossible to pour twenty years of veteran teachers' experience or 800 articles worth of information into early career teachers' brains. Accordingly, the second goal was to develop preservice teachers' habit of mind towards seeking knowledge of content and students as they prepared, implemented, and debriefed their algebra instruction. The intent was for preservice teachers to access the Encyclopedia and Formative Assessment Database as they prepared their lessons so that they could anticipate students' thinking. They would use formative assessments as well as virtual manipulatives during their instruction to access and address challenging areas for students when learning algebra.

To determine the effectiveness of the approach to developing preservice teachers' knowledge of students and content as well as their habit of mind in using the resources, the study included quantitative and qualitative data. Each preservice teacher was given a pre-test and post-test of their algebraic content knowledge as measured by the Algebra Content Knowledge Instrument (ALCKIN, Black, 2008). Each preservice teacher was also given a pre-test and post-test of their orientation towards incorporating students' algebraic thinking into their instructional decision making through the Algebraic Thinking Project Teacher Survey, which was developed by the ATP. Each of the four campuses has one representative for data collection who conducted two case studies of preservice teachers. Each preservice teacher was pre- and post-interviewed regarding their understanding and orientation towards students' algebraic thinking as well as their use of project resources. Video of each preservice teachers' instruction was taken at the beginning, middle, and end of their field experience. The project will analyze the video to identify the nature of preservice teachers' interactions with students in regard to use of their algebraic thinking for instructional decision making using the Teacher Disposition towards Students' Thinking (TDST) instrument, a tool developed by the ATP.

Results and Discussion

In April, 2013 the Algebraic Thinking Project will have preliminary data on the preservice teachers who have taught during their student teaching experience. In early June, 2013 we will have final data and analysis completed in time to share the effectiveness of the approach with PME participants.

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PRESERVICE TEACHERS' EMERGENT CONCEPTIONS OF EQUITY

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Secondary mathematics preservice teachers were asked to interpret and respond to five quotes related to issues of equity in mathematics education. These quotes were designed to elicit the preservice teachers' emergent conceptions of equity in mathematics. They were coded using a framework of power and identity, as defined by Gutierrez (2009). Overall, the preservice teachers noted the importance of real-world contexts, role of motivation in learning, impact teachers can have on their students, and stereotypes can negatively affect both teacher expectations and a student's mathematical identity. Most of the preservice teachers recognized some issues of identity and power, such as combating stereotypes, using students' culture and background when developing lessons, and teaching mathematics for social justice.

Keywords: Equity and Diversity, Teacher Education-Preservice

Objectives

Researchers (e.g., Malloy, 2009) argue that as teachers practice equitable pedagogies, they are able to build relationships, have high expectations, and their students are able to maintain their identities. Teachers who want to practice equitable pedagogies must have knowledge of students' culture, lives, and experiences to make learning relevant (Malloy, 1997). They must also be equipped to meet the needs of diverse learners (Gutierrez, 2009). The purpose of this paper is to describe secondary mathematics PSTs' emergent ideas and beliefs about equity. In order to create curriculum and instruction that supports PSTs in creating equitable classrooms, mathematics educators must understand their PSTs' emerging knowledge and beliefs.

Theoretical Framework

Gutiérrez (2009) calls for the field to include students' identities and power in its conception of equity. Identity includes not just what an individual is, but also what individuals do. Power includes both the power of mathematics and what can be done with it. Mathematics educators should support PSTs in attending to issues of equity, particularly related to power and identity, which are engrained within the system and defines what it means to do mathematics and who is able to do it.

Conversations about equity can be difficult. Most will see mathematics as devoid of cultural, social, and political conversations; therefore, it is unlikely that PSTs have previously had conversations about equity (Aguirre, Kitchen, & Horak, 2005). Additionally, conversations about equity related to race and ethnicity are more difficult, as individuals have been more comfortable traditionally discussing gender, language, and issues of participation rather than race and racism (Parks & Schmeichel, 2012). We examine how PSTs conceive of equitable mathematics teaching using these ideas to frame our study.

Methods

This study took place during the Fall 2012 semester in three different mathematics methods courses at three universities. A total of 39 PSTs participated in the study, with 29 females, 10 males, 5% African American, 5% Asian, 3% Latina, and 87% Caucasian. The participants were

undergraduate and graduate students enrolled in either a middle school or secondary (i.e., high school) mathematics methods course.

The authors administered the *Equity Quote Gallery* (see Table 1), which contained five quotes adapted from Gutstein & Peterson (2006), Tenenbaum & Ruck (2007), and Gonzales, Andrade, Civil, and Moll (2001), in late September to their PSTs. All the PSTs responded to the second quote, as well as two other randomly assigned quotes. In their response, they had to (1) interpret the meaning of the quote in 1-2 paragraphs, and (2) discuss their reactions to the quote in 1-2 paragraphs. Table 1 identifies the quotes and number of respondents for each.

Table 1

Equity Quote Gallery

Equity Quote	Number of Preservice Teachers who responded
Quote 1: Minority and linguistically diverse students have not been construed as visible players within mathematical discourse either in or out of schools.	19
Quote 2: "Teachers have different expectations of their students based off their students' ethnic and socio-economic background."	36
Quote 3: The way teachers teach mathematics does not send any messages; mathematics is free of context."	21
Quote 4: "I thought math was just a subject they implanted on us because they felt like it, but now I realize that you could use math to defend your rights and realize the injustices around you… now I think math is truly necessary and, I have to admit, kinda cool. It's sort of like a pass you could use to try and make the world a better place"	19
Quote 5: "Students can connect math with their own cultural and community histories and can appreciate the contributions that various cultures and people have made to mathematics."	18

We conducted an inductive content analysis (Patton, 2002) to analyze our data. We individually used open coding to code the PSTs' responses and wrote theoretical memos (Grbich, 2007). We then discussed and came to a consensus on the codes. Once a consensus was reached, we identified themes and patterns that emerged within and across the quotes.

Results

Quote #1 Analysis and Discussion

Overall, PSTs' responses focused on the stereotypes that teachers and others have of their students, particularly around viewing some students as capable in mathematics and others not. One PST suggested this quote highlighted that there is "an unlevel playing field in mathematics" for ELs and students from non-dominant races and ethnicities. Further, another PST noted, "These minority and linguistically diverse students are expected to have trouble with and perform poorly in mathematics. Therefore, they have not been considered when discussing mathematical discourse." PSTs highlighted that teachers can either see the deficits or resources with which their students come to class. In order to overcome these negative stereotypes of students, PSTs argued that "mathematics course are not being taught in an equitable manner," and they, as future teachers, would need to consider how to "provide them with the opportunity to learn in an equitable classroom with the same high expectations as other students." The PSTs identified a number of different strategies for supporting their students to be successful: (1) adapting lesson plans to support students' development of language, (2) being aware of student differences, (3) creating safe classroom environments, and (4) becoming better educated about their students' backgrounds.

Quote #2 Analysis and Discussion

Most PSTs responded that many teachers think many teachers have different expectations of their students, which is unfair, unacceptable, and should not happen. One PST noted, "I think this

is absolutely true on average, unfortunately. Many teachers teach based on stereotypes...” Our PSTs suggested that it bothers them that this happens, there is inherent bias in the educational system, and there are negative effects because of these stereotypes. PSTs explained that teachers play a key role in overcoming these stereotypical views. Primarily, PSTs suggested, “teachers should have the same high expectations for all students regardless of their students’ ethnic and socioeconomic backgrounds.” Across most responses, PSTs argued for equality – treating students equally, having equally high expectations for all students, or having the same expectations for all students. PSTs also examined the role of the teacher in fighting these stereotypes and providing opportunities for their students. For example, teachers could provide students extra help, motivate and encourage students, and advocate for their students. PSTs noted repeatedly that a key way to support students is to be aware of their students’ backgrounds and experiences.

Quote #3 Analysis and Discussion

Many PSTs expressed the way teachers teach mathematics does send messages by influencing students’ attitudes toward mathematics. For example a PST commented, “The delivery of a lesson…can make [students] hate math or make them enjoy it.” Some PSTs stated that since mathematics is free of context, teachers can send the wrong messages if teachers do not connect mathematics to real life situations. The PSTs articulated that word problems provide the context needed to connect to students’ lives. However, one PST cautioned, “Math problems do have context, which can reflect the dominant culture if care is not taken to ensure the problems are equitable.” As a result, “mathematics teachers need to be cautious of what they say in the classroom not [to] offend any students.” Other PSTs argued, “Mathematics is a language of numbers and processes and student backgrounds do not matter because mathematics can be taught the same to everyone.” Moreover, “Mathematics is universal. Teaching mathematics will not personally affect any student if they are of a different race, culture, gender, economic status, etc. It does not send any personal harmful (or positive) messages to any student.”

Quote #4 Analysis and Discussion

The interpretation and responses to Quote 4 fell into four general categories: (1) mathematics relating to the real world, (2) affinity and motivation for learning mathematics, (3) teacher impact, and (4) equity. All participating PSTs made some reference to how mathematics is related to the “real world”. Some PSTs suggested, “many students struggle to find the meaning behind mathematics, but there are many places that math is useful. Mathematics goes far beyond using numbers in the classroom.” Over 75% of the PSTs discussed issues of motivation or affinity towards mathematics, such as disliking math. These PSTs argued that “students need to feel that there is a purpose [to] what they are studying otherwise they will lose interest and motivation.” Many PSTs recognized that good teachers can make a difference in students’ motivation and affinity towards mathematics. Over half of the PSTs discussed issues of equity and felt, for example, that “math can be applied on a much larger scale for greater causes like justice and equality.”

Quote #5 Analysis and Discussion

The PSTs were aware of the importance of incorporating students’ identities and backgrounds in their mathematics lessons because it “enables the students to feel more comfortable not only in the classroom but with the material as well.” One PST expressed that she had never previously thought about including students’ backgrounds in the teaching of mathematics, but has now seen some ways “a teacher can expand the diversity of the classroom and increase student achievement by the assimilation of student culture.” Most PSTs

acknowledged students will be more interested, engaged, and inspired to learn mathematics when they learn about people from similar cultures and backgrounds who made contributions to the field of mathematics. A PST remarked,

> Students have created low expectations for themselves based on the socially constructed ideas that their people groups do not achieve in mathematics. However, highlighting the mathematical contributions of various different people groups can inspire students to see themselves differently. Students need to see diversity in mathematics in order to see that all groups can achieve in mathematics.

Moreover, the PSTs expressed that by highlighting the contributions of mathematicians from various cultures, students will develop an appreciation for mathematics, other cultures, and themselves. One PST noted, “Giving other cultures credit helps students appreciate them more and make a connection to them too.

Conclusions

Many similar themes ran across the five quotes. Overall, students noted the importance of real-world contexts, the role of motivation in learning, the impact that teachers can have on their students, and stereotypes can negatively affect both teacher expectations and a student’s mathematical identity. Most PSTs recognized at least some issues of identity and power, such as combating these stereotypes, using students’ culture and background when developing their lessons, and teaching mathematics for social justice. When recognized, the PSTs were eager to teach in a way that created more equitable opportunities for students to learn. However, some PSTs failed to recognize the relationship between culture and mathematics. Other PSTs saw mathematics as useful, but did not comment on the power of learning it from an equity viewpoint. We suspect that the mixed responses have several causes. These ideas are relatively new for the PSTs and it is unreasonable to assume that they would have embedded them into their understanding of teaching and learning so quickly. We also contend the quotes themselves may be a cause for the disparity of responses. Whereas quotes like #2 and #5 are easily understood and have implied “correct interpretations”, the other quotes are more subtle and require a more sophisticated interpretation in order to relate them to equity, identity, and power.

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LEARNING AND TEACHING LINEAR FUNCTIONS

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This paper reports on the methodological approach and preliminary findings from a research study of the Learning and Teaching Linear Functions professional development program. The study is currently in its third year and is in the beginning stages of data analysis. The Linear Functions for Teaching project investigates the research questions: (a) what do teachers learn from participating in the Learning and Teaching Linear Functions video-based professional development? And, (b) what do their students learn? Initial results will be shared regarding teacher learning of mathematical knowledge for teaching.

Keywords: Algebra and Algebraic Thinking, Mathematical Knowledge for Teaching, Teacher Education-Inservice/Professional Development, Teacher Knowledge

Introduction

Over the past two decades mathematics education research has identified ways in which more than a procedures-based approach to algebra is needed (Bednarz, Kieran, and Lee, 1996). For example, research has documented the various challenges students (and their teachers) face with the concept of slope. Students have difficulty interpreting linear functions and their graphs (Schoenfeld, Smith, & Arcavi, 1993), connecting graphs to linear equations (Kerslake, 1981), and connecting graphs to the notion of rate of change (Bell & Janvier, 1981). The mathematics education community has been working towards a goal of having algebra students explore relationships between symbolic expressions, graphs of lines, and to the meaning of intercept and slope. If that understanding is to occur, teachers must figure out how to use tasks and orchestrate discussions around linear functions to push for conceptualizing linear functions.

The Linear Functions for Teaching study focuses its work on the video case materials, *Learning and Teaching Linear Functions* (LTLF) (Seago et al., 2004), which are designed to enable teachers to deepen their understanding of ways to conceptualize and represent algebra content within their teaching practice. LTLF is premised on the idea that using artifacts of practice within a well-structured PD program can promote mathematical knowledge for teaching. This idea is supported by a variety of learner-centered, inquiry-based theoretical traditions, including constructivist and situative perspectives on learning (Cobb, 1994). These perspectives share the notion that engaging in challenging, problem-based, collaborative, and socially shared activities is likely to promote an expanded knowledge base (Borko, et al., 2005). The *Learning and Teaching Linear Functions* materials were designed with all of these features in mind and include an analytic framework, explicit tasks, teacher learning goals, and facilitation supports.\

Theoretical Framework

The theoretical frame for the LTLF video case materials is adapted from the work of Deborah Ball and colleagues (Cohen, Raudenbush & Ball, 2003) that incorporates research on both teaching and learning. The content of the video case materials focuses on the interactions between the teacher, the content (in this case, linear functions tasks), and the students, within the context of an authentic classroom environment (see Figure 1). The materials are designed to be used by a teacher educator who is faced with a similar set of relationships: the interactions between the teacher educator, the content (in this case, teaching and learning of linear functions),

and the teachers he/she works with. To assist the teacher educator in using the PD materials productively with teachers, in-depth resource materials are provided to facilitate teachers' knowledge development. Resource materials include: mathematics content information, probing discussion questions, and other facilitation guidance specific to the materials.

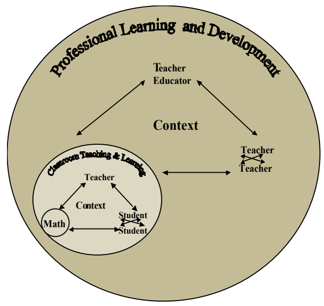

Figure 1: Theoretical Framework (Adapted from Cohen, Raudenbush & Ball, 2003)

As Ball and her colleagues have noted, of central importance with respect to interactions around the content with students is the teachers' mathematical knowledge for teaching (MKT; Ball, Hill & Bass, 2005). Their research has shown that MKT relates to the quality of teachers' classroom work and positively predicts gains in their students' mathematical achievement (Hill, 2010). MKT can be understood as the knowledge that teachers need to effectively carry out the work of teaching. MKT incorporates subject matter knowledge as well as pedagogical content knowledge.

The Learning and Teaching Linear Functions Video Case Materials

Learning and Teaching Linear Functions was designed to enable teachers to deepen their understanding of ways to conceptualize and represent linear functions within their teaching practice. The major goals of these materials are to help teachers deepen their understanding of mathematics content, students' mathematical thinking, and instructional strategies; as well as develop norms and practices for learning about teaching. The first module, *Conceptualizing and Representing Linear Relationships,* is a sequential series of eight 3-hour sessions that are designed to enrich teachers' ability to teach linear relationships and deepen their own detailed knowledge of the distinctions and linkages among the various representations. Each session has at its core one or two digital video clips of a mathematics classroom. These clips are unedited segments selected from real classroom footage of un-staged mathematics lessons, representing a range of grade levels, geographic locations and student populations.

Method

The Linear Functions for Teaching efficacy study consists of 66 teachers from across the state of California. A mixed within-school and between-school random assignment design was used. All potential participants were informed at the outset that they have a 50% chance of being assigned to the intervention group. All interested teachers were randomly assigned to treatment and control groups (34 treatment and 32 control group teachers). The study is currently in its third year and is in the beginning stages of data analysis. The LTLF foundation module was used with 34 treatment teachers in a 5-day institute August 2011. Two instruments were used to assess impacts of the LTLF PD on teachers' knowledge of mathematics for teaching: the Learning Mathematics for Teaching online instrument (TKAS) and the Artifact Analysis Assessment. Both of the assessment items include pre- and post-measures and were administered to both treatment and control teachers. The TKAS assessment focused on grades 6-8 patterns, functions and algebra scales. The Artifact Analysis Assessment consists of analyzing a linear functions

task and solution methods; viewing a 5-minute video clip of 6th grade students presenting solutions to the same linear functions problem and then providing written responses to a series of increasingly detailed questions about the video. The assessment also asks teachers about three samples of student work on the problem (each represented a different kind of common student error). A scoring rubric was developed for participant responses based on a small set of randomly selected pre- and post-pilot data (blind scored. The rubric used to score this assessment includes three dimensions determined to be central to PD intervention goals:

- Justification: Making claims and providing evidence for them or making claims without providing evidence; and
- Interpretation: Focusing on students' potential understandings (vs. deficits in understanding);
- Attention: Focusing on specific mathematics content, generic mathematics content, or non-mathematical content.

Responses for each of the three areas were analyzed separately. We established inter-scorer reliability of 0.82 using percents of agreements among four researchers who used the coding scheme for scoring 34 teacher responses. We blind scored (treatment/control and pre-post) all responses in January 2012.

Preliminary Results

As stated earlier, we are in the initial stages of data analysis. Preliminary results of the impacts of the LTLF intervention on treatment teachers' mathematical knowledge for teaching are somewhat promising. Table 1 shows the TKAS pre-test scores by treatment/control status. Table 2 shows the post-test scores.

Table 1: Pre-test TKAS Scores by Treatment/Control Status

	Treatment M (SD)	Control M(SD)	Difference	p-value	Effect Size	Sample Size
LMT Scores (Pre-test)	0.36 (0.79)	0.53 (0.72)	-0.17	0.38	-0.23	63

Table 2: Post-test TKAS Scores by Treatment/Control Status

	Treatment M (SD)	Control M (SD)	Difference (SE)	p-value	Effect Size[B]	Sample Size
No Pre-test Adjustment						
LMT Scores (Post-test)	0.45 (0.91)	0.44 (0.70)	0.01	0.95	0.02	63
Pre-test Adjustment						
LMT Scores (Post-test)	0.53 (0.91)	0.35 (0.70)	0.18 (0.10)	0.09	0.26	63

Teachers in the treatment group exhibited growth of TKAS scores of 0.09, while control group teachers exhibited a 0.09 decline in scores. The difference in these two trajectories results

in a net increase for the treatment group of 0.18 relative to the control group. This difference in change scores is not statistically significant at conventional levels, but is suggestive of a trend favoring the treatment group.

The Artifact Analysis assessment shows significant results for treatment teachers compared to control teachers. For the student video, student work, and teacher video, no statistically significant treatment/control differences on the pre-test measures were apparent. For the post-test measures, treatment teachers were substantially more likely to indicate an understanding of students' potential than control teachers on the student work task (47% vs. 7%). Treatment teachers were also more likely to focus on the mathematical content of student work (78% vs. 44%). There was also a tendency for treatment teachers to use evidence to justify their inferences with regard to student work ($p = .12$) and the teacher interview ($p = .12$), although these differences were not statistically significant at conventional levels.

Conclusion

Preliminary findings indicate that the treatment provided teachers with significant additional understandings in understanding students' potential, focus on the mathematical content of student work and use evidence to justify their inferences. The teacher learning exhibited in the early results is consistent with some of the learning goals of the LTLF PD intervention. The artifact analysis measure imitates most closely the work that teachers did within the PD (work on a math task, analyze a classroom video of students working on the task, and analyzing student work from the class). Other learning goals are subtle and may require more sensitive measures. As our early analysis emerges, we sometimes feel that the data from our measures capture only part of the story. It is important to find and use responsible measures that can (1) assess the impact of a PD intervention's goals for teaching learning, and (2) accurately measure the subtle and incremental nature that so often characterizes teacher learning. This provides us with a challenge for how to measure dimensions of teacher learning in ways that 1) allow comparisons across studies; 2) seem to capture the essence of the learning; and 3) respectfully characterize the complex and challenging work that teachers undertake when they engage in PD.

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EFFECT OF VIDEO ILLUSTRATIONS ON PRESERVICE TEACHERS' NOTICING

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The study investigated two cohorts of preservice secondary teachers' noticing of two video episodes of children's mathematical practices. The preservice teachers paid attention to the strengths that the children exhibited and were less attentive to their errors in reasoning or to the possible factors contributing to their mathematical practice resulting in a weak analysis of how to organize curriculum and instruction to accommodate such issues.

Keywords: Teacher Education-Preservice, Mathematical Knowledge for Teaching

Introduction

It is generally agreed that understanding children's mathematical thinking and the ability to make sense of children's mathematical ideas is a key aspect of effective practice (Rowland, 2008) and a component of Mathematical Knowledge for Teaching (Ball, Thames, & Phelps, 2008). In helping teachers develop skills and expertise in this particular area, use of illustrations of mathematical practices of school children has been identified as a powerful pedagogical tool (Author). Inspired by the promising results of Cognitively Guided Instruction (Carpenter, & Fennema, 1992) on teachers' practice, greater efforts have been devoted to development and implementation of resources in teacher education that build around children's mathematical thinking. Jacobs and Phillips (2004) called for including children's mathematical artifacts in courses designed for future teachers to ground teachers' development of pedagogical cognition (Hatfield, & Bitter, 1994; Corwin, Price, & Storeygard, 1996). Studies that carefully outline ways in which such resources are used, or those that elaborate on the content of resources and particular domains of pedagogical content that are affected by their use are rare. Missing from the literature is an understanding of how teachers do interact with such materials, make sense of what they observe, and what they gain from them. It is also not clear what challenges might be associated with their use or the particular demands that they may place on facilitators in teacher education. Absence of such reports at the secondary teacher level is particularly paramount.

We examined two issues associated with the use of illustrations of children's mathematical practices in secondary mathematics teacher education as a venue for advancing MKT pertaining to understanding of children's thinking. We documented nature of interactions of two cohorts of preservice secondary mathematics teachers, at two different institutions, with video-illustrations of work produced by high school students in the area of geometry. We studied what issues gained notice (van Es, & Sherin, 2002) and inferences teachers made about pedagogical implications for their own practice. The following question guided our study: What do teachers notice?

Theoretical Framework

Our research was grounded in two genres of scholarly efforts, including situated cognition and "noticing" as lenses for cataloging teachers' interactions with video illustrations. Situated Cognition (Brown, Collins, & Duguid, 1989) offered the umbrella framework for task design and implementation procedure used at each of the research sites. The construct of "noticing" (van Es, & Sherin, 2002) provided a tool for examining teachers' interactions with the illustrations and a venue for tracing implementation challenges. van Es and Sherin (2002) describe three aspects of noticing as follows:

(a) Identifying what aspects of an instructional situation should be deemed important.
(b) Connecting broader principles of teaching and learning to specifics of interactions.
(c) Being able to analyze these interactions.

Methodology

Data was collected from two different cohorts of preservice secondary teachers (total of 30) at two different institutions and at two different phases. During the first phase of data collection, the participants were shown a video-illustration of student work. The discussion of the illustrations followed a designed format in which they were asked to record what they noticed when viewing each of the videos. These individual analytical accounts were collected and coded at the end of each session. A whole group discussion of participants' observations then took place (approximately an hour). Field notes were taken during the discussions.

At the second phase of data collection, approximately two weeks later, a second illustration was used. Although several features of the second video contrasted sharply with the first one, it contained compatible demands for analysis of mathematics of the child, as well as pedagogical constraints demanding contemplation and deliberation. These constraints included the nature of the teacher/student interactions, influence of curriculum conditioning on children's mathematical practices, and children's mathematical problem solving contrasted against skills mastery.

Data was compiled at each site and analysis of each data set was completed independently. This allowed the facilitators to detect patterns in analysis of change in foci of attention of participants from the first video experience to the second session. This process was completed at each of the two research sites independently and results compared to consider similarities or differences between the two cohorts as a means to identify issues that merited attention.

Videos

The first video illustrated a student working on a geometry problem involving areas. The problem required comparison of the areas of two triangles in a rectangle such that they had a common base (one side of the rectangle) and the third vertex was on the opposite edge of the rectangle. The student reacted to the question by asking for the lengths of the sides. When she was given the values, she was still not able to work with them. When asked, what the area of the triangle was, she stated the formula and was able to show why the formula worked in the case of right triangles, but was not too sure about why it worked for other types of triangles. Once she had the formula, she concluded that areas of the two triangles were the same as they shared a common base and height. From an intended design issue, we had considered the illustration to be powerful in raising pedagogical capacity among teacher in four domains. First, why was it that a student, who had taken a course in algebra, did not make use of knowledge assumed to be available to her unless promoted to do so (curricular disconnect)? Second, why had the child been reluctant to pursue the problem when measurements were not provided, assuming there was not enough information to tackle the task (curricular considerations-focus on instances and computations vs. conceptual analysis of meaning behind them)? Third, the teacher's recognition of the child's epistemological need as a means to gain entry to the task and honoring it by providing space for making abstract generalizations. Fourth, how the teacher attempted to make the child make the necessary connections, through sequencing of her questions, so to enable her to solve the task, as well as extend her understanding of structure.

The second video illustrated a student working on a comparison of areas, which required him to compare areas of a square with side of length a, circle of diameter a, and an equilateral triangle of side a. The student began saying that the area of the square was bigger than the area of the circle because the circle fit inside the square. He then picked $a = 3$ and started plugging the number into area formulae and was easily able to calculate the area of a square, but made an error while calculating the area of a circle, and made an approximation for the area of the triangle since he was not sure how to find its height. Based on his calculations, he came to the conclusion that the area of the circle is bigger than the area of the square, while the area of the triangle is the smallest. When asked why he changed his initial answer, he replied that he trusted the numbers that he obtained from the formulae. Although he was able to visualize that the square should have the larger area, he was hesitant to believe it, as the numbers he got did not support that argument. While the final answer of the child was incorrect, throughout the problem-solving phase, he demonstrated several key mathematical insights when comparing lengths, offering

approximations based on theories he knew (Pythagorean theorem, triangle inequality) and heuristics he used when strategizing how to compare areas (working backward, solving a simpler problem, and charting visual information against analytical computations). From a design perspective, the illustration embodied three important mathematical and pedagogical points: it might raise awareness of factors that children might consider when responding to assessment items (choosing a wrong answer does not mean lack of knowledge), influence of the school conditioning on children's thinking (dismissing their own intuitions, privileging numerical values), and the power of providing children to explore as a means to understand their thinking.

Analysis

Analysis of data followed two stages. First, the participating teachers' reactions to what they observed and what they noticed was compared against those intended in the design. That is, we compared to see what, from the videos, received attention. This was done for each episode individually. At the second level of analysis, teachers' responses were classified into four levels using van Es and Sherin's (2002) noticing platform:

Table 1: Trajectory of Development in Learning to Notice (van Es, & Sherin, 2002, p. 582)

	Level 1	Level 2	Level 3	Level 4
Dominant pattern in writing	Describe and Evaluate	Mixture of describe and evaluate and complete analytic chunks	Complete analytic chunks	Complete analytic chunks
		Or	and	Connections among callouts and evidence
			Evaluate	
		Incomplete analytic chunks		Identify pedagogical solutions

Lastly, the participants' analysis of the first episode was compared to that of the second phase so to detect shifts in their noticing due to prior experience and class discussions.

Findings and Discussions

The prospective teachers' noticing stance when viewing the first video, globally, transitioned between levels 2 and 3, averaging 2.3 for both cohorts. The primary focus of the participants' at both sites concerned mathematical proficiency of the child and various ways that they had interpreted what the child knew. This was an exceptionally important outcome of the video-illustration as it enabled us to capitalize on the importance of providing instructional time and space for increasing children's mathematical problem solving capacities. The participants were less concerned with analyzing the nature of interactions depicted in the videos and tended to trivialize the space of inquiry and learning that was provided by the teacher for the children's devolution. They criticized the interviewer for not having provided the child sufficient help and guidance during the problem solving process and having used ambiguous problems confusing to the child. This practice was consistent across both cohorts and both video illustrations. This criticism was less prominent at the site where the interviewer/researcher instructed the course.

The preservice teachers' ideas, for designing follow up instruction, concerned providing corrective opportunities during which the child would apply formulas for calculating areas.

A majority of the teachers' pedagogical solutions were at 2 (Mean 1.9). This indicated that although they showed maturity in noticing what children did, their pedagogical analysis did not reach the same level. Indeed, their analysis was prominently grounded in their understanding of mathematics and knowledge of

how phases of growth of understanding might be articulated. That is, in places where they felt most comfortable making sense of intricacies of the child's work, they tended to make more relevant decisions, drawing from theoretical knowledge, when offering suggestions for future interactions. The teachers generally focused on strengths that children exhibited during the interactional episodes, and were less concerned about errors in reasoning, or factors that could have contributed to their practices. Such a focus tended to weaken their analysis of how curriculum and instruction could be organized to address this issue.

In this work, we adopted van Es and Sherin's (2002) analytical tool to examine prospective teachers' noticing. While this framework provides a global tool for cataloguing teachers' reactions, it was problematic in capturing the depth and relevance of content referenced in their discourse. That is, while episodes of attempts at connecting (Level 4 analysis) were present, connections made were not always mathematically deep or pedagogically grounded in content specific insights.

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COMPUTATIONAL ESTIMATION SKILL OF PRESERVICE TEACHERS: OPERATION TYPE AND TEACHER VIEW

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Despite the importance of computational estimation skill for the improvement of number sense, little research exists on preservice teachers' estimation skill and their view on it. This study examined the proficiency level of computational estimation skill of 58 preservice elementary teachers and its relation to their views of the meaning of estimation and the importance of teaching it. Findings revealed that preservice teachers performed differently depending on the types of operations on the computational test and the estimation test. Four factors including preservice teachers' mathematical knowledge, their confidence on estimation skill, their views of estimation in teaching mathematics, and their self-reported knowledge about calculator were found to correlate with their estimation skill. Implications are discussed in accordance with the findings.

Keywords: Teacher Education-Preservice, Mathematical Knowledge for Teaching, Number Concepts and Operations, Computational Estimation

Introduction

The ability to estimate the answer to an arithmetic problem rather than compute an exact answer with paper and pencil has been considered an essential part of mathematics education (Greeno 1991; McIntosh et al. 1992; NCTM, 2000). In estimating, students demonstrate their understanding of number magnitude, number relationships, and the meaning and effect of operations. Estimation is essential for solving problems, for example, as a strategy for assessing the reasonableness of answers (Dowker 2003; Reys & Reys 1989; Verschaffel et al. 2007). NCTM (2000) makes it clear that computational estimation needs to be integrated to the mathematics curriculum to help students understand the power of number sense and its use in communicating and reasoning mathematically (Greeno 1991; McIntosh et al. 1992).

The purpose of this study is to investigate the proficiency level of preservice elementary teachers (PTs) in estimating answers to computational problems. This study also examines the relationship between PTs' proficiency level in estimation and other factors including their self-reported mathematical knowledge, their views of the meaning of computational estimation, and their views of the importance of teaching it. The purpose of this study is not to add to the collection of studies documenting preservice teacher weaknesses, but rather to inform the design of teacher education in this area. The research questions that guided the study were:

1. What are PTs' views of the meaning of computational estimation and how do they view the importance of teaching estimation?
2. How do they perform computational estimation in comparison to finding an exact answer with paper and pencil? Do PTs perform differently depending on the types of operations (whole numbers, decimal, and fraction operations)?
3. What strategies do they use in doing computational estimation involving whole number operations and fraction operations?
4. What factors are related to success in computational estimation of PTs?

Theoretical Background

Research on Estimation Skill and Strategy

A large portion of research on estimation has focused on students' computational estimation skills (e.g., Reys et al., 1982; Verschaffel et al., 2007), how mathematics abilities relates to estimation (Hogan & Brezinski, 2003; Rubenstein, 1985), estimation strategies (Lemaire & Lecacheur 2002), and the format of estimation tests (Schoen et al., 1990). However, relatively a small number of researchers have

investigated teachers' estimation skills (e.g., Gliner, 1991) and the factors contributing to their success in estimation (e.g., Trafton, 1994). Table 1 presents five key estimation strategies identified from previous studies (e.g., Alajmi, 2009; Hanson & Hogan, 2000), which is used for the analysis of PTs' computational strategies.

Table 1: Five Estimation Strategies Used by Good Estimators

Strategies	Characteristics	Examples
1. Front-end	Focuses only on the digits in the highest place value position. Estimates by adding up the "front-end" digits (the left-most digits).	4.5+6.7 +3.2 +7.8 +9.2 is estimated by 4 + 6 + 3 + 7 + 9 = 29.
2. Rounding	Uses alternative numerical data to make it easy to find an estimate, without changing the structure of the problem, such as rounding to the nearest 10, 100, or 1,000.	The product 36 x 75 might be estimated by 40 x 70 = 2800.
3. Compatible number strategy	Involves choosing numbers which make computation easy but yet give a good estimate of the original problem. An example is adding a group of numbers together.	27 + 49 + 56 + 81; 27 + 81 is almost 100; and 49 + 56 is almost 100. These pairs of numbers are "compatible."
4. Clustering	Involves adding many numbers together when those numbers all "cluster" around a particular value.	70,132 + 67,782 +73,212 is almost 210,000 since all numbers are about 70,000 and there are three of them; multiply 3 by 70,000 for the total estimate.
5. Special numbers	Involves looking for numbers near "special" values. It is often used with fractions, where the students look for fractions near 0, 1/2, or 1	12/13 + 1/17 + 7/15 is near 1.5 since 12/13 is almost 1, 1/17 is almost 0, and 7/15 is almost 1/2.

Research on Teachers' Knowledge and Views of Computational Estimation

Prior research on (preservice) teachers' computational estimation skills reported similar findings with students' computational estimation skills. Trafton (1994), for example, reported that US teachers' views about estimation "are similar to those of students" by limiting estimation to rounding to multiples of 10 (p. 77). In particular, he reported that the most common strategy by teachers was to round both operands down to the closet smaller decades (Alajmi, 2009; Gliner, 1991). In addition, prior research reported the different tendency of teachers' performance on paper-and-pencil tests and on estimation tests. Gliner (1991), for example, reported that preservice teachers did better on estimation problems with applied format (i.e., story form) than on those with computational format (i.e., numerical form); however, they performed better on paper-and-pencil tests with computational format. Reys and Reys (2004) stress that teachers need to see computational estimation as important and worthy of instructional attention in order to help their students appreciate the significance of estimation. Although a considerable amount of attention to number sense is occurring in the literature, the term "computational estimation" is rarely heard in preservice teacher research in the US. Little attention is devoted to PTs' computational skills and their views of computational estimation. Supporting that understanding the level of computational estimation should play an important role in preservice teacher programs, this study intended to address such gap.

Methods

Fifty-eight PTs participated in the study from three elementary mathematics methods classes. Participants were in their internship year of elementary teacher preparation program at a large southwest university in the United States. Data was collected in their second, a year-long methods courses. All three methods classes were taught by the first author (approximately 14 weeks) during the 2011-2012 academic year (2 sections in the fall, 1 section in the spring), designed to support PTs' understanding of approaches, strategies, and issues that are relevant to the teaching and learning of mathematics. Three sets of tasks

were used for the study: (1) Estimation task, (2) Computational test, and (3) Belief survey. Table 2 presents the estimation task used for the study, which was slightly modified from Hanson and Hogan's (2000) instrument. Participants were told that they would be asked to estimate answers to 20 arithmetic problems (see Table 2).

Table 2: Items Used for the Estimation Test

Item	Category	Item		Category
1. 1.1219+217	WN[a] Addition	11.	443.7 + 19.19	Decimal Addition
2. 2.97 × 26	WN Multiplication	12.	7 is what percentage of 65?	Percent
3. 3.2021-519	WN Subtraction	13.	369 - 2.89	Decimal Subtraction
4. 4.123 × 48	WN Multiplication	14.	$6\frac{1}{3} \div 12\frac{1}{2}$	Fraction Division
5. 5.5869 + 642 +8190 + 16	WN Addition	15.	48% of 41 = ___	Percent
6. 6.58.4 × .13	Decimal Multiplication	16.	$\frac{12}{19} + \frac{2}{17}$	Fraction Addition
7. 7.58,795 – 2,989	WN Subtraction	17.	$\frac{9}{32} + \frac{4}{15}$	Fraction Addition
8. 8.6,123 ÷ 54	WN Division	18.	$\frac{20}{21} - \frac{5}{19}$	Fraction Subtraction
9. 9.8402÷ 41	WN Division	19.	$64 \times \frac{14}{17}$	Fraction Multiplication
10. 10.16 is what percent of 38?	Percent	20.	$534\frac{7}{9} \times 0.495$	Fraction Multiplication

Note: WN = Whole number.

Three sets of tasks were administrated to the entire class in three mathematics methods course sections towards the end of the semester. Qualitative and quantitative analyses were conducted. For the estimation task, we determined scores by computing an accuracy score in the same manner as Hanson and Hogan (2000) had done. For the computational test, we scored them by giving 1 point for a correct answer and 0 point for an incorrect answer (correct/incorrect scoring). The second research question involved investigating whether there was a significant difference between the number of problems correct in the estimation task versus the computation format. A *t* test for paired data was performed to answer this question. In order to answer the fifth research question, investigating the relationship between success in mathematical estimation and other variables, first, factor analyses were performed to identify study's constructs that include 'knowledge about math', 'knowledge about estimation', 'knowledge about calculator', 'teacher view on estimation in teaching', 'teacher view on calculators in teaching', 'PD about calculator (I have had adequate training....)' and 'PD about estimation.' Pearson correlation (*r*) was performed to determine the relating factors to estimation skills of PTs.

Summary of Results

We found that our PTs were more successful in solving computation problems than in estimating answers. In particular, they did better on whole number operations than on fraction and decimal operations. Rounding was the dominant strategy for whole number operations, while special number was most frequently used in fraction operations. We also found that PTs' estimation performance was significantly related to their math knowledge and their views of estimation in teaching mathematics. In particular, we found that four factors were identified as significantly related to scores on the estimation test (see Table 3). PTs' ratings of their own mathematical abilities, the ratings of their own estimation abilities, and teacher view on estimation, positively correlated to estimation scores. This finding indicates that good estimators claimed that they developed a deep understanding of mathematics and estimation skills, while strongly supporting the importance of estimation in teaching mathematics. However, interestingly, PTs' ratings of their knowledge about calculator negatively correlated with their performance on estimation.

Table 3: Correlations Among Estimation Cores, Self-ratings, and View on Estimation

Variable	1. Math knowledge	2. Estimation	3. Calculator	4. Views on estimation	5. Performance on estimation
1	1				
2	0.45**	1			
3	0.28*	-0.11	1		
4	0.28*	0.35**	-0.22	1	
5	0.32*	0.34**	-0.27*	0.32*	1

Note: For n=58. **. Correlation is significant at the 0.01 level; *. Correlation is significant at the 0.05.

These findings highlight the challenges that teacher educators and school administrators face in teaching mathematics teachers about computational estimation. Our study showed the same tendency of preservice teachers' knowledge regarding computational estimation, which highlights the challenges that the US faces in reintroducing computational estimation into the curriculum. This study has implications for teacher educators and future studies. For example, teacher educators must provide opportunities for preservice teachers to see computational estimation as important and worthy of instructional attention.

Inservice and preservice teacher educations will be needed to support teachers in understanding and teaching this concept. Training will be needed before re-introducing the concept into the curriculum to help teachers better understand it. The program will also face the challenge of changing teachers' views of the importance of computational estimation in the classroom. Ongoing support will also be needed for problems teachers face as they make these changes in the curriculum. Additional research is needed to provide substantiation in how we can help preservice teachers prepare for the multiple strategies in estimation and its importance of computational estimation for teaching math.

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MULTIPLICATIVE CONCEPTUAL FIELD THEORY: A TRAJECTORY FOR TEACHER PROFESSIONAL DEVELOPMENT IN THE MIDDLE GRADES

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Keywords: Teacher Education-Inservice/Professional Development, Mathematical Knowledge for Teaching, Middle School Education

Professional development for mathematics teachers is often focused on pedagogy, such as methods and strategies for teaching, rather than providing opportunities for them to build deep understandings of the content they teach. Research has shown that a powerful way for improving teaching practices is to engage teachers in meaningful activities that not only deepen their understanding, but also focus their lens on student thinking in the context of mathematics (Silverman & Thompson, 2008). This report presents our initial findings that demonstrate how a curriculum built upon a central mathematical theme – multiplicative reasoning – can help improve teachers' content knowledge and influence their beliefs about teaching mathematics.

Introduction

Past professional development (PD) for teachers of mathematics has largely focused on providing teachers with strategies and methods for teaching. The teaching strategies promoted in many PD programs are typically not discipline-specific and lack a fundamental focus on mathematical content. Research has shown that a powerful way for improving the teaching and learning of mathematics is to engage teachers in meaningful activities that not only deepen their understanding and conceptual knowledge of the mathematics they teach, but also focus their lens on student thinking in the context of mathematics (Silverman & Thompson, 2008). Reflecting this focus, a research-based PD program was implemented with middle school mathematics teachers. A major emphasis of this program is to improve the teaching and learning of middle school mathematics by improving teachers' content knowledge and influencing their beliefs about teaching mathematics. More specifically, we believe that if the participating teachers develop meaningful ways of thinking about mathematics, develop a profound understanding of the fundamental mathematics that they teach, and develop strategies for effective problem solving, then these same teachers will be better prepared to engender these mathematical practices in the students that they teach (Ma, 1999).

In this report we intend to present some of our initial findings that demonstrate how a curriculum built upon a central mathematical theme – multiplicative reasoning – can help improve teachers' content knowledge and influence their beliefs about teaching mathematics. We will present preliminary evidence based on teacher surveys, observations and quantitative instruments that were implemented during 2012-2013 to support our claims.

Why Middle School Mathematics?

Research has shown that middle school is the gateway to high school course taking and college enrollment (Hill, 2007). In particular, completion of Algebra II is strongly correlated

with success in college as well as future earning potential (Adelman, 1999). Yet over recent years, school districts have repeatedly seen student mathematics performance rates decline from grade 4 to grade 8. If students do not succeed in middle school mathematics, they will not have the foundation necessary to succeed in Algebra and other challenging high school math courses (Evan, Gray, & Olchefshe, 2006).We therefore believe that students who possess a solid foundation in middle school mathematics are more likely to experience positive outcomes in high school mathematics, further preparing them for success in mathematics beyond high school.

Theoretical Perspective

Rooted in the theory of pragmatism, the core idea of conceptual field theory is that knowledge has many local facets that are molded and connected by mastering problems that are related conceptually (Vergnaud, 1994). One theory, in particular, that has captured the attention of mathematics education researchers is multiplicative conceptual field (MCF) theory, which addresses the ability to reason multiplicatively. Vergnaud argues that multiplicative reasoning plays a significant role in the development of conceptual understandings of multiplication, division, fractions, proportions, ratios, rates, rational numbers, and linear functions. It also serves as a gateway for building a mature understanding of exponential functions (Strom, 2008). Vergnaud refers to these topics as ingredients for the MCF which he defines as a complex theory that is “simultaneously a bulk of situations and a bulk of concepts” (p. 46) where concepts (tools for analyzing situations) become meaningful through situations (things that require multiplicative operations) and situations are analyzed through these multiple concepts. This reflexive relationship between concepts and situations provides the basis of the conceptual field theory and serves as the mathematical foundation for this investigation with middle school teachers.

Research and Implementation Framework

The framework for this study originates from the network of pedagogical content knowledge described by An, Kulm and Wu (2004, see Figure 1). An et al. integrate teacher beliefs and key student characteristics such as addressing student misconceptions and building on students’ math ideas. The focus of this framework is *students’ learning* from the viewpoint of teaching as a convergent process that “focuses on knowing students’ thinking, which consists of 4 aspects: building on students’ mathematical ideas, addressing students’ misconceptions, engaging students in mathematics learning, and promoting students’ thinking mathematically” (p. 148). This investigation builds teachers’ pedagogical content knowledge by focusing on content and curriculum. Teachers collaborate to develop their mathematical knowledge, and they focus on understanding students’ thinking while keeping students’ prior knowledge in the forefront.

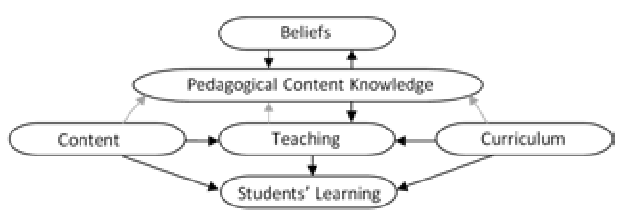

Figure 1. Network of Pedagogical Content Knowledge (abridged) (An et al., 2004).

The primary research questions driving this study are: (a) In what ways does a teacher PD program grounded in MCF promote teachers’ conceptual understanding of mathematics? (b)

What effect does a curriculum focused on multiplicative reasoning have regarding teachers' beliefs about teaching mathematics?

Multiplicative Concepts – A Sample Activity

As an example of the activities that are implemented in the PD program, consider the following situation of the Tree Problem (see Figure 2). Which tree grew more, Tree A or Tree B?

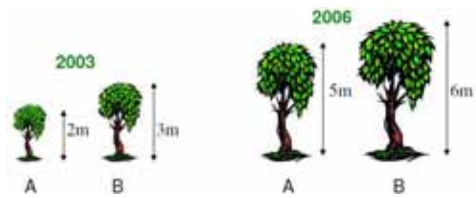

Figure 2. The Tree Problem (Lamon, 2005).

We have found that when teachers first encounter this problem, they feel as though the wording of the question is "vague." Once we explain that this question can be answered in more than one way, they begin to consider the meaning of *more*. If teachers define *more* in the additive sense, they contend that neither tree grew more since each tree grew an additional 3 meters from 2003 to 2006. However, if teachers define *more* in the multiplicative sense, they can then think about Tree A as growing more since its height in 2006 is 2.5 times (or copies of) its height in 2003 while Tree B's height in 2006 is only 2 times (or copies of) its height in 2003. Another important concept is the ability for teachers to be aware of the type of reasoning they engage in and to consciously choose to reason additively and/or multiplicatively based on the context of the problem. Opening teachers' minds to thinking multiplicatively allows them to expand how they compare quantities and operate on numbers. This ability to reason multiplicatively is an important component of the MCF underpinning K-12 mathematics.

Methodology

Sixty-seven middle school teachers from seven school districts in Arizona were recruited to participate in PD focused on increasing mathematical content knowledge during a one-week summer workshop, followed by four workshops over the school year. Teachers were given conceptually rich tasks designed to promote multiplicative thinking and meaning making. These workshops were augmented with school-based CCOLs (Collaborative Community of Learners) facilitated by project staff. The CCOLs met regularly throughout the school year and are designed to support teachers as they deepen their content knowledge and transform their beliefs relative to mathematics. Both quantitative and qualitative data is being collected and analyzed.

Prior to the PD workshops, we administered the Mathematical Knowledge for Teaching (MKT) Instrument developed by the Learning Mathematics for Teaching project (Learning Mathematics for Teaching, 2011) and an adaptation of the Views About Mathematics Survey (Carlson, Buskirk, & Halloun, 1999). Teachers have also provided periodic reflections of their views on the changes they have implemented in the classroom as a result of their learning.

Discussion and Implications

Initial results have shown that teachers who scored high on the MKT (answers 70% or more of the questions correctly) view the role of mathematical instructional practices as helping students develop a broader understanding of mathematical ideas while teachers who scored

poorly (answers less than 70% correctly) feel the focus should be helping students perform a series of steps. Since one of our goals is to help teachers reshape their practice around the Common Core Standards for Mathematics (NGA, 2010), we believe improving their content knowledge while reinforcing the Standards for Mathematical Practice will have an impact on their beliefs about mathematics teaching and thus influence their classroom practices.

By engaging the teachers in a curriculum of number sense, geometry/measurement, and problem solving tied together by the common theme of multiplicative reasoning we are helping them to develop deep, well-connected meanings of mathematics. Focusing on these big mathematical ideas facilitates teachers' development of a cohesive and flexible knowledge base for teaching mathematics. Early evidence suggests that teachers believe they have changed their teaching practices because of their increased content knowledge. The following excerpt provides insight into how one teachers' classroom practice has been modified:

> *Teacher:* As part of the AMP program, in my classroom, I have used more task-oriented lessons and hands-on activities. Taking more effective prompts in my questioning of my students and allowing them to use their knowledge and not me give them the answers, but waiting for them to come up with them. I'm having more discussions on mathematics instead of just a lesson where they're doing all the work…showing ideas and reasoning – not mine – but the students' ideas and reasoning and discussing those.

Additional data from the post-test MKT and the VAMS survey adaptation in summer 2013 will be compared with pre-test data to determine potential shifts in teachers' beliefs and attitudes of mathematics relative to their conceptual shifts in mathematical content.

Acknowledgements

Research reported in this paper is being supported by National Science Foundation Grant No. DUE-1103080. Any conclusions or recommendations made are those of the authors and do not necessarily reflect official positions of NSF.

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CORRELATIONAL STUDY OF MIDDLE SCHOOL MATHEMATICS TEACHERS' CONTENT KNOWLEDGE AND STUDENT ACHIEVEMENT IN RUSSIA

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The study examined the relationship between three different cognitive types of teachers' content knowledge and student achievement in lower secondary schools in Russia. The first cognitive type assessed participants in their ability to understand basic facts and procedures. The second cognitive type measured teachers' knowledge of concepts and connections. The third cognitive type gauged teachers' knowledge of mathematical models and generalizations. Teachers (n =87) from grades 5-9 completed the Teacher Content Knowledge Survey (TCKS) instrument for data collection purpose. The study comprised two levels of data analysis. First, we explored each cognitive type of content knowledge and the overall TCKS score as they related to student achievement. Second, we studied the correlation between each cognitive type of teacher content knowledge to deepen the understanding of content associations.

Keywords: Teacher Knowledge, Student Achievement, Middle School Education

Significance of the Study

Although there are numerous studies that focus on the understanding of teachers' knowledge, there is a dearth of studies that provide an in-depth examination on the various facets of such knowledge and its relationship to effective teaching and learning. A number of studies (Hill, Shilling, & Ball, 2004; Rowland, Huckstep, & Thwaites, 2005; Davis & Simmt, 2006) emphasize the importance of various aspects of teacher knowledge. Most of those studies relate to participants within the United States and a few other European countries. This study is intended to contribute to the field of inquiry related to teacher knowledge with participants selected from Russian lower secondary schools.

For the purpose of this study, we will focus on the examination of teacher content knowledge. Bransford, Brown, and Cocking (2000) state that teacher content knowledge requires three features: 1) "a deep foundation of factual knowledge", 2) understanding of the "facts and ideas in the context of a conceptual framework", and 3) organization of the knowledge "in ways that facilitate retrieval and application" (p. 16). According to Hill, Ball, and Schilling (2008), it includes the knowledge "that allows teachers to engage in particular teaching tasks, including how to accurately represent mathematical ideas, provide mathematical explanations for common rules and procedures, and examine and understand unusual solution methods to problems" (p. 377-378). To know what kinds of teacher content knowledge impact student achievement is a critical point. Due to the multifaceted nature of student achievement, few scholars have analyzed how the different types of teacher content knowledge are associated with student learning. Stein, Smith, Henningsen, and Silver (2000) claim that student learning depends on the level of cognitive demand established in the mathematics classroom. Furthermore, Tchoshanov, Lesser, & Salazar (2008) point out that the cognitive demand is a function of the cognitive type of teacher content knowledge. Hence, this study considered three cognitive types of teacher content knowledge that were identified: knowledge of facts and procedures (T1), knowledge of connections and concepts (T2), and knowledge of generalizations and models (T3) (Tchoshanov, 2011). The analysis of the study was guided by the following *research questions*: (1) to what

extent are the different cognitive types of middle school Russian mathematics teachers' content knowledge related to student achievement? (2) is there a significant correlation among the cognitive types of teachers' content knowledge?

The Russian educational system does not apply state-mandated standardized tests at the lower secondary school level. In this study, we refer to the term of student achievement as the grade obtained in a common cumulative assessment composed by a set of open-ended multi-step problems at the end of a term. Specifically, student achievement is based on the percentage of students who receive highest grades (corresponding to grades of "A" or "B" in the U.S. system).

Methodology

Participants

The sample consisted of 87 Russian middle school teachers. Participating teachers had attained a secondary mathematics teacher preparation Specialist's degree, which allowed them to teach in secondary schools (grades 5-11). In Russia, the secondary school consists of lower and upper secondary schools: the lower secondary school includes grades from 5 to 9, and grades 10-11 are part of the upper secondary school. Eight surveyed teachers were males, and the rest were female teachers. This sample was composed of 78% of teachers who have more than 10 years of teaching experience. Approximately 64% of surveyed teachers had students who participated in Mathematics Olympiads at least at the school level. Most participants (77%) did not teach a particular grade level, but were rather teaching different grade levels.

Research Design

The quantitative research design consisted of an analysis of variance (ANOVA) based on the research questions. The analysis encompassed two levels of investigation. The first level explored each cognitive type of content knowledge and the overall Teacher Content Knowledge Survey (TCKS) scores of teachers as they related to student achievement. The second level studied the correlation between each cognitive type of teacher content knowledge to deepen the understanding of mathematical content associations.

Instrument. Teacher Content Knowledge Survey is the instrument that was designed to assess the teacher content knowledge based on the three cognitive types identified above (Author, 2011). The survey consisted of 33 multiple choice-items about relevant topics for secondary grades teachers' knowledge: Number Sense, Algebra, Geometry and Measurement, Probability and Statistics. There were 10 items that measured the cognitive type 1. The cognitive type 2 was measured by 13 items. The rest of the 10 items measured the cognitive type 3. The teacher content knowledge survey does not have any division per cognitive type. All the items are located randomly throughout the survey. The instrument was developed by interdisciplinary faculty with expertise in the following domains: mathematics, mathematics education, statistics and statistics education, representing various institutions such as university, community college and local schools. The main steps were the selection of items for the survey, the classification of items by cognitive type, and modifying an item in another cognitive type. The instrument was field-tested during 2005-06 (Tchoshanov, Lesser, & Salazar, 2008). The alpha coefficient technique (Cronbach, 1951) was utilized to evaluate the reliability of the teacher content knowledge survey instrument. "The value of the coefficient of .839 suggests that the items comprising the TCKS are internally consistent" (Tchoshanov, 2011, p.148).

Data Analysis

In order to answer the first research question posed, the following variables were considered as independent: T1, T2, T3, and the total score of TCKS. The dependent variable used was

student achievement. To answer the second research question, we made three correlational considerations. The first makes a comparison using the cognitive type 2 as the dependent variable and cognitive type 1 as independent variable. For the second correlation, we use cognitive type 3 as the dependent variable and cognitive type 1 as the independent variable. Lastly, we do a correlation among cognitive type 3 and cognitive type 2, using this last one as the independent variable. Data analysis was performed using ANOVA to determine how the cognitive types of teacher content knowledge total scores related to student achievement, as well as the relationship among the three cognitive types.

Results and Discussion

The following data are representative of the results obtained in this study, which examined the relationship between cognitive types of teacher content knowledge (T1, T2, T3, and total score of the TCKS) and student achievement. Results of the study show correlation between different cognitive types of teacher content knowledge and student achievement approaching significance: for T1 Pearson's r =*.20*, p =*.058*; T2 – r =*.19*, p =*.069*; T3 – r =*.19*, p =*.067*). Thus, implying that different cognitive types of teachers' content knowledge have a certain impact on student achievement.

The most substantial finding was the correlation between teachers' total score on TCKS and student achievement presented in Table 1. The analysis of variance showed a significant correlation between teacher content knowledge measured as the total score on the TCKS and student achievement (Pearson's r =*.27*, p =*.013* < .05). The corresponding F value was $F\ (1,85) = 6.451$, p < .05. In other words, the teachers' performance on the TCKS is significantly related to student achievement, implying that teachers who performed better in the content test overall, had a higher student success rate.

Table 1: Relationship between Teacher Content Knowledge and Student Achievement

ANOVA	*Df*	*SS*	*MS*	*F*	*Significance F*
Regression	1	797.1361	797.1361	6.451445	0.012904967
Residual	85	10502.54	123.5593		
Total	86	11299.68			

It is apparent that when separated, each cognitive type of teacher content knowledge only has a certain, yet non-significant influence on students' performance. At the same time, teacher's overall mastery of all cognitive types of content knowledge can significantly impact the students' success attaining As and Bs in mathematics classes. Therefore, we can conclude that teachers' content knowledge plays a significant role in teaching and learning middle school mathematics in Russia.

Furthermore, analysis of the relationships between the cognitive types of content knowledge (T1-T2, T1-T3, T2-T3) produced significant results. We conducted three correlational analyses. The first analysis examined T1 and T2. The results for this comparison showed Pearson's correlation r =*.27*, p =*.012*. Therefore, we can deduce that a teacher who possesses the knowledge of concepts and connections has a sound foundational knowledge of facts and procedures. The second analysis focused on comparing T1 and T3. For this comparison, the relationship between cognitive types was also significant at a level of p <.01 (r =*.31*, p =*.003*). This finding implies that a teacher who is able to do generalizations and mathematical models is also able to know the procedures and facts of mathematical content. Finally, the most significant

finding showed a strong correlation between cognitive types 2 and 3 ($r = .42$, $p < .001$). Thus, we can presume that a teacher who knows mathematical models and generalizations is expected to also know the concepts and connections among mathematical topics. We can construe that each cognitive type of content knowledge is a kind of a building block for the other two. In other words, for a teacher to be able to develop connections in mathematics, a teacher needs to first be able to understand the basic facts and procedures needed to solve mathematical problems. Similarly, a teacher who has an understanding of development of models and generalizations must first be able to understand the connections between mathematical ideas.

Conclusion

The results of this study indicate that teachers' content knowledge does have an impact on student achievement in middle school mathematics in Russia. Moreover, there is a significant correlation between cognitive types of teacher knowledge making it hierarchical: high cognitive type requires the low type of content knowledge as a foundation. Overall, teachers with a strong background of content knowledge will be able to prepare students more effectively. Correspondingly, a teacher with poor content knowledge might be limited and unable to produce a high rate of student achievement. We are cognizant on the limitations of the non-uniform assessment (e.g., cumulative exam) used in Russia as an indicator of student achievement. Nonetheless, "a teacher with content knowledge limited to mathematical procedures only has less opportunity to influence students' success than a teacher who conceptually understands the subject" (Tchoshanov, 2011, p.144).

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TEACHING TEACHERS TO LEAD MATHEMATICALLY RICH DISCUSSIONS

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Despite available strategies for leading discussions, maintaining mathematical quality is challenging, especially for teachers with limited mathematical knowledge. This paper analyzes the effects of a professional development design that provides teachers with two novel mathematical resources for leading mathematically rich discussions. Using evidence from teachers' practice, students' contributions, and the mathematical quality of instruction, we found that having teachers focus the setup of the problem on mathematical conditions given in the problem provided students with language for making mathematically productive contributions to discussions and that providing teachers with sample student explanations, idealized to capture the mathematical structure and language of a range of common student approaches, helped teachers hear, respond to, and build connections among student contributions.

Keywords: Teacher Education-Inservice/Professional Development, Elementary School Education, Instructional activities and practices, Mathematical Knowledge for Teaching

The Problem

Leading a discussion in a mathematics lesson is challenging work. To support teachers, several scholars have proposed strategies or frameworks for leading discussions (Chapin, O'Connor, & Anderson, 2009; Lampert, 2001; Leinhardt & Steele, 2005; Michaels, O'Connor, & Resnick, 2007; Stein, Engle, Smith, & Hughes, 2008). Several of these frameworks have emphasized the role of language and of other discourse moves—revoicing, wait time, repeating what someone else has said—that together provide opportunities for teachers and students to attend more closely to how mathematics is discussed. By using these moves, scholars say, teachers will be able to make students' thinking more visible and then be more proactive in helping them shape their mathematical ideas. Although all of these scholars are concerned with the treatment of subject matter, discursive concerns have been the primary conceptual tool in most approaches to leading discussions.

Two crucial challenges arise in existing proposals for supporting teachers to learn to lead discussions. One is that, although recommendations (and admonitions) are given for attending to content, what a teacher needs to do in order to attend to content is unclear. In other words, the practice of leading a discussion, as currently conceptualized in the literature, is underspecified in relation to the key role of content inside the work of leading mathematically rich discussions. A second issue is that meaningful engagement in any proposed teaching moves or practices, however well specified, requires significant mathematical knowledge, which is often lacking. Many pre-service and in-service teachers are underprepared for the kind of mathematical work involved in leading mathematically rich discussions, and when teachers lack content knowledge for leading a discussion, discourse moves are unlikely to lead to productive discussions.

This study investigates potential effects of using resources that combine pedagogical need and mathematical content on teachers' learning to lead mathematically rich discussions. It contrasts effects of professional development focused on these additional pedagogically informed mathematical resources with professional development focused only on discursive resources. Both designs attend to important discursive resources for leading discussions, but the

former considers these to be necessary though insufficient. The designs are the same in all other aspects. Thus, the contrast can be used to examine ways in which these mathematical resources might additionally support learning to lead discussions. Our research questions, then, are:

1. Might the use of mathematical instructional resources (focused on problem conditions and idealize explanations) contribute to the mathematical quality of discussions?
2. In what ways might discussions of teachers who use mathematical instructional resources and discursive instructional resources differ in terms of teaching practice, student contributions, and mathematical quality?

In these questions, our use of "might" instead of "does" is deliberate. We do not seek to make strong claims of causality. Instead, we want to know whether it is possible to achieve greater success in supporting teachers in leading discussions by providing targeted mathematical tools. And, if so, we want to know what such tools might afford.

Method

As part of a larger investigation of teacher and student talk in classrooms, we designed and conducted a six-week, CEU-granting, online professional development course for teachers interested in increasing student talk. In the first year, we piloted a basic design in which teachers read Chapin, O'Connor, and Anderson (2009), watched short video clips of discussions, and reflected on their own practice and ways they might take up ideas suggested by the readings and observations. In the second year, we additionally provided a protocol for leading a mathematics discussions and a set of resources and activities designed to support teachers' incorporation of talk moves into their practice. In the third year, we replaced the additional talk-move resources and activities with mathematical resources and activities. Each year, we collected video of teachers leading discussions in their mathematics lessons, both at the start of the professional development and at the end. In years two and three, we provided a lesson for all teachers to teach (one for grades 1 and 2, another for grades 3 and 4). Baseline videos of classroom discussions (involving varied topics and lesson formats taught at the start) together with videos of a common culminating classroom discussion lesson (taught at the end) provided a basis for examining both changes in individual teacher's practice as well as differences in changes between the learning of teachers in years two and three. To characterize both changes in teachers and differences between the two designs, we are currently scoring baseline and culminating lesson videos using an existing measure of mathematical quality of instruction (MQI) and developing a set of inductive codes for teaching practice and student contributions (blind to treatment).

One of the two problem-specific mathematical resources we developed is the explicit identification of the conditions of a problem and the use of conditions in setting up the problem. In teaching her students how to study in her classroom, Lampert (2001) explicitly identifies a practice of "finding or articulating the 'conditions' or assumptions in problem situations that must be taken into account in making a judgment about whether a solution strategy is appropriate" (p. 66). Some mathematics problems amount to giving a set of conditions that a solution must satisfy. In such problems, the conditions provide a structure for the mathematical justification of a solution and can be used as an additional rhetorical device for structuring the content of a crucial type of contribution to make to a mathematically productive discussion.

The second problem-specific mathematical resource is a small set of idealized solutions (with explanations) that provides a road map of the conceptual terrain across which a discussion is likely to travel. As many scholars have pointed out, anticipation of student thinking is an important component of preparation for leading a discussion (Hiebert & Wearne, 1993; Stein et

al., 2009). Anticipation of student thinking prepares teachers to more readily hear and correctly interpret student thinking and to respond appropriately. For leading a discussion, it is important that teachers anticipate, not only students' thinking and solution strategies, but also the explanations they are likely to produce (spontaneously or with prompting). What it takes to explain a solution is different from what it takes to solve a problem—and requires different support. In addition, the resource of idealized explanations is organized in mathematically structured ways. This structure helps teachers recognize important types of explanations and important types of mathematical similarity and difference. The idealized explanations also provide language that can help teachers hear their students' explanations, but also connect student language to important mathematical terms and ways of talking.

Table 1 describes the sample of teachers we had in each year of the program. Each year there were several repetitions of the program with different cohorts of teachers.

Table 1: Number of Participants in the Online Program per Year and per Cohort with the Range of Grades Taught

Year (# of cohorts)	Number of Teachers	Grades
1 (4)	5+3+5+4=17	1-5
2 (4)	4+6+6+5=21	1-4
3 (2*)	4+8=12	1-3
Note: * The data collection for this year is still in progress.		

The *Mathematical Quality of Instruction* (MQI) instrument was designed at the University of Michigan to provide scores on important dimensions of classroom mathematics instruction (Learning Mathematics for Teaching Project, 2010). We are using a revised version of the instrument (available from the National Center for Teacher Effectiveness at Harvard University). This instrument includes 18 codes scored for 7.5-minute segments of instruction throughout lessons and organized into five categories: (a) classroom work is connected to mathematics, (b) richness of the mathematics, (c) working with students and mathematics, (d) errors and imprecision, and (e) student participation in meaning-making and reasoning. It also includes two whole-lesson codes. Obtaining 80% inter-rater reliability in training, each lesson is being individually coded by two raters, with disagreements reconciled by the pair. Raters are being randomly assigned to one another and to lessons. These codes provide a detailed account to use in examining differences in the mathematical quality of the discussions both between baseline videos and common, culminating-lesson videos for each teacher and between classes in the second and third years. These codes will be augmented by a set of inductive codes focused specifically on distinctive features of teaching practice and student contributions evident across the full data set.

Preliminary Results

From initial analysis, we can report on a few preliminary results. First, we are finding that explicit use of the conditions of the problem during the setup helps students understand the meaning of the task and begin work on the task and it provides them with language for presenting their solutions to the problem at the start of a class discussion. We are also finding that idealized sample explanations that provide a conceptual map of likely explanations, what we are calling the "explanation space," help teachers hear and respond to student solutions, make

connections among and comparisons between different student solutions, and make decisions about the direction and conclusion of a discussion.

We have also found evidence of the importance of targeting very specific content-discursive moves in the setup of the problem. For example, lessons in which teachers have a student read the problem aloud to the class, have students express the problem in their own words, and make sure students can identify the conditions of the problem and how the conditions can be used for testing whether any given answer is correct or not, not only help students begin to work on the problem, they also support more engaged and focused mathematical discussions. For instance, when a teacher begins the whole-class discussion, asking if anyone has something to share, when a student goes to the board, instead of simply writing a number with nothing more to say, immediately begins using the conditions to explain why the number constitutes a solution and uses this as a springboard for explaining more about his or her thinking and overall approach. Likewise, having used the sample explanations handout to prepare a lesson, after two students have presented different approaches to the problem, teachers seem more readily able to focus a discussion of differences on mathematically important features.

Conclusion

Koellner et al. (2007) suggest that one way to bootstrap the joint improvement of teaching and teacher content knowledge is to build professional development around a specific task and to incorporate mathematical work on that task as part of the work of the professional development. Doing so makes it more likely that teachers know the specific mathematics needed for the problem they will teach and that they are in a position to be successful with and learn to use the pedagogical tools provided well enough to explore their broader use. Koellner et al. have teachers do the task as a way to establish basic content knowledge for the work of leading a discussion about the task. In addition, because teachers then begin to plan a lesson using the task and eventually to teach the lesson, the facilitated engagement in leading a discussion using the task serves to prompt and structure continued mathematical work on the task. For instance, as teachers work to anticipate different ways students might do the problem, they may, with facilitation, investigate further the key mathematical ideas afforded by the task.

Our work continues with Koellner et al.'s bootstrapping approach, integrating work on content and pedagogical practice. We are finding that development in this direction, specifically the use of content-specific pedagogical resources, is improving our ability to support the increase of teachers' capacity to lead mathematically rich discussions.

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TASK DESIGN IN MATHEMATICS CONTENT COURSES FOR PRESERVICE ELEMENTARY TEACHERS: A COLLABORATIVE APPROACH

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The authors describe their collaborative approach to modifying a rational number task (involving fraction comparison) from the elementary mathematics curricula, Investigations, for use with preservice elementary teachers (PSTs). The research team examines how the task has evolved through multiple cycles of (a) modifying and planning, (b) implementing the task in their own mathematics content courses for PSTs, and (c) reflecting upon their implementations. In addition to designing a task with the potential to develop PSTs' mathematical knowledge for teaching, the group aims to identify generalizable principles to guide future task design for PSTs.

Keywords: Teacher Education-Preservice, Elementary School Education, Rational Numbers, Curriculum

Objectives and Purposes of the Study

The purpose of this study is to examine a collaborative process for task design and implementation within mathematics content courses for preservice elementary teachers (PSTs). Specifically, this study seeks to explicate the process of: (a) selecting a task designed for elementary school children; (b) modifying it for use with PSTs; (c) implementing the modified task in content courses for PSTs; (d) engaging in focused reflections on the implementations; and (e) reevaluating and modifying the task based on the implementations. This cyclical process will continue over the course of multiple enactments across content courses at several institutions in the United States. Our goal in explicating a process of iterative cycles of task design (Liljedahl, Chernoff, & Zazkis, 2007) is to provide a tool to help mathematics teacher educators (MTEs) modify tasks designed for children for use in their own content courses with PSTs.

In addition to an explication of this process, our study also provides MTEs with an example task, which originated from the elementary mathematics curriculum, *Investigations,* and has been redesigned to develop PSTs' understanding of how to compare fractions through sense-making. Through a collaborative process of writing, reflecting, and modifying, our research team has designed a task with high cognitive demand (Smith & Stein, 1998) to achieve the multiple goals of: (a) addressing PSTs' incoming conceptions of fractions (Bransford, Brown, & Cocking, 1999); (b) helping PSTs develop number sense as it relates to fractions; and (c) allowing PSTs to deepen their understanding of fractions while at the same time making sense of and learning about children's mathematical thinking. The research team's joint experience enacting fraction comparison tasks in content courses for PSTs and its knowledge of the literature on PSTs' knowledge of fractions (e.g. Livy, 2011; Yang, Reys, & Reys, 2009) continues to guide the development and enactment of this task.

Perspective(s)

Research in mathematics education has shown that the tasks in which students have opportunities to engage influence the mathematics they learn (e.g. Stein & Lane, 1996). A current conception of the term *task* includes more than a set of questions or prompts; it also incorporates ways in which the task can be presented, and ways in which follow-up activities can be organized (Watson & Mason, 2007). Given the importance of tasks for student learning, task design has been a recent focus of the mathematics education community. For example, the *Journal of Mathematics Teacher Education* (JMTE) recently published a triple special issue on task design (2007), and the International Congress on Mathematical Education's current study group is focusing on task design. Furthermore, in her editorial opening to a recent issue of JMTE, Chapman (2013) highlights the importance of what she refers to as, mathematical-task knowledge for teaching.

One framework for the task-development cycle consists of four phases: *design*, *enact, reflect*, and *modify/re-design* (Liljedahl et al., 2007). According to Liljedahl, the initial design should build on PSTs' incoming conceptions and a hypothesis of how those conceptions develop. Enactment then occurs within content courses for PSTs. After the enactment phase, the reflection cycle allows for consideration of the elements of the task that contributed to student learning, reasons accounting for the occurrence of any unexpected events, and modifications that may improve the task for future implementations. These modifications are then considered in a new task-development cycle of *redesign*, *enactment*, *reflection*, and *modification*.

In our study, instead of writing an original task, the initial design phase of our task-development cycle involved modifying a pre-existing elementary school mathematics task. Our decision to do so was based upon several factors, including wanting to provide MTEs an opportunity to address PSTs' MKT as well as elementary school children's conceptions of mathematics content (Yackel, Underwood, & Elias, 2007). Teachers need to "know and understand deeply the mathematics they are teaching and be able to draw on that knowledge with flexibility in their teaching tasks" (NCTM, 2000, p. 17). They also need to have a good understanding of the mathematical knowledge their students bring with them to the classroom (Fennema et al., 1996). Moreover, it is generally accepted that teachers need to know more than their students do. Therefore, we decided to examine elementary curricula to first determine what children need to know about a certain topic, and then consider the additional knowledge that their teachers would need to know to effectively teach that topic. In our case, we know that children need to understand a variety of procedures in order to be able to compare fractions effectively, but we also would like them to develop number sense about the relative size of fractions and numbers in general. We believe that it is this number sense that needs to be deeply developed in teachers so that they can flexibly perform fraction tasks with their students, as well as attend to student thinking.

A number of studies have examined PSTs' knowledge of fractions. However, the majority of these examine PSTs' understandings of fraction operations, particularly multiplication and division. Few studies have examined PSTs' general number sense as it deals with fraction comparison. The research that has been done suggests that PSTs rely mainly on two procedures for comparing fractions, finding a common denominator and converting the fractions to decimals (Livy, 2011; Yang, et al., 2009). While some PSTs have demonstrated number sense in terms of being able to compare a given fraction to a benchmark, such as 1 or ½ (Author(s), Yang et al., 2009), others have demonstrated faulty logic in their fraction comparisons (e.g., comparing the differences between the numerators and denominators) (Author(s), Zazkis & Chernoff, 2008).

Methodology

Six mathematics teacher educators (MTEs) from six institutions of higher education across the United States collaboratively designed a fraction-related task to use with the PSTs in their mathematics content courses. The topic of fractions was selected for two reasons: (a) it is central to the mathematics that is studied in the elementary grades (National Governors Association & Council of Chief State School Officers, 2010); and (b) it is a topic that is notoriously difficult for elementary students and PSTs (e.g. Author(s); Yang et al., 2009). After discussing multiple sub-topics within fractions (such as multiplication and division), we narrowed our topic down to fraction comparison with the goal of providing PSTs an opportunity to develop strong underlying understanding of fractions as numbers. After examining fraction tasks from three (relatively) widely-used elementary mathematics curricula (*Everyday Mathematics*, *Investigations*, and *Trailblazers*), the research team selected the following fifth-grade-level tasks from *Investigations* (citation needed: left: Session (#), p. #; right: Session (#), p. #):

Solve the problems below and explain or show how you determined the answer.

1.	Which is greater?	$\frac{7}{10}$	or	$\frac{3}{5}$
2.	Which is greater?	$\frac{7}{8}$	or	$\frac{9}{10}$
3.	Which is greater?	$\frac{4}{3}$	or	$\frac{3}{4}$
4.	Which is greater?	$\frac{3}{8}$	or	$\frac{1}{3}$

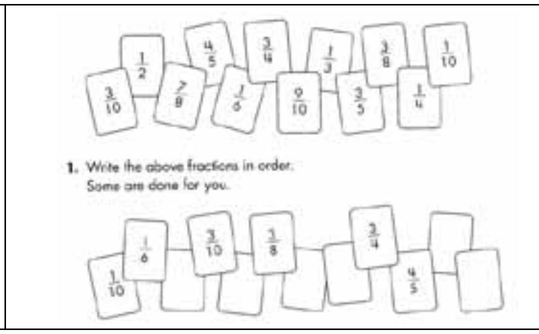

Figure 1: *Investigations* Tasks for Children Utilized in Task Design for PSTs

After selecting the task from *Investigations,* the research team modified the task for their audience of PSTs (the modified task will be shared at PME-NA). Modifications were based on: (a) the MTEs' prior experience teaching content courses for PSTs; (b) literature on task design; and (c) research on PSTs' understanding of fractions. For example, one modification was to change some of the fractions used in the original task to ones with denominators that would make using the common denominators strategy less practical, e.g., 2/17 and 2/19. Our goal was to create a task that would encourage PSTs to seek out and develop additional and more efficient strategies, in this case, utilizing the fact that the fractions have common numerators.

During the spring of 2013, two members of the research team enacted the task in their classes and reflected on the enactment by completing a post-lesson protocol that prompts the instructor to specify any changes made to the planned task implementation and to identify any suggestions for revising the task. Data to be collected includes the completed post-lesson protocols, PSTs' written work on the task, and artifacts from the enactment (e.g., notes from document cameras or whiteboards). The research team will meet after this first round of enactments to reflect upon them and use the data that was collected to modify the task before two additional members of the group enact the task in their classes. This cycle will be repeated until all six MTEs have had the opportunity to enact the task.

Results

By the end of the Spring 2013 semester, the research team will have completed two cycles of task modification, implementation, and reflection (by the PME-NA annual meeting the authors

expect to complete at least one additional cycle). Data analysis will occur during the Summer of 2013. At the PME-NA annual meeting, the authors plan to: (a) share the task they created, in multiple stages of its development; (b) describe the evolution of the task through repeated cycles of modification, implementation, and reflection; and (c) discuss any design principles that emerge from their work that may be generalizable beyond the particular mathematical content of rational numbers. Two examples of design principles that have already emerged include (a) using a task designed for children, from an elementary mathematics curriculum, as an initial starting point for developing tasks for PSTs; and (b) modifying problems for PSTs in ways that will reduce the usefulness of the strategies they most-likely remember, to encourage the consideration of new and more effective strategies.

Discussions and Conclusions

The purpose of this study was to bring together a set of diverse MTEs to collaboratively develop a meaningful task for PSTs, related to the mathematics their future students will study, and explore the task development process. By collaborating in the *selection* of mathematical content on which to focus and an elementary task to adapt, the *modification* of a task designed for children for use with PSTs, the *implementation* of a task, and the *reflection* on multiple implementations of a task (with several cycles of the last three phases occurring), the research team has developed a task they believe has the potential to significantly impact PSTs' MKT (future work is planed to explore such changes). Finally, by further examining the developmental process of their task, the group aims to identify principles to guide future task design for PSTs.

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THE DEVELOPMENT OF MATHEMATICAL KNOWLEDGE IN A STEM-FOCUSED ELEMENTARY TEACHER PREPARATION PROGRAM

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The purpose of this study was to examine the specialized content knowledge, pedagogical content knowledge, and common content knowledge in mathematics of preservice elementary teachers enrolled in a STEM-focused teacher preparation program. Preservice teachers of two academic levels demonstrated differences in their knowledge (before and midway through teacher preparation program). Likewise, statistically significant positive changes occurred during the senior year for a cohort of preservice teachers who were engaged in STEM-focused tasks and a yearlong field placement.

Keywords: Teacher Education-Preservice, Teacher Knowledge, Elementary School Education

Teacher education researchers agree that a teacher's mathematical knowledge is essential for effective instruction (e.g., National Mathematics Advisory Panel [NMAP], 2008; Mewborn, 2001). This agreement stems from research indicating a unique contribution of teacher's mathematical knowledge to student achievement in mathematics (Hill, Rowan, & Ball, 2005) and to instructional practices (Sowder, Philipp, Armstrong, & Schappelle, 1998). Therefore, in recent years, elementary teacher preparation programs have been called to an increased focus on developing mathematical knowledge of preservice teachers (NMAP, 2008).

The purpose of this study was to examine the mathematical knowledge of preservice teachers in a STEM-focused teacher preparation as measured by two of the most commonly used assessments in the United States: (1) the Learning Mathematics for Teaching: Mathematical Knowledge for Teaching Measures (LMT-MKT) (LMT, 2004); and (2) the Diagnostic Teacher Assessment in Mathematics and Science (DTAMS) (Center for Research in Mathematics and Science Teacher Development, 2008). Specifically, the two research questions were:
(1) Do preservice elementary teachers of two academic levels (beginning of versus midway through teacher preparation program) differ significantly in their mathematics specialized content, pedagogical content, and common content knowledge, after controlling for aptitude?
(2) What changes in preservice elementary teachers' mathematics specialized content, pedagogical content, and common content knowledge took place over the course of an academic year (beginning to the end of their senior year)?

Theoretical Framework and Related Literature

Some of the earliest literature on teachers' content knowledge in mathematics can be found in Begle's (1979) meta-analysis of studies between 1960 and 1976 on the effects of teachers on student performance. Begle found neither the number of courses mathematics teachers had taken nor a major or minor in mathematics had an impact on student performance; in contrast, the greatest number of positive main effects in his analysis was the relationship between the number of mathematics methods courses and student performance. This set the stage for Shulman's (1986) introduction of pedagogical content knowledge.

Once Shulman brought his idea to the attention of researchers, the construct of mathematical knowledge for teaching emerged. Mathematical knowledge for teaching (MKT), a framework outlined by Ball and colleagues (2008) and used in the context of this study, includes *common*

content knowledge (CCK), *specialized content knowledge (SCK)*, and *pedagogical content knowledge (PCK)*. *CCK* includes mathematical knowledge and skills held by most people and used in settings beyond teaching; for example, knowing how to multiply 24 and 13 requires common content knowledge. *SCK* is mathematical knowledge unique to teaching such as interpreting solution strategies. For example, being able to evaluate alternative algorithms for multiplying 24 by 13 requires specialized content knowledge. *PCK,* distinct from *SCK*, involves a link to students and pedagogy such as being familiar with common conceptions and misconceptions among students. In the example of 24 times 13, knowledge of common errors for multiplication would be *PCK*.

To assess this special mathematical knowledge for teaching, using traditional mathematics tests or the number of completed mathematics courses will not capture the multifaceted, complex phenomenon (Mewborn, 2001). Descriptive studies have indicated preservice elementary teachers' knowledge is fragmented (e.g., Simon, 1993), but many of these studies have neither examined change over time in preservice teachers' knowledge nor analyzed the preservice teachers' general content knowledge (i.e., *CCK*) compared to the special mathematical knowledge needed for teaching (i.e., *SCK* and *PCK*). The current study, using two assessments specifically developed to assess mathematical knowledge of teachers, addresses a need in the literature by examining the three aforementioned knowledge types over time for a group of preservice elementary teachers enrolled in a STEM-focused teacher preparation program.

Methods

Context

For the first two years, preservice teachers in this STEM-focused program are engaged in their general studies program that includes 27 credit hours of coursework in science, mathematics, and engineering design. During their junior year, they complete the first half of their elementary teacher preparation program that includes, among other courses, one engineering methods, two science methods, and two mathematics methods courses. For each of these courses, they complete field-based assignments in elementary classrooms. These field placements include weekly visits along with two full-time weeks each semester (86 hours per semester). During their senior year, they participate in a STEM-focused seminar, conduct in-depth research on a mathematics or science topic in the elementary curriculum, and develop integrated STEM units. Simultaneously, they engage in a yearlong field placement. In the fall, the field placement is weekly along with three full-time weeks (121 hours). The spring involves 15 full-time weeks (525 hours).

Participants

Participants were 110 preservice elementary teachers; 53 were starting their full-time teacher preparation program (end of sophomore year) and 57 were midway through the program (end of junior year). Nearly all participants were female (n=108). Their mean SAT score was 1100 (SD=105), and their mean GPA at the end of sophomore year was 3.59 (SD=.371).

Data Sources

LMT-MKT assessment in Number and Operations (K-6). The LMT-MKT in Number and Operations (K-6) consists of 16 items designed to measure teachers' mathematical knowledge for teaching number and operations (whole and rational numbers). Items are multiple-choice and include such tasks as evaluating unusual solution methods and representing mathematical content to students. The LMT-MKT results were reported as standardized scores in standard deviation units from the mean based upon equating work the authors have conducted. In the development of the LMT-MKT, the authors worked to capture mathematics knowledge

specific to teaching. Using the MKT framework (Ball, Thames, & Phelps, 2008), our own analysis of the items indicates the assessment measures *SCK* necessary for teachers.

DTAMS (two assessments: Elementary School Whole Numbers and Computation and Elementary School Rational Numbers and Computation). The DTAMS instruments, consisting of 20 items each, were designed to measure teachers' mathematics content knowledge and to a lesser extent, their *PCK*. They were scored on a scale of 40 points and converted to a percentage for this study. The first half of each assessment includes multiple-choice items, and the second half includes open-response items. Both of the DTAMS instruments measure four types of knowledge, each worth 10 points of the assessment score: (1) memorized/factual knowledge; (2) conceptual understanding; (3) reasoning/problem solving; and (4) PCK. These subscales were used to measure *CCK* (Subscales 1, 2, 3) and *PCK* (Subscale 4).

Analysis

To address the first research question, a one-way analysis of covariance (ANCOVA) was conducted for each of the dependent variables. The independent variable was academic level (beginning of versus midway through teacher preparation program). The dependent variables included scores on the following: LMT-MKT assessment in Number and Operations, subscales of DTAMS Whole Number assessment, and subscales of DTAMS Rational Number assessment. The covariate was the students' SAT scores. To address the second research question, paired sample T-tests were conducted to compare scores on the LMT-MKT and DTAMS assessments (to include subscales) at the beginning and end of senior year for a subsample of study participants (n= 51, 31, or 29; sample size varied per assessment).

Results

Research Question #1

Prior to evaluating the differences between the two academic levels, several assumptions were examined. Normality was supported by the Shapiro-Wilk tests, and Levene's test of homogeneity of variance indicated variances of the two groups were not statistically different. An evaluation the homogeneity-of-regression (slopes) assumption indicated that the relationship between SAT and assessment scores did not differ significantly as a function of academic level.

The ANCOVA analyses were significant for all constructs and content foci except *PCK* as measured by the DTAMS in Rational Numbers; the results on this assessment showed differences that could be considered marginally significant ($p = 0.067$). In all cases, the preservice teachers who were midway through the program scored higher than those at the beginning. A small amount of the total variance in assessment scores (3% to 9%; ω^2 ranged from .03 to .09) was accounted for by the two academic levels when controlling for SAT scores.

Research Question #2

Paired sample T-tests indicated significant differences in scores from the beginning to the end of the senior year on three components of DTAMS. The preservice teachers scored significantly higher on the *PCK* subscale of the Whole Numbers assessment at the end of the senior year ($M = .51$, $SD = .16$) than at the beginning ($M = .36$; $SD = .15$), $t(30) = 4.04$, $p < .001$, $d = .73$. They also scored significantly higher on the *PCK* subscale of the Rational Numbers assessment at the end ($M = .66$, $SD = .17$) than at the beginning ($M = .58$, $SD = .20$), $t(28) = 2.40$, $p = .023$, $d = .45$. Finally, the preservice teachers scored significantly higher on the *CCK* subscale of the Rational Numbers assessment at the end of the senior year ($M = .84$, $SD = .76$) than at the start ($M = .76$, $SD = .14$), $t(28) = 3.84$, $p = .001$, $d = .74$. No significant differences were found for *CCK* in Whole Numbers and Operations or for *SCK* in Number and Operations.

Discussion

The results of this study indicate two significant findings: (1) preservice teachers' mathematical knowledge appeared to develop throughout the STEM-focused teacher preparation program; and (2) field placements seem to be important for preservice teacher growth. First, a comparison of two groups of preservice teachers at two academic levels showed significant differences in *SCK*, *PCK*, and *CCK*. As expected, the group that was halfway through the program scored higher than the group just starting the program. This is plausible because the group midway into the program had already completed two mathematics methods courses aimed at deepening these three types of knowledge. Further investigation is needed to determine which specific features of the program during the junior year are the most instrumental in impacting knowledge. Also, it is important to note that the cohort of senior preservice teachers experienced growth across one year. Perhaps, their engagement in STEM-focused activities (e.g., seminar, research project, integrated unit) had an impact on their knowledge growth.

Second, preservice teachers scored statistically higher on both measures of *PCK* (whole and rational numbers) at the end of their senior year than they did at the beginning. It is likely their growth can be explained by their experiences in a yearlong field placement. At the time of the post-assessment, they had just completed their field experiences in elementary school classrooms. Their full-time teaching gave them opportunities to better anticipate student responses, make sense of student errors, and respond to student misconceptions. They also demonstrated statistically significant positive differences in their common content knowledge in rational numbers, perhaps due to their experiences planning lessons and working to better understand the mathematics themselves. Collectively, these findings support the importance of both coursework and field placements for preservice teacher growth, an important implication in the current era of criticism of teacher preparation programs.

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VIEWING PROSPECTIVE ELEMENTARY TEACHERS' PRIOR KNOWLEDGE AS A RESOURCE IN THEIR NUMBER SENSE DEVELOPMENT

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The mathematics education community is dissatisfied with prospective elementary teachers' content knowledge. The literature emphasizes what prospective elementary teachers do not know. At the same time, what prospective elementary teachers do know (e.g., the standard algorithms) does not seem to be valued. Mathematics educators want prospective elementary teachers to understand why the standard algorithms work, as well as to understand and use nonstandard strategies. In the spirit of viewing students' prior knowledge as a resource in their learning, I suggest reconsidering the prior knowledge of mathematics that prospective elementary teachers bring to the table. Examples from a recent study illustrate how prospective elementary teachers' procedural knowledge and knowledge of commonplace mathematical ideas can serve as resources in their number sense development.

Keywords: Teacher Education-Preservice, Teacher knowledge, Number Concepts and Operations

I report findings from a study of of number sense development in a mathematics content course for prospective elementary teachers. Drawing from analyses of collective activity and individuals' learning, I highlight results that illustrate productive roles for prospective elementary teachers' prior knowledge of procedures and commonplaces ideas in their learning.

Theoretical Perspective

This report is influenced by a knowledge-as-resources view: students' prior knowledge is seen as a resource in the development of more advanced understandings (Hammer, 1996; Smith, diSessa, & Roschelle, 1993); this perspective constrast with a misconceptions view, from which prior knowledge may be regarded as incorrect, unwanted, and in need of removal or replacement.

Background

Prospective elementary teachers tend to come to teacher education with an understanding of mathematics that is largely procedural (Ball, 1990) and has been characterized as "disconnected" (Simon, 1993). A popular goal in mathematics teacher education is to enhance prospective elementary teachers' conceptual understanding of elementary mathematics (e.g., Hill & Ball, 2004). For example, prospective elementary teachers typically know the standard written algorithms that they were taught in school, but they do not understand why these algorithms work (e.g., Simon, 1993; Thanheiser, 2010; Zazkis & Campbell, 1996).

An influential perspective in mathematics and science education is the view of students' prior knowledge as a resource in their learning (Hammer, 1996; Smith, diSessa, & Roschelle, 1993). In the literature concerning prospective elementary teachers mathematics content knowledge, however, it is difficult to find evidence of the influence of such a perspective. In elementary mathematics content courses, prospective elementary teachers revisit elementary mathematics and are expected to come to understand that mathematics in new ways (e.g., Sowder, Sowder, & Nickerson, 2010). Prospective elementary teachers come to these courses already familiar with

standard algorithms of elementary mathematics. This procedural knowledge tends not to be valued (e.g., Yang, Reys, & Reys, 2009). Prospective elementary teachers have been described as dependent or overreliant on these algorithms (Ball, 1990; Nickerson & Whitacre, 2010; Yang, 2007). Prospective elementary teachers also tend to overgeneralize from their experiences with whole numbers (e.g., Simon, 1993; Tirosh & Graeber, 1989). These overgeneralizations have been characterized as misconceptions and regarded as an impediment to the development of more sophisticated conceptions (e.g., Graeber, Tirosh, & Glover, 1989). With the best of intentions, mathematics educators may conceive of the goals of content courses in terms of moving prospective elementary teachers away from their procedural knowledge and imprecise generalizations and toward thinking about and understanding mathematics in new ways. The danger is that mathematics educators may undervalue prospective elementary teachers' prior knowledge and fail to recognize its productive role potentially in their learning.

Methods

The setting for this study was an elementary mathematics content course taught at a large, urban university in the southwestern United States. Data collection took place during Fall Semester 2010. There were 39 students enrolled in the course, and 38 of them were female. The majority of the students were freshmen. Prospective elementary teachers' number sense development was investigated on both the collective and individual levels. For reasons of length, this report focuses on individual learning. Individual learning was analyzed through qualitative case studies (Merriam, 1998; Yin, 1994) of the number sense development of two of the prospective elementary teachers enrolled in the course. These drew on interview data, as well as the participants' written work. The details of these analyses appear elsewhere (Whitacre, 2012). The present report focuses on a particular question: What role does prospective elementary teachers' prior knowledge—especially their knowledge of procedures and commonplace mathematical ideas—serve in their number sense development?

Rather than "misconceptions" or other terminology, I prefer the term *commonplace mathematical ideas*. These may be correct or incorrect from an expert perspective. What makes these ideas commonplace is that they are common to lay people and are not typically valued by experts. One example is reasoning about multiplication in terms of repeated addition. Tirosh and Graeber (1989) describe prospective elementary teachers as holding the misconception that "multiplication always makes bigger." This phrasing, especially the inclusion of the word *always*, casts this idea as being decidedly incorrect and undesirable. However, reasoning about multiplication in terms of repeated addition is something that children and adults—even mathematicians—all do at times. There are other ways of reasoning about multiplication that are also important and useful, but reasoning in terms of repeated addition is a valid and desirable way. True, multiplication does not always make bigger. However, the related way of reasoning—reasoning about multiplication in terms of repeated addition—can serve as a productive resource in prospective elementary teachers' number sense development.

Results

This brief report offers one of many examples of the productive role of a prospective elementary teachers' prior knowledge in her number sense development.

The Case of Valerie's Way of Accounting for Partial Products

Valerie, who was the subject of a case study, initially relied on an invalid multiplication strategy. For example, she (along with many others in the class) computed the product of 23 and 23 mentally as 20 x 20 + 3 x 3. Valerie was eventually able to correctly account for partial

products in double-digit multiplication. She did this by making use of tools that helped her to organize her thinking. On one occasion, she used rectangular area to represent a multidigit product, and this helped her to account for all of the partial products. On another occasion, Valerie used her procedural knowledge of the standard multiplication algorithm for the same purpose. She was able to correctly account for the four partial products in 12 x 45 by reasoning about pairing up numbers of tens and ones, while at the same time enacting the sequence of steps in the standard multiplication algorithm. She tapped her finger on each of the digits as she talked about pairing them. Doing this seemed to help her to keep track of which digits had been paired.

In Valerie's *finger accounting* for partial products in multidigit multiplication, she drew on the commonplace idea of reasoning about multiplication in terms of repeated addition, coordinated with her procedural knowledge of the standard multiplication algorithm. In this way, Valerie's procedural knowledge and knowledge of a commonplace idea served as a resource as she worked to make sense of and appropriately account for partial products. Over time, Valerie came to reason more flexibly about multiplication, to understand that the strategy she had relied on previously (e.g., 20 x 20 + 3 x 3) was invalid, to be able to justify the standard algorithm, and to be able to account explicitly for partial products.

Discussion

Making sense of and accounting for partial products in multidigit multiplication is a challenge for prospective elementary teachers. It is, therefore, important for mathematics teacher educators to understand ways in which prospective elementary teachers are able to learn to correctly and sensibly account for partial products. In designing instruction for the course, the instructor had envisioned that rectangular area would become a model for reasoning about products in terms of partial products. For the class and for many individual students, rectangular area did seem to serve that purpose. At the same time, students found other ways of accounting for partial products. Valerie's way is one such example. She accounted for partial products by reasoning about multiplication in terms of repeated addition, together with exercising her procedural knowledge of digit pairing in the standard multiplication algorithm. In this act of reasoning, Valerie used the standard multiplication algorithm as a tool that helped her to organize her thinking (Zandieh & Rasmussen, 2010), much the same way that she and other students used rectangular area on other occasions. Thus, Valerie's knowledge of a relevant procedure and a commonplace idea served as resources that helped her to make sense of elementary mathematics in new ways, and thus contributed to her number sense development.

Implications

Many in the mathematics education community agree that it is important for teachers to take into account students' prior knowledge. However, when it comes to prospective elementary teachers, little attention has been given to their prior knowledge from a knowledge-as-resources perspective. We know that prospective elementary teachers come to teacher education familiar with standard algorithms and commonplace ideas. It is easy to undervalue this prior knowledge as we endeavor to foster new learning. However, knowledge of procedures and commonplace ideas should be recognized as useful for their potential to serve as productive resources in prospective elementary teachers' number sense development.

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PRACTICE-BASED CLASSROOM COLLABORATIONS FOR MATHEMATICS TEACHER DEVELOPMENT

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Keywords: High School Education, Teacher Education-Inservice/Professional Development, Standards (broadly defined), Teacher Knowledge

As we move toward adaption of the Common Core State Standards (Council of Chief State School Officers, 2010), teacher educators are faced with the challenge of supporting teachers in understanding and implementing the new requirements. In this poster we illustrate a model for teacher development that engages teachers and teacher educators around the question of how classroom lessons can leverage key mathematical concepts and practices to improve student understanding of the ideas included in the CCSSM and increase student engagement in the development of these ideas.

In order to overcome the perceived challenges of adopting the CCSSM, with its list of individual standards and standards for mathematical practice, the CCSSM must be understood in terms of the connections between these standards and appreciated in terms of the power of supporting students to learn mathematical content through engaging in mathematical practices. We describe collaborations among teachers and teacher educators to design instruction around key mathematical content and practices.

We are interested in methods of engaging both novice and experienced teachers in the work of teaching in ways that will develop their facility with instructional practices associated with ambitious mathematics teaching (Kazemi, Franke, & Lampert, 2009). To do this we engage in 'boundary practices' (Wenger, 1998) that take advantage of resources from both university and school settings. Examples of these boundary practices are co-planning of classroom lessons, coaching, co-development of assessments, and debriefing of lessons.

Through these collaborations, classroom teachers and teacher educators develop goals for instructional units and outline the mathematical content and practices that will be targeted in each lesson. During the enactment of the unit the teacher educator and novice teacher observe the lessons and assist the experienced classroom teacher in listening to the mathematical contributions of the students, connecting those contributions to clusters of standards, and integrating those contributions in to the mathematical discussion. We describe these collaborations and discuss the teacher development that results.

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STUDENTS' IMAGINATION OF ALTERNATIVE COURSES OF ACTION WITHIN AN INSTRUCTIONAL EPISODE

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Keywords: Teacher Education – Preservice, High School Education, Technology, Reasoning & Proof

Discussions of teaching making use of representations of classroom episodes, such as written and video cases (for example, Lampert & Ball, 1998), have become an integral part of teacher education. The current study examines one possible focus for these discussions, *alternativity*. Alternativity describes how teachers take up and examine possible courses of action not displayed in a classroom episode. We provide more detail on what we mean by alternativity and how it can be useful in teacher education. Further, we describe a study that tests the hypothesis that animated representations of classroom episodes better support alternativity than video representations.

Two common foci for discussions around representations are personal experience (see, Santagata & Angelici, 2010) and student thinking (see, Sherin & van Es, 2005). When participants engage with an episode from an alternativity stance, they consider the courses of action that could have been taken within an episode. Participants are asked to imagine courses of action not included in the scenario or to pinpoint the moments where the action could have occurred differently.

We argue that focusing participants' interactions with episodes on alternativity is beneficial for developing their teaching practice. A long-standing problem in teacher education has been to support teachers in developing a teaching style that is different from the style in which they were taught (Lortie, 1975). Through considering the alternative actions, and pinpointing the moments at which these alternatives could come to pass, we claim that teachers can build a vision and feel for how teaching could look differently than the teaching that they experienced as students.

We interviewed sixty pre-service secondary teachers. Participants viewed one of three representations of a classroom episode and responded to questions, including questions about alternatives to the actions shown in the representation. One of the representations was a classroom video and the other two representations were animations of the same episode. The two animations differed in the visual resources used to depict the episode. Using Systemic Functional Linguistics (Martin & Rose, 2003) we examined transcripts of these interviews to compare the number and quality of the alternatives suggested by the participants in response to each representation.

In this poster we present our findings, showing how the linguistic resources used by participants varied according to representation and how participants were able to imagine alternative scenarios in response to the questions posed in the interview. These finding support the claim that animations are useful tools for aiding participants in developing new visions of teaching practice.

To explain the difference in teachers' engagement with animations and videos, we note that video portrays a scenario in which all the choices have been made and the events have already occurred. In contrast to this, animated representations of classroom episodes portray a series of events that *could* occur and invite viewers to consider other possible courses of action.

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TOWARD AN UNDERSTANDING OF MATHEMATICS TEACHER CHANGE: CHARACTERIZING TEACHERS' EQUITY-RELATED PEDAGOGICAL REASONING

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Keywords: Equity and Diversity, Professional Development, Teacher Beliefs

Objectives

This study examines teachers' pedagogical reasoning as they participate in conversations in the context of an extensive, equity-focused mathematics PD program. Evidence of teacher reasoning is gathered from teachers' conversations with coaches. Analysis seeks to (a) characterize teachers' in-the-moment reasoning as they engage with coaches in making sense of their practice, (b) identify patterns of reasoning for teachers who have engaged in the PD for different amounts of time, and (c) begin to sketch possible learning trajectories for teachers working to build equitable teaching practices. Findings contribute to the field's understanding of mathematics teacher learning for equity and suggest directions for future research.

Research Questions

This study aims to build our understanding of how teachers who are engaged in learning about an equity pedagogy reason about their practice and how this reasoning develops. (1) How do teachers engaged in an equity-oriented PD reason in conversation with others about the successes and challenges of their practice? (2) In what ways do teachers' patterns of reasoning differ after different tenures of engagement with the PD? (3) To what extent does this reasoning appear to develop in ways that support equitable teaching practice?

Methods

Two teachers are observed and recorded as they teach and as they converse with their CI coach. Conversations are transcribed and coded to capture aspects of pedagogical reasoning. Each coded conversation is represented visually with a *code profile*, and codes are counted. Change over time is identified using counted codes and observed change in patterns of codes.

Findings

The two teachers show different patterns of reasoning. The teacher who is less experienced shows evidence of learning over the course of the study.

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"I HATE THAT KIDS HATE MATH": THE POWER OF CONTENT PASSION IN STUDENT TEACHING PRACTICES

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Most teachers view student teaching as the most valuable part of their preparation, claiming that most of what they know comes from first-hand teaching experience (Feiman-Nemser & Buchmann, 1985). However, there is considerable research showing that student teachers tend to "survive" instead of enacting strategies learned in teacher education programs (i.e. Korthagen & Kessels, 1999) or developing feelings of ownership and empowerment (Lane, Lacefield-Parachini, Isken, 2003). Hoping to better understand these common pitfalls of teacher education (Feiman-Nemser & Buchman, 1985) from a different perspective, our study aimed to examine the student teaching experiences of two high-achieving preservice mathematics teachers placed in high-needs urban middle schools.

Based upon overall grade point average and faculty feedback, Brett and Julie were identified as high achieving in their teacher education coursework and invited to participate in the study. Brett and Julie were placed in urban middle schools with similar settings in the Southeastern region of the US. Data sources included interviews, observations, and coursework artifacts (lesson plans, written reflections, and position papers). At the completion of data collection, which spanned Fall 2011 through Spring 2012, individual case studies (Yin, 1994) were created for both Brett and Julie. Comparing across cases, we then used natural generalizations (Stake 1995) to explore the similarities and differences revealed within the contexts of the two classrooms.

The results of the study showed that both Brett and Julie wanted to make a difference in their student teaching placement, but, similar to most of the student teachers in the program, they initially felt overwhelmed by their student teaching placements. Despite similar struggles, our data revealed differences in Brett and Julie's classroom practices related to their: (1) responses to challenges in practice, (2) primary focus for instructional decisions-making, and (3) motivation for developing relationships with students. Looking across both cases carefully to understand the differences, we found that math appeared to be central in all that Julie did in the classroom while it was more "on the side" for Brett. Our findings point to the centrality of mathematics in field placements as leveraging power. Tied to her confidence in and love of the subject, Julie focused on what she had the power to change in her internship—namely the mathematics instructional strategies—and focused less on the things she did not have the immediate power to change. These results point to preservice teachers' content knowledge and passion for mathematics as some of their most powerful tools.

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AMBITIOUS TEACHING PRACTICES: LEARNING TO CONDUCT NUMBER STRINGS

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Researchers have identified several ambitious teaching practices that can provide teachers with familiar discussion structures, enabling them to focus less on *teaching* and more on *exploring* the mathematics and *facilitating* conversations about students' solutions and reasoning (Lampert, Beasley, Ghousseini, Kazemi, & Franke, 2010). One such routine is number strings (Lampert et al.; Fosnot & Dolk, 2001; Parrish, 2011). This study addresses the following research questions: (a) After learning the components of number strings in their methods course, to what extent do teacher candidates enact the elements of this ambitious practice in an initial number string lesson? (b) What are their successes and struggles in enacting number strings?

Participants, Setting, and Methods

The four teacher candidates in this study all taught number string lessons in third grade classrooms, as part of an elementary education teacher training program in California. During their mathematics methods class, they learned how to use number string lessons to elicit multiple strategies, probe students' explanations, use incorrect answers as learning opportunities, represent students' thinking, and make connections among students' strategies.

Three candidates used two problems for their number string: 4+7 and 44+7. Their goals were to encourage students to break apart and regroup numbers and notice similarities between the amounts in the ones place. The fourth candidate wanted her students to think about the common difference strategy for subtraction and used four problems: 25-3, 25-13, 26-14, 28-16.

Results

The results suggest that number strings are indeed ambitious practices, but are powerful teaching and learning experiences that are realistic for beginning teachers to implement. The four teacher candidates incorporated all elements of number strings, but the depth with which they did so varied. They elicited 3 to 6 unique solutions per problem and asked students to elaborate at least once per minute on average. They probed student responses often (e.g., "How did you know to add one more after you added three?"), but students sometimes had difficulty responding. Although the teacher candidates were able to represent student solutions in various ways and verbally highlight connections among problems, all of them struggled to help students articulate the main relationship underlying the string and use representations to do so. These are areas where mathematics methods courses could focus additional attention.

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SIMULATING CLINICAL EXPERIENCES: PRE and IN-SERVICE TEACHERS' EFFORTS TO CREATE ANIMATED SIMULATIONS OF TEACHING SCENARIOS

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Following the recommendation of the Report of the Blue Ribbon Panel on Clinical Preparation and Partnerships for Improved Student Learning (NCATE, 2012), many teacher training programs have made a concerted effort to transition from traditional content classes to clinically based teacher preparation models. The goal of our project has been to develop and study the effectiveness of a video authoring tool, called the Dynalogue, for engaging preservice teachers (PSTs) in the process of creating animations to simulate clinical tutoring experiences. The Dynalogue software is a web-based interface that allows users to type dialogue between a teacher and a student *and* to animate how they might use a whiteboard to model their work. The idea of scripting a lesson (as opposed to just creating a lesson plan) is not new. In fact, Zazkis, L & Sinclair (2009) have written extensively about the merits of this approach. Our inclusion of a white board provides users with the additional challenge of thinking about how they would use graphics and mathematical language to lay out a conversation with a student.

The Study

The Dynalogue software was used in two settings: a mathematics education course for undergraduate prospective teachers (n = 18) and a PD course for teachers in grades 3-8 (n = 20). Both groups studied various approaches to solving proportional reasoning problems and were then shown videos of two students who demonstrated additive rather than multiplicative reasoning to solve the 'oranginess" task (See Figure 1).

Results

Analyses of the pre-service teachers' videos showed much greater homogeneity than did those of the in-service teachers. In particular, most of the pre-service teachers chose to have the student use equivalent fractions as the only strategy to solve the task. In contrast, the inservice teacher groups used a variety of strategies including ratio tables, forming composite units, and forming unit rates. This variety dovetails with the preservice teachers' main objection to the assignment: while they felt it provided some "real life experience", they were unsure about how a young child would react. The inservice teachers were more comfortable with such predictions and their input will help with the next phase of this project: building "webots" that will simsulate students' answers based on pre-serice teacher prompts.

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A "NO-CEILING" APPROACH TO YOUNG CHILDREN'S MATHEMATICS: PRELIMINARY RESULTS OF AN INNOVATIVE PROFESSIONAL LEARNING PROGRAM

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Poster Summary

There is limited research on what young children can do in mathematically stimulating environments. Our Canadian research team initiated a professional learning study of young children's (ages 3-7) mathematics. We adapted lesson study to include task-based interviews, exploratory lessons, and web-based resource development with 12 teacher teams who explored specific mathematics content with particular emphasis on early geometry and spatial reasoning (an underserved area of mathematics in this age range). Student data included pre and post task-based interviews, video documentation, observation field notes, and annotated student work samples; teacher data included field notes from meetings and classroom visits, video of classroom activity, focus group interviews, pre-post content mapping, and a pre-post efficacy survey. Analysis involved a multiple case study method (Yin, 2009) with descriptive open coding (Charmaz, 2003), and cross case analyses to examine the effects overall. Preliminary outcomes of the program included deep teacher learning of precise mathematics content, in parallel with increased estimation of young students' mathematical abilities, leading to an asset-oriented "no-ceiling" approach to math programming. Additionally, researchers found increases in teacher confidence and risk taking. Finally, teachers engaged in co-creation of digital resources and toolkits based on classroom tools, strategies, structures and learning activities refined throughout the project. The result of this toolkit development was a high level of commitment to sharing their learning.

Acknowledgements

This research was funded through SSHRC and partnerships with the Literacy and Numeracy Secretariat of Ontario Canada, Trent University and the Trent Mathematics Education Research Collaborative, OISE of the University of Toronto and the Robertson Foundation. School and District Partnerships include the following Ontario School Boards: KPRDSB, SMCDSB, TDSB, TCDSB, WCDSB. Independent School partnerships include: The Eric Jackman Institute of Child Study and The Bishop Strachan School.

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ENGENDERING SPECIALIZED CONTENT KNOWLEDGE IN PRE-SERVICE ELEMENTARY TEACHERS: RESPONDING TO STUDENT CLAIMS

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Keywords: Elementary School Education, Mathematical Knowledge for Teaching, Measurement, Teacher Education-Preservice

Theoretical Perspectives

For a time it was assumed that if a teacher knew the mathematical content well enough, he or she would be able to transfer that content with ease to the students. However, traditional knowledge of mathematical content (often referred to as subject matter knowledge) alone is not enough; as researchers, Ball, Thames, and Phelps (2008) propose that subject matter knowledge be broken down into two 'big picture' categories: that of common content knowledge (CCK), which can be equated with the mathematics they need to teach, and specialized content knowledge (SCK). SCK is a specialized type of knowledge that is mathematical in nature, but outside the realm of knowledge held by a generally well-educated adult. For example, a well-educated adult would be expected to know how to use 'long division' to divide numbers, but a teacher would have specialized knowledge about this algorithm including why it works, other ways you can do long division, how long division relates mathematically to different types of representations, and common errors students make when doing long division. Due to its specialized nature, this knowledge is often outside the realm of that gained in general or traditional mathematics courses; however, it is fundamental in teaching mathematics. The goals of this research are to investigate what SCK is revealed when pre-service teachers analyze student claims as well as how different task features influence knowledge accessed.

Methods

The focus of this study is pre-service elementary teachers enrolled in a three-credit geometry and measurement course offered at a midsized Rocky Mountain University. Within the context of the course, various student claims related to measurement are presented to pre-service teachers. Each claim is presented four times in the same sequence: first on a homework, then in small group discussions, third in a whole class discussion, and lastly on a final written assessment. All discussions are audio recorded and written work collected and copied before it is returned to the students. Responses to claims are analyzed using a constant comparison, case study method (where each claim represents an individual claim).

Results

While data collection is continuing (to be finished prior to the poster presentation), preliminary results indicate that pre-service teachers often focus solely on whether a statement is right or wrong, and even in doing this, often do not have sufficient mathematical background to accurately assess the claim.

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ERROR ANALYSIS REVEALS PRESERVICE SECONDARY TEACHERS' MATHEMATICAL KNOWLEDGE FOR TEACHING

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Introduction and Rationale

The mathematics education community continues to call for greater understanding of teachers' knowledge of mathematics for teaching, as part of the knowledge base for improving mathematics education and, ultimately, student achievement (National Science Board, 2007).

The goal of this study was to investigate how preservice secondary teachers (PSTs) draw on two components of their mathematical knowledge for teaching: (a) *specialized knowledge of mathematics* and, (b) *knowledge of mathematics content and students* (Hill & Ball, 2009) in the analysis of students' mathematical work in the context of a precalculus class. Precalculus was chosen because the content calls upon the PSTs algebraic fluency, as well as depth of understanding of the function concept (including trigonometric functions). Analysis of student work provided the means for uncovering the PSTs knowledge of the relevant mathematics and their ability to use this knowledge to assess student thinking and identify misconceptions.

Results and Conclusions

Twenty-two preservice secondary teachers, enrolled in a capstone mathematics content course, took a pre- and post-test to evaluate their overall procedural fluency on a range of precalculus topics chosen from a university precalculus final exam. Fourteen of the preservice teachers agreed to participate in a series of interviews that asked them to solve precalculus problems and to analyze hypothetical students' solutions to these problems.

A class mean below 50% on the pretest and similar difficulties solving tasks during the interviews demonstrated that a majority of the PSTs had limited content knowledge. A qualitative analysis of their work suggests that this was primarily procedural knowledge. Several PSTs questioned their own solutions during the analysis of student work, which provided additional evidence that their content knowledge was relatively fragile. In analyzing student work, most PSTs focused heavily on the procedural correctness of students' solutions, and seemed to ignore correct attempts on some aspects of the problem. A closer look at the error analysis tasks revealed that many PSTs were able to identify potential student errors based on their own experiences with the task or their experience as mathematics tutors. On the other hand, only one PST was able to call on his deep content knowledge to make sense of student errors. A post-test mean of 68% shows statistically significant growth of the PSTs content knowledge.

This study gives insight into how preservice teachers draw on their knowledge of content and students in one particular aspect of practice, analysis of student work, and could inform the development of more appropriate experiences in the preservice classroom.

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EXPERIENCING INDIGENOUS MATHEMATICAL KNOWLEDGE SYSTEMS THROUGH IMMERSION-BASED LEARNING IN TRANSCULTURAL SETTINGS

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Investigating the impact of using indigenous techniques and practices to investigate relevant concepts in the mathematics classroom is an emerging research interest. Ample research has shown the importance of integrating cultural practices that resonate with students' ethnic and background experiences in everyday instruction (Ascher, 2002; Presmeg, 2007; Noss, Hoyles, & Pozzi, 2000). Furthermore, extensive research has urged the utilization of cultural immersion experiences as means of increasing cultural sensitivity for students across disciplines. Particularly in teacher education, the value of using community-based learning has been heightened in light of the diverse demographics that thrive in schools today. In mathematics education, the yet unseen value of immersion experiences transpires inadvertently when participants consciously view, actively participate, and reflect upon how other cultures use and develop their own mathematics to respond and make sense of their world.

Notwithstanding the extensive literature that urged the utilization of cultural immersion experiences as means of increasing cultural sensitivity for students across disciplines, little effort has been documented about the usefulness of immersion for teaching culturally-responsive mathematics. In this research we investigate firsthand field experiences of graduate mathematics education students while engaged in exploring ethnomathematical ideas by immersion in the daily experiences of indigenous cultures in Morocco and South Africa.

This study is part of a newly designed course uniquely offered at an urban, highly diverse university in the southeast. The course particularly focuses on teaching ethnomathematics through cultural immersion in indigenous contexts. The primary purpose of the course is to examine how different cultural groups interpret mathematical concepts in ways that are quite different from what we might expect from typical mathematical texts. This instructional experience dovetails two components:

Component 1: Class work (Location: on campus). In this component, students and in-service teachers investigate mathematical ideas inherent in activities such as creating calendars, art and decoration, divination, and counting schemes. Connections between ethnomathematics and mathematics education in schools are also emphasized.

Component 2: Cultural immersion (Location(s): South Africa; Morocco). This component affords firsthand field experiences to explore ethnomathematical ideas by immersing students in the authentic daily experiences of master craftsmen and knowledge holders of vital indigenous technologies. During the site visit, students explore the integration of hand and mind tools that indigenous cultures continually employ to plan, conceptualize, visualize and execute myriad activities as part of their daily practice. To motivate critical thinking, we employed various educational modalities in the on-campus portion of the course including: cubing, small group inquiries, whole-group discussion, problem-based investigations and tiered assignments. For the immersion component, students engage in participative forms of inquiry investigating ideas that transpire in out-of-school settings. Throughout the course students are encouraged to experiment with innovative ideas, to make their experimental thinking public, and to develop new epistemologies that guide the teaching and learning of nonconventional mathematics. Students' effort was evaluated based on the intellectual merit of their research projects, the investigative procedures they developed during field work, validity of their claims, and communicability of their research accounts.

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COMPARATIVE ANALYSIS OF MATHEMATICS TEACHER PREPARATION IN MEXICO AND THE UNITED STATES

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The emphasis of the study will be partitioned in three main domains: a chronological history of the Mexican educational system, mathematics curriculum at the secondary level, and secondary mathematics teacher training and qualifications. A comparison between Mexico's teacher preparation and one south-western U.S. University will also be discussed as a reference point to enrich the arguments. Degree plans and mathematics curricula for teacher education will be used to support the analysis.

The intention of the proposed analysis of secondary mathematics teacher preparation in Mexico is to re-think teacher education in the U.S. Therefore, it is critical that we examine any possibilities that can lead to improving mathematics teacher preparation programs as a way to enhance mathematics teaching and learning, as well as to maintain a competitive status in mathematics education with other countries around the world. According to Robitaille and Travers (1992), a comparative study provides opportunities for sharing, discussing, and debating important issues in an international context. Stigler and Perry (1988) point out that a

> cross-cultural comparison also leads researchers and educators to a more explicit understanding of their own implicit theories about how children learn mathematics. Without comparison, teachers tend not to question their own traditional teaching practices and are not aware of the better choices in constructing the teaching process. (p. 199)

As stated by our findings, the educational system in Mexico involves intense teaching practice as well as content knowledge preparation through two major types of institutions: Normal Superior and Pedagogical Institutions. It is important to note that the number of course credits obtained in Mexico are calculated differently than that of the parallel preparation program in the United States. There are 392 credits required under the teacher preparation program in Mexico, which are equivalent to 224 semester hours; that is, each semester hour is comparable to 1.75 credit hours. Similarly, in the U.S. teacher preparation program requires 120 semester hours; that is, each semester hour is comparable to 1 credit hour. Correspondingly, the bachelor degree in Mexico is more extensive with 45 courses needed for completion, compared to 40 courses in the U. S. The Mexican teacher preparation at the master's level consists of 84 credits equivalent to 42 contact hours, versus 36 credits in the U.S. equivalent to 36 contact hours. Additionally, pre-service teachers in Mexico must also complete a thesis. Nonetheless, both programs offer twelve courses at the Master's level.

Learning about secondary mathematics teacher preparation in Mexico could present different conceptions and diverse pedagogical approaches. With this diversity, mathematics teacher educators could reflect on their own practices and try to use effective strategies that can be incorporated in their own teacher preparation system.

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FROM STEM PROFESSIONAL TO STEM TEACHER: NEW PERSPECTIVES ON TEACHING AND STUDENT LEARNING

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Keywords: Teacher Education-Preservice

The call for Science, Technology, Engineering, and Mathematics (STEM) teachers who have deep mathematics knowledge and a mastery of pedagogical skills was renewed in this year's State of the Union address in a call for new efforts to recruit and prepare 100,000 STEM teachers. Preparing more STEM teachers has become an increasingly critical need. Two-thirds of the nation's K-12 teachers are expected to retire or leave the professional in the next 10 years (The National Commission on Mathematics and Science, 2000). Research suggests that effective mathematics teachers, who have deep content knowledge and a mastery of pedagogical skills, can engage students in cognitive demanding activities that will empower them in a way that deepens their understanding of mathematics (Hill, Rowan, & Ball, 2005).

In the ongoing effort to improve classroom teaching of STEM in secondary schools, the National Science Foundation (NSF) has funded a number of projects devoted to improving the K-12 mathematics achievement of students by preparing STEM undergraduate students and STEM professionals to become certified to teach mathematics. One such NSF supported project is the Robert Noyce Increasing Mathematics Teachers for ALL Students (IMTAS) program, a two-year Master's of Arts in Teaching (MAT) degree program which leads to initial teacher certification. Those enrolled in the IMTAS program receive training in working with students who are English learners (EL) and from culturally-diverse backgrounds. The program has an aim of addressing a shortage of mathematics teachers who have deep content knowledge by encouraging STEM students and STEM professionals to pursue teaching careers in high needs school districts.

The focus of this qualitative study was on STEM students and professionals who completed the MAT degree/IMTAS program, and are currently teaching mathematics in a high needs school district. The study investigated their perspectives on teaching secondary mathematics and on student learning while they were enrolled in the program and during their first three years of teaching. Preliminary findings indicate that prior to entering the classroom as fulltime mathematics teachers, the participants held the perspective that teaching mathematics involved posing problems and combining problem solving with project-based learning across disciplines. They envisioned themselves working together with students on activities to develop students' critical thinking, communication, problem-solving, and reasoning skills. However, once they transitioned from STEM professional to secondary mathematics teacher, the participants' perspectives on teaching mathematics and student learning evolved. The participants expressed concern that too often students lack enthusiasm and curiosity, and are unable to engage in the high cognitive demanding mathematical tasks that will prepare them for college or career. The majority of participants came to believe that while deep knowledge of mathematics is a necessary part of effective teaching, deep knowledge of mathematics alone is not sufficient to ensure student learning will take place. The participants indicated that knowing their students well, understanding their students' culture, and using their students' cultural backgrounds in their planning and instruction can increase motivation and improve mathematics learning.

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BRIDGING COMMUNITIES ONLINE: THE ROLE OF BROKERS IN PROFESSIONAL DEVELOPMENT AT THE MATH FORUM

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Summary

In this presentation, we share the preliminary results from the first phase of a project focused on supporting the emergence of an online community of mathematics educators built upon using student thinking as the vehicle to drive the development of shared formative assessment tools. Our work builds on Wenger's (1998) three aspects of communities of practice—mutual engagement, joint enterprise and shared repertoire. Applying these indications to educational settings, Cobb, Zhao and Dean (2009) developed a learning trajectory that took into account the specifics of teachers' institutional setting. Examining the school using Wenger's (1998) notion of a lived organization enabled them to identify various communities within the institution, including teachers, school leaders and district leaders. Despite a joint enterprise to improve students' mathematical performance, the actions and beliefs of the different groups were not always aligned. Building on the approach of Cobb et al. (2009) and Wenger's (1998) notion of boundaries and boundary crossing, this project seeks to understand and document the characteristics and connections between the participants, in order to create an inter-institutional and largely virtual community.

Through a process of iterative coding and constant comparative methods (Corbin & Strauss, 2008), the following themes emerged with regards to the teachers' predispositions:

- Formative assessment
- Perceptions of who would benefit from their participation
- Participants interest level in technology

We explored how these dispositions and practices supported various components of community development, learning trajectories and participation patterns, and examined the development of a shared repertoire of tools and how those tools evolved and changed over the course of the community's growth with a particular focus on respondents' dispositions and beliefs. We will discuss evidence of changes in attitudes, the use of artifacts, participation rate and role in the community over time, and explore the relationships between these changes and the themes identified above.

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THE MATHEMATICS HISTORY JOURNAL: USING THE HISTORY OF MATHEMATICS IN TEACHING PRESERVICE MATHEMATICS TEACHERS

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Keywords: Teacher Education-Preservice, Mathematical Knowledge for Teaching

At Governors State University in Illinois, USA we offer a program leading to a mathematics degree and teacher licensure. For this study, preservice mathematics teachers enrolled in a secondary education course developed an electronic journal of reflections based on peer presentations in the history of mathematics. One objective of the Mathematics History Journal was to measure the content knowledge and pedagogical content knowledge of the preservice mathematics teachers for accreditation purposes. The National Council of Teachers of Mathematics (NCTM, 2003) expects programs that educate preservice mathematics teachers and seek NCTM program recognition include instruction in the historical development of mathematics and the contributions of diverse cultures.

In this study, the reflective (electronic) journals written by preservice mathematics teachers after viewing peer presentations were analyzed for new knowledge of mathematics content and new knowledge of pedagogical strategies. Two research questions were asked. Was there evidence that using the history of mathematics for instruction can strengthen the mathematics content knowledge of preservice mathematics teachers? Was there evidence that using the history of mathematics for instruction can strengthen the pedagogical content knowledge of preservice mathematics teachers?

The data was taken from (electronic) journal entries for the history of algebra, the history of geometry, the history of measurement, and the history of probability. The nine participants were enrolled in the second of two secondary mathematics methods classes. Students researched and presented a topic in one of the four content areas for 30 minutes. Presenters were required to also develop mathematics problems related to their selected topic and lead the class through a problem-solving session. All students were required to submit a one-page, single-spaced electronic reflection within one week after the presentation based on the activity, its appropriateness for the high school classroom, and the important mathematics presented. Results of the study indicated that by allowing preservice teachers to investigate and present topics in the history of mathematics, preservice teachers believed they strengthened their mathematics content knowledge and were introduced to a new avenue for teaching secondary students about mathematics. In the poster I describe the assessment in more detail and include excerpts of students' comments from the electronic journals.

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NEGOTIATING NOS/OTR@S RELATIONSHIPS IN AN AFTER SCHOOL MATHEMATICS CLUB

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After school mathematics clubs can be ideal spaces for pre-service teachers to facilitate engaging activities, negotiate relationships with marginalized students, and have mathematical discussions (Vomvoridi-Ivanovic et al., 2011). Yet the majority of this research has been conducted with prospective elementary teachers, not pre-service secondary mathematics teachers (PSMTs) (Turner et al., 2011). In addition, little attention has been paid to the complexity of PSMTs negotiating relationships with students in order to facilitate mathematical discussions.

Our analysis is informed by a framework of *political conocimiento* (Author 2, 2012), which assumes that it is important for mathematics teachers to develop a political stance and be in solidarity with marginalized students (*nos/otr@s*). This stance can facilitate relationships that will allow marginalized students to engage in rigorous mathematics and develop robust mathematical identities as it does not presume a hierarchy of expert teacher and novice learner.

This study is part of a longitudinal study documenting the development of PSMTs (Authors, 2013) who are committed to teaching at a high needs schools upon completing undergraduate/certification requirements in an R1 Scholars Program. We highlight two female PSMTs and their efforts to negotiate relationships with marginalized middle school students at an after school mathematics club, *I Do Mathematics* (IDM). The data was obtained from individual interviews that occurred after their first year of being an R1 Scholars. They were asked about their approaches for negotiating relationships, how they elicited mathematical discussions, and challenges they encountered and how they responded to these challenges.

The two focal PSTMs believed that the IDM allowed them to effectively engage middle school students in a challenging modular origami activity. Many goals competed for their attention: establishing relationships, maintaining relationships, and discussing mathematical concepts. Their efforts ranged from first making a personal connection with the students to using the modular origami to initiate their relationships with students. Both grappled with how to maintain these relationships when the difficulty of the task increased and how to discuss the mathematics involved without overwhelming the students.

After school mathematics clubs provide a unique space for PSMTs to develop *political conocimiento* and *nos/otr@s* relationships with marginalized students. Without this lens PSMTs may overlook that mathematics teaching is a political act and how negotiating relationships can humanize the doing mathematics for marginalized students as these studntse take on the identity as doers-of-mathematics.

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INVESTIGATION OF MULTIPLE REPRESENTATIONS OF MATHEMATICAL TASKS

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Keywords: Classroom Discourse, Curriculum, Instructional Activities and Practices, Teacher Knowledge

Poster Summary

The ways in which mathematical ideas are represented is fundamental to how people can understand and use those ideas (CCSS, 2012). To support student learning, teachers are often asked to provide mathematical tasks that provide "multiple representations." However, the meaning of such expressions is not always well understood, and as a result, the charge to do so is too vague to guide teachers in planning math tasks that lead to student learning. The challenge is that teachers may interpret this call as just "solving the task in different ways", and ignore the question of how multiple representations connect to the development of a mathematical concept. This poster presents an analysis of a classroom activity in an elementary pre-service content and methods class, shown on the right. Our intent was to investigate the teacher candidates' understanding of multiple representations when solving a mathematical task focused on finding and extending a pattern. In this case, the complete solution to the problem asked the candidates to do more than just find a numeric answer. The work samples shown on the poster highlight the different strategies, as well as descriptions of how the representations were shown to be connected.

The Lobster Trap Problem
Linda Greenlaw always places her lobster traps in a square array such that the number of traps in each row is the same as the number of rows. This year, she put out 23 more traps than she used last year. How many lobster traps did she have out this year?

An important aspect of learning mathematics is to develop explicit strategies to pass from one representation to another. One of the questions addressed by our research was:

- What are the strategies that pre-service teachers use to establish correspondence between their different representations of a problem?

In response, this study was completed as part of ongoing research into preservice teachers' learning about the mathematics needed to teach effectively. The setting for the data collection was a graduate content and methods class for K-5 teachers. The eighteen students in the class (16 female and 2 male) are graduate students enrolled in a teacher licensure program in an eastern state university. The student desks were grouped in pods of four to accommodate discussion and sharing of ideas, although each student was responsible for completing and turning in individual work on the assigned problem. The students had approximately 50 minutes to complete the activity. All were able to provide the requested two representations, although about half of the students did not complete the third part of the task, which asked them to explain how the two solution methods were connected to one another.

We think there are many important findings and lessons to be learned through unpacking student work in cases such as this. While we hoped to see how the preservice teacher candidates would explain how mathematical ideas in one system were represented in the other system, most responses were limited to a verification of the two solutions, not on describing how these approaches were connected mathematically. Although, they solved the task and showed (at least) two ways to do so, there were instances where representations using different semiotic resources (diagrams, symbols, or numeric patterns) were either identified as one, or provided a correct answer but utilized a flawed mathematical argument.

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MATHEMATICAL KNOWLEDGE FOR TEACHING AND EQUITY: DESIGNING NEW OPPORTUNITIES FOR DEVELOPING EQUITABLE MATHEMATICS TEACHING PRACTICES

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Purpose and Overview

This study explores what constitutes ambitious and equitable mathematics instruction. This poster and presentation describes efforts to design a math methods course that enables mathematics teachers to identify, and later design and enact, equitable teaching practices, and details the development of the instrument used as the framework for this work. Specific instructional practices are evaluated to determine whether and how particular teaching practices provide leverage and create access to the mathematics content in elementary and middle school classrooms for different groups of learners. This poster and presentation details both the content and structure of the observational tool entitled Mathematical Quality and Equity video codes and describes results from its use with pre-service elementary and middle school teachers. As such, this presentation focuses on instructional interventions as the teachers learn to develop practices associated with equitable teaching. The goal of this study extends current work on defining ambitious and equitable instructional practices in elementary and middle school classrooms and helps to further refine a theory of how teachers use their mathematical knowledge to create access to the mathematics content for diverse students, thus promoting equity. Results from this study indicate that using the MQE observational tool serves as an important resource for teachers' learning to develop ambitious and equitable mathematics teaching practices.

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CROSSING THE MATHEMATICS-SPECIAL EDUCATION DIVIDE IN THE ERA OF THE COMMON CORE STANDARDS

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In the era of the Common Core Standards for School Mathematics (CCSSM) (2010), nearly all students with learning disabilities are expected to demonstrate the same rich understanding of challenging mathematical ideas as other students (NCLB, 2002). While both math and special educators generally agree that this is a worthwhile and attainable goal (Conference Board of Mathematical Sciences [CBMS], 2012; National Council of Teachers of Mathematics [NCTM], 2000; Woodward and Montague, 2002), research on how to prepare teachers to meet this goal is limited. Specifically, little has been written about how to best prepare special and math educators to co-teach in inclusive classrooms. This poster reports on the initial stage of a larger research collaboration between its authors—a math education professor and a special education professor—who teach preservice methods courses at the same institution. The larger design study examines the effectiveness of the researchers' effort to help teachers from both backgrounds develop their capacity for teaching all students in the era of the CCSSM. The poster reports the researchers' efforts to develop common understanding around two questions: (a) what is the nature of the mathematics teaching required to meet CCSSM goals, and (b) what are instructional needs of students with learning difficulties with respect to reform-oriented teaching? A theoretical frame of professional learning through collaborative examination of practice (Ball & Cohen, 1999) is used. Data include transcripts of audiotapes made as both educators discuss video cases of reform-oriented mathematics lessons. Findings are reported as the key areas of agreement and tension that two educators with different knowledge bases bring to the analysis of constructivist teaching.

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TEACHERS' PERSPECTIVES OF CHANGE IN DISCURSIVE PRACTICES LEARNING DYNAMIC MATHEMATICS COLLABORATIVELY

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The past decade's research on discourse in mathematics education contains few investigations on teacher professional learning in the use of dynamic mathematics software. We agree with Sfard (2008) that mathematics learning is discursive and participatory and we believe that teachers gradually develop their Technological Pedagogical Content Knowledge (TPCK) by interacting discursively in small groups (Mishra & Koehler, 2006). In our study, 32 middle and high school teachers participated in a 10-week course in which they interacted in small teams in an online collaborative environment, known as Virtual Math Teams with GeoGebra (VMTwG) (Stahl, 2009) focusing on mathematical practices and discourse. Teachers engaged in asynchronous individual preparation and synchronous collaboration to solve open-ended mathematics problems and then reflected on the logs of their synchronous collaborative interactions to identify successful and unsuccessful discourse moves as well as instances of mathematical practices (Common Core State Standards Initiative, 2010).

Guided by our question—what are teachers' perspectives of differences in their discursive, mathematical, and collaborative practices from earlier VMTwG sessions to a session in the latter half of the professional development course?—we analyzed the data from the course using conventional content analysis (Hsieh & Shannon, 2005). Analysis indicated that comparing transcripts of their own discourse from earlier VMTwG sessions to a session in the latter half of the course teachers provided evidence of shifts in their practices in each of three themes—discursive, mathematical, and collaborative. As these changes occurred in natural and symbolic languages, the shifts they described are improvements in their overall discourse. Teachers also noted that the persistent feature of their discursive interactions in the chat messages contrasts with the ephemeral nature of classroom discourse.

Acknowledgement

This work is based upon research supported by the National Science Foundation, DRK-12 program, under award DRL-1118888. The findings and opinions reported are those of the authors and do not necessarily reflect the views of the funding agency.

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PRE-SERVICE MIDDLE SCHOOL TEACHERS' NOTICING OF STUDENTS' ALGEBRAIC THINKING

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Algebra plays a prominent role in mathematics education reform efforts because it is valued as an important subject in mathematics. Stein, Kaufman, Sherman, and Hillen (2011) found an increasing number of K-12 students are enrolling in algebra courses, and an increasing number of these students are unprepared to take these courses. Consequently, the mathematics education community must address issues related to broadening perspectives on algebraic thinking and learning, including support for future mathematics teachers.

Sherin and van Es (2002, 2003) argued that developing pre-service teachers' capacity to notice students' mathematical thinking is a way to support and sustain their continued learning to teach mathematics after teacher preparation. Developing *professional noticing of children's mathematical thinking* (Jacobs, Lamb, & Philipp, 2010) in teacher preparation is a way to leverage effective teaching by enabling teachers to inform their practice by using students' mathematical thinking. Although professional noticing skills develop with teaching experience, the demands of current reform push for shaping these skills during teacher preparation.

This study seeks to better understand pre-service middle school teachers' noticing of students' algebraic thinking. Participants include three pre-service middle school teachers from a teacher preparation program that places an emphasis on student thinking. Through one-on-one interviews, I presented a task-based video clip for participants to interpret and respond to students' understandings of equivalence and variable. With a focused video clip and content-specific questions, I intended to concentrate noticing on algebraic concepts. Additionally, I asked participants to self-report experiences inside and outside teacher preparation that have contributed to their own noticing skills. I examined levels of noticing students' algebraic thinking, adapted from van Es's (2011) framework for learning to notice student thinking. Results suggest that participants felt comfortable interpreting students' thinking and attributed their noticing skills to experiences both inside and outside of their teacher preparation program. In this poster, I will report on a design to identify pre-service middle school teachers' noticing of students' algebraic thinking related to equivalence and variable, findings regarding participants' levels of noticing, and specific self-reported experiences that contribute to their noticing.

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Preservice Mathematics Teacher Perceptions of Student Engagement in a Classroom

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Keywords: Teacher Education-Preservice, Teacher Beliefs, Affect

According to the 2008 Final Report of the National Mathematics Advisory Council "experimental studies have demonstrated that…increased emphasis on the importance of effort is related to greater engagement in mathematics learning and, through this engagement, improved mathematics grades and achievement" (p. 59). Mathematical engagement is dynamic, and is often described as consisting of cognitive, behavioral and/or affective components (Fredericks, Blumenfield, & Paris, 2004). Central to our research, is the premise that it is necessary for teachers, both pre- and inservice, to understand and usefully address the many and complex influences on student engagement with mathematics and the conditions under which they occur. In the case of preservice teachers, the subjects of our study, we believe that a good starting point involves understanding how they view engagement and how their ideas evolve over time. Thus, we hope to expand the knowledge base regarding how preservice teachers understand engagement, and how that understanding broadens their perspectives on teaching math.

The research questions addressed in this study include the following: 1. How do math majors enrolled in a practicum course identify engagement while observing middle school students? 2. How do their ideas change over the course the semester? 3. What factors appear to influence the changes? 4. How does this impact the decisions they make while student teaching?

Data is being collected on five math majors enrolled in a course in which they observe middle (and high school) classrooms several times a week (this course is taken prior to student teaching). A pilot study conducted the previous year provided preliminary information that informed data collection for this study. As part of the current study, we are collecting audio-recorded focus group and interview sessions, weekly engagement observation logs and other written work from the preservice teachers, researcher field notes and reflective comments.

Preliminary findings indicate, not surprisingly, that the pre-service teachers began the semester by focusing on behavioral types of engagement. For example, in their week-one observations, all noted that they could tell if students were engaged by looking at how often they took notes or raised their hands. Several weeks into the course, their reflections appeared to include more cognitively based cues. For example, one preservice teacher wrote: "At first engagement seemed quite simple because I based it more on all students doing their work. However engagement isn't only physical but also mental. Students also need to be mentally engaged so that learning actually occurs." This preservice teacher also began to notice that some of the students that she thought were engaged at the beginning of the semester, may not have been, and vice versa. We will describe, in greater detail results for all five of the pre-service teachers as well as when and how these changes occurred and the implications of these changes.

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CONVERSIONS WITHIN THE METRIC SYSTEM: A MEANINGFUL APPROACH

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Preservice mathematics teachers (PSMT) play an integral role in preparing students for future careers in STEM fields. Commonly used methods to solve measurement conversion problems are proportional reasoning and dimensional analysis (DeMeo, 2008). However, a less often used approach to conversions involves using the meaning of the prefixes, where the grouping formed by a prefix symbol attached to a unit symbol constitutes a new inseparable symbol... which can be raised to a positive or negative power and which can be combined with other unit symbols to form compound unit symbols…for example, 2.3 cm^3 = 2.3 $(cm)^3$ = 2.3 $(10^{-2}\ m)^3$ = 2.3 x $10^{-6}\ m^3$. (Thompson & Taylor, p. 14)

Prior research showed that PSMT could not solve conversions within the metric system accurately (author), therefore this research examined the question, do PSMT understand and apply a novel approach, using the meaning of metric prefixes and the multiplicative identity, to solving conversion problems within the metric system?

Participants solved the problem in ways taught in high school using mnemonics, proportional reasoning, or dimensional analysis. Then another method, using the meaning of metric prefixes and the multiplicative identity method was demonstrated. Examples included: Problem A: **Express 26 Gm as nm**. 26 x 10^9 x (10^9 x 10^{-9}) m = 26 x (10^9 x 10^9) x 10^{-9} m = 26 x 10^{18} nm. Problem B: **Given 450 mg of a substance occupying a volume of 50 mL, calculate the density of the substance in grams per liter.** (450 x 10^{-3} g)/(50 x 10^{-3} L) = 450 g/50 L = 9 g/L. Consistent with a models and modeling perspective of teacher development (Doerr and Lesh, 2003), participants practiced this method, and discussed and shared the merits of various solution methods.

Initially, PSMT often solved the problems incorrectly because of incorrect mnemonics, or division mistakes during a dimensional analysis approach. During the intervention, participants solved the problems in a variety of ways. The only method used during the intervention that yielded consistent results was the method illustrated above using the meaning of the metric prefixes. The study demonstrated that it is possible to promote PSMT accuracy, repertoire of solutions, and deeper understanding of the metric system and the meaning of metric prefixes. PSMT often commented that using this method yielded a correct solution that made sense. In this poster session, examples of PSMT solutions and comments will be shared.

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MATH FOR YOUNG CHILDREN: A PROFESSIONAL DEVELOPMENT MODEL THAT BRIDGES COGNITIVE SCIENCE AND MATHEMATICS EDUCATION

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Keywords: Cognition, Geometry and Geometrical and Spatial Thinking, Teacher Education-Inservice/Professional Development

A gap exists in the application of cognitive science to mathematics instruction (Laski, Reeves, Ganley, & Mitchell, 2013). A growing body of evidence shows that knowledge of children's mathematical thinking (e.g., cognitive requirements of a given math problem) has a powerful influence on teachers, as they plan, design, carry out lessons, and reflect on student learning (Laski et al., 2013; Saxe, Gearhart, & Nasir, 2001). Evidence shows teacher professional development (PD) that includes a focus on student thinking and knowledge of cognitive research improves teacher's mathematics instruction (Fennema et al., 1996; Saxe et al., 2001). Innovative methods are required to bridge the gap between cognitive science and mathematics education in PD.

In this poster, we outline a PD model that has been effective in having teachers, researchers, and administrators work together in learning about and applying cognitive science findings to early year's math instruction. Our model is rooted in principles of Lesson Study (Lewis, Perry, & Hurd, 2004), but adapted to place more emphasis on the role of developmental cognitive science when considering children's thinking and when planning lessons and activities. To date, this PD model – known as the Math for Young Children project – has been carried out with more than 50 early year's teachers and well over a 1000 Pre-K – Gr. 2 elementary school students across Ontario. In this poster, we provide a case study of one team's efforts to gain insight into young children's understanding of 3D geometry and spatial reasoning.

More specifically, we will highlight how learning about 3D mental rotation, a topic deeply rooted in cognitive science (Shepard & Metzler, 1971; Casey, Andrews, Schindler, Kersh, Samper, & Copley, 2008), inspired an innovative and iterative approach to lesson design. We present a PD model of how to incorporate cognitive science findings to the practice of mathematics. Furthermore, we show that not only can teachers use cognitive science to inform practice, but also demonstrate the valuable role teachers can play in contributing to cognitive science.

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BUILDING MATH TEACHERS' LEADERSHIP CAPACITY

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In the 21st century world of diminishing funds for public education, more is being asked of mathematics teachers than ever before. Not only are they expected to hit ambitious learning targets for all students, but they increasingly take on teacher-leader roles. Research on teacher leadership supports the concept that effective teacher leaders need to be strong in subject area content as well as experts in pedagogical practices that support student learning (Elliot, Kazemi, Lesseig, Mumme, Carroll, & Kelley-Peterson, 2009; Manno & Firestone, 2007). They also need to know how to facilitate groups effectively, deal with conflicts, provide feedback to peers, and be comfortable as the liaison between teachers and the official administrators (Heck & Hallinger, 2010). Despite documentation of some of the skills of effective teacher leaders, at the present time there is little research about the breadth of these school-based roles and how teachers develop capacity to effectively enact these roles, and thus support students' learning.

In an effort to enhance math teacher leadership capacity, we partnered with four school districts to create the Math Leadership Academy (MLA). The focus of the program was to build the participants' confidence and expertise in math content, pedagogy, and leadership. An emphasis was placed on *mathematics collaborative leadership,* which we defined as any activity that impacts the instructional capacity of colleagues, a department, the school or the larger field to support *students' learning of mathematics*. Twenty experienced middle and high school math teachers participated in MLA, earning 12 graduate credits during the 2011-12 school year.

Results from the larger study of the impact of MLA indicate that the goals of the program – to enhance confidence and expertise in content, pedagogy and leadership—were met. This poster presentation presents an analysis of interview data from eight participating teachers to answer the following research questions: (1) What leadership roles do secondary mathematics teachers report taking on in their schools? and (2) To what degree do math teachers feel they are prepared to take on these leadership roles? An interview guide approach (Rossman & Rallis, 2003) was employed to elicit participants' views on specific topics, which also allowed for additional elaboration.

Analysis is ongoing. Preliminarily results suggest that secondary mathematics teachers are in fact taking on many leadership roles. Some are logistical, but others focus on influencing peers to enhance instruction that supports student learning. Generally, the roles that involve affecting change in pedagogy, which is an important goal of collaborative leadership, are the ones that teachers report feeling the least confident with and for which they desire more support.

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HITTING THE CONTENT KNOWLEDGE WALL: SUPPORTING PRE-SERVICE TEACHERS AT-RISK IN MATHEMATICS

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It has been widely accepted that teaching mathematics requires a specialized knowledge of mathematics (Ball, Thames, & Phelps, 2008; Silverman & Thompson, 2008). Our experience with pre-service teachers (PSTs) has shown that some enter the teacher education program with a deep rooted fear of mathematics as well as a lack of such knowledge (Kajander, 2010), and that reform-based methodologies are often very different from their own classroom learning experiences. During the methods course in our one year teacher education program (the current standard in Ontario), we strive to support both pedagogical as well as mathematical content development.

As part of an on-going project to document the mathematical development of our PSTs, we have noticed that each year we have a few participants who are significantly disengaged with mathematics to the extent that their lack of success on our required mathematics-for-teaching exam holds them back from graduating. The current sub-project focuses on PSTs who are significantly at-risk in mathematics. In this poster, a narrative case study of the experiences of two such research participants who were forced to repeat the methods course due to weak mathematics knowledge is presented. Our guiding research question is: What are the experiences of pre-service teachers who have struggled with mathematics in the past, as they navigate a reform-oriented methods course for the second time? Data included transcripts of interviews conducted with each participant at the beginning and end of their second year of the methods course, as well as class assignments and examination responses.

Our poster focuses on two specific pre-service teachers, Grace and Stuart (pseudonyms), who self-identified as being struggling mathematics students in the past, and were unable to successfully complete the methods course in their first year. Grace was ready to withdraw from the program during the first year due to her lack of confidence in her mathematics abilities but remained in it as a result of intense support from the instructor. While unsuccessful, she decided to continue and saw the second year of the methods course to be a chance to fill in gaps in her understanding. She engaged enthusiastically during the repeated year and deepened her content knowledge significantly. Stuart, on the other hand, saw having to repeat the methods course as an unfair punishment, and approached his second year with a very negative attitude. Despite differing views on repeating the course, both pre-service teachers were successful in completing the second year including the mathematics content exam, and showed immense gains in their understanding of mathematics. Grace and Stuart's own words and work samples are used to support the results. The experiences of these two pre-service teachers highlight the challenges encountered by potential teachers at-risk in mathematics.

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AFFORDANCES AND CHALLENGES ON COMMON CORE STATE STANDARDS PERCEIVED BY PRESERVICE TEACHERS

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Keywords: Teacher Education-Preservice, Teacher Beliefs, Teacher Knowledge

To help accelerate the successful implementation of the Common Core State Standards for Mathematics (NGA Center & CCSSO, 2010), this study probes the challenges preservice teachers perceive in the implementation process. In response to the achievement gaps both within the U.S. and between U.S. students and high achievers worldwide, as reflected in PISA 2006 and 2009 (Organization for Economic Co-operation and Development [OECD], 2010), the Common Core State Standards (CCSS) document was developed to help improve students' mathematical abilities such as problem solving and mathematical reasoning. CCSS is highly regarded as better than the existing state standards in cultivating students' expected mathematical abilities (Porter, et al., 2011). However, it has been reported that teachers are not prepared to implement CCSS (EPE research center, 2013). Given the potential gap between the intended curriculum (CCSS standards) and curriculum as implemented (materials implemented by classroom teachers), students may not benefit from this document if classroom teachers do not change their teaching practice to align with the standards. This study may illuminate factors affecting teachers' implementation of CCSS and provide insights into improving preservice teacher education and in-service teacher professional training.

A survey was developed to investigate preservice teachers' general background information (e.g., years of teaching experience, teaching load, number of math courses taken); confidence and knowledge of mathematics (e.g., math content knowledge, pedagogical knowledge, pedagogical content knowledge); confidence and knowledge of CCSS (e.g., familiarity with CCSS goals, familiarity with standards for mathematical content, familiarity with standards for mathematical practice); barriers to implement CCSS (e.g., lack of materials, lack of administrative support); beliefs about CCSS (e.g., belief of CCSS in improving students' academic performance). The survey included both five-point likert scale items and open-ended items.

Survey results showed that preservice teachers lack confidence in both their mathematical knowledge and their knowledge of CCSS. Even though they learned CCSS in their method class and compared CCSS with existing state standards, their understanding of how to implement CCSS is still limited. Most of them believe CCSS helps in improving students' academic achievements, but barriers such as lack of teaching materials, lack of professional training, and lack of school, or district support, make them doubt their ability to implement CCSS in classrooms.

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SECONDARY SCHOOL MATHEMATICS TEACHERS' CONCEPTIONS OF PROOF IN TRINIDAD AND TOBAGO

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Keywords: Reasoning and Proof, Teacher Beliefs

Recent reform efforts in Trinidad and Tobago demand that secondary school mathematics teachers provide students with opportunities that reflect the nature and role of proof in mathematics (Republic of Trinidad and Tobago Ministry of Education, 2003). A teacher's propensity to enhance the role of proof in the classroom and to respond to curriculum demands depends on the nature of their own conceptions of proof and their beliefs about its role in mathematics (Knuth, 2002). By "conceptions of proof," I refer to teachers' mathematics content knowledge, beliefs about what constitutes proof, and the role of proof in mathematics (Knuth, 1999).

In this poster, I present findings from interviews with teachers from Trinidad and Tobago. I ask the following questions: (1) *how do teachers in Trinidad and Tobago view the roles that proof plays in mathematics teaching?* (2) *What value do teachers place on proof in mathematics teaching?* I interview 10 in-service secondary school mathematics teachers in Trinidad and Tobago. I apply the five-role theoretical framework in Knuth's 2002 study to guide my analysis of teacher's conceptions about the various roles that proof plays in mathematics teaching.

My analysis of the interviews suggests that teachers identified roles that reflect those in previous studies conducted with in-service teachers (e.g. Dickerson, 2008; Knuth, 2002). These roles included: to *verify* that a statement is true, to *explain* why a statement is true, to *communicate* mathematical knowledge, and to *discover and create* new mathematics. However, my analysis did not find evidence to support the identification of one of the defining roles of proof in the five-role theoretical framework- to *build* an axiomatic system of mathematical results. Teachers also identified additional roles: to *show* that a statement is false and to *provide* intellectual challenge. My findings suggest that the teachers in Trinidad and Tobago understand what constitutes a valid proof and they value its usefulness in mathematics teaching. The teachers see the importance of proof in mathematics; but perceive that their efforts to highlight its value in their classes are limited by curriculum and time constraints. The results of this study can inform future research and professional development efforts in Trinidad and Tobago.

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CONSTRUCTING PRESERVICE TEACHERS' IDENTITIES THROUGH ONLINE JOURNALING

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The literature has shown that having a firm sense of teacher identity helps educators understand and navigate complex disciplinary situations (Knowles, 1992) and that there is a connection between development of a teachers' professional identity and their efficacy in the profession (Alsup, 2005). Therefore, pre-service teachers (PSTs) will need to have a strong sense of teacher identity with which to confront complex teaching situations (Bullough, Knowles, & Crow, 1992). While the teacher education literature illustrates the importance of self-reflection in developing teacher identity, it does not consider the contents of the reflection pieces themselves. We suggest that learning journals might prove useful as a means by which PSTs can document and reflect on their changing teacher identities. We define a learning journal to be a series of reflections/essays written by the PSTs in which they can articulate and reflect on their thoughts and ideas of what it means to be a teacher. We focus on how PSTs make sense of themselves as future teachers through reflections on their process of becoming a teacher.

This study investigated the ability for online journals to support PSTs developing identity as teachers. The PSTs were enrolled in a mathematics methods course in their first year of a two-year teacher training program. In that course, PSTs were asked to write seven reflective journals throughout the semester. The PSTs were provided with examples and suggestions for creating useful entries and communicating with others about their journals. Qualitative analysis was used to code the journals and aggregate codes into larger themes; quantitative analysis and PST interviews will be used in the second phase of this mixed-methods work.

Findings show that online journals provide an opportunity for PSTs to develop their teacher identity. Within their journals, PSTs wrote about their personal experiences as K-12 math students, experiences in practice teaching blocks, interpretations of course and other readings, reflections on other PSTs' journals, and course activities. Our results suggest that as a result of writing online journals and reading their fellow PSTs journals, a community of learners has been fostered. This is in line with the work of Nasir and Cooks (2009) who argue that learning is not only a cognitive process but also a social and cultural process. Additionally, the PSTs identities and their expression of these identities are central for participation in social and cultural practices (Lave & Wenger, 1991).

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INTEGRATION OF PRE-SERVICE TEACHER KNOWLEDGE AND *CCSSM*: MODELING IN ALGEBRA

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As most states currently require completion of an algebra course for high school graduation (Teuscher, Dingman, Nevels, & Reys, 2008), there is a national emphasis on teaching algebra to a diverse population of students. However, the high failure rate for students taking algebra courses raises concerns about how teachers are prepared to teach algebra (Loveless, 2008). Modeling may gain attention as an area of emphasis in algebra courses as the *Common Core State Standards for Mathematics* (*CCSSM*) includes modeling as a strand of high school mathematics and "Model with Mathematics" as one of the standards for mathematical practices (National Governor's Association Center for Best Practices & Council of Chief State School Officers, 2010). With consideration of the importance of teaching algebra and modeling, this study seeks to answer the question "What are the opportunities for secondary mathematics pre-service teachers (PSTs) to learn about modeling and how to teach modeling in algebra?"

Our data sources included transcripts of instructor interviews and corresponding course materials collected from five universities that house secondary mathematics teacher education programs. The transcripts and course materials were coded based on a framework that focuses on learning modeling and learning to teach modeling, as well as instructional activities used to accomplish both of these types of learning. The data were also coded using the processes described in the modeling strand in *CCSSM* (e.g., interpret mathematical results in the context of the problem).

Preliminary Results and Discussion

In all cases, PSTs learned about modeling real-world situations with algebraic concepts in their teacher education programs. To be specific, they identified variables from a real world context and formulated a model by using representations; these components are recommended by *CCSSM*. PSTs also learned about how to teach modeling. For example, they learned how to facilitate a classroom where their students learned to solve real world algebraic problems by modeling. Specific instructional strategies, provided in both interviews and course materials, were largely consistent across universities, but important differences surfaced. Unique experiences of PSTs from each university will be shared in this poster session to inform other educators who prepare PSTs to teach algebra in general, and modeling in particular, in their programs.

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EXAMINING RELATIONSHIPS BETWEEN PROFESSIONAL DEVELOPMENT CHARACTERISTICS AND TEACHERS' MATHEMATICAL KNOWLEDGE FOR TEACHING

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Loucks-Horsley et al.(2003) defined important contextual factors of professional development (PD), including standards, student learning needs, practices regarding curriculum, instruction, and assessment, and education policies. Research has not yet determined how much and what kind of professional development is enough to improve learning and teaching. This study examines several MSP projects across one state that implemented professional development programs (PDPs) designed to increase teachers' content and pedagogical knowledge. Researchers looked for relationships between the features of the PDPs and teachers' gains in Math Knowledge for Teaching (MKT).

Projects in the study were implementing research-based mathematics professional development, and were predicted to demonstrate positive gains in MKT across all projects. Researchers explored whether differences in content gains were correlated with differences in hours and types of professional development. In this statewide MSP program, all projects are required to offer at minimum 40 hours of algebra content, but the exact timelines and offerings are decided by each project. Within each project it is common to see a combination of locally developed PD (dependent on the expertise of project leaders), curriculum based PD (such as supporting teachers' knowledge of state and national standards), additional PD from other content strands and nationally developed PD. Thus, the programming within each project resembles what is typical when research-based practices for professional development are scaled up and disseminated to large populations of teachers.

Teachers' pre/post-test scores (n=1030) from the Patterns Functions and Algebra LMT assessment were used to measure MKT gains in terms of effect size by project. For each program year, positive significant gains were found with the following average effect sizes: Y1(.16); Y2(.30); Y3(.45); Y4(.36). Individual projects' effect sizes ranged from -.04 to 1.09. PD sessions within each project were coded either *algebra* or *not algebra* to calculate each projects' average number of algebra hours. No significant correlation was found between number of hours of algebra PD and effect size (Y1: r(9)= .27, p=.43; Y2: r(12)=.33. p=.25; Y3:r(11)=.40, p=.17).

In 2011-12, session coding was expanded to include contextual features discussed in the literature (Garet et al, 2001). PD sessions were coded with five contextual factors: content, student thinking, instruction, curriculum and standards, and professional community. Analysis of projects shows that professional development programs vary in two ways. First, in some projects teachers receive the same PD experience, while for others teachers' individual experiences vary widely. Second, projects devoted similar amounts of time to instruction, curriculum and standards and professional community. PD offerings coded as direct content and student thinking varied and were not always present in PDPs. Contextual descriptions of PD provide clues explaining how teachers gain content knowledge, and why teachers' gains vary between projects.

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PROSPECTIVE PRIMARY TEACHERS' VIEWS ON THE NATURE OF MATHEMATICS

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Descriptions of Study

The purpose of this study was to describe prospective primary teachers' beliefs about the nature of mathematics using Raymond's (1997) criteria. Twelve primary teacher candidates were interviewed in the first semester of a mathematics methods course.

Findings

Two of the participants held a traditional view of the nature of mathematics using Raymond's classification, eight showed primarily traditional, and two held an even mix of traditional and nontraditional views. All of the participants mentioned "basics" or "foundations" in describing mathematics and saw these as *given* and *connected*. However, the purpose of foundations or basic concepts in mathematics varied among participants. Table 1 summarizes the views of the participants.

Table 1: The Participants' Description of Interconnections in Mathematics

Types of connection		Description
Mathematical foundations/basics are connected	hierarchical	Prior mathematical foundations are stepping stones for the more advanced level of mathematics learning.
	as certain steps	Mathematical foundations represent steps in a procedure, and the steps produce a final answer.

Implication and Conclusion

It is promising that the prospective primary teachers recognized that the basics of mathematics are connected. However, they considered the concepts as given and as connected in predetermined ways. Thus teacher educators might need to consider the prospective teachers' views on the nature of mathematics when designing teacher education programs and create experiences that help prospective teachers to see mathematics in a broader perspective (Bahr, Shaha, & Monroe, 2013).

Acknowledgement

Data collection was funded by the Spencer Foundation under grant number 200000266.

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TEACHERS' CRITIQUES OF FOCUS IN HIGH SCHOOL MATHEMATICS: REASONING AND SENSE MAKING

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Recommendations for improving the nature of teaching and learning mathematics have been ongoing over the past thirty years, traced back to 1980 with the National Council of Teachers of Mathematics (NCTM) publication of *An Agenda for Action*. A recent iteration of recommendations for improving mathematics teaching specifically addressed high school mathematics classrooms. *Focus in High School Mathematics: Reasoning and Sense Making* (NCTM, 2009) argued that participation in reasoning and sense making is critical for providing students the opportunity to develop a conceptual understanding of mathematical concepts and procedures, and to recognize the connections between concepts, rather than simply memorizing procedures.

In these recommendations, the practices of reasoning and sense making are framed as necessary components of all mathematics learning. Reasoning is defined as "the process of drawing conclusions on the basis of evidence or stated assumptions," (NCTM, 2009, p. 4). Rather than defining reasoning narrowly as an informal precursor to mathematical proof (e.g., NCTM, 2000; Stylianides, 2010), reasoning is conceptualized as a form of mathematical logic that is necessary in all mathematical activity (e.g., Sternberg, 1999). The authors argue that reasoning and sense making are the basis of what has been missing from high school mathematics classrooms.

This research is part of a larger study that investigated the experiences of seven mathematics teachers responding to the recommendations of *Reasoning and Sense Making* through action research. Teachers read and discussed the recommendations and selected aspects to enact in their teaching practice. This study used narrative inquiry (Clandinin & Connelly, 2000) to understand teachers' experiences enacting change. Data collected included teachers' conversations, journal reflections, interviews, and classroom observations. This poster presentation will focus on the ways teachers found the document to be supportive of their work to align their practice with it, and their critiques of how it could be improved.

Findings suggest that teachers agreed with the goals of *Reasoning and Sense Making* and it was a productive focus for their changes in practice. Teachers critiqued, however, the artificiality of the teaching vignettes included. The vignettes portrayed students conversing in ways very different from the examples of students' messy in-progress reasoning they valued in their classroom. The teachers did not find the document to be supportive of the transitional process they engaged in as the grassroots initiators at their school to engage students in reasoning and sense making. Their critiques are informative for the writing of future recommendations, and for mathematics teacher educators seeking to support teachers engaging in similar work.

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PRE-SERVICE ELEMENTARY TEACHERS' ANCHORS FOR GENERALIZATION

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Teachers need to understand deeply the content they are to teach (Ball, Thames, & Phelps, 2008; Ma, 1999). Although number patterning tasks (i.e., generalization) present a means to develop students' algebraic thinking (Bednarz, Kieran, & Lee, 1996), research on pre-service elementary teachers' understanding of numerical patterning is limited. Richardson, Berenson, and Staley (2009) found that pre-service teachers symbolically generalized linear relationships and improved their justification ability with repeated exposure to similar tasks. Zazkis and Liljedahl (2002) found that participants were able to generalize patterns verbally but not symbolically when working with patterns present in an array of numbers. This study investigated pre-service elementary teachers' symbolic generalizations (i.e., explicit, closed, algebraic forms) of an exponential relationship presented pictorially and the justification of their generalizations. Utilizing representations as tools to justify understanding conceptualized this study.

Data were collected from 23 pre-service elementary teachers enrolled in a content course on algebra and probability/statistics. The task asked students to find the relationship between the number of shaded triangles at each iteration of the Sierpinski Triangle. Participants were asked to develop and connect pictorial, verbal, tabular, and graphical representations before symbolically generalizing the relationship. This data was analyzed using thematic analysis (Glesne, 2011) to identify participants' generalizations and the associated justifications.

Analysis revealed that pre-service elementary teachers anchored their symbolic generalization to varied representations of the relationship. For example, participants argued their symbolic generalization through patterns in tabular values, pictorial changes, and graphical shapes. Additionally, this anchoring was often tied to representations that participants had understood through recursive reasoning. Analysis also revealed that participants possessed misconceptions regarding restrictions of the context. Furthermore, the participants struggled with the process of generalizing, explaining their developed generalization, and attending to co-varying quantities of the relationship. Building upon Richardson et al. (2009), these findings suggest that pre-service teachers' use of recursive reasoning and connections between representations may aid them in developing a symbolic representation. Misconceptions may also interact with students' reasoning while developing a symbolic representation.

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PRE-SERVICE TEACHERS' ENGAGEMENT IN SOCIOPOLITICAL PROBLEMS

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Introduction

This case study examines conflicts that pre-service elementary teachers encounter as they engage and solve problems featuring classical, community, and critical mathematics content.

Theoretical Perspective

This work draws on Gutstein's (2008) framework of teaching mathematics for social justice that integrates community, critical, and classical mathematics knowledge bases into problem identification and problem-solving. Community knowledge is defined as "how people understand their lives, their communities, power, relationships, and their society" (Gutstein, 2008, p. 155). Classical knowledge includes mathematical skills, competencies, and practices outlined in national standards. Critical knowledge refers to "knowledge about the sociopolitical conditions of one's immediate and broader existence" (Gutstein, 2008, p. 156).

Methods

Pre-serivce teachers (PSTs, n=21) worked in groups to design and present a problem in class integrating classical, community, and critical mathematics content. The data analyzed included artifacts developed by each group as they designed and presented their problem, as well as exit slip comments and lesson reflections. A deductive analysis identified instances when PSTs noted (a) classical, (b) community, or (c) critical mathematics content in lesson artifacts and responses. Subsequent inductive analysis then examined how the PSTs' responses indicated understanding of the three knowledge bases in conjunction with and in relation to one another.

Results

Findings suggest PSTs confronted three conflicts as they engaged in problem-solving: (a) tension attending to the integration of community and critical mathematics content with classical mathematics content, (b) difficulty representing the community and critical mathematics content with classical mathematics content, and (c) trouble communicating the importance of classical, community, and critical content as interrelated mathematics knowledge bases.

Discussion

The results from this study reveal that although PSTs encountered conflict when integrating, representing, and communicating classical, community, and critical mathematics content in problem-solving, PSTs did recognize the importance of including all three knowledge bases in their problems.

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PRE-SERVICE TEACHERS' PERCEPTIONS OF REAL-LIFE CONNECTIONS

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This study aimed to explore pre-service elementary teachers' perspectives of real-life connections (RLC) in the case of story problems. The intent was to gain insights into their understanding of what constitutes RLC in mathematics education. The data were collected from a two-part task: (a) providing criteria for exemplary RLC and colleting two story problems that best represent RLC in the participants' opinions, and (b) evaluating the quality of sample story problems. Several noted findings and remaining issues for further discussions are as follows.

Beliefs vs. actual practice: There appeared to be strong support for the use of RLC among the participants. However, most of the problems posed by participants were "standard problems", which can be solved by applying one or more straightforward operation with the given numbers. This contrasted with their overly positive feelings toward the effectiveness of incorporating RLC. Given this situation, one immediate challenge for mathematics teacher educators is determining how to unpack the widespread recommendations regarding the use of RLC for prospective teachers and how to help them critically evaluate and internalize these recommendations.

Varied interpretation of reality: While participants' awareness of the importance of RLC was apparent, it is not clear how the participants thought RLC would contribute to the development of mathematical knowledge. A question still remains: What would be the desirable balance between mathematics and reality as story problems present the relationship between them (Sowder, 1989)?

Usefulness: The majority of participants perceived that RLC has a value because of its usefulness. This pervasive view, which is in contrast to some researchers' view (e.g., Carraher & Schliemann, 2002), is ultimately related to teachers' beliefs of the nature of school mathematics beyond the case of story problems. Teacher educators need to provide prospective teachers with opportunities to reflect upon this issue.

Overall, the findings suggest that prospective teachers are aware of the importance of utilizing RLC in their teacher education programs at the general level, yet they are not fully exposed to the teachers' thinking process for the implementation of real-life contexts in mathematics education. As the most fundamental implication, there is a need for clarification of how RLC is presented and referenced in a mathematics teacher education program. This leads to the need for providing prospective teachers with explicit opportunities to examine the justifications behind their potential teaching strategies.

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PROSPECTIVE ELEMENTARY TEACHERS' THINKING TRAJECTORY ON MULTIPLICATION

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This poster reports the findings of 56 prospective elementary teachers' works on writing and visualizing multiplicative problems from a trajectory perspective. The following research question is answered: What kind of semantic structures do prospective teachers use to interpret symbolic expressions of multiplication ranging from finding the product of whole numbers to finding the product of decimals?

The theoretical framework for this research is built on specialized content knowledge (SCK) proposed by Ball, Thames, and Phelps (2008) and Piaget's theory of assimilation and accommodation. SCK, a central idea in the proposed theoretical framework, refers to mathematical knowledge and skills needed uniquely by teachers. In this research, preservice teachers' SCK can be determined through accessing and analyzing the semantic structures that they used to interpret symbolic multiplication. Learning about whole numbers has been assumed to be different from learning about non-whole numbers (Siegler et al. 2011). Piaget's theory of assimilation occurs when individuals use their existing schema to respond new conceptions or phenomenon. Piaget's theory on accommodation, on the other hand, asserts that one's existing schema must change in order to suit more complicated and challenging conceptions or phenomena. While progressing from whole-number multiplication to non-whole number multiplication, it was unknown whether or not preservice teachers would use their existing semantic structures for interpreting whole-number multiplication to interpret the more complicated and challenging multiplication, or accommodate it with new semantic structures.

Participants of this research comprised 56 prospective elementary teachers from a U.S. university's elementary teacher education. For data coding and analysis, multiplicative situations created by participants were classified into four semantic structures: Repeated Addition or Partial Group (RA/PG), Multiplicative Comparison (MC), Combinations or Probability (C/B), and Two-Dimensional Units (TDU). The development of this classification scheme considers not only previous work by mathematics educators (e.g., Greer, 1992; Taber, 2002), but also the need to connect whole-number multiplication with non-whole number multiplication.

Over a half of the prospective teachers explained multiplication as repeated addition or partial group (RA/PG) of a quantity, but the percentage declines while progressing from a whole number to a decimal as the multiplier. The percentage that the prospective teachers explained multiplication as two-dimensional units (TDU) increases while progressing from a whole number to a decimal as the multiplier. The findings can help mathematics education community recognize prospective teachers' understanding and future teaching tendency in multiplication.

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INVESTIGATING PROSPECTIVE SECONDARY TEACHERS' MATHEMATICAL KNOWLEDGE FOR TEACHING EQUATIONS AND INEQUALITIES

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Keywords: Algebra and Algebraic Thinking, Mathematical Knowledge for Teaching, Modeling, Teacher Education-Preservice

Although researchers have devoted much attention to developing a conceptual framework for the knowledge needed for teaching mathematics at the elementary level (Hill, Ball, & Schilling, 2008), less attention has been given to the knowledge needed to teach secondary mathematics, with the exception of the MET II report (Conference Board of the Mathematical Sciences, 2012). In this study, we focused on knowledge for teaching algebra, investigating prospective secondary teachers' (PSTs) knowledge of the Reasoning with Equations and Inequalities cluster of the *Common Core State Standards for Mathematics*. Relatively little research has been dedicated to secondary teachers' knowledge of algebra (e.g., Asquith, Stephens, Knuth, & Alibali, 2007; McCrory, Floden, Ferrini-Mundy, Reckase, & Senk, 2012).

To investigate PSTs' knowledge for teaching equations and inequalities, we conducted problem-based, semi-structured interviews with 12 PSTs from one Midwestern university. Although our interview protocol consisted of 12 items; this analyses focuses on the findings from two items addressing linear equations. The first item asked PSTs to solve the equation $5x+9=2+7(x-3)$, to explain their thinking, and to respond to a student that asked them why they can "do the same thing to both sides." The second item (adapted from McCrory, et al., 2012) displayed a pan balance that could be used to model the equation $3B+4=10$. PSTs were asked to use the pan balance to solve the equation and identify benefits and limitations of the model. Our preliminary findings suggest that most PSTs could solve the linear equation algebraically but when using the pan balance their reasoning showed great variability. When completing Item 1, some PSTs used mathematical language, such as "Distributive Property" or "simplify," while other PSTs used less mathematical language, such as "cancel," "plug in," and referenced order of operations by the acronym PEMDAS. All PSTs could use the pan balance to model addition and subtraction, but almost all struggled with modeling division with the balance. In addition, PSTs tended to see the model as something for visual learners to be used after students learned to solve equations, rather than as a means for helping students learn to solve the equation.

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AN EMERGING FRAMEWORK TO CHARACTERIZE TEACHERS' PEDAGOGICAL GOALS FOR STUDENT LEARNING

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Keywords: Mathematical Knowledge for Teaching, Teacher Education-Inservice/Professional Development, Teacher Knowledge, Teacher Beliefs

In this study, two secondary school teachers were asked to reflect on their pedagogical goals for student learning for a series of lessons using a research-based Precalculus curriculum (Carlson & Oehrtman, 2012) from the Pathways project for the first time. Project Pathways is an initiative that focuses on professional development to improve teacher's key developmental understandings (Simon, 2006) of the mathematics they teach in order to improve content knowledge through fostering a rich connection of mathematical ideas and relationships. Part of this initiative led to project leaders developing a research-based conceptually oriented curriculum that teachers involved in the project could use in their classroom.

I used Silverman and Thompson's (2008) construct of mathematical knowledge for teaching as a lens for examining how a teacher understands ideas and connections among ideas, and how this influences her pedagogical decisions and actions. From the participants' stated goals, a framework emerged to characterize the developmental stages of a teacher's goals for student learning as her teaching orientation shifts from calculational to conceptual (Thompson, Philipp, Thompson, & Boyd, 1994).

In this presentation, I will share my findings of how the framework emerged in this study, and how it was then used as both a research tool to characterize teachers' goals for student learning and as a tool to promote professional growth in subsequent studies.

Acknowledgement

Research reported in this presentation was supported by National Science Foundation Grant No. EHR-1050721. Any conclusions or recommendations stated here are those of the author and do not necessarily reflect official positions of NSF.

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SIMULATED INTERACTION MODEL (SIM): AN INNOVATIVE APPROACH FOR PREPARING PRESERVICE MATHEMATICS TEACHERS

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Teachers often face *problems of practice* – content or pedagogical challenges that have the potential to impact student learning (Lampert, 2001, 2010; Shulman, 1986). Preservice teachers in formal teacher preparation programs have the luxury of time to consult resources, reflect in writing, and carefully ponder the professional problems they *anticipate* in future classroom practice. Practicing teachers, in contrast, do not experience this luxury of time, and instead must enact and apply professional knowledge to a wide variety of content and pedagogical challenges in real time. As a result, practicing teachers often cite a gap between the readings, reflections, and case studies of teacher preparation programs and the in-the-trenches decisions, actions, and pace of teacher practice (Ball, 2000; Putnam & Borko, 2000). Teacher education scholars continue to seek innovative approaches that challenge novice teachers – either at the preservice or induction (Years 1-3) career stage – to enact and apply their professional knowledge and skills in authentic situations.

Utilizing a pedagogy common to medical education contexts, we are designing simulated interactions, where preservice mathematics and science teachers engage in live, one-to-one simulations with standardized individuals to address content-specific problems of practice. Our research aim is to identify and address strengths and gaps in preservice teacher understanding and practice in order to enhance preservice preparation. The underlying premise of the Simulated Interaction Model (SIM) is that identifying and addressing these strengths and gaps will advance teacher instructional capacity and, in turn, enhance student learning. The SIM rests on the theoretical tenets of situated cognition.

In the first phase of this study, we interviewed 25 veteran mathematics and science teachers to ask them to reflect on common problems of practice, challenging content to teach, common student struggles, examples of interesting student understandings, and difficulties experienced by novice teachers. We analyzed these interview transcripts using four simulation design tenets as guiding codes – Prevalence, Instructional Importance, Clinical Impact, and Social Impact. We also reviewed research literature to identify misconceptions or conceptual struggles that are common and used the interview data and our literature review to design simulations, which include resources for both the learner (preservice teacher) and the standardized individual (actor). During 2012-2013, we are in the process of engaging preservice teachers with some of these simulations. Our poster will present examples of two mathematics simulations and our analysis of the data collected from preservice teachers' engagement with these simulations.

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INTEGRATING CULTURALLY RELEVANT PEDAGOGY IN A MATHEMATICS PROBLEM-SOLVING COURSE: PRE-SERVICE TEACHERS' DIVERSITY AWARENESS GROWTH TRAJECTORIES

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In the past decades, researchers have accumulated rich evidence of U.S. middle grade students' mediocre performance on national and international mathematics assessments (e.g., Mullis, Martin, Gonzalez, Gregory, Garden, O'Connor, Chrostowski, Smith 2008). Among varied efforts to improve student achievement, increasing attention has been given to the importance of developing teachers' mathematical knowledge as a key to preparing effective teachers (Hill, Rowan, & Ball, 2005). Prospective teachers' beliefs and perceptions about racial, cultural, ethnic, social, and linguistic diversities are equally important and should be addressed (Ladson-Billings, 1991). As pointed out by Gay (2010, p. 143), prospective teachers' "racial, ethnic, and cultural attitudes and beliefs are always present, often problematic, and profoundly significant in shaping teaching conceptions and actions."

We designed a study to analyze pre-service teachers' diversity awareness growth trajectory in a mathematics problem-solving course that integrated culturally relevant teaching pedagogy and used three instructional activities. A total of 35 pre-service teachers participated and completed a 13-question, Diversity Preparedness Response Inventory (DPRI). We found statistically significant overall growth in pre-service teachers' diversity awareness level and when we decomposed the effects based on the instructional activities, we found that their diversity awareness was enhanced when they received traditional and Second Life instruction. Although some of the traditional activities were more effective than others, the pre-service teachers experienced a linear growth in their diversity awareness with the Second Life activities. The results suggest that pre-service middle grades mathematics teachers' diversity awareness can be enhanced integrating culturally relevant pedagogy.

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AN INVESTIGATION INTO ELEMENTARY MATHEMATICS TEACHERS' KNOWLEDGE BUILDING THROUGH COLLABORATIVE INQUIRY

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Inservice teachers need professional development in order to enhance students' achievement in mathematics (Wei, et. al., 2009). Cross-national research shows the impact of inservice teachers' collaborative work on teachers' professional development (Ma, 1999; Stigler & Hiebert, 1999). Because of the differences of teaching culture in schools and challenges that new teachers encounter (Fantilli & McDougall, 2009), Canadian schools need to effective ways to upgrade teachers' knowledge. This study investigated how Grade 8 teachers build mathematical knowledge through a professional development project employing collaborative inquiry.

The theoretical frameworks that grounded the collaborative inquiry design are (1) collective cognitive responsibility (Scardamalia, 2002) where all members take the responsibility of group success, and (2) situated approach to teachers' professional development (Putnam & Borko, 2000) where " knowing and learning are situated in physical and social contexts, social in nature, and distributed across persons and tool" (p. 12). The central idea of the inquiry project was to integrate technology into teacher's pedagogical content knowledge development. The Ten Dimensions of Mathematics Education (McDougall, 2004) was employed for program content design, which includes a Likert-like scale survey with questions to explore teachers' beliefs and attitudes with mathematics. The teachers selected five dimensions for professional development: student tasks, constructing knowledge, manipulative and technology, assessment, and teacher comfort and attitude towards mathematics. The participants were 29 eighth grade mathematics teachers from eight different schools from a large urban school district in Ontario.

The statistical T-test analysis and qualitative results indicated that the combination of online resources, peer teacher discussion and sharing, and collaboration are the best ways for teachers to improve. This study will provide educational institutions and researchers with strategies for creating effective programs for teachers.

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PREPARING TO TEACH ALGEBRA WITH EQUITY IN MIND

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The National Council of Teachers of Mathematics *Equity Principle* (2000) recommends that all students be supported to learn mathematics in meaningful ways and have high expectations for success with appropriate learning environments, resources, and accommodations. Moses and colleagues (1989) posited that algebra is a gatekeeper or a filter for more advanced mathematics courses and a wide variety of college opportunities. Furthermore, with the increase in diverse learners, along with growing failure rates in algebra courses (Loveless, 2008) it is important to investigate the pre-service teachers' (PSTs) opportunities to learn about equity in algebra. We present results from an investigation of opportunities that PSTs have to learn about equity in algebra and how they are prepared to teach diverse student populations with these equity issues in mind.

In order to investigate PSTs' Opportunities to Learn (OTL) about equity issues related to algebra, we utilized an analytic framework that focuses on what PSTs learn about equity issues related to algebra (e.g., algebra as a gatekeeper or civil right, achievement gap in algebra) and how they learn about these issues (e.g., instructional activities, assignments, discussions). The framework also focuses on what PSTs learn about how to teach with these equity issues in mind (e.g., supporting diverse learners, various pedagogical strategies, culturally relevant pedagogy) and how they learn to teach with equity-mindedness based on recommended strategies, methods, and research findings, (e.g., experiencing culturally relevant teaching, case studies, microteaching).

As part of a larger study, we analyzed 15 instructor interview transcripts from two institutions. In preliminary analysis of this interview data, the instructors reported a variety of opportunities for PSTs to learn about equity issues in education and in mathematics, but had more limited opportunities to learn about equity issues specifically related to algebra. For example, PSTs at one university learned about equity issues such as the disparity between women and men represented in mathematics, exclusion of non-white mathematicians from a common historical narrative, social inequities, and economic disparities in social and employment settings. Some PSTs also learned about the achievement gap, marginalized student populations, and differentiated instruction in algebra. In the participating program PSTs learned about these issues in ways that include semester projects, lesson plans, discussions, lectures, readings, and reflexive activities.

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EXAMINING INTEGRATED PROFESSIONAL DEVELOPMENT TO SUPPORT MATHEMATICS TEACHERS OF ENGLISH LEARNERS

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While the percentage of students who are English learners (ELs) in US public schools continues to grow, less than a third of teachers report receiving training to effectively teach ELs (Ballantyne, Sanderman, & Levy, 2008). The Mathematics Coaching Supporting English Learners project (MCSEL) seeks to address this mismatch by developing and studying professional development materials to support middle-grades mathematics teachers whose classes include ELs. To provide teachers with opportunities to build content and pedagogical knowledge that directly addresses the work of teaching, MCSEL combines monthly seminars and coach-led classroom inquiry cycles (CICs), where each CIC includes a pre-conference, a coach-observed lesson based on specific instructional routines, and a post-conference. Research on instruction of ELs and empirical research on mathematical problem solving inform three design principles that shape the seminars and CICs: attention to challenging mathematical tasks, multimodal mathematical representation and communication, and developing academic language. Also, an emphasis on repeated structured practices supports integration of learning into practice. Data were analyzed to understand shifts in participating teachers' practice across seminars and CICs: In what ways do participating teachers increase their attention to cognitive demand by using challenging mathematical tasks, to the use of multi-modal communication and representation, and to academic language development in lesson planning and instruction for ELs? Data from six participating teachers across three districts included seminar and CIC observer notes and reflections, teacher and coach feedback, and five interviews per teacher. Discourse analysis focused on teacher talk and indicators related to three design principles.

The poster will highlight shifts in teacher indicators across MCSEL participation as related to the three design principles, identifying changes in practice and relations between MCSEL and those changes. Preliminary analysis reveals two trends: (1) Teachers' increased attention to mathematical diagramming was related to how they understood diagramming to support their own mathematical thinking, as well as progress they noted in their students' diagramming, and (2) Across CICs, teachers' engagement with the MCSEL math tasks and language development strategies increased, which may suggest that integrated MCSEL seminars and CICs support teachers' attention to students' academic language in their mathematics classrooms.

Acknowledgments

Research reported here was supported by Institute of Education Sciences, U.S. Department of Education, through Grant R305A110076 to EDC. The opinions expressed are those of the authors and do not represent views of the Institute or U.S. Department of Education.

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RECOMMENDATIONS FROM *MET II*: INVESTIGATING THE ALIGNMENT OF SECONDARY MATHEMATICS TEACHER EDUCATION PROGRAMS

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The Mathematical Education of Teachers II (MET II) (CBMS, 2012), the most recent set of recommendations about teacher preparation from the mathematics societies, was written in response to *Common Core State Standards for Mathematics* (*CCSSM*) (National Governor's Association & Council of Chief State School Officers, 2010). The purpose of this study was to investigate how current secondary mathematics teacher education program course requirements align with the *MET II* recommendations. Survey responses related to the course requirements of 78 secondary mathematics teacher education programs were considered in this analysis; when classified by the certification resulting from their largest program, 2 were middle grades only, 16 were high school only, and 60 awarded both middle and high school certification. Thus, 62 programs (2 middle grades only and 60 combined programs) were examined using *MET II's* middle grades recommendations and 76 programs (16 high school only and 60 combined programs) were examined using *MET II's* high school recommendations.

Table 1 summarizes the programs' alignment with *MET II*'s recommendations.

Table 1: Programs' Alignment with *MET II's* Recommendations

Middle Grades Programs	
MET II Recommendation	Percentage of Programs That Met the Recommendation (n=62)
At least 24 semester-hours of mathematics	100%
At least 15 semester-hours of mathematics for middle grades teachers	0%
At least 9 semester-hours of advanced mathematics	100%
At least 2 middle grades methods courses	0%
High School Programs	
MET II Recommendation	Percentage of Programs That Met the Recommendation (n=76)
Three-course calculus sequence	80%
Introductory statistics course	89%
Introductory linear algebra course	97%
At least 9 semester-hours of high school mathematics from an advanced perspective	11%
At least 9 additional semester-hours of advanced mathematics	100%
Mathematics methods courses	95%

Results indicated that the vast majority of surveyed programs are meeting the mathematics recommendations but are falling short in their requirements for courses that focus on school mathematics.

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USING SELF-SELECTED VIDEO IN PROFESSIONAL DEVELOPMENT TO ENCOURAGE PRODUCTIVE DISCUSSION

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In the past decade, several researchers have investigated the use of video in professional development as a motivator of teacher discussion (i.e. Sherin, 2004; Borko, Jacobs, Eiteljorg & Pittman, 2008). In most of these projects, videos were captured using outside videographers and the professional development team chose the segments and facilitated the discussion. As a professional developer, I was intrigued by the prospect of the use of participant classroom video, but in a way more accessible to most projects, and designed and implemented a mathematics professional development program (CMAP) with this in mind. CMAP focused its year-long, intensive professional development program on number concepts and operations as well as the CCSS Standards for Mathematical Practice. During monthly sessions, 25 teachers collaboratively planned lessons to incorporate new content into their classroom practice. Teachers captured their implementation of that lesson with video cameras. Following the taping, each teacher was asked to select a three to five minute section of video. During the next PD session, teachers reassembled in their small groups and viewed and discussed the selected excerpts using a protocol for watching the videos that included first reconstructing and then interpreting the video (Coles, 2010). The question driving this work was: Would the teachers engage in reflective and productive discussion of their video clips when the group discussion was not facilitated by the professional developer but rather guided by a written protocol?

Fifty-six teacher discussions were captured on video and classified as being *productive* or *not productive*. *Productive discussion* was defined as conversation between teachers connected to the teaching and learning in the participating teacher's classroom. With the belief that teacher change results from productive discussion in a professional community (Borko, Jacobs, Eiteljorg & Pittman, 2008) this poster will explore the nature of these segments, both productive and not productive, to provide insight into the conditions needed for teachers to use self-recorded and self-selected video as stimuli for productive discussions about mathematics lessons.

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DEVELOPING TEACHERS' KNOWLEDGE OF COMPUTATIONAL ESTIMATION

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Mathematics educators in recent decades have understood that computational estimation is important for elementary students to learn; at the same time it is difficult to teach. Researchers have made headway toward understanding how students learn computational estimation, but less is known about mathematical knowledge for teaching computational estimation or how professional developers might offer opportunities for teachers to learn it.

On this poster, I illustrate a theoretical framework for mathematical knowledge for teaching computational estimation, developed by bringing together literatures on conceptual and procedural knowledge for estimating (e.g., LeFevre et al., 1993) and mathematical knowledge for teaching (MKT) (Ball, Thames, & Phelps, 2008). I then describe three activity structures routinely employed during a two-year professional development (PD) program on MKT. The activity structures are: solving and discussing math problems related to the elementary curriculum, investigating records of classroom practice, and investigating elementary curriculum materials. Each activity structure has a research base, which I used as a heuristic to map the activities onto the initial framework. The resulting map illustrates a theoretical argument for the potential affordances of the PD activities for enhancing teachers' knowledge of MKT for computational estimation.

I tested the model by analyzing the discourse in three consecutive PD sessions focused on computational estimation in seven PD groups, each led by one of four different facilitators. Unsurprisingly, although the facilitators co-planned the sessions and used the same activities and materials, facilitation differed across groups. The analysis focuses on two comparisons: teacher talk across activity structures within groups, and across groups within activity structures. The results complexify the original map of the PD model. For example, the original map demonstrated the potential for investigations of elementary curriculum materials to engage teachers in learning specialized content knowledge, horizon content knowledge, knowledge of content and teaching, and knowledge of content and curriculum. However, in some groups teachers' discussion focused on issues of common content knowledge and specialized content knowledge, while other groups focused on knowledge of content and teaching.

The interplay between the design and facilitation of the activities and teacher talk has implications for designers and facilitators of PD focused on MKT. The study also contributes a theory of knowledge of computational estimation for teaching.

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TEACHERS LEARNING ABOUT STUDENTS' MATHEMATICAL REASONING IN AN ONLINE COURSE

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This study reports on an experimental online course that was developed through design research as a context for teachers to build knowledge through studying videos and related literature about how a classroom of 4th graders developed ideas about fractional numbers and relationships. The instructional intervention is timely in light of what has become widespread adoption of Common Core State Standards, which has generated a need to support teachers in their efforts to facilitate the engagement of their students in mathematical learning activities that are aligned with the Standards of Mathematical Practice (NGACBP & CCSSO, 2010).

The poster describes the online course design and its content, which utilized the resources of the Video Mosaic Collaborative (VMC), an open-access repository that makes available videos and related metadata from more than two decades research on the development of students' mathematical ideas and forms of reasoning (www.videomosaic.org). The course was designed to provoke consideration of how fractions are introduced in school, what may be some obstacles to fraction learning, and how a particular approach to building rational number sense by engaging children in tasks with Cuisenaire rods provides them with a flexible manipulative for building models to support conjectures and claims about fraction ideas (Steencken & Maher, 2003).

Teacher learning was investigated through discourse analysis of online discussion data from two implementations of the course with 25 total participants (Palius, 2013). Data were first analyzed with discussion group as the unit of analysis to examine what patterns of discourse emerged among the small groups. Detailed discourse analysis was performed to code the contents of teachers' posts to the online discussions.

Results show that teachers attended to specific details of students' mathematical activities, connected what they viewed on videos to theoretical ideas from the literature, discussed pedagogical actions, and considered implications for student learning while reflecting on teaching practices. The poster illustrates these findings with examples of teachers' discourse.

Acknowledgements

The author gratefully acknowledges the support of the National Science Foundation (award DRL-0822204) for this work as part of the Video Mosaic Collaborative. The views expressed in this paper are those of the author and do not necessarily reflect those of the funding agency

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TEACHERS' BELIEFS REGARDING REFORM STANDARDS, EQUITY AND SELF-EFFICACY FOR TEACHING

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Proponents of equitable reform-based mathematics suggest that empowering the nation's diverse population of students to excel mathematically requires teachers to possess not only a strong mathematics knowledge-base, but also pedagogical beliefs that advance critical thinking and equity in mathematics education. This exploratory study examines twenty in-service K-3 teachers' reform-based pedagogical beliefs, beliefs about equity, and self-efficacy for teaching mathematics. The study reveals important alignments among their beliefs and describes the role and impact of an intensive professional development program designed to strengthen these vital belief structures that have been associated with teachers' reform-based instructional practice.

A growing body of research links teacher knowledge, opinions, beliefs, and proclivities concerning mathematics with teacher instructional practice (e.g., Beilock, Gunderson, Ramierez, & Levine, 2010; Jackson, 2011; Scrinzi, 2011). Researchers have called for studies to understand teachers' beliefs regarding reform-based mathematics and equitable instruction (Gutiérrez, 2002). Particularly, studies are needed to learn more about the pedagogical beliefs teachers of early childhood education possess and ways to support them in moving towards reform-based instructional practice (Scrinzi, 2011). The questions guiding this exploratory study are: To what extent do K-3 teachers espouse reform-based mathematics beliefs? How do K-3 teachers' beliefs about reform align with their beliefs about equity and relate to their self-efficacy for teaching?

A battery of pre-program surveys was administered online during the month prior to the program and post-program surveys were administered online one day after the completion of class instruction. Included were a demographic questionnaire, *The Standards Belief Instrument* (SBI) and the *Teachers' Sense of Efficacy Scale* (TSES). Paired-samples *t*-tests were conducted to determine changes in the teachers' pre- and post-program scores on the SBI and TSES, which were normally distributed. Pearson's Product-Moment Correlation Coefficient was used to assess the direction and strength of relationships among teachers' pre- and post-program SBI and pre- and post-program TSES scores to explore connections between teachers' reform-based beliefs and their level sense of self-efficacy before and after the program.

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THE PRACTICE OF ELEMENTARY SCHOOL TRAINEE TEACHERS AND THEIR CONSTRUCTION OF THE NOTION OF FRACTIONS

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Basic education in Mexico today is experiencing the effects of reform where teachers are the main focus of attention. A point of analysis concerning these actors of education is their knowledge. We have seen that this knowledge consists of a large network where there are components that should be included in the training process. Under this premise, we consider the necessity of studying, from a socio-epistemological viewpoint, how elementary school trainee teachers (PPF) construct the notion of fractions in their classroom practice and under this idea we ask, based on learning practice and knowledge acquired through their teacher training, what practices and meaning construction of fractions are PPFs prepared to develop? We followed a group of trainee teachers from a teacher college in Mexico who we will observe and interview with respect to their mathematical training in the subject of fractions. Since we consider teacher training as a process of holistic integration of pedagogical and disciplinary knowledge, we have considered contributions from: Shulman (1987) on his categorization of the pedagogical reasoning process. From Lezama and Mariscal (2008) we consider some factors that hinder the activity of mathematics teachers. Regarding Ball, Thames and Phelps (2008) we are interested in their categorization of mathematical knowledge for teaching. From Ball (1993), the dilemmas teachers face in teaching mathematics and from Askew (2008), the awareness aspects for encouraging new facets of the discipline in teachers. All this in the framework of the knowledge needed for teacher training. Regarding the disciplinary dimension, the contributions considered are from Lamon (2007) and Flores (2011), who make a clear distinction between the various meanings associated with fractions. Likewise, we will attempt to identify the deep understanding that is addressed, concerning fractions, in the observed group, taking into account the vision of Ma (2010).

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DEEPENING TEACHERS' KNOWLEDGE OF FUNCTIONS THROUGH ON-LINE PROFESSIONAL DEVELOPMENT

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Developing a deep understanding of the mathematics they teach their students is one way teachers can become more effective practitioners (Ball, Thames, & Phelps, 2008). However, limited time and financial resources can be obstacles to accessing such professional development opportunities for many teachers. The widespread availability of Internet access and a growing number of platforms available for inexpensively hosting online courses, therefore, represent a promising area for the development of opportunities for teachers' to deepen their mathematical content knowledge.

This poster presents the results of a study of 20 in-service mathematics teachers enrolled in an online course designed to deepen their understanding of functions through examination of middle-school-level curricular approaches to function and formal mathematical conceptions of function. Both the course and the study were designed based on the Lesh Translation Model (Lesh & Doerr, 2003), using it as a framework for describing conceptual understanding of functions. Within this perspective, conceptual understanding is measured by the flexibility with which an individual can translate between and among different representations of the concept.

The goals of this study were to (a) describe the modes of representation chosen by teachers as they participated in various course activities, and to (b) examine the ways in which the modes chosen changed during the course. Data used in the analysis included participants' work on pre- and post-tests, discussion posts, and work uploaded to the group as part of those discussions. Pre- and post-tests were specifically designed to elicit thinking about functions in different modes of representation within the Lesh Translation Model and were the primary source of data on how such thinking changed during the course. All data were coded for evidence of different modes of representational thinking and for translations between those representations. Themes that emerged are presented and analyzed. Implications for how conceptual understanding of functions may be deepened through online professional development courses are included as well.

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DEVELOPMENT OF TEACHERS' COMPUTATIONAL THINKING

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iMPaCT Project and Professional Development

The iMPaCT-Math team has developed three modules of activities that use programming on TI-83/84 calculators as a means to engage students to explore foundational concepts in Algebra 1 (Kranz et al, 2012). These modules allow students to make connections across multiple representations: (a) statements in a program, (b) computational process; (c) graphical output, and (d) underlying mathematical concepts. For example, by making connections between the rise and run of dot sequences and the concept of slope (Module 3), students recognize the invariance of the rise/run ratio as the defining characteristic of a linear function (Tabor, in press). Kranz et al. (2013) reported that students were highly engaged in activities involving Cartesian coordinates (Module 1) and enjoyed the creativity in activities such as *drawing a picture* using the *point-on* command; these experiences eliminate students' confusion between the x- and the y-coordinates.

Eleven Algebra 1 teachers attended a 3-day professional development program (PDP) in Summer 2012. They experienced the core lessons, identified and analyzed math and programming concepts, and discussed implementation issues. Teachers took a pre-and-post assessment with 11 conceptual items on Algebra and 8 analytical items involving programs. The paired t-tests showed a significant improvement on programming items ($p = 0.001$), but not mathematics items ($p = 0.296$). Teachers improved their reasoning with programs involving the while-end loop, keeping track of the variable-values, and determining the number of points plotted. For example, this program ":2→X :1→Y :While X≤10 :Pt-on(X,Y) :X+4→X :Y+2→Y :End" plots these three points: (2, 1), (6, 3), and (10, 5).

In this poster presentation, we shall outline key activities in the three modules, highlight the PDP training for these activities, and illustrate how the PDP could have contributed to the improvement in certain programming concepts.

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CLASSROOM DISCUSSION OBSERVATION INSTRUMENT

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We have been involved in an effort to research the effectiveness of a professional development program by evaluating the classroom activities of our participants, elementary school teachers. In this poster session, we will share the instrument that we developed and its connections to previous research. At the heart of our perspective was Stein, Grover and Hennigsen (1996)'s analysis framing students "doing mathematics" as the ultimate goal of mathematics instruction. We adopted their focus on meaningful explanations, multiple representations and multiple solution strategies. From Hufferd-Ackles, Fuson and Sherin (2004), we took up consideration for a student-centered environment. From Kazemi and Stipek (2001), we adopted attention to students' defending and justifying their answers. Finally, because we were committed to students' developing conceptual understanding, we looked for the degree to which they were making connections among ideas (Hiebert, Gallimore, Garnier, Givvin, Hollingsworth, Jacobs, et al., 2003).

Since we needed to observe in over 20 classrooms with limited personnel, we required a tool that could be used consistently across classrooms that was designed in advance of data collection. We explored the variety of instruments other researchers have used in these circumstances. We considered Stein et al.'s (1994) approach but their focus on task demands was inappropriate for our sample since our participants had limited choice regarding lesson content. The closest that we found to our circumstances was Akkus and Hand's (2011) Mathematics Reasoning Approach (MRA) used to observe and compare activities in three high school classrooms. This tool differs from many other methods that code individual utterances based on videotaped records of discussion. Our instrument, Classroom Discussion Observation Instrument (CDOI), is distinct from the MRA because our unit of analysis is the discussion as a whole, rather than the moves of the teacher. The CDOI has proven effective at capturing differences between discussions in our sample and includes five categories: opportunities for students to speak, equitable participation, multiple solution strategies and resources, conceptual explanations and connections between concepts.

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HIGH SCHOOL MATHEMATICS TEACHERS' REFLECTIONS ON VIDEOS OF THEIR TEACHING: EXPLORING TEACHER KNOWLEDGE AND INSTRUCTION

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Background

Mathematics teachers use a specialized knowledge of mathematics in the classroom—called mathematical knowledge for teaching (MKT; e.g., Ball, Thames, & Phelps, 2008). The purpose of this study was to understand the MKT used in high-quality mathematics teaching. Recognizing that MKT is situated in classroom practice (Rhoads & Weber, 2013), the focus was on teachers' reflections on videos of their teaching.

Methodology

High school mathematics teachers with advanced training or teaching recognition were recruited, and mathematics teachers from the same schools were also invited to participate. As of December 2012, participants were five teachers from two schools in New Jersey. Two teachers had advanced training: One held National Board Certification in adolescent and young adulthood mathematics (National Board for Professional Teaching Standards, 2010), and one held a doctorate in mathematics education. Each teacher was observed and video-recorded in one class period for three consecutive days. Before each lesson, the teacher was interviewed about lesson planning and anticipated events. After the three observations, the teacher participated in a post-interview, watching approximately six 5-minute video clips of their teaching. Teachers were asked to reflect on their thinking and explain their classroom decision-making.

To identify high-quality teaching, videos were coded for mathematical quality of instruction (MQI, Hill et al., 2008). Teachers with advanced training had higher MQI, on average, than their colleagues. Using grounded theory (Strauss & Corbin, 1990), interviews were coded according to the nature and depth of mathematics that was discussed and the factors that teachers considered when determining what content to develop with students and how.

Preliminary Findings

Data analysis is ongoing. Teachers with advanced training illustrated more sophisticated knowledge of content and more complex considerations of enacting content in the classroom than their colleagues. Discussion about possibilities for further analysis will be welcome.

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PRE-SERVICE TEACHERS' REFLECTIONS AND EXPERIENCES WITH PROOF AND ARGUING AS STUDENTS AND AS TEACHERS

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Keywords: Reasoning and Proof, Classroom Discourse, High School Education

This poster presents preliminary findings from a study of pre-service secondary mathematics teachers' past and present experiences, and future plans with proof and argumentation during their student teaching experience. The value of engaging in proof and argumentation is clear from the literature (e.g., Stylianides, 2009; Wood, 1999). Unfortunately, little is known about the teaching of proof in schools (Stylianides, 2007; Stylianou, Blanton, & Knuth, 2011).

The purpose of this multiple case study was to investigate four pre-service secondary mathematics prior experiences with proving and argumentation in mathematics, and their plans for incorporating proving and arguing in their teaching (Yin, 2003). I conducted semi-structured interviews with participants during their student teaching experience to explore their past and current experiences, as well as plans for possible future experiences with proof and argumentation. Two frameworks- Knuth and Peressini's (2001) and Stylianides' (2010) Reasoning-and-Proving framework- are the theoretical foundation for this research study. One of the three guiding research questions is: How do they perceive proof and argumentation in mathematics?

The process of analyzing the data followed Corbin and Strauss' (2008) grounded theory. Interview data suggest a clear distinction between the student teachers pre-college and college mathematical experiences with proof. There are striking similarities in the student teachers' pre-college experiences with respect to proof and argumentation; however, there are significant differences between student teachers in how they engaged and participated in proving and arguing activities in college. The descriptions given by the student teachers of their pre-college experiences suggest the primary discourse in their mathematical discourse was univocal, while college experiences had at least more opportunities to be dialogical.

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"WHY DO I HAVE TO LEARN THIS?" DEVELOPING PROSPECTIVE SECONDARY MATHEMATICS TEACHERS' VIEW OF MATHEMATICS

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In college mathematics courses for prospective secondary mathematics teachers (PSTs), the way we help students understand *why* they are learning what they are learning takes on a particular significance because PSTs need to learn mathematical content and practices, as well as how to make explicit connections between high school and college mathematics to be prepared to teach (CBMS, 2012, p. 54). Intended to address this dual purpose, capstone mathematics courses for PSTs are structured to "focus on high school mathematics from an advanced viewpoint" (CBMS, p. 18). Instructors of these courses, however, may experience a tension between maintaining the mathematical rigor and incorporating explicit connections to high school mathematics because PSTs may view explorations of higher-level topics as less relevant based on their high school experiences. How can mathematics education research inform instructional practice and motivate PSTs who are learning advanced mathematical perspectives?

Mathematics education researchers agree that effective teachers need to draw upon a deep and broad knowledge base, referred to as *mathematical knowledge for teaching* (MKT) (e.g., Ball, Thames, & Phelps, 2008). The CBMS supports this notion of MKT, stating that PSTs "need to understand the fundamental principles that underlie school mathematics, so that they can teach it to . . . students as a coherent, reasoned activity" (p. 17). Since capstone courses are designed around this idea, I examined: (1) How can MKT terms be made more explicit in a PSTs' capstone mathematics course? (2) How does the explicit use of MKT terms affect PSTs' attitudes about learning mathematics?

I will present the design and outcomes of a capstone mathematics courses for PSTs (taught by the author) where MKT terms were explicitly discussed. For question one, I analyze the nature of class discussions where MKT was introduced and how PSTs incorporated MKT terms on assignments throughout the semester. For question two, I used a constant comparative method (Glaser & Strauss, 1967) to analyze PSTs' responses on course evaluations and course papers, coding for how they articulated whether learning course content was useful. Without being prompted, PSTs made explicit statements about how they believed the design of the course increased aspects of their MKT. They talked positively about the utility of learning a concept or proof even when they considered it unlikely they would directly teach it to high school students. For example, referring to a proof for the Fundamental Theorem of Algebra, a PST said it was "more specialized content knowledge because [high school] students do not necessarily need to be taught it . . . ; however, a teacher should be aware of this" (Journal Entry 2). Contrasting these reactions with course evaluations from previous semesters teaching this type of course suggests that features of this course design and the explicit use of MKT terms could help PSTs make connections between high school and college mathematics content and practices.

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APPROXIMATING PRACTICE: DOES THE MEDIUM MATTER? A COMPARISON OF SECONDARY MATHEMATICS PRESERVICE TEACHERS' ANTICIPATED LESSONS

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The move towards practice based teacher education requires a move towards activities that afford preservice teachers (PSTs) opportunities to "approximate practice" (Grossman et al., 2009). One such activity that scaffolds PSTs approximating practice as a teacher might, engages PSTs in anticipating a classroom interaction around particular mathematical content (Zazkis et al., 2013). Around this core goal of anticipating a classroom interaction, the activity can be instantiated in various ways. In particular, PSTs could be asked to either script a classroom interaction in text-form or create a comic-based representation, using the *Depict* (Herbst, Chazan, Chen, Chieu, & Weiss, 2011) tool, to depict the interaction. These two mediums (text-form and comic-based) have different semiotic resources that suggest the possibility that representations created will differ. In the comic-based medium, PSTs have additional semiotic resources to draw from to attend to the *multimodality*—the many modes of communication—in classroom interaction such as inscriptions, facial expressions, and gestures. Additionally, the use of images allows PSTs to notice that there are many students and can cue PSTs to attend to the *multivocality*—the diverse, and sometimes divergent, individual ideas students may express—of teaching (Herbst, et al., 2011, p. 94). These additional semiotic resources are expected to produce differences in the content of the classroom interactions created in *Depict* versus in the text-format, which are the focus of this study. This study sought to answer: What are the characteristics of anticipated lessons created in the two mediums that can be contrasted? How do these identified characteristics differ between the two mediums?

To answer these, we did a secondary data analysis of student work collected by the instructor of a secondary mathematics methods course at a large Mid-Western University in which 16 PSTs were enrolled. In the course, the PSTs had completed homework in text-format (that included prompts to "script" interactions) during the first half of the semester and using the *Depict* tool during the second half. This homework was analyzed with qualitative and quantitative methods, focusing on the nature of the teacher and students dialogue (the triadic dialogue) as well as their actions. The analysis uncovered differences between the mediums. In particular, the nature of the students' interactions differed in interesting ways, with PSTs anticipating more student involvement in their comic-based representation. Additionally, the comic-based representations included more mathematical representations. These results provide impetus for considering the medium in which approximations of practice are enacted.

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A NEW DEVICE IN TEACHERS TRAINING WHICH AIMS AT WORKING PRECISE COMPONENTS OF TEACHING PRACTICE

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The steady team-work we have done with teachers and prospective teachers over the years witnesses what research says about students at their first experiences of teaching. Indeed, referring to those researches, which analyze teaching practice, we can assess that they mainly focus on surviving in their new environment and its rules. Therefore, energy and work on components related to subject matter are then at their lowest! Considering that all the components (social, environment, mathematic, curricular,…) equally contribute to good conditions of teaching and learning, teacher trainers must work for this equilibrium in practice. The main goal faced by didactic of mathematic trainers is then that mathematic components, which they have worked on with their students prior to the teaching immersion, survive and contribute to ease the pressure on other components. This is the aim of the activity "exposés oraux".

This new device was put on in the nineties in the course *Didactic I*. It is the first of a sequence of five didactic of mathematic courses and is immediately followed by the first internship. Consequently, mathematic subjects covered in the course are mainly those taught in the first and second year of high school. The same subjects are also in large part those that students will be asked to teach. All the work involved on "exposés oraux" in *Didactic I* aims at developing and acquiring skills, abilities, habits and even automatisms, which will naturally be called upon when teaching the same subjects in the classroom. Each of these, recorded on video, features didactic principles and target elements of the conceptual analysis on the studied subject.

Didactical principles are drawn from a well defined list built as the course goes on: use good adequately natural language to develop reasoning, have recourse to visual representation, promote reasoning, provoke conflict,… Conceptual analyses are pedagogical documents written for or by students on various mathematical topics in order to summarize findings of research regarding their teaching and learning. They bring light on: reasoning, abilities to develop, conceptions, errors, difficulties…. to be considered in the learning of the subject.

In order to measure the impact of the activity "exposés oraux", groups of students were filmed while in their working sessions and they filled a questionnaire two months later. It comes to evidence that students have been brought to build for themselves the didactical principles, topics findings and the related practice encompassed in the "exposés oraux".

This new device in teachers training falls well within the analytical perspectives of the works of Robert and Rogalski (2002) and Lenoir (2009). It comes out through first analysis that five out of eleven components of the teacher's practice raised by Lenoir (2009) are significantly worked upon: historic (constructivism and importance of language), curricular (giving sense), epistemic (a relationship with knowledge based on what we know about our students), didactic (relationship with knowledge), and socio-affective (relationship with the professional identity).

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FEEDBACK AND FORTH: WHAT CAN NOVICES LEARN ABOUT TEACHING MATHEMATICS FROM FEEDBACK CONFERENCES?

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Feedback given to novices, also called teaching interns, most often aims to improve the intern's teaching practice (Brinko, 1993, and Coe, 1998). While feedback can be given in multiple ways, this research examines a unique feedback form called feedback-and-forth. In feedback-and-forth, teacher educators, who in this research are called course instructors, and interns sit down together and have a conversation about the intern's teaching performance (Le & Vásquez, 2011; Soslau, 2012). Feedback-and-forth conferences offer multiple affordances for instructors and interns. First, feedback given verbally, rather than in writing, allows the instructor to be more congenial and convey information using body language, tone of voice, and eye contact. Giving feedback in a shared interaction also allows the instructor to read the intern's body language and clarify, restate, or reinforce particular aspects of the feedback being given. Second, the intern has a chance to react to the feedback given by the instructor by contributing her thinking and asking for clarification. Last, feedback conferences are live, and are conducted within 30 minutes of the intern's teaching episode so memories are recent and fresh in the mind.

In this exploratory study the researcher investigates the content and purpose of feedback given to teaching interns during an elementary school based mathematics methods course. The methods course is taught in conjunction with an after school program for elementary students in grades 3-5 who struggle with mathematics fluency. The stated purpose of feedback from the course was twofold: (a) for instructors to give intern's feedback that is useful and timely, and (b) for the feedback-giving process to be manageable (e.g., multiple instances of feedback given each week for each of the 28 interns enrolled in the course) for instructors. The researcher hypothesized that the feedback given would inform and improve intern teaching in two important ways: (a) raise the intern's awareness of teaching errors and facets of practice that can be addressed through continued work, (b) support the intern in making changes to her own teaching practice so that this practice promotes student learning and achievement.

The data consisted of audio recordings of feedback conferences and post-conference interviews with individual interns. Conferences were coded for instances of specific feedback (tied to the intern's teaching episode observed by the instructor) and more general feedback about the interns practice (including what the intern said, her tone and manner, and the presentation and explanation of the mathematics). Interviews were segmented into individual pieces of feedback and coded for usefulness (useful, not useful, unknown) and actionability (will act on, won't act on, unknown). The preliminary findings suggest that the two hypothesized features, raising intern awareness and support in making positive changes, were present in the feedback given. An unexpected finding indicates that feedback conferences included multiple instances of positive feedback given by the instructor. Equally surprising was the number of instances intern's mentioned this positive feedback as useful (deemed important and/or a source of learning) and actionable (will impact intern's future teaching practice).

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INVESTIGATING PROVIDERS AND RECIPIENTS PERCEPTIONS OF EARLY CHILDHOOD PROFESSIONAL DEVELOPMENT IN MATHEMATICS

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High-quality mathematics professional development for K-12 teachers has been linked to positive effects on student achievement and dispositions towards mathematics (Blank & de la Alas, 2009). However, there are few studies that examine the impact of professional development in mathematics education for teachers of students aged birth to five in educational settings prior to formal kindergarten. But of these few studies, researchers have noted a positive effect on the mathematical practices of educators (Björklund, 2012) and children's mathematical abilities (Sophian, 2004).

Upon analyzing the results from a larger study examining the nature of professional development for early childhood educators (birth to five) in a southeastern region in the United States, the researchers noted a lack of emphasis and concern from participants in regards to professional development related to mathematics. To examine this phenomenon more in depth, the researchers posed the following question: How and to what extent are preservice and inservice early childhood educators being prepared to work with young children aged birth to five to develop early mathematics skills and processes in one southeastern state? The research team extracted data from a multi-phase mixed methods study (Creswell & Plano Clark, 2011) and obtained additional data from a follow up survey to answer this central research question.

As determined from the data, early childhood preservice educators in this sample are not being prepared to teach students the content and processes of the CCSSM. Specifically, of the 44 institutions analyzed, fifteen met all five of the standards categories. Further, when recipients were asked to list the content covered in professional development sessions, a mere 33 of the 1127 (3.2%) survey participants mentioned mathematics; additionally, only 16 (1.8%) stated they would like mathematics to be offered as a professional development session. It appears as though professional development in mathematics is not a priority or a concern for early childhood educators.

Even in the event that professional development sessions in mathematics were offered, the majority of the sessions were short hourly sessions with no follow up, thus making it difficult to determine if providers' expectations are being implemented by recipients as part of their mathematical practices. Additionally, topics in specific mathematical content strands, such as number and operations and measurement, are rarely being offered to those working with children aged birth to five (17%) and acquiring knowledge in mathematical content was not expressed by providers as an expectation or a goal for their sessions. The lack of developing mathematical content knowledge of those working with children age birth to five could potentially be detrimental.

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TEACHERS' NOTICING OF STUDENT THINKING: THE CASE OF MATHEMATICAL JUSTIFICATION

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Teachers' *professional noticing of children's mathematical thinking* (Jacobs, Lamb & Philipp, 2010) is hypothesized as an important skill needed for the enactment of sophisticated pedagogies such as supporting student participation in justification and argumentation. We examined teachers' *professional noticing of children's mathematical thinking* as teachers reviewed videos of their own students working on justification tasks. Our aim was to better understand differences in how teachers "attend to noteworthy aspects of complex situations" (Jacobs et al., 2010, p. 172) and the relationship between teachers' professional noticing of children's mathematical thinking and students' mathematical activity. We pursued the following research questions: 1) Do teachers who consistently support student justification differ in their professional noticing of students' mathematical thinking? 2) Are differences in teachers' professional noticing related to differences in students' justification activity in their classrooms?

Data were collected as part of a larger NSF-funded project, JAGUAR (*Justification and Argumentation Grows Understanding of Algebraic Reasoning*), which focused on the nature of justification in middle school classrooms. The main data sources were two interviews, conducted at the end of each project year (76 to 144 minutes in duration), with each of 12 teachers. The interviews elicited teachers' professional noticing of students' mathematical thinking including their pedagogical reasoning with respect to organizing and supporting student participation in justification. Typical questions posed included: *What do you notice about students' thinking? What do you see yourself trying to do? What has the student justified, if anything?*

For this inquiry, we drew significantly on Jacobs et al.'s (2010) *professional noticing of children's mathematical thinking* and the work of Sherin & van Es (2009) on teacher noticing. We examined two interrelated ideas: *how teachers attend to students' thinking* and *teachers' pedagogical reasoning.* Using qualitative techniques, the following four categories were used to capture how teachers attended to students' thinking*: Does not describe, Restates/Interprets, Investigates* and *Synthesizes*. Three categories were used to describe how teachers' pedagogical reasoning accounted for students' thinking: *Does not describe, Generic* and *Tightly Coupled*.

Initial findings demonstrated that teachers, on the main, fell into one descriptor for each category and, while some variation in responses was observed, teachers were fairly consistent in the nature of their responses. We also found that teachers whose responses were categorized as *Investigates/ Synthesizes* and *Tightly Coupled* had more extensive justification activity in their classrooms. This finding lends support to the connection between teachers' noticing and their enactment of sophisticated pedagogies, such as those that engage students in justification.

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MATHEMATICAL MISCONCEPTIONS OF A DIFFERENT KIND: WOMEN PRESERVICE TEACHERS' WORKING THEORIES OF MATHEMATICS TEACHING

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Background

Mathematical anxiety and confidence in women who are entering the elementary teaching field is a subject that has captured the interest of mathematics teacher educators. Previous research has revealed that women who pursue elementary teaching careers are often individuals who themselves have confronted anxiety and low confidence in mathematics during their own K-12 experiences (Brady & Bowd, 2006; McGlynn-Stewart, 2010; Sloan, 2010).

Prior studies in mathematics education reveal that individuals' experiences with mathematics shape how they think about doing and teaching mathematics (Ball, 1988; Rodríguez & Kitchen, 2005). Long before preservice teachers step foot into their teacher education program, their student experiences have shaped how they view mathematics as well as how they perceive their own mathematics abilities (Ball, 1988). "In short, prospective teachers do not arrive at formal teacher education "empty-headed." Ball, 1988, p.40). Instead, they have already begun to develop a plan or a program of action (Kounin, 2009) of how teachers should teach mathematics. These teaching ideas are derived primarily from their personal experiences as mathematics students (Ball, 1988). Through the use of narratives, teacher educators can gain access to a better understanding of the sense making that preservice teachers have about what qualities and characteristics are important for a mathematics teacher to possess.

Methods

This study examined eighty-two narratives written by women elementary preservice teachers that revolved around the theme of anxiety and low confidence in mathematics.

Findings

Finding 1: Isolated "I" Imperatives

One major finding revealed in this study was that participants focused their understanding of what it means to be a mathematics teacher by isolating what they believed teachers should not do while teaching mathematics. These behaviors came from places of specific personal mathematics burdens the preservice teachers wanted to resolve or "fix."

Finding 2: Fragmented Frameworks

Another major finding discovered in this study focused on preservice teachers' understanding of what it means to be a mathematics teacher by stressing broader ideas or concepts that teachers should employ when teaching mathematics. These comments included such statements as teachers need to implement policies in the mathematics classroom that motivate their students, teachers should help their students to find the resources they need to learn new concepts and to work with peers to promote all students' learning, and teachers must be passionate about teaching mathematics in order to create a greater impact on students' mathematical learning.

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MARKERS OF ENGAGEMENT IN MATHEMATICS CLASSROOMS: PROSPECTIVE TEACHERS' VIEWS

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Engagement in mathematics classrooms may be considered on two inter-related planes—engagement with the classroom activities and engagement with mathematics. Engagement with mathematics is the cognitive engagement that maintains or extends one's mathematical understanding. Engagement with the classroom activities focuses on affective and behavioral patterns of participation that include patterns of participation in classroom discussions, hands on activities, and other tools for mathematical thinking.

Engaging mathematics classrooms create a context in which students are able to construct mathematical meanings, apply mathematical ideas, develop positive dispositions towards mathematics and consequently have access to higher education and better jobs. However, it is a challenge for teachers to plan and enact instructional practices (using challenging and compelling instructional tasks, orchestrating discussion purposefully, and supporting making and justification of conjectures, e.t. c) that foster engagement. Furthermore, notions of engagement that do not pay attention to mathematics being learned are not uncommon. For example, some may believe that whenever students look active, they are engaged with the mathematics. Hence, lack of engagement in mathematics classrooms is a persistent problem and a central theme for school reform internationally (Marks, 2000).Breaking the persistent lack of engagement in mathematics classrooms significantly lies on teacher education programs—to prepare teachers who can foster and monitor engagement. As in this study, exploring prospective teachers' notions of engagement in mathematics classrooms presents an opportunity to examine the strengths and weaknesses of those notions and plan their educational experiences accordingly.

Forty-nine prospective teachers participated in the study. They free listed markers of engagement in mathematics classrooms. Some of the participants (n =17) watched videos of two classrooms and discussed markers of engagement, and lack of engagement. Content analysis of the participants' free lists showed that 73.5% of the responses focused on use of manipulatives and/or group work. 32.6% of participants mentioned both manipulatives and group work, 38.5% mentioned manipulatives only, and 34.5% mentioned group work only. Additionally, 14/17 students who analyzed classroom videos explained that off-task behavior (e.g., side talk) was a marker of lack of engagement. Few participants (n=19) free listed characteristics of tasks. Of the 19 who mentioned tasks, 16 described realistic and problem based tasks while the rest described tasks that required students to create something new. Only 4 out of the 49 participants mentioned making and/or justifying conjectures. Interestingly, 10 of the 17 participants of classroom video analyses considered giving correct responses as a marker of engagement.

From these results, prospective teachers' notions of engagement are mostly based on participation and not engagement with mathematics. Teacher education should aim at building an understanding of the complexity of engagement in mathematics classrooms.

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TEACHERS' LAUNCH OF A DISCOURSE-RICH MATHEMATICS LESSON USING THE THINK ALOUD STRATEGY

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Mathematics learning has suffered from a lack of meaningful modeling practices in launching a problem solving lesson because teachers often explicitly model a problem solving situation, work, and solution before releasing students to solve similar problems. As a consequence what is rewarded as doing mathematics during the student work time is adherence to the steps and procedures modeled by the teacher. This type of learning exchange falls drastically short of demands for students to, "Construct viable arguments and critique the reasoning of others" (CCSS-M, 2009). In an effort to promote discourse-rich mathematics lessons that prompt students to verbally "construct viable arguments and critique the reasoning of others," we have borrowed the strategy of Think Aloud from literacy to purposefully launch a mathematics lesson.

The purpose of this poster is to describe a professional learning task (PLT) using Think Aloud and how it informed a larger professional development research and development project. The following two research questions were addressed: How did teachers implement Think Aloud in their classrooms? What emerged as successes and challenges of implementing the strategy?

A group of 26 second-grade teachers participated in a forty hour yearlong professional development in Project XXX. Around the middle of the year teachers engaged in a Professional Learning Task (PLT) with the Think Aloud strategy. Our choice of operating at the strategy level is analogue to Boerst et al.'s (2011) notion of decomposition of practice. That is, one can break down the work of teaching into learnable parts for the purpose of teaching and learning. This PLT was couched in the launch phase of a lesson as we wanted teachers to consider ways to model thinking, before the students engage in their own discourse-rich tasks. The purpose of Think Aloud is to make thinking explicit to others and show students how the teacher approaches problems and makes sense of them.

Teachers' reflections described Think Aloud as a way to purposefully launch a discourse-rich lesson. Evidence of successes and challenges with implementing this strategy will be shared along with examples of Think Alouds that were used to introduce teachers to the strategy in the professional development project.

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LESSON STUDY WITH PRESERVICE TEACHERS: THE INCLUSION OF PROFESSIONAL NOTICING

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One method for developing pedagogical content knowledge is through cultivating the ability to professionally notice, which involves attending to, interpreting, and responding to students based on their mathematical thinking (Jacobs, Lamb, & Philipp, 2010). Preservice teachers need explicit opportunities to learn to professionally notice, and teacher education programs should provide these opportunities (van Es, 2011). The following research question was explored: How do preservice teachers professionally notice through participation in lesson study?

Data from this study come from a larger project termed Iterative Model Building that incorporates an iterative cycle for a teacher preparation program in which preservice elementary teachers engage in weekly formative assessment interviews, model building, whole-class lesson planning/teaching, and lesson study. Participants included 24 preservice teachers in one field experience course in a teacher preparation program. Data for this study consist of video recordings of four different groups of preservice teachers each engaging in six lesson study meetings. Data were analyzed using a noticing framework adapted from van Es (2011).

The overall combined percentages from the four different groups of preservice teachers for each of four levels of noticing were determined, with Level 1 (baseline) involving the least in-depth noticing and Level 4 (extended) involving the most advanced types of noticing. A plurality of the statements (70.83%) made by preservice teachers for each meeting were coded at a Level 1, which signifies that most of the preservice teachers' statements were general, descriptive impressions of the enacted lesson. Statements coded at Level 2 were common, yielding 24.07%. Fewer preservice teacher statements were coded at Level 3 (4.57%) or Level 4 (0.52%).

Analysis of data across the six weeks of lesson study indicate that one code at Level 2 had a correlation coefficient that was statistically significant from zero. However, the general trends in the data by level show that there was moderate decrease in statements made at the Level 1 data. Similarly, there were moderate increases in the statements made at Levels 2 and 3. The paucity of responses at Level 4 did not indicate any relative change at this level. Our data supports the notion that such weekly cycles may impact teacher noticing. This structure provided a context for preservice teachers to focus on student thinking, analyze evidence from a lesson collectively, and make evidence-based decisions for future teaching (Murata & Pothen, 2011).

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This material is based upon work supported by the National Science Foundation under Grant No. 0732143. Any opinions, findings, and conclusions or recommendations expressed in this materials are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

TEACHING TO TEACH WITHOUT HAVING TAUGHT: MATHEMATICS TEACHER EDUCATORS PREPARING ELEMENTARY TEACHERS OF MATHEMATICS

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Keywords: Teacher Education-Preservice, Elementary School Education

Purpose of the Study

In this poster, we share results from a study in which we seek to understand the knowledge, dispositions, and experiences of Mathematics Teacher Educators (MTEs) that may contribute to their ability to successfully prepare elementary teachers. Of specific interest to us are MTEs who do not have fulltime elementary teaching experience. Thus, the first of many goals of this work is to understand how MTEs who contribute to the preparation of elementary teachers make sense of their own lack of experience teaching mathematics to elementary-aged children. Our study is guided by two research questions:

1. In lieu of elementary teaching experience, what type of professional learning experiences do new MTEs have or believe contribute to their preparation for working with preservice elementary teachers?
2. In addition to particular experiences, what additional factors do new MTEs believe contribute to a person's preparation for working with preservice elementary teachers?

By learning about the experiences of novice MTEs who are responsible for the mathematical and pedagogical preparation of elementary teachers, we intend to contribute to efforts to improve our field's ability to prepare MTEs and, ultimately, highly-qualified elementary teachers of mathematics.

Methods

We began our data collection by constructing an online survey based on the work of Reys, Cox, Dingman, and Newton (2009) and a literature review of challenges faced by beginning faculty. The survey was sent to participants of an NSF-funded mentorship program for recent graduates of mathematics education doctoral programs, all within their first three years of post-doctoral work as faculty members. Sixty-nine of the 81 program participants responded to the survey. Analysis of survey responses revealed 13 participants who identified "elementary" as the primary focus of their current teacher preparation work, yet who themselves had 1) no elementary teaching certification or licensure, 2) no fulltime elementary teaching experience, and 3) a doctoral focus outside of mathematics at the elementary-school level. From this group of 13, eight were willing and able to participate in semi-structured focus group interviews. We conducted these interviews by distance (online) during the summer of 2012, asking the participants open-ended questions about their professional experiences working with elementary-aged children and prospective or practicing elementary teachers of mathematics. It is from analysis of these interviews that themes have been identified; findings will be shared in our poster.

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THE EFFECTS OF LEARNING TRAJECTORY BASED PROFESSIONAL DEVELOPMENT ON TEACHERS' NOTICING

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Keywords: Learning Trajectories (or Progressions), Teacher Education-Inservice/Professional Development, Teacher Knowledge

It has been established in the mathematics education community that identifying and understanding children's mathematical thinking is a key factor for teachers' instructional decision making. Researchers have hypothesized that knowledge of hypothetical learning trajectories could be a way for teachers to develop a deep and connected knowledge base of student thinking across grade levels (Daro, Mosher, & Corocoran 2011; Sarama & Clements, 2009) to inform instructional decisions. Initial studies investigating teachers' uses of learning trajectories indicated changes in both teachers' mathematical and pedagogical content knowledge over time, but more research is needed to investigate the impact of learning trajectory based professional development on their ability to interpret and respond to students' thinking.

The data in this presentation comes from a two-year Math and Science Partnership (MSP) with 24 elementary teachers in an urban midwestern school district. The purpose of the professional development was to introduce teachers to learning trajectories on measurement (Sarama & Clements, 2009) as a way to formatively assess students' thinking. Following the first year of professional development, the participant teachers and a control group of teachers took part in a noticing assessment activity where they watched a video of a student attempting a measurement task. They were then asked to reflect on the student's strategies and understanding as well as propose an appropriate instructional response. Using Jacobs and her colleagues' framework (Jacobs, Lamb, and Philipp, 2010) the researchers analyzed how the teachers attended to, interpreted, and responded to student thinking based on the student's response in the video.

From this assessment activity, the findings indicate that the learning trajectory based professional development acted as a lens or a filter for teachers to describe student actions and student understandings within the video setting. The participating teachers were able to discuss the student's measurement strategies and thinking with greater depth and detail compared to the control group. Secondly, the returning teachers were better able to propose and justify student-centered tasks to follow the response from the video. The findings that emerged provide an initial glimpse into how learning trajectory based professional development may effect how teachers think about and respond to individual student thinking.

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INVESTIGATING MATHEMATICS TEACHER EDUCATORS' PRACTICES IN THE CONTEXT OF COMMON CORE STATE STANDARDS FOR MATHEMATICS

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Historically, there has been substantial variation in the design and content of courses and experiences intended to prepare teachers to teach mathematics (Taylor & Ronau, 2006). The recently-adopted *Common Core State Standards for Mathematics* (*CCSSM*) (NGA & CCSSO, 2010) has prompted a national-level conversation about mathematics standards, providing an important opportunity for convergence in key components of the mathematics teacher education curricula as mathematics teacher educators (MTEs) incorporate *CCSSM* into their courses (e.g. Heck, Weiss, & Pasley, 2011; Krupa, 2011). However, studies of prior standards movements suggest that the relationship between policy and implementation is complex and not always what is intended by policy (e.g., Hill, 2001; Spillane, 2006). We seek to capitalize on this unique moment in mathematics teacher education by exploring: (a) How MTEs have changed their teacher preparation coursework to connect to *CCSSM* and (b) MTEs' rationales for these changes and for the particular *CCSSM*-related activities they are implementing in their courses.

This study is part of a larger project examining *CCSSM* in mathematics teacher preparation. This poster presents findings from our initial data collection including a survey of MTEs. While our data collection and analysis are still in preliminary stages, our initial analysis suggests that MTEs are modifying their courses and teacher preparation programs in response to *CCSSM*. However, most of these modifications are minor adjustments such as explicit mention of *CCSSM*, reference to *CCSSM* in lesson plans, or changes in course content to match *CCSSM* content. While MTEs are discussing *CCSSM* in conference presentations and publications, our initial findings suggest that *CCSSM* are not yet provoking the intended national-level conversation about mathematics teacher education curricula.

Acknowledgment

This material is based upon work supported by the National Science Foundation under Grant No. ESI-0333879.

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MATCH BETWEEN A PRE-SERVICE ELEMENTARY MATHEMATICS PEDAGOGY MODULE AND THE CONTENT OF ELEMENTARY SCHOOL MATHEMATICS

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Research has revealed that there can be a mismatch between what is taught to undergraduate elementary pre-service teachers during their teacher preparation and what they subsequently teach in elementary school mathematics (Greenberg & Walsh, 2008; Sparrow & Frid, 2001). Moreover, there is a need to study the curriculum for teacher preparation because compared with the school curriculum "much less has been written on the professional curriculum for teacher preparation" (Stuart et al, 2000, 493).

To investigate the match between the content of a mathematics pedagogy module (MPM) taught at one major university in Saudi Arabia (KSA) and the content of elementary school mathematics curriculum in KSA, an observation method was developed to investigate how much emphasis during the MPM sessions was given to content topics of the elementary school mathematics curriculum.

The MPM is one of the modules provided for undergraduate pre-service mathematics teachers during the fourth year of their university level preparation program. Each of the nineteen weekly sessions of the MPM is 120 minutes long. Each session covers mathematical teaching skills and techniques in order to enable pre-service elementary mathematics teachers to teach the mathematics curriculum in elementary schools.

Recorded observations were made of all the taught sessions of the MPM. In terms of the four mathematical areas of elementary mathematics school curriculum, that is number, geometry, algebra, and data, this took place during nine of the MPM sessions (out of 19). Data from the observations showed that during the MPM there was heavily emphasis and high match on topics related to elementary school *number* (883 minutes was allocated our of 1920 minutes), a moderate emphasis was on some topics related to elementary school geometry (422 minutes allocated of 1920 minutes). However, there was little emphasis related to topics in elementary school *algebr*a (162 minutes was allocated out of 1920 minutes) and elementary school *data* (46 minutes was allocated out of 1920 minutes). Further research is needed on how to decide how much emphasis to provide on the various school mathematics topics during teacher preparation.

Acknowledgement

This research was sponsored by the KSA Ministry of Higher Education. The paper reflects solely the views of the authors.

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TEACHERS' PERCEPTIONS OF EFFECTIVE PROFESSIONAL DEVELOPMENT: A CASE STUDY

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A shortcoming of many professional development programs is that they don't adequately meet the needs of teachers (Guskey, 1986; 2002). Professional development programs must address the needs of teachers in order to be effective (Darling-Hammond, 2010a; 2010b). This study explores the following research questions:

1. What motivates teacher attendance at professional development sessions?
2. What types of knowledge are teachers seeking and which kinds of knowledge do they value?
3. According to teachers, how does the context of work influence teacher professional development?

The data came from a Mathematics Partnership (MP) project, a collaborative endeavor among five school corporations in one rural midwestern county. At least one teacher from K-6 grade level and the math coach participated in this study. The data included here relied on teacher interviews as the main source of data collection. Additionally, follow up interviews and observations during monthly professional development sessions and planning meetings were conducted. Strauss and Corbin's (1990) open coding technique was used to examine the data collected.

Renyi's (1996) broad survey cited time, teachers' role in developing professional development opportunities, community organizations, and funding as issues that impact the quality of professional development. Findings from this study align with those of that report. However, this study also revealed that the context of work had a major influence on professional development. In line with previous studies (Pintrich and Schunk, 2002), the data underscores the bidirectional interaction between motivation and professional development as well as between work context and professional development.

Figure 1: Professional Development Schematic

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PEDAGOGICAL CONTENT KNOWLEDGE OF THREE TEACHER EDUCATORS

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Keywords: Post-secondary Education, Teacher Knowledge

Although extensive scholarly energy has been devoted to exploring the structure and content of pedagogical content knowledge of classroom teachers (Hill, Ball, & Schilling, 2008), little has been done to examine the epistemic structure of knowledge of those responsible for teaching teachers. Additionally, from a pragmatic viewpoint, little information exists regarding how teacher educators determine and reinforce essential components of mathematics teacher education programs (Chauvot, 2008). As a result, a shared knowledge base regarding mathematics teacher educators is yet to be developed. To contribute to the development of this knowledge base, the following questions were addressed: 1) What knowledge domains do mathematics teacher educators draw from and use when providing content specific pedagogical experiences for teachers?; 2) How do these knowledge domains influence the activities of mathematics teacher educators as they design and implement pedagogical experiences for teachers? This presentation will report results from the study.

Data was collected using in-depth interviews, classroom and professional development session observations, and existing documents prepared and used by the participants when organizing activities for preservice and inservice teachers. Participants included three faculty members in mathematics education who teach in teacher preparation programs and lead professional development sessions.

Data for each participant was analyzed at three levels. At the first level of analysis, abstracts from the past five years of three journals including 1) *Journal for Research in Mathematics Education* (JRME), 2) *Journal of Mathematics Teacher Education* (JMTE), and 3) *Teaching and Teacher Education* (TTE) were reviewed with the intent of developing a "big picture" of the teacher educators practice based on prominent themes obtained from the abstracts. For the second level of analysis, Shulman's (1986) categories of *content knowledge* for teaching and the *practical knowledge* of teachers as described by Elbaz (1983) provided the initial framework for coding of data. This level provided a more detailed analysis of the focus and influences with regard to the knowledge base and decisions of participants. At the third level of analysis, a cross examination of all cases was conducted to generate a comprehensive list of knowledge domains expressed to be used, or used in practice by the participants. Preliminary results indicate that two prominent forces on teacher educators' decision making appear to be one's own theoretical (research) interests along with personal experiences as classroom teachers and teacher educators.

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Research Reports

Brief Research Reports

Poster Presentations

RETHINKING MATHEMATICS INSTRUCTION: AN ANALYSIS OF RELATIONAL INTERACTIONS AND MATHEMATICS ACHIEVEMENT IN ELEMENTARY CLASSROOMS

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The research examined how teacher-student relational interactions affect students' mathematics achievement and learning. Namely, the paper highlights the different ways that mathematics teachers relate to their students and the subsequent impact on outcomes for mathematics success. Analyzing 7 second and third-grade classrooms in a large urban district within a low SES community, the study found that relational interactions explained a significant portion of the variance in mathematics achievement, even when controlling for prior achievement. The results speak to a needed reconceptualization of mathematics instruction as both an academic and social mechanism affecting equitable opportunities. A call for future research is made to investigate how student demographics (i.e. ethnicity, gender) warrant varying forms of relational interactions in the mathematics classroom.

Keywords: Equity and Diversity, Elementary School Education, Instructional activities and practices

Introduction

While the lack of quality of instruction in urban mathematics teaching has received much attention (see Ladson-Billings, 1997), other mechanisms that affect students learning in mathematics have not (Lubienski, 2002). One mechanism that the broader educational literature points to as a critical feature of classrooms is teacher-student relationships (Jerome, Hamre, & Pianta, 2009; Pianta, La Paro, Payne, Cox, & Bradley, 2002; Pianta, Nimetz, & Bennett, 1997). Scholars outside of mathematics education have found that teachers' interactions, particularly with students of color, can result in students' disengagement, misbehavior, or dropping out (Feagin, Vera, & Imani, 2001; Solórzano, Allen & Carroll, 2002). By looking at relational interactions as they play out in mathematics classrooms, research should be able to document the quality of these relational interactions between students and teachers in better understanding them as mechanisms that influence mathematics learning. These relationships go above and beyond content instruction in conveying messages about who is mathematically able, whose mathematical contributions are valid, and whose cultural and linguistic practices are legitimized in mathematics classrooms. This paper reports on a study of relational interactions across 7 second and third-grade classrooms in one urban district to understand their impact on mathematics achievement and learning for students of color in a low SES community.[i]

Literature Review

A number of scholars have documented mathematics instructional practices that differ by socioeconomic status (SES) or ethnicity in elementary grades. They have found that mathematics teachers of children of color and lower SES communities were more likely to disconnect taught procedures from students' thinking, teach fragmented or unexplained procedures, decontextualize mathematics vocabulary, assess students based on following steps rather than

mathematical reasoning or even correct/incorrect answers, and use less resources such as manipulatives even when available (Anyon, 1981; Ladson-Billings, 1997; Lubienski, 2002; Means & Knapp, 1991). As the forms of mathematics instruction available to students of color and those in poverty are often impoverished, they serve to limit the relationships that these students develop with the field of mathematics.

A separate literature documents the often-limited relationships that students of color develop with their teachers. Research on teacher-child relationships has found them to be a critical component of both psychosocial and academic development for students (Jerome, Hamre, & Pianta, 2009; Pianta, La Paro, Payne, Cox, & Bradley, 2002; Pianta, Nimetz, & Bennett, 1997). This work has compared teacher and student ratings of closeness and conflict for white, African American, and Latino students. Teacher ratings of their closeness and conflict with white students' ratings are very accurate. However, teacher ratings of relationships with African American and Latino students are decidedly inaccurate (Murray, Waas, & Murray, 2008). Jerome, Hamre & Pianta (2009) determined that teachers rate relationships with African American and Latino students as more conflictual than the students. These authors also note in this longitudinal work (kindergarten through 6th grade), that teachers rate their relationships with black students as becoming more conflictual over time in comparison to white students. So while many teachers do not have accurate notions of their relationships with African American and Latino students, they perceive these relationships as more conflictual, and this tends to worsen as students of color continue their schooling. This work however, was developed through questionnaires rather than direct observation and is not specific to mathematics content.

Within mathematics classrooms, the field is only beginning to understand the development of teacher-student relationships and their impact on the quality of instruction and the mathematics learned. In prior work, we found that the quality of instruction, as measured by an observational protocol, did not directly relate to the quality of relational interactions as determined through observation of classroom behavior (Battey, 2013; Battey & Neal, under review). The observation protocol measured commonly-held values about what quality instruction looks like in mathematics classrooms such as cognitive depth, classroom discourse, the nature of explanation and justification, and the types of problem solving in which students were engaged (Stecher et al. 2005; 2007). What this speaks to is that the relational interactions that are being measured are outside what the field typically considers to be quality mathematics instruction.

However, what we do not yet understand in this developing work is the impact of relationships on student learning and achievement. Lubienski (2002) makes the case that the field understands little about classroom mechanisms other than instruction that not only provide a lower quality mathematics learning environment, but also negatively affect achievement outcomes for students of color and those in poverty. While good mathematics instruction is requisite, it might not manifest itself in learning if teachers do not develop meaningful relationships with students in mathematics classrooms.

Theoretical Framework

We define relational interactions as a communicative action or episode of moment-to-moment interaction between teachers and students, occurring through verbal and nonverbal behavior that conveys meaning and can mediate student learning (Battey, 2013). These interactions can be either positive or negative in nature and can range in intensity. In conceptualizing this in the realm of mathematics, previous work documented five dimensions of relational interactions: addressing behavior, framing mathematics ability, acknowledging student

contributions, attending to culture and language, and setting the emotional tone (Battey, 2013). We briefly review each dimension below.

Addressing behavior captures the ways in which teachers respond to various forms of student behavior. *Framing mathematics ability* is more specific to mathematics. Mathematics ability can be framed as innate or changeable and can pass on various messages about students' capabilities to succeed mathematically (Battey & Stark, 2009). *Acknowledging student contributions* entails an episode between teachers and students that responds directly to the mathematical thinking of students. While a teacher's decision to address the mathematics within a student contribution would typically fall under the category of content instruction, the form of acknowledging, valuing/devaluing, or praising/disparaging shows relational aspects outside this realm. *Attending to culture and language* denotes the inclusion or omission of cultural and linguistic practices in mathematics instruction. It can be an important communicator of whose ideas are valued, what forms of mathematical thinking are acceptable, and who can participate in formal mathematics. These interactions are typified by how student language is dealt with by teachers or how a teacher draws on cultural resources, designs mathematics instruction to integrate home practices, or embeds students' home language. *Setting the emotional tone* doe not have to be a specific response to a student contribution. This dimension speaks to teachers establishing expectations of what it means to do math. For instance, it might be a teacher sharing a personal story about struggling in mathematics or an interaction that sends a message that students just need to rotely learn the mathematics content.

These five dimensions of relational interactions in the mathematics classroom serve as the framework used to operationalize the in the moment behaviors of students and teachers. While the interactions have been established and related to the quality to instruction in prior work, we do not have a sense of their relationship to students achievement and learning in mathematics. The study explored two research questions: 1) How do relational interactions explain variance in student achievement in elementary mathematics classrooms? and 2) How do relational interactions differ based on demographic variables such as sex, English language proficiency, and ethnicity?

Methods

Participants

Four second-grade and three third-grade elementary mathematics classrooms (137 students) from one large, urban district in Southern California were the focus of this research study. The district served students who are predominantly (99%) African-American and Latino (82% are Latino), receive free or reduced lunch (93%), and many were enrolled in an English Language Learner program (52%). The district, in its 2nd year of new leadership, had a history of poor performance and a long-standing sense from those outside the district that it would never do well. According to the state's ranking system and standardized test scores, it was one of the lowest performing school districts in California (lowest decile ranking, 1 out of 10, on California's Academic Performance Index during data collection) bringing the school district to be ranked as one of the lowest performing in California. At the start of the research study, only 57% of the teachers in the school district held credentials and 30% of the teachers were in their first or second year of teaching.

Teacher Professional Development

Prior to the research study, administrators and teachers acknowledged the importance of elementary mathematics students' engagement with algebraic reasoning. This led to the seven

observed teachers' participation in a large-scale study and on-site professional development series regarding students' algebraic reasoning skills and its affordances in meaningfully understanding arithmetic. The professional development intervention was based on *Thinking Mathematically: Integrating Arithmetic and Algebra in the Elementary School* (Carpenter, Franke & Levi, 2003) with a specific focus on *relational thinking* including concepts such as (i) understanding the equal sign as an indicator of a relation, (ii) using number relations to simplify calculations, and (iii) generating, representing, and justifying conjectures about fundamental properties of number operations. This intervention included school-based workgroup meetings and on-site support measures for the establishment of a learning community among classroom teachers, mathematics coaches, and professional development facilitators (Jacobs, Franke, Carpenter, Levi, & Battey, 2007). All seven teachers were observed for approximately 2 hours each month for 8 months after the end of the professional development intervention.

Data Sources

Each teacher's classroom was videotaped on two occasions within a one-week period. To capture video and audio for as many students as possible (12 in each classroom), we used two cameras (to record 8 students) and six audio recorders (to record an additional 4 students). Each video camera had two audiofeeds connected to flat microphones (four flat mics in all), so that 8 students could be recorded simultaneously. Each flat mic was positioned between a pair of students. For the students audiotaped only, in addition to the flat mic between 2 students, each student had an individual lapel mic (each attached to a different audio recorder). Triangulating the recordings from the flat mic and the individual mics helped identify the speaker for students who were audiotaped only.

In addition to the video and audiotaping, the data includes the state standardized tests, namely the California Standards Test (CST) and the California Achievement Test (CAT6). The CST is initially taken in second grade (allowing for a control of prior achievement in third grade only) while the CAT6 is taken starting in third grade. Briefly, the CST is based on the state standards while the CAT6 is a norm-referenced examination. Scaled scores were used for this study to compare students uniformly across grades based on state means. Additionally, we had current grade level scores reflecting student performance on each of the district's benchmark quarterly assessments. This allowed for a measure of mathematics learning rather than achievement as in the case of the state standardized tests. Finally, the dataset included demographic data on sex (for both grades), as well as English language proficiency and ethnicity for third graders alone.

Analysis

Relational interactions were coded in four layers on the video specific to each student. First, we identified relational interactions and coded their dimension as discussed in the theoretical framework. Second, we coded forms of emphasis to determine intensity. Forms of emphasis referred to both verbal and nonverbal communication that accentuated an interaction such as stressing one's voice or gesturing. Next, we identified interactions as positive/negative based on the tone of the interactions and the message it sent about student thinking or behavior. Fourth, we coded the intensity of the interaction (low-1, medium-2, high-3). If an interaction did not contain any form of emphasis, it was coded low. If an interaction contained one or more emphases, we classified it as medium or high depending on the extent of the emphasis. Inter-rater reliability was 92% in identifying relational interactions, but the researchers came to agreement on 99% of coded episodes. Interactions without agreement were eliminated.

After coding the relational interactions, we calculated a relational interaction rate for each dimension for each student. This consisted of summing the interactions (as positive and negative), multiplying by their intensity, and dividing by the minutes of instruction across the two-videotaped lessons. The rates were then multiplied by 100 to make the decimals more manageable. After examining descriptive statistics for each relational interaction dimension, we entered them into linear regression models to see the amount of variance they explained with respect to mathematics achievement and learning to respond to the first research question. Finally, we ran T-Tests and one-way ANOVAs to examine the quality of interactions with respect to demographic variables in light of the second research question.

Results

The following results first examine descriptive statistics for relational interactions and their subsequent impact on all students' mathematics achievement on the CST. The results are then parsed to explore the impact of relational interactions for third grade and second grade students separately. Finally, the results end by examining the relationships of demographic variables to the quality of the relational interactions.

Relational Interactions: Descriptive Statistics

Dimension Rate	Mean	Standard Deviation	Minimum	Maximum
Addressing Behavior	-6.32	13.11	-46.60	8.55
Framing Mathematics Ability	1.66	1.52	0.00	7.34
Acknowledging Student Contributions	9.38	7.32	-1.20	31.20
Attending to Language and Culture	0.04	0.25	0.00	2.40
Setting the Emotional Tone	4.96	4.16	0.00	13.10

The most common, negative, and intense dimension of relational interactions was addressing behavior across classrooms. The other dimensions skewed positive with acknowledging student contribution as the second most common type of interaction. Attending to language and culture was very uncommon in these classrooms.

Students' Mathematics Achievement

We entered all of the relational interactions rates and sex (the only available demographic variable for both grades) as independent variables with the CST score as the dependent variable into a linear regression. Across all of the students, the only significant relationship was Acknowledging Student Contributions (F = 21.57, $p < .01$). It explained 13.4% of the variance in students' mathematics test scores.

Third Grade Mathematics Achievement and Learning

Linear regression was used to analyze third grade CST scores employing all of the relational interaction rates, sex, English language proficiency, and ethnicity. Additionally, the prior school year's CST scores were included as independent variables. Prior CST scores accounted for 61.4% of the variance of students' subsequent CST score and were significant. The only other variable that accounted for a statistically significant part of the variance was setting the emotional tone, accounting for 12.6% of the variance in CST scores. The effect size for the model was high (F = 63.63, $p < .01$).

Additionally, we ran a linear regression on CAT6 scores for third grade. We first ran the regression with the relational interaction variables, sex, language, and ethnicity. Addressing behavior significantly accounted for 13.8% of the variance on CAT6 scores (F = 8.96, $p < .01$). No other variables were significant. We then added CST scores from the previous school year to the regression model as a measure of prior mathematics achievement. When that was included, it significantly explained 56.5% of the variance in third grade CAT6 scores. Setting the emotional tone also significantly accounted for 11.2% of the variance in CAT6 scores with no other variables explaining a significant part of the variance (F = 47.06, $p < .01$).

Finally, we ran a regression on students' fourth-quarter scores on the district mathematics assessment (dependent variable) as a measure of mathematics learning rather than achievement. Again, all of the relational interaction rates, sex, language, and ethnicity were entered along with the prior three quarterly assessment scores as measures of prior mathematics achievement. The third quarter assessment accounted for 68.1% of the variance (F = 104.63, $p < .01$) while a second model included setting the emotional tone, which explained an additional 3.0% of the variance (F = 58.75, $p < .05$). Both were statistically significant though no other variables were.

Second Grade Mathematics Achievement and Learning

Since second grade did not take the CST in first grade, we used the first-quarter district assessment scores as measures of prior mathematics achievement in addition to relational interactions and sex. The first quarterly assessment accounted for 58.3% of the variance in CST scores (F = 102.97, $p < .01$). A second model was significant that included acknowledging student contributions, which explained an additional 7.7% of the variance in CST scores (F = 66.85, $p < .01$). However, no other variable was significant in the model.

Lastly, we ran a regression on the fourth-quarter district assessment, similar to that executed for the third grade, as an assessment of mathematics learning. The first quarter assessment accounted for 57.7% of the variance while the third quarter test accounted for an additional 7.2% of the variance (F = 63.07, $p < .01$). These were the only significant findings of the analysis.

Relationships between Demographic Variables and Relational Interactions

We ran *t-tests* for sex and ethnicity since there were only two ethnic groups in the data – namely, African American and Latino. We also ran one-way ANOVAs for language and sex by ethnicity. However, whenever ethnicity and language were part of the statistical analysis, only third graders were included due to the available demographic data. The only analysis that found a significant relationship was the T-Test on sex. Teachers engaged in acknowledging student contributions at a higher rate for girls than boys across both grades ($p < .0$). While this was the only significant finding, we should note that the gender by ethnicity analysis only included 7 African American girls and 9 African American boys. While not statistically significant, there were notable differences in addressing behavior (advantaging Latinas (mean = -8.49) and disadvantaging African American girls (-18.54) and acknowledging student contributions (advantaging African American girls (14.83) and disadvantaging African American boys (6.03). This result raises the need for future research with more students to see if this pattern is sustained.

Discussion

Acknowledging student contributions and setting the emotional tone both explained statistically significant portions of the variance for both state standardized tests and district assessments, even when controlling for prior mathematics achievement. These variables accounted for between 6 and 15% of the variance that was unaccounted for by prior achievement. This suggests that relational interactions are measuring a classroom mechanism not explained through prior achievement and learning in mathematics.

Addressing behavior accounted for a significant portion of the variance on CAT6 scores for third grade students. When prior achievement was factored into the analysis however, this relationship was no longer significant. This suggests that addressing behavior is related to prior mathematics achievement scores in this dataset. One way to interpret this is that lower achievement or prior negative framing of ability is leading students to resist or engage in non-compliant behaviors to current mathematics schooling. A longitudinal analysis would be needed to look at the relationship of behavior, framing ability, and achievement across multiple grades to see the social construction of noncompliant behavior.

It is not surprising that attending to culture and language and framing mathematics ability did not produce any significant results. This was probably due in part to two possible sources: (a) the overall infrequency of these relational interactions and (b) the teachers' tendencies to engage in these dimensions of relational interaction with the whole class rather than with individual students, which results in a lack of variance for the students in the study. Additionally, since framing mathematics ability correlated highly with setting the emotional tone, it was unlikely that it would explain more of the variance in achievement.

Additionally, teachers acknowledged the mathematical contributions of female students in the classrooms more frequently and intensely than boys. This raises concerns about whether the mathematical contributions of Latino and African American boys are missed in classrooms.

Conclusion

We think these results point to the need for future research in understanding relational interactions. In our prior work, we found that more positive relational interactions do not directly relate to higher quality mathematics instruction (Battey, 2013; Battey & Neal, under review). In this analysis, both setting the tone and acknowledging student contributions significantly explained variance in mathematics achievement and learning. Further research is needed to see if this relationship exists outside of the classroom contexts and students in this study.

Additionally, it is important to understand if students of color are treated differently than their white counterparts. This would mean that relational interactions are a mechanism through which students are treated differentially in mathematics classrooms and could be a factor in producing racial differences in achievement. The research literature has found that teachers perceive their relationships with students of color as more conflictual than their relationships with white students. They also rate their relationships as more conflictual than their African American and Latino students. Taken together, this prior research could explain the negative bent to addressing behavior in the current study. Future work could examine the relationship between perceived and observed behaviors of teachers and students.

The relational interaction framework used here seems to be an important facet of instruction in understanding student achievement and learning. We think the findings of this study provide

evidence that relational interactions are an important construct to measure in future research in mathematics classrooms.

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Investigating the Relationship Between Teacher Professional Vision and Classroom Practices: A Case of Misalignment

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This paper investigates the relationship between teachers' professional vision and their classroom practices. In particular, the study finds that teachers with similar professional vision can have different classroom practices related to noticing and responding to student thinking. Data is drawn from a study in which teachers reflected on video online as well as collected video from their own classrooms. Analysis revealed two teachers with similar attention to student mathematical thinking when watching classroom videos. However, these teachers recalled noticing interesting moments of student thinking, made space for student thinking to emerge, and probed students' underlying understandings with different frequency in their classrooms.

Keywords: Classroom Discourse; Teacher Knowledge; Teacher Education-Inservice/Professional Development, High School Education

Researchers have proposed that a key component of teaching is the ability to notice and interpret what happens in the classroom. Classrooms are complex environments with many events unfolding simultaneously, thus interpreting important features of classrooms requires skill. This ability has been called teachers' *professional vision* (Goodwin, 1994), which is described as teachers' ability to "make sense of what is happening in their classrooms" (Sherin, 2007, p. 384). From the perspective of mathematics education reform, developing teachers' in-the-moment professional vision with respect to student thinking is necessary in order to adopt a responsive or student-centered approach to teaching mathematics.

New methodologies for investigating teacher professional vision have emerged, which have allowed researchers to investigate teachers' in-the-moment noticing (Colestock & Russ, 2010; Sherin & Sherin, 2010). Previously, teacher professional vision was primarily measured through reflection on video, which is likely to be different than teachers' in-the-moment noticing (Sherin, Jacobs, & Philipp, 2011). To distinguish between these two types of professional vision, I will call noticing that occurs while teaching *online professional vision* and noticing that does not occur while teaching (e.g. reflection on video) *offline professional vision*. We might expect these two types of professional vision to differ because the online context includes the "blooming buzzing confusion of sensory data" found in the classroom in addition to requiring teachers to complete many other cognitive tasks at the same time (Sherin & Star, 2011, p. 69).

A variety of research studies have shown that developing teacher professional vision can impact teachers' classroom practice (Frederiksen, Sipusic, Sherin, & Wolfe, 1998; Sherin & Han, 2004; Sherin & van Es, 2005; van Es & Sherin 2010). These studies hypothesize that tuning teachers' offline professional vision to notice students' mathematical ideas increases their focus on attending to and responding to student ideas while teaching. However, these studies have used video as the material for supporting the development of professional vision. Therefore, they claim that development of offline professional vision can impact teachers' classroom practice, which seems to indicate that offline professional vision is closely related to teaching practices. In this paper, I will investigate a case of two teachers with similar offline professional vision but different online professional vision and classroom practices with respect to noticing and

responding to student thinking. I will begin by describing the participants and data sources I will draw upon, and explain the methods used to analyze this data. I will then describe the findings related teachers' offline professional vision, online professional vision, and classroom practices. Finally, I will discuss several explanations for the findings and implications for research on professional vision and teacher education designed to develop teacher professional vision in ways that impact classroom practices.

Methods

Participants and Data Sources

The data used in this paper come from an investigation of teachers' efforts to collect video clips that can support productive video club discussions about student thinking (Sherin, Linsenmeier, & van Es, 2009) . The participants were four high school mathematics teachers from a suburban Midwestern school. All names referenced in this paper are pseudonyms. Each teacher chose one of his or her classes to participate in the study. The teachers attended five 40-minute meetings over the course of 10 weeks. In the meetings, teachers participated in video clubs and discussed what types of clips can support productive video clubs. After each meeting, the teachers collected two to three video clips from their classrooms that they thought would support productive video club discussions and explained their reasons for selecting the clips as well as moments they would suggest discussing in a video club.

To see if participants' judgment of the likelihood of video clips to support productive video clubs was influenced by the study, participants completed an online task where they rated video clips before and after all of the meetings. Participants watched four different 4- to 6-minute video clips of high school mathematics classrooms. The videos were chosen to include a variety of likelihoods to support discussions about student mathematical thinking and different classroom structures (e.g. whole-class discussion, student presentations, small group work). For each video, participants were asked to rate the likelihood that the clip would "support a fruitful discussion of student mathematical thinking" on a four-point scale of very unlikely, somewhat unlikely, somewhat likely, very likely. They were also asked to explain their rating using at least three sentences and describe two moments or topics from the video that they would recommend discussing during a video club about student mathematical thinking.

Participants also completed an online task where they indicated (or "tagged") moments of interesting student mathematical thinking at the beginning and end of the study to see if participants' noticing of student mathematical thinking was influenced by the study. Participants watched two 4- to 6- minute video clips of high school mathematics classrooms. The video were selected because they provided several moments of interesting mathematical thinking expressed by students with varying degrees of clarity. For each video, participants were asked to tag moments of interesting student mathematical thinking while watching the video and describe the student thinking they tagged. If they tagged more than three moments, participants were asked to pick the three most interesting moments and explain why they picked those moments.

The participants were interviewed over the phone shortly after each time they collected video from their classrooms and before they had a chance to review the video. These interviews were designed to capture the thinking process teachers went through in capturing clips. These interviews focused on what participants intended to capture, whether they thought they captured what they intended to capture, and any moments they wish they would have captured.
Finally, participants' classrooms were observed two times during the beginning of the study and one or two times when they were collecting video clips after the final two meetings. The teachers suggested days that would be good to observe their classrooms. Each classroom was videotaped

using two cameras. One camera focused on the whole class, including the chalkboard or screen if used. The second camera captured the work of one group in the class, often chosen by asking the teacher what group would be good to capture. If the students were not working in groups, the second camera was focused on the whole class. The teachers were asked if the lessons observed were typical, and the teachers indicated that the lessons were typical for them with the exception of one lesson. One of Sam's lessons was longer than usual, which caused him to add an activity to the beginning of the lesson, which he would normally not have students complete. The findings are similar qualitatively with or without this activity. In fact, the findings are stronger when excluding this atypical activity.

Analysis of Professional Vision

To analyze professional vision, both administrations of online video tasks were examined as well as comments teachers made during interviews about moments of student thinking they noticed in their own classrooms. In the online tasks common moments teachers mentioned or tagged at the beginning and end of the study were treated as one moment.

I adopted the framework used by Sherin and van Es (2009), who define professional vision of teachers in terms of two aspects: *selective attention* and *knowledge-based reasoning*. Selective attention refers to what teachers pay attention to. In my analysis of selective attention in the clip-rating task, I coded the explanation of the clip rating and productive moments or topics to discuss during a video club in terms of actor and topic codes used by Sherin and van Es (2009) with the addition of the topic code *mathematics* for topics that are purely mathematical that do not explicitly reference student thinking found in the video. These codes are summarized in Table 1. Additionally, I coded for the specific moments of the clip or topics that were referenced.

The analysis of the interesting moments tagged by teachers followed the procedure used in Walkoe (2013), which focuses on the selective attention of teachers. To find instances when teachers tagged the same moment in the video, tags within a 10 second range were identified. Only tags that were identified as one of the three most interesting were used. The moment that the teachers were referencing was determined by their description of the student thinking they tagged. Instances when teachers referenced the same moment but different aspects of that moment were treated as different moments. Eight different moments were identified in one video and 15 different moments were identified in the other video. To look for similarity in selective attention, the proportion of moments identified in common among all possible pairs of teachers was calculated by dividing the number of common moments for both teachers by the sum of the number of coded moments from each teacher. This proportion can range from 0 (no common moments) to .5 (all common moments).

Knowledge-based reasoning refers to how teachers interpret what they attend to. Sherin and van Es (2009) restrict their analysis of reasoning related to student thinking, and analyzed these data based on two dimensions: stance and strategy used to explore student math thinking. This analysis of knowledge-based reasoning deviated from that used in Sherin and van Es (2009) primarily because the interpretation of the video was done in a different context. In this data, teachers were asked to rate the clip in terms of how likely it would be to support discussions of students' mathematical thinking, which corresponds with their interpretation of the clip. After collapsing the clip rating scale into two categories (supportive and unsupportive), teachers that rated clips similarly were seen as having similar knowledge-based reasoning. Additionally, I coded the productive topics responses in terms of the strategy used to explore student math thinking coding scheme that was used in Sherin and van Es (2009). Only instances that were coded as *student math thinking* for topic were included in this analysis because the coding

scheme did not apply to the other topics. These codes are summarized in Table 1. This provides another measure of teachers' interpretation of the clips.

Table 1: Summary of Coding Categories for Professional Vision

Selective Attention	*Knowledge-based Reasoning*
Actor	*Strategy*
Student	Restate student ideas
Teacher	Interpret student ideas
Topic	Generalize student ideas
Student math thinking	
Management	
Climate	
Pedagogy	
Mathematics	

The interviews completed with teachers were analyzed to find references to selective attention that happened while teaching. Interview transcripts were coded using a top-down approach to identify instances when teachers recalled in-the-moment noticing of moments or episodes that, if captured on video, would support video clubs about student thinking. Because interpretations of these moments were not elicited during interviews, these instances were analyzed only in terms of selective attention.

The online video tasks will be used as a measure of offline professional vision because they are not completed while teaching. The interview comments will be used as a measure of online professional vision because they reference noticing completed while teaching.

Analysis of Classroom Practice

To analyze classroom practice related to noticing student thinking, I adapted the analysis used in van Es and Sherin (2010). Only data from the first two observations were used because the final observations were completed when teachers were capturing video so they would be less likely to reflect typical teaching practice. The video was segmented into 2-minutes sections. Using the same categories in van Es and Sherin (2010), each segment was coded for (a) making space for student thinking, and (b) probing student thinking. A summary of the sub-codes for each category is given in Table 2.

Table 2: Summary of Coding Categories for Classroom Practices

Making space for student thinking	*Probing student thinking*
Publicly recognizing unsolicited student ideas	Asking for explanations
Providing extended opportunities for student thinking	Probing explanations
Eliciting multiple methods or solutions from students	

In addition to looking at instances of classroom practices associated with noticing student thinking, the extent to which there were sections of video that would support video club discussions was also investigated. Segments of video that would be likely to support productive video club discussions about student thinking (as determined by Sherin et al., 2009) at least two minutes in length were identified. Segments about a similar topic that were longer than six minutes were split into two different segments because video clips used in video clubs are normally 4-6 minutes in length.

Findings

The findings are separated into sections about offline professional vision, online professional vision and classroom practices with respect to noticing and responding to student thinking.

Offline Professional Vision

Two teachers were found to have similar offline selective attention and knowledge-based reasoning in my sample, Sam and Matt. For both teachers, almost all comments made about the videos had students as the actor and their mathematical thinking as the topic. Sam and Matt also mentioned similar topics and strategies when asked what to discuss about the video, with both teachers focusing on student mathematical thinking. Sam had 11 topic instances about student math thinking, two about mathematics, and one about pedagogy. Sam had all 15 topic instances about student math thinking. Additionally, in three of the four clips, the teachers mentioned similar aspects of the video in their explanations. Given the complexity of classrooms and video of classrooms, attending to similar elements of the video is strong evidence that the teachers had similar selective attention. The clip that they did not notice similar elements in was also rated (i.e. interpreted) differently.

This result is unsurprising because the participants were asked to rate the likelihood that the clips would support discussion of student mathematical thinking. However, the other two teachers' selective attention was not as similar. Gwen's responses show much more attention to the teacher and pedagogy, particularly when describing productive topics to discuss in a video club of student thinking. Harry's responses are more similar to those of Sam and Matt, but he referenced fewer aspects of student thinking in his explanations for clip ratings and he paid more attention to the teacher and pedagogy when describing productive topics. Therefore, Sam and Matt seem to be the teachers with the most similar selective attention among the four. This is confirmed when examining the findings from the online tagging tool as well, which is discussed below.

Table 3: Frequency of Actor and Topic Codes in Clip Rating Task

	Actor		Topic			
	Student	Teacher	Student math thinking	Pedagogy	Mathematics	Climate
Explanations						
Sam	22	0	21	0	0	1
Matt	20	2	20	2	0	0
Gwen	13	1	11	3	0	0
Harry	16	1	14	0	0	0
Productive Topics						
Sam	14	0	11	1	2	0
Matt	15	0	15	0	0	0
Gwen	11	16	6	21	0	0
Harry	13	3	13	3	0	0

The analysis of the video tagging task supports the conclusions made previously that Sam and Matt have the most similar professional vision. They had the highest proportion of moments that they both noticed. Table 4 shows how Sam and Matt had over half of the moments they noticed in common, while all other pairings of teachers had less than half of the moments they noticed in common.

Table 4: Proportion of Common Moments Identified in the Video Tagging Task

	Sam	Matt	Gwen
Matt	.28 (5/18)		
Gwen	.21 (4/19)	.21 (4/19)	
Harry	.12 (2/17)	.18 (3/17)	.17 (3/18)

In my analysis of knowledge-based reasoning, I will focus only on the two teachers previously identified as having similar selective attention. Sam and Matt also had similar knowledge-based reasoning. They rated three of the four clips similarly, including identical initial ratings for two of the clips. The strategies used to explore student thinking that were used in the productive topics was predominantly restating student ideas for both teachers. Matt seems to use the interpreting student ideas strategy more than Sam, using it six times compared to two. These findings are summarized in Table 5.

Table 5: Frequency of Codes for Strategy Used to Explore Student Thinking

	Restate student ideas	Interpret student ideas	Generalize student ideas
Sam	9	2	1
Matt	8	6	0

From these analyses, we see that not only are the two teachers attending to similar events in the videos, but they are interpreting those events similarly as well.

Online Professional Vision

Although Sam and Matt had similar offline professional vision, the two teachers recalled having different online selective attention of student thinking in their classrooms. In the five interviews conducted with Sam, he only mentioned noticing student thinking that would support video club discussions one time. He remarked in one of the interviews that “I'm sure there were moments where students had amazing discussions, I just don't know when they were,” which indicates that he realized that he was not noticing the moments he was trying to capture while they happened. For the one instance that he recalled noticing, his comments were more general instead of describing specific ideas that he noticed the students had. He said, “there's another group that I felt had amazing conversations...and there were a bunch of these sort of aha moments that the kids had where I wish that I had been filming those…I-I can't think of a specific thing…” In this example, Sam provided general descriptions of the moment he remembered, like “amazing conversations” and “aha moments”, but says he is unable to remember the specific things that students were doing as evidence for these descriptions. This interview was completed a day after he videotaped, which could have impacted his ability to remember the specific student ideas he noticed while teaching.

In contrast, in all five of the interviews Matt recalled noticing student groups that he visited having good discussions and when interesting whole class discussions happened. He could regularly remember specific things students said and why they were interesting. For example, he remembered an interesting idea students had about finding the number of ways that the letters in a word can be rearranged when he recalled,

> The class came to a very odd understanding of what it meant for order to matter…they were saying, what the problem says, that we can switch the letter of the orders around. Therefore …the order of the letters must not be important.

In this example, we see that he could remember the specific way the students thought that he found interesting rather than just recalling that something interesting happened. Although four of the five interviews were conducted the same day he videotaped his classroom, some of the specific ideas he recalled were from previous lessons that he wished he had videotaped.

Classroom Practices

Although both teachers had similar structures to their class, their practices with respect to noticing and responding to student ideas was very different. Both teachers had students work in small groups on problems and then discuss the problems as a whole class. However, both the small group work and the large-group discussions provided different opportunities for making space for student thinking and for probing student thinking.

Both teachers' segments reflected the teachers making space for student thinking, although Matt used these practices more frequently. There are larger differences in their practices related to probing student thinking. None of Sam's segments show him asking for explanations or probing student explanations. In contrast, Matt's segments have him asking for explanations, and probing student explanations about as frequently as his practices related to making space for student thinking. These findings are summarized in Table 6. If we normalize based on the number of minutes of instruction, these differences increase because Sam was observed for 110 instructional minutes while Matt was observed for 94 instructional minutes.

Table 6: Frequency of Segments of Noticing-Related Classroom Practices

	Making Space for Student Thinking			Probing Student Thinking	
	Publicly recognizing unsolicited student ideas	Providing extended opportunities for student thinking	Eliciting multiple methods or solutions	Asking for explanations	Probing explanations
Sam	4	2	3	0	0
Matt	6	4	6	4	7

Sam's classroom also had fewer segments that would support productive discussions than Matt. Sam's observations contained 5 segments, while Matt's observations contained 8 segments. If we normalize based on the number of minutes of instruction, Sam's observations contained .45 segments per 10 minutes of instruction, while Matt's observations contained .85 segments per 10 minutes of instruction, almost twice as many. Additionally four of Sam's five segments happened during an activity he would not have given normally (which he gave due to extra instructional time), so the difference in the frequency of segments may actually be underestimated by the observations.

It may be that these instructional practices related to probing student explanations are linked with the frequency with which clips that support video club discussions happen. We might imagine that probing student thinking makes it more likely that student thinking will be visible, a key aspect of clips that support video club discussions.

Discussion

In this paper I have shown that while two teachers have similar offline professional vision, their instructional practices related to attending to student thinking differ remarkably. Teacher professional vision is primarily developed through offline activities, so these results may indicate that online activities designed to develop professional vision may be necessary in order to impact classroom practices of some teachers. However, a similar misalignment between online professional vision and classroom practices may exist for teachers.

There may be several reasons why the difference in offline professional vision and classroom practices were observed in this study. First, Sam is a first-year teacher, while Matt has more than five years of teaching experience. Sam may have developed sophisticated offline professional vision, but it might take more experience teaching in order to develop instructional practices that are similarly sophisticated. Many of the instructional practices that Sam did show related to making space for student thinking could be easily planned for in advance of instruction, while the practices related to probing student thinking require action in-the-moment and cannot be as easily planned for.

The findings also indicate that offline and online professional vision are potentially different for teachers. Sam's online professional vision was not as developed as his offline professional vision. As a beginning teacher there may be more non-routine cognitive work required, so they may not have the necessary cognitive resources available to devote to noticing. Additionally, it may require significant practice or experience in order to develop online professional vision. While I have hypothesized that the measurement of teacher professional in this study differed between offline and online, there are other ways of interpreting this difference. In particular, the differences in the measures of professional vision could be described as professional vision of other teachers' classroom versus professional vision of teachers' own classrooms. Future studies could investigate whether teacher professional vision when watching videos differed depending on whether the video was from another teachers' classroom or a teacher's own classroom.

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THE LIMIT NOTATION AS A MEDIATOR OF MATHEMATICAL DISCOURSE

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Symbols play important roles in higher-level mathematical thinking by providing flexibility and reducing cognitive lo ad. However, they often have a dual nature since they can signify both processes and products of mathematics. The limit notation is considered to be a visual and symbolic mediator that reflects such duality, which presents challenges for students. This study focuses on one instructor's utilization of the limit notation and examines how it mediates his discourse on limits. The findings indicate that the process and product aspects of limit were both present in the instructor's use of the limit notation. Although he differentiated between these two aspects depending on the mathematical context, the distinction remained implicit for the students in the classroom. The study concludes that it is important for teachers to unpack the meanings inherent in symbols in the classroom to enhance mathematical communication.

Keywords: Classroom Discourse, Post-Secondary Education, Advanced Mathematical Thinking

Introduction

Mathematics is often viewed as an embodied activity where learners generate mathematical meaning through their perceptions, senses, and experiences in the real world (Lakoff & Núñez, 2000). Although initial learning of mathematical objects may be based on learners' sensual perceptions of physical objects, such an approach can hinder learning of higher-level mathematical skills (Gray & Tall, 2001). Indeed, a significant characteristic of mathematical sophistication is generalizing context-bound thinking to context-free (abstract) thinking.

Symbols play important roles in such generalizations and the transition to higher order mathematical thinking. There is a reflexive relationship between symbolizing and mathematical meaning (Cobb, 2000) indicating that the meanings learners attribute to the signs are as critical as the syntactical aspects of symbolism. Radford and Puig (2007) argue that signs are deposited with historical cognitive activity and the social practices they mediate are not transparent to the students. According to them, learners' awareness of the historical intelligence embedded in symbols requires their active participation in a sense-making process.

Learners' struggles with the sense-making process, as they interact with symbols, may be due to the ambiguous nature of symbols. Cobb (2000) mentions that a concrete mathematical symbol "can serve different symbolizing functions and might, in fact, be used non-symbolically" (p. 18). Similarly, Gray and Tall (1994) point to the ambiguity of symbols by highlighting that symbols can signify both processes and products of mathematical activity. They argue that, although the ambiguity of symbols may present students with challenges, it is also an essential element of mathematical thinking since it supports flexible interpretation of mathematical notations by simplifying "the cognitive complexity of process-concept duality by the notational convenience of process-product ambiguity" (Gray & Tall, 1994, p. 121).

Limit is a mathematical concept that reflects this process-product ambiguity. Research shows that the informal and formal aspects of limit are based on different metaphors, supporting different realizations of the concept as a process or a product (e.g., Cornu, 1991; Güçler, 2013; Lakoff & Núñez, 2000; Tall & Vinner, 1981). This ambiguity can also interfere with thinking

about the representational aspects of the concept such as the limit notation (Bagni, 2005; Gray & Tall, 1994; Tall & Vinner, 1981). Gray and Tall (1994) indicate that the issue is not whether learners consider limits as processes or products, but whether they can consider limit as both, depending on the mathematical context. They argue that such a utilization of notation requires that the difference "between process and concept is maintained at all times" in the realization of limits (Gray & Tall, 1994, p. 121).

Teachers' discourse in the classroom can play critical roles in students' thinking about limit given that symbols may play ambiguous roles in mathematics and that the syntax as well as the meaning of symbols may not be transparent to students. Teachers may also help students in differentiating and amalgamating the process and product aspects of limit. However, in order for this to happen, it may not be enough for teachers to only use mathematical language and symbols accurately; they need to also make such elements of their discourse transparent to their students in the classroom (Güçler, 2013). This study uses a discursive approach to address the following question: How does the limit notation mediate one instructor's discourse on limits in a beginning-level undergraduate calculus classroom? Particular sub-questions include: How does the instructor utilize the limit notation? What does the limit notation signify in the instructor's discourse? Are the distinctions between the process and product aspects of limit transparent in the instructor's discourse?

Theoretical Framework

The process-product duality of symbolism and learners' need to form conceptual entities from dynamic processes are addressed through different conceptualizations in the literature such as encapsulation (Dubinsky, 1991; Gray & Tall, 1994, 2001) and reification (Sfard, 1991, 1992). These approaches provide significant insights about the nature of mathematical objects and some of the challenges related to mathematical learning, but do not take into account the socio-cultural nature of generating signs and their meaning. The assumption of this study is that signs and artifacts are cultural products; they support semiotic mediation as their meanings are generated and negotiated by communities engaged in mathematical discourse (Vygotsky, 1978; Radford & Puig, 2007).

Sfard's (2008) more recent approach integrates mathematical cognition with mathematical communication with the aim of dissolving the dichotomy between the individual and social aspects of learning. She formulates thinking as an individualized form of communication and considers learning as change in one's discourse through increased participation in communities of practice. The main unit of analysis in her approach is discourse, which is defined as "different types of communication set apart by their objects, the kinds of mediators used, and the rules followed by participants and thus defining different communities of communicating actors" (Sfard, 2008, p. 93). This work is based on her discursive approach that formulates mathematics as a particular discourse that is distinguishable by its word use, visual mediators, routines, and endorsed narratives (Sfard, 2008). Word use refers to participants' use of mathematical words in their discourse. Visual mediators refer to all the visible objects that are created and acted on for mathematical communication. Routines refer to the repetitive meta-level rules in the actions of participants as they substantiate their narratives about mathematics. Endorsed narratives refer to the set of utterances the participants consider as true given their word use, visual mediators, and routines.

In the context of her discursive approach, Sfard (2008) extends the notion of reification to objectification. She notes that a critical feature of word use is objectification, which occurs

through reification and alienation. Reification "is the act of replacing sentences about processes and actions with propositions about states and objects" (Sfard, 2008, p. 44), whereas alienation refers to "using discursive forms that present phenomena in an impersonal way, as if they were occurring of themselves, without the participation of human beings" (Sfard, 2008, p. 295). Said differently, objectification changes the talk about processes to the talk about end-states or mathematical entities. In this study, the instructor's use of the limit notation as a visual mediator is examined in relation to his word use, routines, and endorsed narratives.

Methodology

This work is part of a case study that investigated one instructor's and his students' discourses on limits in a beginning-level undergraduate calculus classroom at a large mid-western university in the U.S. The data sources for the larger study included eight video-taped classroom sessions during which the instructor talked about limits and continuity; a survey given to all the students at the end of their classroom discussions on limits; and four students' responses to task-based interview sessions on limits. For the purposes of this paper, the data consists of the instructor's classroom sessions (see Güçler (2013) for a detailed analysis of the entire study). The classroom observations were transcribed both with respect to the instructor's utterances and actions. The transcripts included the snapshots of everything the instructor wrote and drew on the board.

For the analysis of the visual mediators, an inventory of all the visual mediators the instructor used was collated from the transcripts. They were then compiled into three categories: graphs, written words, and symbolic representation. The focus of this study is on one visual mediator, namely the limit notation as a symbolic mediator of mathematical discourse. Note that the description of what type of mediator used gives little information about mathematical discourse, so the discussion of visual mediators needs to also include how and when they were used, what kinds of narratives about limit the notation supported, and how the instructor talked about the notation. Therefore, the analysis of visual mediators is presented with the analyses of the instructor's word use, routines, and endorsed narratives in the context of the limit notation.

The instructor's word use was analyzed in terms of the degree of objectification. When talking about the limit notation, his word use was considered operational if he referred to limit as a dynamic process, and objectified if he referred to limit as a distinct mathematical object—a static number that is obtained at the end of the limit process. For the analysis of routines, the focus was on the instructor's actions to elicit when and how he used the limit notation (Sfard, 2008). Finally, for the analysis of the endorsed narratives, the focus was on two meta-level narratives that were substantiated by the instructor through his word use and routines of using the limit notation: limit is a process and limit is a number.

Results

Dr. Brenner (a pseudonym) had 775 utterances about limits throughout the eight lessons. About 82% of those utterances were coded as objectified, indicating that he mainly talked about limit as a static mathematical entity. There were two limit-related contexts in which Dr. Brenner's operational word use took place: the informal definition of limit and computing limits. In these two contexts, he shifted his word use from operational to objectified and vice versa. The analysis in this section examines how he communicated in these contexts with a particular attention to the role of the limit notation in his discourse.

When Dr. Brenner introduced the informal definition of limit, he also introduced the limit notation as shown in Figure 1 and Table 1.

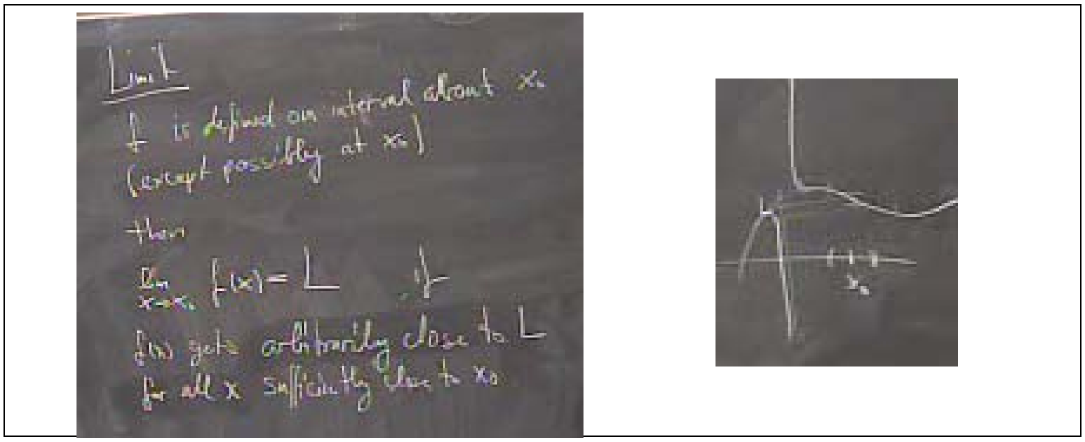

Figure 1: Video Snapshot of Dr. Brenner's Informal Definition of Limit

Table 1: Dr. Brenner's Introduction to the Informal Definition of Limit

What is said	What is done
[1] What is a limit?	He writes "f is defined on an interval about x_0" on the board and starts drawing a graph (See Figure 1).
a) x_0 is here. I have some function and I want to make sure it is defined at least near x_0.	
b) So if I take a small interval here, this function is defined.	He draws an open interval around x_0.
c) So we don't ask it to be defined at x_0 but at least nearby.	He writes "except possibly at x_0" on the board.
[2] Then we say as x approaches x_0 of the function $f(x)$ equals some number L if…(does not finish his sentence)	He writes $\lim_{x \to x_0} f(x)$ when he says these and then pauses.
[3] What does it mean that the limit of the function as x approaches x_0 is L?	He shows the notation and turns back to the graph he drew.
[4] It means that the function value, if I get closer and closer to x_0, should approach some number L.	He puts L on the graph he drew and then completes writing $\lim_{x \to x_0} f(x)$ as being equal to L. (See Figure 1).
[5] I want to say it [$f(x)$] is arbitrarily close to L for all x sufficiently close to x_0.	He writes these on the board (See Figure 1).
[6] So this is what I would want to call the intuitive definition of a limit.	
[7] So we say the limit is L if I can make the function values to be arbitrarily close to L if I choose my values of x sufficiently close to x_0.	

There are a couple observations that can be made from this introduction. Dr. Brenner introduced the limit notation quite tacitly (Table 1, [2], [3]), as if it was an unproblematic mediator in the discourse on limits. When he referred to $\lim_{x \to x_0} f(x)$, he mentioned the x values approaching x_0 (Table 1, [2], [3]), which signified limit as a process through dynamic motion[1]. He paused at this stage and then focused on the behavior of the function of which he drew an example (Figure 1). In other words, the signs $\lim_{x \to x_0} f(x)$ prompted the instructor to examine the behavior of the function, which he did so using words signifying motion, such as getting "closer and closer to" and "approach" (Table 1, [4]), signifying limit as a process. After the examination of the function's behavior, Dr. Brenner completed the right-hand side of the limit notation as being equal to the number L (writing $\lim_{x \to x_0} f(x) = L$) that signifies the number (product) obtained at the end of the limit process. He then shifted his operational word use signifying motion—the process view of limit—to those signifying proximity (Table 1, [5], [7]) consistent with the formal theory of limits—the static and product view of limit—but these utterances did not necessarily cohere with his action of finding the limit. Said differently, although both a process and a product view of limit were present in his discourse when attending to the limit notation, he did not explain these two aspects explicitly in the classroom.

Similar patterns emerged in the instructor's use of the limit notation as a routine when computing limits. Table 2 shows Dr. Brenner's utterances when he computed $\lim_{x \to 2} \frac{1}{x-1}$.

Table 2: Dr. Brenner's Utterances When Computing $\lim_{x \to 2} \frac{1}{x-1}$.

What is said	What is done
[1] What is this limit?	He writes $\lim_{x \to 2} \frac{1}{x-1}$ on the board.
[2] Let's see. If x gets closer and closer to two, this quantity gets closer and closer to one over two minus one.	He shows $x \to 2$ and then shows $\frac{1}{x-1}$. He says these verbally and does not write anything on the board.
[3] It [the function] is very close to one over one.	
[4] So the closer x gets to two, the closer this will get to one.	He shows $\frac{1}{x-1}$.
[5] This limit is one.	He writes $\lim_{x \to 2} \frac{1}{x-1} = 1$; no graph is drawn.

In this computation, Dr. Brenner's initial use of the limit notation prompted him to explore the behavior of the function near the limit point. While doing so, he uttered operational phrases like getting "closer and closer to" and getting to the limit value (Table 2, [2], [4]) and talked about limit as a process. At the end of his investigation, his word use was objectified as he referred to limit as the product (a number) of the process (Table 2, [5]) and wrote the limit being equal to 1 in the limit notation. Note also that his deictic references were about the function values (Table 2, [2-4]) when he attended to the behavior of the function whereas his reference

was about the limit in the end (Table 2, [5]), which was similar to the patterns observed in the context of the informal definition of limit. Dr. Brenner's deictic utterances in Table 1 ([4], [5]) are about the function whereas his final utterance (Table 1, [7]) is about the limit of the function.

These two examples are representative of the instructor's use of the limit notation throughout the 64 limit computation problems on which he worked. In the contexts of the informal definition and computing limits, his routine of using the limit notation can be summarized as follows:

1. The symbol $\lim_{x \to x_0} f(x)$ prompted him to examine the behavior of $f(x)$. When reading $x \to x_0$ in the limit notation, he mentioned x values approaching x_0, consistent with the textbook's introduction of the limit notation.
2. He then examined the behavior of the functions uttering phrases of the form: What does the function values "approach/get closer and closer to/become/go to" as the x values "approach/get closer and closer to/become/go to" x_0 ? (Symbolically represented as $f(x) \to ?$ as $x \to x_0$).
3. At the end of the examination of the behavior of the functions, he focused on the end result of the process and talked about the value L (if it existed) as being *equal* to the limit of the function (Symbolically represented as $\lim_{x \to x_0} f(x) = L$).

The stages of Dr. Brenner's routines and word use suggest that he endorsed the narrative *limit is a process* when he referred to the left-hand side of the limit notation and when he explored the behavior of functions near the limit points. In contrast, he endorsed the narrative *limit is a number*—a product view of limit—when he completed his examination of the behavior of functions and completed the right-hand side of the limit notation.

The routines in the instructor's use of the limit notation can be elicited from his actions and word use, but a critical characteristic of these routines is that they are meta-level, i.e., the metarules forming the routines of discourse are often tacit (Sfard, 2008). That the instructor did not make his discourse on the limit notation an explicit topic of discussion in the classroom is consistent with the tacit nature routines.

Although the students' use of the limit notation is not a main focus of this study due to space constraints, a brief summary is provided to give some information about the role of the notation in their discourse. The analyses of the surveys and the interview sessions showed the following stages in students' routines of using the limit notation:

1. The symbol $\lim_{x \to x_0} f(x)$ prompted the students to examine the behavior of $f(x)$. In most cases, the students did not explicitly talk about $x \to x_0$ in the limit notation. When they did, they mentioned x values approaching/getting closer and closer to x_0 .
2. They then examined the behavior of the functions uttering phrases of the form: What does the function values "approach/get closer and closer to/become/go to" as the x values "approach/get closer and closer to/become/go to" x_0 ? (Symbolically represented as $f(x) \to ?$ as $x \to x_0$).
3. At the end of the examination of the behavior of the functions, the students did not focus on the end result of the process. Instead, their word use remained operational. When asked what the limit of $f(x)$ is, they uttered phrases of the form: "It approaches/gets closer and closer to L" indicating that they did not differentiate the function values approaching L from the limit value being *equal to L*.

Students' routines as well as word use suggested that, in the context of the limit notation, they only endorsed the narrative *limit is a process* and did not consider *limit as a number*. This finding provides some evidence that, unlike the instructor, the students did not consider the process and product aspect of limit as inherent in the symbolism. During the surveys and the interview sessions, there was no clear evidence in students' discourse indicating their awareness of the different meanings the limit notation supported.

These results do not suggest that the instructor's discourse caused the students' difficulties with the limit notation (for which the study has no evidence). Yet, the findings suggest that there was clearly some miscommunication between Dr. Brenner and his students in terms of the limit notation and the different meanings of limit it supported.

Dr. Brenner's use of the limit notation took into account both the process and product aspect of limit. Further, he kept the two aspects of the notation distinct throughout his discourse, as evidenced by his word use and the different narratives he endorsed. However, accurate use of language and symbol use did not seem to be enough for his students to infer the subtleties of symbol use. Dr. Brenner may have enhanced classroom communication had he spent more time introducing the limit notation and unpacking the meaning of the notation in different contexts related to limits.

Conclusion and Discussion

In Dr. Brenner's discourse, the instances he used operational and objectified utterances about limit were clearly distinguished when referring to the limit notation as a visual mediator of discourse. In other words, the instructor maintained the differences between the process and product aspects of the limit at all times as indicated by Gray and Tall (1994). However, such a distinction may not be present in the students' discourse, especially during the initial stages of their learning.

The ambiguity of process-product duality is present in the limit notation. Such ambiguity has its advantages in mathematical thinking in terms of the flexible use of symbols and reduction of cognitive load when thinking about mathematical objects (Gray & Tall 1994, 2001; Sfard, 1991). On the other hand, researchers also argue that such duality creates many challenges for learners. The findings of the study indicate that it is important not to take the communicative power of symbols, or any visual mediator, for granted. In other words, teachers need to unpack the duality inherent in symbolism and the different mathematical meanings that can be generated from the same symbol.

Experts of mathematics are often aware of the dual nature of symbols. We see in Dr. Brenner's discourse how he differentiated between the process and product aspects of limit depending on the mathematical situation, indicating the flexibility of his use of the limit notation. The routines with which he substantiated the endorsed narratives about limit (*limit is a process* and *limit is a number*), however, remained implicit to the students in his classroom. Making the tacit metarules constituting the routines explicit topics in the classroom can be useful in unpacking the meaning-making and negotiation processes behind mathematical symbolism. In addition, it can help highlight the instances of communication and miscommunication between students and teachers.

Note that, although the focus of this study was on the limit notation, the discussions about the instructor's discourse were meaningful only when the visual mediator was analyzed in conjunction with his word use, routines, and endorsed narratives. Sfard's (2008) framework is useful to show the interrelations among all the elements of mathematical discourse, indicating

that successful participation in the discourse on mathematics requires the orchestration of all these elements. These elements of discourse can also help teachers in terms of what they may need to focus on in order to enhance mathematical communication in their classrooms.

Endnote

[1] The instructor's utterances about the limit notation were consistent with those of the textbook he used (Thomas' Calculus, 11th edition).

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PROMOTING STUDENT QUESTIONS IN MATHEMATICS CLASSROOMS

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Question asking is an important part of the learning process. One of our goals as educators is to encourage students to ask questions that focus on developing conceptual knowledge. To do this, we first need to better understand the factors involved in student question asking. This study looks in depth at one high school mathematics teacher who promotes rich, mathematical question asking among her students. She does this by creating a safe learning environment, shifting responsibility to learn onto her students, by modeling good question asking, and through explicitly valuing sense-making. This study hopes to help teachers and educators think about how to promote more student questions in mathematics classrooms.

Keywords: Classroom Discourse, High School Education

Objectives

Question asking is an important part of the learning process. The NCTM *Principles and Standards for School Mathematics* (2000) states that "students gain insights into their thinking when they formulate a question about something that is puzzling to them." The process of asking a question allows the student to verbalize her current understanding of a topic and connect that knowledge with other ideas (Marbach-Ad & Sokolove, 2000). Stipek and colleagues (1998) found that when a student asks a question, she is motivated to learn. Through student questions, teachers can determine how students are thinking about the content of instruction, giving them an opportunity to tune instruction to the intellectual needs of their students (Rop, 2002). Teachers identify student question asking as critical to successful participation in the educational setting (Salend & Lutz, 1984). Clearly, one of our goals as educators is to encourage students to be inquisitive and ask questions. To do this, we first need to better understand the factors involved in student question asking.

There are multiple studies on student question asking in science education (see Marbach-Ad & Sokolove, 2000; Kelling, Polacek, & Ingram, 2009; Rop, 2002 and 2003 for examples) and in English Language Arts instruction (see Rosenshine, Meister, & Chapman, 1996 for a review), and there are studies that look at *teacher* questioning patterns in mathematics classrooms (Boaler & Brodie, 2004) but few studies specifically address *student* question asking in mathematics classrooms. This study attempts to understand what questions students are asking in their mathematics classroom as well as how mathematics teachers can promote question asking among students.

Webb et al. (2009) showed that teachers' instructional practices are related to the students' classroom dialogue. They write, "It is imperative not only to analyze the dialogue among students, but also to examine student participation in relation to teacher participation and the context of the classroom." Hufferd-Ackles et al. (2004) showed that math talk among students changes as teachers change their teaching. Similarly, my study attempts to understand how a teacher's practice affects the student questioning in her classroom. Specifically, I ask how does one high school mathematics teacher promote question asking among her students?

Methods

A Case Study Approach

In the spirit of Milo Small (2009), I choose a unique teacher for my case study, rather than attempting to find an average teacher or a representative case. According to Small, unique cases are especially interesting because they provide ways of developing or extending theories. For this study, I choose to analyze Susan Santiago's[1] teaching practice. Ms. Santiago is a National Board Certified Teacher, a Presidential Awardee for Excellence in Mathematics Teaching, and a member of the state board of the California Mathematics Council. At the time of the videotaping, Ms. Santiago had been teaching high school mathematics for 22 years. In addition to her excellent reputation and vitae, I choose Ms. Santiago because of her curriculum: the *Integrated Mathematics Program* (IMP). This curriculum encourages students "to experiment, investigate, *ask questions*, make and test conjectures, reflect, and accurately communicate their ideas and conclusions [emphasis mine]" (Webb, 2003). Most U.S. high school mathematics teachers use a more "traditional" curriculum that may or may not promote student question asking. Because the IMP curriculum was specifically designed to encourage student question asking, Ms. Santiago's classroom is an ideal place for my research.

Data Sources

One way of capturing the many details of teaching is through videotaping instruction. Video observations allow researchers to go back and observe a class over and over again, a process that helps researchers become more deeply aware of what is going on each time through the videos. This "progressive refinement of hypotheses" (Derry, 2010) enables researchers to modify, hone, and solidify ideas each time through the data. Engle (2007) wrote that going through multiple iterations of hypothesis generation and evaluation leads to greater robustness of findings and increases the likelihood that findings will be replicated in other contexts.

A research team videotaped Susan Santiago's mathematics Algebra and Pre-Calculus classes throughout the 2000-2001 school year. One camera stood in the back of the room and Ms. Santiago wore a microphone throughout the videotaping. Of the approximately 50 video tapes available, I transcribed fourteen lessons where there was full-class instruction and discussion (vs. small group work). I transcribed six videos from the beginning of the school year in order to see how norms around student question asking were established. Then I chose eight more videos so that they would be well spaced throughout the year.

Student Question Categories

While question asking is critical to the learning process, not all questions are of equal learning value. There are students who use questions to disrupt the learning process when they try to show off or derail a lesson. There are students who ask questions for procedural information or just to get the right answer. Then, there are students who ask questions as they seek conceptual clarification and understanding. To begin to study the variety of questions Ms. Santiago's students ask, I used a student question categorization developed by Good (1987) and his colleagues (see Table 1). Good's categories are meant to capture the wide range of questions students ask as well as be a tool that researchers can use to measure the quantity and quality of student questions in a classroom. I include an example of each type of student question in the table. I also added a category for questions about technology.

Table 1: Coding Schema for Student Questions in Ms. Santiago's Mathematics Classes

Type of Question	Example from transcripts

[1] Pseudonym

Explanation: This type of question requests meaning or reasons that help in understanding a mathematical concept, idea, task, or procedure.	Example: “Why are you finding the inverse?”
Information: This type of question seeks specific, factual, academic information.	Example: “Are all the lines called curves on a graph?”
Clarification: This type of question requests clarification of information, procedures, comments, or tasks provided by the teacher or others.	Example: “I don’t understand that last part. Why do we need the 225?”
Confirmation: This type of question seeks confirmation of a completed student response, procedure, or task.	Example: “Don’t we need two measures to find the minor arc and don’t we only have one?”
Procedural: This type of question concerns classroom procedures. This category does not identify questions about procedural mathematics. Rather, this category identifies questions that have to do with the way the class is organized, performance measures, grading procedures, etc.	Example: “Would you put this kind on the test?”
Diversion: This type of question diverts the teacher’s or others’ attention from the task at hand.	Examples: “Do colleges prefer cursive or printing?” or “Can I go to the bathroom please?”
Attention: This type of question draws attention to the question asker.	Examples: “Do you want me to try to explain?” or “Can I blow on [the dice]?”
Technology: This type of question requests help in using technology to solve the problem at hand.	Example: “How do you type in [to calculator] the fourth root?”
On-task curiosity: This type of question displays curiosity tangential to the immediate task.	Examples: “Are you going to do one with like 36 sides?” or “I have a question about when I would use a t-table, like at a job.”
Unknown: This code was used when a student question was inaudible or undecipherable from the videotape.	

Coding

For each of these videos, I went through two levels of transcription. The first level’s purpose was to transcribe as much of the dialogue as possible and to timestamp and highlight all student questions that occurred during whole class discussions. The second level was to more carefully capture the exact wording of each student question as well as the dialogue approximately one minute before and one minute after each question. Next, I counted and categorized all student questions from the transcripts (see Figure 1 for results). A fellow researcher counted and categorized student questions in two of the transcripts and our reliability was 71%. Next, I open coded (Glaser, 1978) each line of the transcripts. Soon I began to see patterns and themes and began to engage in meaning making (Geertz, 1973) of the data. As I went back and read through the transcripts again and again, I revised and solidified my thinking about what components of Ms. Santiago’s teaching promote question asking among her students. As these hypotheses developed, I continually asked myself whether my inferences were logical (Small, 2009). All this led me to hypothesize four important components of Ms. Santiago’s classroom that encourage

students to comfortably participate in inquisitive mathematical communication. These components are 1) a safe classroom environment, 2) a shift in authority from teacher to student, 3) teacher modeling of question asking, and 4) explicit value of sense-making in the classroom.

Results

The following table shows the types and quantity of student questions in Ms. Santiago's mathematics classroom. The sheer number of student questions is remarkable. Each transcribed lesson is only 50 minutes long, and thus Ms. Santiago's students ask questions, on average, every two minutes throughout instruction. It is also interesting to note that the Pre-Calculus students asked more *Explanation*, *Information*, and *Clarification* questions (70% vs. 43%), indicating that they were more engaged in the mathematical material at hand than the Algebra students. The Algebra students asked more confirmation questions (10% vs. 5%), indicating that they were more interested in the answer (vs. process of getting there). The Algebra students also asked more procedural questions (16% vs. 9%), an indication that they were more interested in classroom organization, grading procedures, upcoming tests, etc. than the more advanced students. The Algebra students asked more diversion questions (13% vs. 3%), indicating that they were more easily distracted than the Pre-Calculus group, and they also asked more attention questions (6% vs. 1%) which, when looking closely at the transcripts, indicates an eagerness to participate in learning activities. There were also important qualitative differences between the IMP1 and IMP4 classes. For example, in the Algebra class, confirmation questions often indicated that the student asking the question *assumed he was wrong* and sought verification of this. Two examples of such questions are: "It's not nine, is it?" and "I'm just wondering because you told me we didn't have to go up past six sides and…I'm just wondering if we did it wrong." In the Pre-Calculus class, however, the student asking the confirmation question often *assumed he was right* and sought verification of this. Two examples of such questions are: "Wouldn't 7.74 minus 10.88 be the period?" and "So it's just 96 divided by 2?"

Question Category	Algebra Average	Percentage	Pre-Calculus Average	Percentage
Explanation	4.8	18%	5.9	22%
Information	2	8%	3.6	13%
Clarification	4.5	17%	9.5	35%
Confirmation	2.5	10%	1.4	5%
Procedural	4.3	16%	2.4	9%
Diversion	3.3	13%	0.8	3%
Technology	0.3	1%	1.4	5%
Attention	1.5	6%	0.3	1%
On-task curiosity	1.3	5%	1.1	4%
Unknown	1.8	7%	0.8	3%
Total	26	100%	27.2	100%

Figure 1: Student Questions in Ms. Santiago's Mathematics Classes

Counting and categorizing Ms. Santiago's students' questions helped me understand *what* kinds of questions her students were asking, but I was also interested in *how* Ms. Santiago is able to promote question asking among her students. As mentioned in the Methods section, I found

four main components of Ms. Santiago's instruction that seemed to promote student questions asking. First, I explain each component and give examples from the transcripts. Next, I provide Figure 2 that contains the breakdown of the four components in Ms. Santiago's classroom.

Safe Classroom Environment

In order to learn something new, a student must enter unknown intellectual terrain, and it is unrealistic to expect that the student will immediately understand a new concept or idea. Throughout her instruction, Ms. Santiago tries to make her students feel safe as they engage in the often messy and unpredictable process of learning. In fact, Ms. Santiago makes reference to the importance of making mistakes, of being comfortable with intellectual ambiguity, and the value of asking questions on average 4.9 times per hour. When a student is up at the board presenting her solution to a problem, another student asks, "Wouldn't that not work because it's seven feet above the fence?" Ms. Santiago responds by asking the class, "What do we think?" The student at the board says, "I don't know. That's what I did." Then Ms. Santiago says, "Her answer is coming out close to the correct answer. Why do you think that is?" No one answers, so then Ms. Santiago says the following:

> Okay, you know what, my philosophy is. She's got a lot of good math going on here and he's got an excellent question. So I love it when people go up and present. She's very close to the right answer and we're able to ask ourselves is it right and what's going on here, so very good job. Did anyone do it a different way?

This statement promotes a classroom culture where it is safe to make mistakes, not know things, and ask questions. After another student has been at the board and has explained her solution and answered other students' questions about her solution, Ms. Santiago says, "so that was a very nice job…some people go up and do some of the problems and some people ask questions and that is what we want to do." This safe learning environment allows students to publicly convey their only partially developed understanding without suffering negative consequences.

Shift in Authority

Another way that Ms. Santiago promotes student question asking in her classroom is through a shift in authority from herself to her students. Throughout her classes, Ms. Santiago conveys to her students that it is their responsibility to internalize mathematical concepts. Instead of standing in front of the classroom and lecturing, Ms. Santiago guides and supports her students as they develop the confidence to learn for themselves. On average, Ms. Santiago mentions something of this sort 3 times per hour. The following are two examples from the transcripts of how Ms. Santiago shifts the authority to her students to learn and make sense of the mathematics.

> *Ms. Santiago*: So now that this group is finished, we need to go over minimum and maximum.
> *Student*: Can I do it?
> *Ms. Santiago*: Okay, and *the rest of you need to decide* if the numbers are right or wrong.
> *Student*: What do we work up to?
> *Ms. Santiago*: Well *you decide*. You know the game.

This shift in authority encourages student questions because as students become more active in their own learning, they will have questions—the very act of trying to learn something new means that there will be things a student does not understand, and whether stated aloud or not, questions form in the students' minds.

Modeling of Good Questions

Another way Ms. Santiago promotes question asking among her students is through asking good questions herself. On average, Ms. Santiago asks 57 questions per hour. Her questions not only indicate what kind of thinking and learning she values, but also model what kinds of

questions she expects from her students. Table 2 shows a few of the 775 questions I listened to Ms. Santiago ask while doing this research.

Table 2: Examples of Ms. Santiago's Questions

- How would we know? What will it look like?
- Okay what am I trying to find?
- Why isn't that negative?
- Why do I know this is constant acceleration? Can you prove to me that you are going faster than 25 miles per hour?
- How do you know that? Who has something to add to the discussion?
- So how am I going to write this mathematically? Is there another way she could have written the answer instead of y=n times n?
- Okay, what do you think is the purpose of the worksheet? Earlier in the year you heard me say this was a bad bad bad mistake. (She circles the answer.) Why is that?

These questions indicate that while Ms. Santiago does care about mathematical correctness, she values mathematical understanding even more. Ms. Santiago's students are exposed to rich mathematical question asking, and through this, they learn to ask good questions themselves.

Sense-making Valued

Yet another way Ms. Santiago promotes student question asking in her classroom is through valuing sense-making. On average, Ms. Santiago refers to the importance of making sense of the mathematics 4.25 times per hour. This frequency makes her intention clear: she does not want her students to simply memorize formulas or procedures. Instead, she wants them to internalize and make sense of the mathematics. The following are some quotes from her classes that show how she expresses this expectation to her students:

> "This particular problem you'll find again and again and again. So you *if you don't understand how to do it, it's really important that you ask questions* …okay, very nice job"

Later in the same class, Ms. Santiago uses a visual representation of the mathematics and says,

> I'm trying to show you where it comes from in the picture. Do you see where this comes from the picture? *I want to show you how things are coming from the picture, not just give you a formula to plug in.* Do you see how it relates to this picture?

Ms. Santiago wants her students to understand mathematics. It is not enough for her students to blindly use formulas. She teaches them to ask themselves whether or not things make sense.

The following table shows the frequency of these four components in Ms. Santiago's classrooms:

Incident	Algebra Average (times per hour)	Pre-Calculus Average (times per hour)	Total Average (times per hour)
Safe-learning environment	6	3.75	4.88
Shift in authority	3.5	2.5	3
Ms. Santiago's questions	63.5	50.5	57
Sense-making emphasized	4.25	4.25	4.25

Figure 2: Breakdown of the Four Components in Ms. Santiago's Classrooms

Discussion and Significance

Traditionally, the school teacher has been seen as the source of knowledge and the one responsible to ensure that her students actually learn. In science and ELA classes, it has been shown that if a teacher is able to create a shift that places the source of knowledge in the subject itself and places the responsibility to learn and understand onto individual students, then students ask more questions as well as a wider variety of questions (Chin & Brown, 2002; Oyler, 1999). Ms. Santiago's classroom shows us that this is true in mathematics classrooms as well.

Shifting responsibility to learn onto students pushes them to engage in sense-making, a process that cultivates questions. In order for students to actually vocalize these questions, however, they must feel safe in the classroom and must know that it is productive to ask questions. Pearson and West (1991) found that students often don't ask questions because they fear a negative reaction from their teacher. Ashcroft (2002) found that how a teacher responds to a student question can either heighten or allay students' math anxiety. In Ms. Santiago's classroom, she values student questions and carefully and thoughtfully answers them. This safe learning environment makes it easier for students to reveal misconceptions and gaps in their understanding.

We see that shifting authority to learn onto students' shoulders compels them to engage in sense-making, a process that generates questions, which are vocalized if a safe learning environment exists. Ms. Santiago pushes this a step further by literally showing her students *how* to ask good questions. Her students are continually bombarded with her mathematical questions, and her students respond to this environment by asking many questions of their own.

Some teachers attempt to tightly control everything that happens in their classrooms. They carefully plan their lessons and rarely deviate off course. There is minimal room for autonomous learning in these classrooms (Greeno, 1991). Because students are inured to transmission style teaching and learning, they are not likely to ask as many questions in class because they are not likely to see themselves as inquisitive participants in the learning experience. Unfortunately, this type of teaching is particularly prevalent in U.S. mathematics classrooms. The case of Ms. Santiago, however, shows different possibilities for promoting participation and engagement in the classroom that will likely lead to more conceptual student questions.

Limitations

This study reports on the experience of only one teacher, and future studies are needed to ascertain whether the four components proposed in this study lead to increased question-asking in other classrooms with different teachers and students. A larger study could look at a diverse sample of teachers and classrooms to determine if there are reliable associations between the four components and student question asking. Also, we need a categorization for student questions designed specifically for mathematics classrooms. While Good's (1987) categorization provided a place to start looking at quantity and quality of student questions, it became very clear while I was doing this research that a more specific categorization is needed.

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INNOVATION IN EARLY MATH EDUCATION: THE WHOLE TEACHER APPROACH TO PROFESSIONAL DEVELOPMENT

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In this presentation, we introduce a conceptual framework for in-service professional development—the Whole Teacher approach, which attends simultaneously to the attitudes, knowledge, and practice of a teacher's growth. Putting the framework in operation, we describe a project designed to improve teachers' competence and increase children's performance in early mathematics. Utilizing a quasi-experimental design, pre- and post-measures with intervention and comparison groups have been collected. The results indicated that significant growth in children's mathematical performance favored to the intervention group. The discussion focuses on the significance of the Whole Teacher approach to teacher professional development.

Keywords: Early Childhood Education, Mathematical Knowledge for Teaching, Teacher Education/In-service Professional Development

Objectives of the Study

A large body of literature specifies what constitutes high quality in-service professional development (PD) (Borko, 2004; Darling-Hammond, et al., 2009; Desimone, 2009; Guskey, 2003; Wei, Darling-Hammond, & Adamson, 2010). Among the major factors noted are (1) the use of an on-going process instead of a one-shot, cursory workshop, (2) emphasizing collaborative participation rather than teachers working in isolation, (3) tailoring training to meet the specific needs of teachers as opposed to such general goals as improving teaching and learning, (4) providing hands-on opportunities to construct new knowledge instead of lectures that target knowledge transmission, and (5) connecting workshop-developed knowledge to classroom practice through coaching or follow-up sessions.

Augmenting to the effective PD strategies in the literature, this presentation describes a conceptual framework for in-service PD, namely, the Whole Teacher approach (Chen & McCray, 2012). As a conceptual framework, it provides an over-arching understanding of "what works" and "why works" in addition to "how to ensure PD works". To articulate the Whole Teacher approach, we first define its three major components: attitudes, knowledge, and practice. We then describe a PD program aimed at improving the quality of early math education in urban school settings utilizing the approach. We further present outcome data that demonstrates its positive impact on children's math achievement. The presentation concludes with a discussion of the significance of the study to the field of teacher professional development.

Theoretical Framework

The Whole Teacher approach attends simultaneously to the social/emotional, cognitive, and behavioral aspects of a teacher's growth. A significant departure from the traditional approach to PD that speaks primarily to teachers' acquisition of knowledge and skills, the Whole Teacher framework emphasizes promoting all aspects of a teacher's development, including her attitudes, knowledge, and practice (Author, 2006a, 2006b; Author, 2012).

Teacher attitudes about a content area or an instructional practice are rarely addressed in PD sessions despite of the fact that they are closely related to teachers' knowledge acquisition and classroom practice (Author, 2006b; Pianta, et al., 2005; Vartuli, 2005, Wilkins, 2008). In the field of early mathematics, a large portion of early childhood teachers describe themselves as math phobic (Copley, 2010). Such attitude toward mathematics leaves a strong imprint on children's minds (Beilock, Gunderson, Ramirez, & Levine, 2009). By explicitly addressing early childhood teachers' attitudes toward math, PD is more likely to affect positive changes in teaching (Clements & Sarama, 2009).

Knowledge is the primary focus of most PD programs. Early mathematics—the mathematics that precedes arithmetic and the use of symbol systems for describing mathematical operations—is not widely understood (Author, 2011). Early childhood teachers are trained as generalists. It is unlikely that they can provide quality early math education without the understanding of foundational math concepts in relation to young children's development (Sarama & DiBiase, 2009).

An ultimate measure of PD effectiveness is the classroom teaching. As teachers apply knowledge and methods learned through PD programs, they inevitably encounter unexpected challenges that require adaptations to make the practices effective (Darling-Hammond, et al., 2009). When early childhood teachers are further asked to change practice and address a weak spot such as math that they have long felt unconfident and under-prepared in their teaching, ongoing and individual support is vital (Copley, 2004).

Modes of Inquiry

Participants and Context

In partnership with Chicago Public Schools (CPS), Author has provided PD training in mathematics to approximately 80 Head Start, pre-kindergarten and kindergarten teachers each year for the last four years. These teachers came from 150 different schools and they serve primarily low-income and minority children.

The Author's PD program includes three components: (1) Learning labs—early math instructors lead these yearlong, interactive learning sessions focusing on teachers' understanding of foundational mathematics; (2) On-site coaching—between learning labs, teachers work with a math coach in their classrooms to plan and reflect on their teaching; and (3) Guided classroom implementation—teachers practice "mathematizing" classroom experiences under the guidance of coaches (see Figure 1).

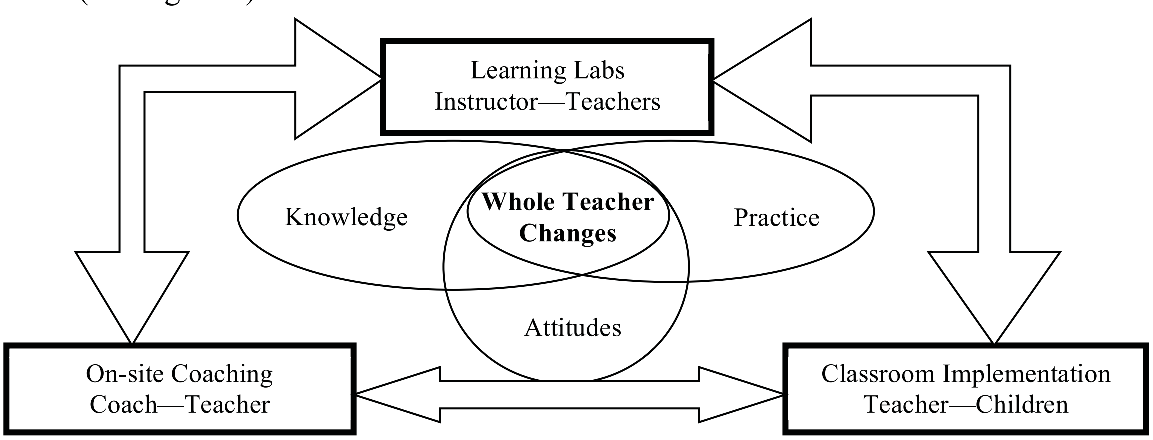

Figure 1. PD Program Conceptual Framework and Components

In response to the special needs of early childhood teachers, numerous strategies are used throughout the PD components, such as engaging in adult learning experience in mathematics investigation, using children's book as an entry point for math learning, introducing structured math research lessons for classroom implementation, and forming a community of learners for collaborative learning and reflective practice, to name a few. Central to all strategies is teachers' understanding of the Big Ideas in early mathematics and make use of them in classroom teaching. Big Ideas are "clusters of concepts and skills that are mathematically central and coherent, consistent with children's thinking, and generative of future learning" (Clements & Sarama, 2009). For example, attributes can be used to sort collections into sets is a Big Idea for algebraic thinking in early math. All measurement involves fair comparison is a Big Idea for measurement. An essential tool for mathematical understanding, teachers can use Big Ideas to organize the classroom environment, plan meaningful activities, engage in curriculum analysis, and articulate the underlying purpose of students' work (NRC, 2009).

Data Collection

The Whole Teacher framework guides our program evaluation. Figure 2 illustrates the logic model of the EMC PD program, which addresses two basic research questions of the program: To what extent does the EMC PD change teachers' attitudes, knowledge, and practice in early mathematics and how do these changes impact child outcomes? Utilizing a quasi-experimental design, pre- and post-measures with intervention and comparison groups have been collected.

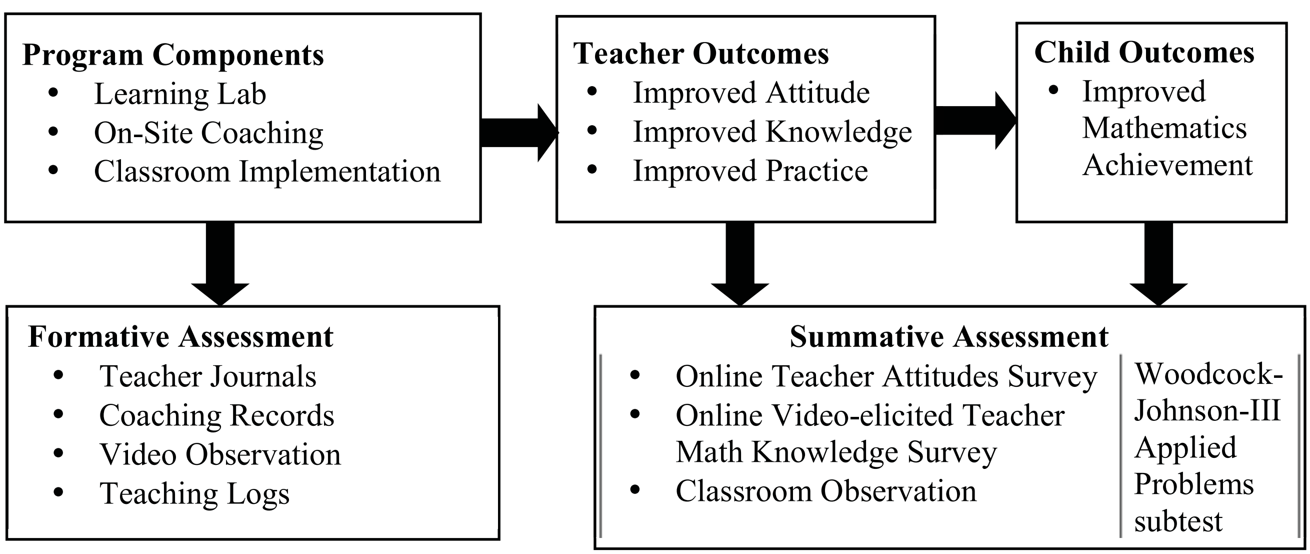

Figure 2. Logic Model of Professional Development

For teacher change, three instruments are used: (1) a teacher survey of Attitudes, Beliefs, and Confidence in Early Math (ABC-EM); (2) an online survey of Pedagogical Content Knowledge in Early Math (PCK-EM) utilizing video stimuli, and (3) a classroom observation of High Impact Strategies in Early Math (HIS-EM). Together, the three tools are used to assess changes in teachers' attitudes, knowledge, and practice in early math. Due to space limitation, the results are reported in this presentation.

For child learning outcome, children's mathematical abilities were measured using Subtest 10 of the Woodcock-Johnson III (WJ-III) Achievement Battery. A total of 154 three- to five-

year-olds participated in the study. Of these children, 91 were randomly selected from 12 participating classrooms and served as the intervention group. An additional 63 children randomly selected from matched classrooms served as the comparison group.

Data Analysis

For purposes of report, scores on WJ-III subtest 10 are converted to age estimate scores; that is, scores are converted to the average age in months of children in the normed sample who achieved them. Changes in WJ-III estimated age were calculated for each child. Two-level Hierarchical Linear Modeling (HLM) was used to determine how much of the variance among these changes could be attributed to teacher participation in the intervention. The HLM model is presented below

Level-1 Model

$$Y = \beta_0 + \beta_1(\text{T1 WJ Age Estimate}) + \beta_2(\text{Hispanic}) + r$$

Level-2 Model

$$\beta_0 = \gamma_{00} + \gamma_{01}\,(\text{Intervention}) + \mu_0$$
$$\beta_1 = \gamma_{10}$$
$$\beta_2 = \gamma_{20}$$

Results

Results showed that participation in the intervention significantly predicted changes in WJ-III age estimates (see Table 1). Compared to comparison classrooms, children in intervention classrooms showed an average of 3 months additional growth in WJ age estimate score over the intervention year (p<.03). The growth of children who began the school year behind national norms was closer to five additional months of learning. These results point to the positive impact of the program on children's learning and its particularly significant effects on the children most in need of help.

Table 1: Results of HLM on WJ III Age Estimates (in months), Controlling for Pre-test Scores and Ethnicity, and Using Participation as a Predictor

Final Estimates of Fixed Effects	Coefficient	*SE*	*T*	*df*	*p*
For Intercept 1, β_0					
Intercept 2, γ_{00}	21.7635	3.1903	6.822	18	0.000
Participation, γ_{01}	3.0609	1.309	2.337	18	0.031
For Pre-Test WJ-III, β_1					
Intercept 2, γ_{10}	-0.3184	0.0686	-4.642	147	0.000
For Ethnicity, β_2					
Intercept 2, γ_{20}	-1.5718	0.9160	-1.716	147	0.088

Discussion

Many factors contributed to the success of our early math PD program. At the top is our clearly defined conceptual framework, which provides a basis for us to set goals, select instructional strategies, and evaluate outcomes. In the Whole Teacher approach attitudes, knowledge, and practices play equally important roles in teacher professional development. The focus on multiple dimensions offers teachers multiple pathways to learning. For some teachers, attitudes will be the most important first step; for example, overcoming fear of failing in teaching mathematics. For others, classroom practice will be the key, as when children's excitement and

interest in learning mathematics affects teachers' attitudes. Knowledge, too, can play a pivotal role, as an "aha" moment in a PD session makes a teacher feel competent enough to try something new. Accessing multiple learning pathways allows PD to build on teachers' motivations and respond to their needs, rather than requiring that all teachers follow the same course of learning.

For early childhood teachers, there is one additional benefit to the explicit adoption of the Whole Teacher framework for PD. It is readily understood and meaningful to early childhood teachers because it resembles a widely accepted principle in early education; namely, the importance of addressing the development of the "whole child" (Copple & Bredekamp, 2009). Familiarity with the whole child concept helps teachers reorient how they see themselves and welcome and integrate shifts in their attitudes, knowledge, and practices that will make them effective early math teachers.

Guskey (1995) succinctly states the value and necessity of PD: "Never before in the history of education has there been greater recognition of the importance of professional development. Every modern proposal to reform, restructure, or transform schools emphasizes professional development as a primary vehicle in efforts to bring about needed change" (p.1). Effective PD updates teachers' content knowledge, exposes them to new teaching strategies, sustains their teaching effectiveness, and prompts continuous growth (Desimone, 2009; Hawley & Valli, 2001). For PD to deliver on its promise in education, the field needs not only evidence-based effective strategies, but also conceptual frameworks that are grounded in theories of teacher change and help explain and predict for what works in teacher professional development.

The Whole Teacher framework is one such attempt. Our experience speaks of its promising future. The framework is based on the premise that teacher attitudes, knowledge, and practices interact and influence each other. It promotes PD strategies that build on the interrelationships and offers teachers multiple ways of learning, doing, and succeeding. Our work focuses primarily on math education during early childhood years; we believe however that the framework applies for other content areas across the age range. Funded by the Department of Education, we now engage in a multi-year PD in math education with teachers of pre-kindergarten through grade 3 in eight CPS public schools. The longitudinal data of teacher change in attitudes, knowledge, and practice as well as child outcomes will provide more empirical evidence to test the power of the Whole Teacher approach to teacher professional development.

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INCIDENTS OF INTRUSION: DISRUPTIONS OF MATHEMATICS TEACHING AND LEARNING BY THE TRADITIONAL WESTERN WORLDVIEW

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This paper reports on a theme, the intrusion of the Traditional Western worldview, emerging from an ongoing study of the impact of teachers' engagement in the Transreform approach to the teaching and learning of mathematics on students' affective and cognitive responses to and achievement in mathematics. Newly theorized (Russell & Chernoff, 2011, 2012), the Transreform approach occurs when the teaching and learning of mathematics is grounded within an Indigenous worldview. This paper identifies and analyzes incidents captured within the study's data that demonstrate ways in which the Traditional Western worldview can intrude on an Indigenous worldview and upon the teaching and learning of mathematics.

Keywords: Elementary School Education, Teacher Beliefs

A recently emerging thread of research has theorized about the relationship between aspects of mathematics education and two different worldviews: the Traditional Western worldview and an Indigenous worldview. In particular, this research has theorized about the relationships between the two worldviews and constructivist teaching and learning in mathematics (Russell & Chernoff, 2011); the math wars (Russell & Chernoff, 2012); and the marginalization of Indigenous students within mathematics teaching and learning (Russell & Chernoff, online first). Emerging from this body of work, is the proposition that there exists a third option within the math wars context, sitting outside of the spectrum of the two camps of traditional and reform approaches to mathematics teaching and learning, that of the Transreform approach (Russell & Chernoff, 2012, online first). The Transreform approach to the teaching and learning of mathematics (described in detail below) theorizes that it is possible to move beyond the dichotomy of the math wars, and in doing so, provide a space in which the teaching and learning of mathematics becomes accessible to all students.

The purpose of this paper is to report on some initial findings of a study designed to determine the impact of the Transreform approach to the teaching and learning of mathematics on both the affective and cognitive domains of students in elementary mathematics classrooms. The findings presented in this paper however, focus not on the results of the change, but on how teachers experience the change to the Transreform approach. In particular, this paper will report on instances where the Traditional Western worldview has disrupted the teaching and learning processes, and the struggle of teachers to recognize these intrusions for what they are and to determine how to respond to them in ways that are grounded within an Indigenous worldview.

In upcoming sections of this paper, the setting of this study, methods used to collect data and analysis of that data will be presented. Prior to these discussions however, a quick explanation of three key concepts, that of the Traditional Western worldview, an Indigenous worldview and the Transreform approach to the teaching and learning of mathematics, which make up the theoretical framework for this study, is in order.

Theoretical Framework

Worldviews, which are the basis of the theoretical framework for this study, define what knowledge and ways of knowing are (most) valued by those who hold the particular worldviews. This does not imply that everyone within a group identified by a worldview's name holds that particular worldview, rather that it is commonly held by most in the group. With this understanding of worldviews, the foundational characteristics of the two worldviews considered in this study, the Traditional Western and an Indigenous worldview, are now presented.

The Traditional Western Worldview

Within the Traditional Western worldview, the characteristics of knowledge and ways of knowing that are valued focus on absolutes and truth. Of most importance, the Traditional Western worldview values knowledge that is linear, singular, static and objective in nature. As a consequence, the Traditional Western worldview seeks one correct answer and one right action, resulting in specialization and abstraction being indicators of greater knowledge. With respect to how to find that one correct answer or action, methods whose results can be replicated and measured for validity, such as the Scientific method, are to be used. Moreover, as such methods require the isolation of what is being studied, the resulting knowledge is also compartmentalized, isolated, and categorized to maintain its validity. Finally, because the written word does not change over time (without intentional outside interference), it is held to be the way to preserve truth (Kovach, 2009; Little Bear, 2000). The Traditional Western worldview, because of its emphasis on singularity of truth, answers and methods, has very little variance in its enactment within Western societies.

An Indigenous Worldview

An Indigenous worldview however, does not have the restriction of singularity; rather it values diversity. As a result, there is no one Indigenous worldview, but many different Indigenous worldviews, each responsive to the conditions, places and times of the people that hold those particular worldviews (Kovach, 2009; Little Bear, 2000). Despite the natural occurrence and possibility for variation within Indigenous worldviews, there are common characteristics that can be identified and used to define an overarching Indigenous worldview. This common Indigenous worldview is grounded within the importance of relationships (physical, social, emotional, and intellectual) to people and to the physical and spiritual world. As a result of the importance of relationships, knowledge that is valued in an Indigenous worldview is connected to the place in and for which it is gained, with abstraction and compartmentalization being possible, but not the ultimate goal. As well, subjective knowledge, with its strong ties to the emotional and social realms, is knowledge that is valued, along with the objective knowledge that the Traditional Western worldview seeks. The valuing of subjective knowledge allows personal experience, observation and intuition to be considered valid sources of that knowledge. Diversity in ways of knowing and knowledge are not only valued, but sought, and as a result the individual becomes important because of what they can contribute to the group. Thus, knowledge is to be sought in order to give back to the greater good of the community, the world, and the cosmos. Finally, within an Indigenous worldview, truth, which changes with time and place, can be captured and held within oral language (Kovach, 2009; Little Bear, 2000).

Research referred to at the beginning of this paper used these two worldviews as lenses through which to view and analyze different facets of mathematics education. For example, the math wars, and in particular, the two approaches to the teaching and learning of mathematics which have come to represent a dichotomy within those wars (traditional and reform), have been viewed through these two lenses (Russell & Chernoff, online first). The result of this analysis has

been the theorizing of the possibility of a third approach to the teaching and learning of mathematics: the Transreform approach, which allows for the nullification of the dichotomy that has driven the math wars for more than two decades. This approach is discussed in more detail below.

The Transreform Approach to Teaching and Learning Mathematics

Prior research (Russell & Chernoff, online first) has posited that when considered through the lenses of the Traditional Western worldview and an Indigenous worldview, the two opposing camps on the teaching and learning of mathematics in the math wars (traditional and reform) form a strong alignment with alternate worldviews. In particular, it was found that the approaches of the traditional camp strongly position themselves within the Traditional Western worldview, while the approaches of the reform camp are reflective of the values of an Indigenous worldview. Moreover, moving beyond the question of which worldview best matches each of the math wars camps, it was also noted that within an Indigenous worldview the approaches of the traditional camp in the math wars are not negated, but already exist as one of many possibilities.

Alternatively, there is no room within the Traditional Western worldview for the approaches to teaching and learning espoused by the reform camp. As a result, it is theorized that, if the teaching and learning of mathematics is grounded within an Indigenous worldview, the approaches used by both camps (traditional and reform) are viable and appropriate. In other words, the approaches to teaching and learning of mathematics found within both the traditional and reform camps of the math wars are possible within a mathematics classroom grounded in an Indigenous worldview. Thus, within an Indigenous worldview, the two camps of the math wars become two of a set of possible approaches, and the dichotomy of the two camps is eliminated. Teaching and learning of mathematics that is grounded within an Indigenous worldview is what has been termed the *Transreform* approach to the teaching and learning of mathematics. “In the Transreform approach, the diversity of mathematical knowing and the precision of mathematical knowledge exist in harmonious exchange” (Russell & Chernoff, online first).

These three concepts, the Traditional Western and an Indigenous worldview and the Transreform approach to the teaching and learning of mathematics, form the theoretical framework for this research study. Next, the participants, research methods and methods of data collection used in the study are described in final preparation for the analysis of a limited number of examples (due to page limitations) from the study.

The School and Participants

This study is being held in a small rural elementary (K-6) community school in western Canada. The town in which the school is located is adjacent to a First Nations reservation and at least 90% of the students at the school are of First Nations or Métis descent.

There are four elementary teachers involved in this study, Sharon, Elizabeth, Lena, and April, who teach grade 2, 4, 6 and an ungraded class respectively. The ungraded class consists of a maximum of 7 students, primarily between the ages of 6 and 8, with the students being identified by the end of Kindergarten as not yet ready to enter grade 1. As the students in April’s class progress in their learning (for up to a maximum of 2 years), they are gradually integrated into the grade 1 or 2 classrooms. Of the four teachers, only Elizabeth is not of First Nations or Métis descent.

The teachers involved in this study reported two reasons for wanting to participate: to improve their students’ mathematics achievement (by grade 6, a majority of the students are still learning outcomes from the grades 3 and 4 mathematics curricula) and to support their students in developing a positive affective response to mathematics and mathematics learning.

Method

The teachers involved in this study were initially given an introduction to the concepts of the Traditional Western worldview, an Indigenous worldview and the Transreform approach to the teaching and learning of mathematics. On a weekly basis, the researcher conducts in-class observations. In addition, monthly full-day meetings with the teachers and researcher are held in which experiences over the past month are shared, questions arising from those experiences are explored, and plans are made for future teaching and learning activities. During these meetings, the teachers and researcher shape their ideas and work within the theoretical framework of the study.

Data Collection

During in-class visits, data in the form of observational notes, photographs, audio recordings and surveys of the students are collected. At the full-day meetings with the teachers, data in the form of observational notes, audio recordings, and teacher questionnaires are collected. Portions of both the in-class and full-day meeting audio recordings were selected, transcribed and verified by the participants for use in this paper.

The teachers also designed and implemented pre- and post- assessments for determining their students' affective responses to mathematics and level of academic performance related to the mathematics they were learning. The instruments (and results) were discussed with the researcher, with attention paid to the worldview in which the instruments were grounded.

From the data collected, a selection of four incidents, one for each of the teachers, was chosen as a representative example of a disruption of their teaching and learning of mathematics by the Traditional Western worldview. These incidents are now analyzed for evidence of such an intrusion.

Analysis

Due to the limited space of this paper, the four incidents presented for analysis must necessarily be considered without the data 'story' that surrounded them. That is, evidence of how the approach to the teaching and learning of mathematics by the participant teachers prior to each incident had been grounded in an Indigenous worldview, and thus was representative of the Transreform approach. However, speaking in generalizations, each incident was preceded by the teacher focusing on the building of relationships, looking for diversity in approaches and solutions, and contextualizing of the teaching and learning. Although these four incidents are a very limited number of all the data collected, they and their analysis are representative of the entirety of all such incidents within the data collected to date.

Sharon and the Hundred Chart

On a meeting day with the teachers, Sharon stated "I can't wait to find out how my class did on the hundred chart puzzle task today". Sharon explained that the task involved the students being given portions of a hundred chart, cut out like puzzle pieces, on which only one or two numbers were pre-recorded for the students. The students were to fill in the blank squares on the puzzle with the appropriate numbers (as it would be on the hundred chart). During a short break in the meeting, Sharon went to her classroom to check how her students had done with the task.

Upon returning to the meeting, Sharon expressed disappointment because her students had been unable to do the puzzle task. As Sharon explained what the substitute teacher had said happened, she interjected with a personal note: "I can't blame the students for finding the task hard. The hundred chart is so confusing – I don't know why it was made that way – it's not the

way I would have made it. I would have made it made it like a *Snakes and Ladders* game. Start at the bottom left and wind back and forth until you get to the top left. That makes more sense". Although the other group members agreed that the hundred chart was not arranged the way they would do it, all of the teachers agreed with Lena when she said "But – that's the way the hundred chart has to be, so the students just need to know it". Sharon decided that she would need to spend more time repeating some of the things she had done previously with her students so that they "got it".

There are two aspects of this incident that provide evidence of the Traditional Western worldview asserting its authority over the situation. First, the group immediately rejected the possibility of alternate hundred chart arrangements in favour of the one correct one; "the way the hundred chart has to be". If the teachers, at that point, had in fact been completely grounded in an Indigenous worldview, the discussion of alternative hundred chart designs would not have been curtailed by the existence of a standard hundred chart that is commonly used.

The second point at which the Traditional Western worldview disrupted the discussion was with Sharon's explanation of what she felt she needed to do next. Her plan was not based upon contextualization, building of relationships or diversity of knowledge and ways of knowing, as one grounded in an Indigenous worldview would be. Instead, Sharon was planning to repeat what has been done before in order to have her students gain the knowledge needed. The singular nature of her response, there is one right way to teach and learn this topic, and since it has been done already it needs to be done again, is also an example of the Traditional Western worldview interfering with the teaching and learning of mathematics being grounded in an Indigenous worldview.

Elizabeth and Arrays

At the beginning of a particular class observation, Elizabeth informed the researcher that her students had been exploring representing multiplication statements using arrays. The task that she gave the class, originating from an intensive professional development that Elizabeth had been involved in a few years before, was one that she had used "with success in previous grade 4 classes". After a series of activities in which Elizabeth had asked students to identify the multiplication statements represented by arrays (comprised of dots) that were shown using an overhead projector, Elizabeth then gave the students the task of creating an array to represent the multiplication statement of 3 x 4.

Prior to the start of the class, Elizabeth had told the researcher that she was looking forward to the discussion of "which way should we draw the array". However, the discussion did not happen. Elizabeth reported that "the activity collapsed" as she had to deal with a large number of students who were starting to draw puppies, Sponge Bobs, and pineapples. Elizabeth came to me and said: "What do I do, they are all off task – drawing pictures – not dots. None of the previous classes ever did this. Perhaps I should move off of math for a while and then come back to the arrays". During the recess break that followed the class, the researcher had the opportunity to talk to some of the 'off task' students and it was revealed that, in fact, the students had been drawing arrays, but not arrays of dots – arrays of puppies, Sponge Bobs, and pineapples.

Because her students were not creating arrays of dots, Elizabeth concluded that they did not want to, or could not, do the task. She inferred that this group of students was deficient in its understanding of the mathematics because what the students were doing was not right according to her past experiences. This expectation of singularity of approach to a task is foundational to the Traditional Western worldview, but not an Indigenous worldview. In this way, the teaching and learning of mathematics in Elizabeth's class had been intruded upon by the Traditional

Western worldview.

April and the Bag of Beads

During a recess break on a day that the researcher was doing in-class observations, April told the researcher that she was struggling to get her students to understand the partitioning of whole numbers to 10. She explained that she had the students working on a task in which they were shown a bag and were told that it contained a certain number of beads (some yellow and some blue), say 8. The students were then asked questions such as "if there are 3 yellow beads in the bag, how many blue beads are there"? April explained: "they just can't do it … it's been like this with every class I've ever tried this activity with. They just never get it. We do it over and over, but they just can't do it. They never get partitioning, no matter how many times we do this activity".

Similar to Sharon and Elizabeth, April was struggling with her students not being a carry out a task she has given to them. What is different in the case of April's task was that it has never been successfully completed in any of April's classes, yet it still carries the same kind of assumed authority that Sharon's and Elizabeth's tasks also held. Thus, this too is an example of the Traditional Western worldview, and in particular the belief in the one right way, intruding into, perhaps even taking over, a mathematics classroom.

Lena and the Curriculum

Part way through this research study, the teachers were also given an unrelated opportunity (as part of an in-school professional development plan) to start or join an existing professional learning community (PLC). In response, Elizabeth, April, Sharon and Lena decided that it made sense for them to make their PLC part of the research study, preferring to "continue to work with math rather than start something else as well and not get anything done in either case" (Lena).

After setting a required SMART goal for the PLC, the teachers began planning units for the teaching and learning of mathematics using Backwards Design. As they were developing their plans, Sharon, Elizabeth, and April all discussed their struggles with creating their assessment tools based on the outcomes they had chosen, but Lena's struggle began even sooner. The instant Lena had to look into the curriculum document to determine the outcome and indicators she and her students would be focusing on a dilemma emerged. For Lena, the curriculum outcomes and indicators were limiting and even judgmental: "When someone comes into my class, they don't look and say 'look at what they can do with multiplication, factors and multiples', they say 'where are the words multiplication, factors and multiples'? We use the words eventually, but I'm more concerned that my students understand what they are doing". Pointing at the outcome she selected from the curriculum document, Lena continued, "but this is what I will be judged on by someone coming in my class from outside – on the words, not on what my students understand".

Written in single, and as a result often convoluted, sentences, Lena felt the curriculum outcomes and indicators were telling her that there was one right way to know mathematics. In essence, she was struggling with a Traditional Western worldview interpretation of the mandate of the curriculum outcomes (probably not without good reason) in contrast to her Indigenous worldview approach to the teaching and learning of mathematics. Lena's resolution to the conflict was to write the outcome in the space provided on the planning template, and then to never look back at it again. Thus, Lena did not fully reject this intrusion of the Traditional Western worldview into her teaching and learning of mathematics, but unlike the other three teachers, she was able to deny it absolute authority with respect to her planning for the teaching and learning of mathematics in the classroom.

Conclusion

As demonstrated in the analysis above, when attempting to engage in the Transreform approach to the teaching and learning of mathematics, that is, when attempting to ground oneself in an Indigenous worldview, there are many ways and times in which the Traditional Western worldview can disrupt the situation; asserting the authority of its values of singularity, correctness, and validity over those of the values of an Indigenous worldview. As has been demonstrated, such intrusions of the Traditional Western worldview can happen when student use and understanding of pedagogical tools (such as the hundred chart) do not produce expected results; when students engage in unexpected ways with tasks perceived to be 'tried and true'; when students are unable to engage with tasks that are assumed to be effective; and when teachers are asked to merge outside forms of authority (such as a curriculum document) with their approach to the teaching and learning of mathematics. Moreover, with the possible exception of Lena's encounter, the examples considered in this paper demonstrate that not only can the Traditional Western worldview intrude upon the teaching and learning of mathematics grounded within an Indigenous worldview – it can derail a teacher's attempts to use such a Transreform approach.

Discussion

Not included in the analysis above are the 'after' stories. In each instance sited, the researcher and teacher(s) were able to engage in discussions through which it was possible to eliminate, or at least divert, the specific disruptions by the Traditional Western worldview. For example, when Sharon presented her specific problem to the teachers, the researcher suggested that each teacher explain how they would organize the hundred chart if it was their choice. Sharon decided that she would try such an activity with her class in order to determine how her students were thinking about the whole numbers to 100. One week later, Sharon's students were completing a hundred chart that showed only the multiples of 5 and which was spiraling clockwise from the center of the chart (a pattern that the students determined for themselves). Moreover, the students were creating their own hundred chart puzzle pieces for their classmates where the organizational scheme of the originating hundred chart had to be determined by the student who got the particular puzzle piece. Sharon was pleased with the resulting confidence her students demonstrated, stating: "and they can even use the regular hundred chart".

In the case of Elizabeth and the arrays, the researcher and Elizabeth discussed what the purpose of the task was, after which, Elizabeth decided that restricting her students to using dots had nothing to do with understanding multiplication. Moreover, she came to view the restriction of using dots as being characteristic of the Traditional Western worldview, highly abstracted and devoid of context. Elizabeth later reported that by allowing the students to use representations of their own choice, and through the sharing and discussing of the representations that the students created, her class progressed rapidly in their attainment of the multiplication outcome.

April was able to overcome the assumed authority of the 'beads in a bag' task by discussing what partitioning is and where it could be found in daily activities. The result was that April returned to her classroom, provided her students blue and yellow beads and had them answer (in a way that made sense to each particular student) the question, "if you were putting 8 of these beads in the bag, how many would be yellow and how many would be blue"? Since then, April shared with the teachers in the study how her students are "really getting a good handle on partitioning". She now views the original task as "okay, but not important".

Finally, Lena, by refusing to acknowledge the curriculum outcome beyond writing it down, had not allowed the intrusion of the Traditional Western worldview to completely disrupt her planning, has since reported finding a place for that outcome in her Transreform approach to teaching and learning mathematics. In her struggle with the words of the outcome, Lena came to embrace those words within her larger and diverse understanding of the mathematics, and consequently her students are now also "saying multiple and factor all the time when they are talking to each other".

Thus, because of having an opportunity to talk about what was happening at the moment of an incident of intrusion by the Traditional Western worldview, the teachers were able to find room for the point of disruption within their grounding in an Indigenous worldview without having to sacrifice that worldview. The teachers did report however, that had they not been afforded such opportunities they were not confident that they would have overcome these moments of disruption. At a recent meeting, Elizabeth said to the researcher: "You always know what to ask us, what to make us think about, so that we see that we are being Traditional Western. I never see it myself". However, later in that same meeting, Sharon said: "You know, I was just thinking about a class I had yesterday, and I realized that I had unintentionally shut down the possibility for the students to bring in their own ideas. I'm going to go back to that tomorrow and try something different". In this instance, and without outside interference, Sharon was recognizing and questioning a disruption by the Traditional Western worldview.

For the teachers in this study, the change from the Traditional Western worldview to an Indigenous worldview is one that "just makes sense" (April), but it is also "challenging at times" (Elizabeth). The assumed singular authority of the Traditional Western worldview by the worldview itself (and thus by much of the society in which these teachers live and work) makes it possible for the Traditional Western worldview to "slip in and take over" (Lena). Whether these four teachers will be able to recognize (and overcome) such future incidents of disruption on their own remains to be seen.

Acknowledgements

The authors of this paper acknowledge partial funding of this research by the Dr. Stirling McDowell Foundation for Research Into Teaching.

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CHARACTERIZING PIVOTAL TEACHING MOMENTS IN EXPERIENCED MATHEMATICS TEACHERS' PRACTICE

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This report presents the findings of a study that was designed to characterize the pivotal teaching moments (PTMs), as defined by Stockero & Van Zoest, (2012), faced by experienced mathematics teachers. To better understand how experienced mathematics teachers identify the PTMs and appropriately respond to them, the mathematics teaching videos from Annenberg Learner's multimedia resources were analyzed. Implications for teacher education are discussed.

Keywords: Algebra and Algebraic Thinking, Classroom Discourse, Mathematical Knowledge for Teaching

Introduction

Students' mathematical achievement is influenced by many factors. One such factor is the instructional practice of their teachers. Practices such as engaging students in rich mathematical tasks, encouraging students to explain their thinking processes and building students' mathematical understandings from prior knowledge have been reported to increase students' mathematical achievement. (Henningsen and Stein 1997; Stein and Lane 1996; Superfine 2008).

This study examines what Stockero and Van Zoest (2012) call Pivotal Teaching Moments- "interruptions in the flow of a lesson [which] provide an opportunity to modify instruction to improve students' mathematical understanding" (p.3)-observed in videos of algebra instruction. This report shares the findings of that study, offering implications for teacher education.

Literature Review

Many researchers have documented how teachers could implement these recognized instructional practices in their classroom to advance students' mathematical learning. Chi, Leeuw, Chiu and Lavancher (1994) claimed that encouraging students to generate more self-explanations promotes greater learning and understanding of new knowledge. Cengiz, Kline and Grant (2011) explored the teaching of six experienced elementary school mathematics teachers and identified their individual instructional actions to extend student thinking. They emphasized that the first step in extending student thinking is to recognize the potential of a particular situation which requires careful listening to student thinking and having clear goals about the mathematical ideas and concepts they are to pursue. Van Es (2011) refers to classrooms as complex settings in which all kinds of interactions take place at one time and teachers need to decide on what to pay attention to and how to respond to the events and interactions. Ball and Cohen (1999) also suggested that teachers should learn to carefully look at or think about a situation and then decide how to act from moment to moment. Van Es and Sherin (2002) propose that the skill of noticing for teaching consists of three main aspects: identifying what is important in a teaching situation; using what one knows about the context to reason about a situation; and making connections between specific events and broader principles of teaching and learning.

How can teachers be supported in learning to recognize the most important elements in the classroom practice? Stockero and Van Zoest (2012) analyzed videos of beginning secondary

school mathematics teachers' instruction to identify and characterize what they called *pivotal teaching moments* (PTMs) in mathematics lessons. PTMs are hi gh-leverage moments that can significantly impact student learning. They defined a PTM to be an instance in a classroom lesson in which a student-generated interruption in the flow of the lesson provides the teacher an opportunity to modify instruction in order to extend or change the nature of students' mathematical understanding. In their exploratory study, they identified five circumstances that led to PTMs: (a) extending: when students make a comment or ask a question that is grounded in, but goes beyond, the mathematics that the teacher had planned to discuss; (b) incorrect mathematics: when incorrect mathematical thinking or an incorrect solution is made public; (c) sense-making: when students are trying to make sense of the mathematics in the lesson; (d) mathematical contradiction: when two different answers to a problem that clearly should have only one answer are provided or two competing interpretations of a mathematical situation; (e) mathematical confusion: a student's expression of mathematical confusion.

This initial PTM framework is of potential usefulness to the teacher educators as well as teachers to improve mathematical noticing ability and decision-making. However, Stockero and Van Zoest (2012) also suggest that in order to fully understand PTMs and teacher decisions in response to them, experienced mathematics teachers' practice need to be included into the data set. Therefore, this paper is supposed to fill this gap. The following two research questions were explored:

1. What are characteristics of PTMs faced by experienced secondary school mathematics teachers during classroom instruction?
2. What types of decisions do they make when a PTM occurs during their instruction?

Perspectives

In considering the above questions, the researchers turned to previous work on students' mathematical thinking. The National Council of Teachers of Mathematics (2000) calls for mathematical instruction that builds on children's mathematical thinking and many research studies show that such practices result in a richer instructional environment and greater student achievement (Carpenter, Gennema, Peterson, Chiang, & Loef, 1989; Sowder, 2007; Wilson & Berne, 1999). However, to learn and practice such instruction is complex and difficult (Ball & Cohen, 1999). In many cases, teachers either missed the opportunities to use student thinking to further their mathematical understanding or did not properly act upon them (Peterson & Leatham, 2009; Stockero & Van Zoest, 2012).

An important first step in helping teachers capitalize on important mathematical moments is to recognize that such moments exit (Stockero & Van Zoest, 2012). Providing a framework for teachers to use helps them to pick up easily mathematically valuable moments that occur during instruction and appropriately act upon them. This paper aims to characterize the PTMs faced by experienced mathematics teachers and the types of decisions that they make in response to those critical moments.

Methodology

In this exploratory, descriptive study of PTMs, the researchers focused on experienced teachers who had more than five years teaching experience in secondary-school mathematics.

Data Collection

For this study, videos from Annenberg Learner's multimedia resources (http://www.learner.org/) were used. Annenberg Learner's goal is to use media and

telecommunications to advance excellent teaching in American schools. The statement that "All Annenberg Learner videos exemplify excellent teaching" (http://www.learner.org/about/) implies the mathematics teachers demonstrating their teaching in those videos are expert teachers. Hence, their existing body of video data best served the purposes of this study.

The mathematics video resources featuring real classrooms on the Annenberg Learner website are kept in four series: Teaching Math: A Video Library, K-4; Teaching Math: A Video Library, 5-8; Teaching Math: A Video Library, 9-12 and Insights Into Algebra I: Teaching for Learning. Each series contains videos that vary in length (some as short as 15 minutes) and number. "Insights Into Algebra I: Teaching for Learning" offers the most in-depth look at classroom practice, containing 8 one-hour video programs. This set of videos features nine experienced middle and high school mathematics teachers teaching 16 topics found in most Algebra I programs. Each of the eight sessions contains two half-hour videos displaying effective mathematical teaching strategies (http://www.learner.org/resources/series196.html). Because the longer videos provided more opportunities to examine effective teaching practices, the videos from this series were selected for this study.

Data Analysis

This analysis was made up of two stages. The first stage was to reduce the video data to potential episodes for PTMs. The second stage was to identify the PTMs and characterize them.

Stage one: video reduction. The PTM definition provided by Stockero and Van Zoest (2012) was used to pick out PTM episodes. In the definition there are three important components. First, the definition emphasizes "an interruption in the flow of the lesson" (p.3), which means that it is an unanticipated event such as a student's question or comment. Second, the definition offers an opportunity for the teacher to "modify instruction" (p.3). The third component is that there is potential to "extend or change the nature of students' mathematical understanding" (p. 3). Accordingly, the researchers paid close attention to students' questions or comments that provided teachers a chance to extend or change their mathematical understanding. The two researchers then individually watched the videos and marked the time when a PTM occurred. Through discussion that aimed toward consensus, 29 episodes remained in the data base and were transcribed for further analysis.

Stage two: characterizing PTMs. After identifying all those PTMs episodes ,the researchers used Stockero and Van Zoest (2012)'s characteristics of PTMs in beginning mathematics teachers' practice as a guide to characterize the PTMs in the experienced mathematics teachers' classroom. During this phase of analysis, the researchers aimed at two things: to identify the PTM type, and to code the teacher decision action. If the identified PTM is beyond the framework created by Stockero and Van Zoest (2012), the researchers used open coding (Carspecken, 1996) to label the feature. Each code was discussed between the researchers until an agreement was reached.

Results

The coding process resulted in three PTM types: incorrect mathematics, sense-making and confusion. The following four teacher decision actions were identified: extend/ make connections, pursue student thinking, emphasize mathematical meaning and wait to allow student explore first (see table 1).

Pivotal Teaching Moment Characteristics

Incorrect mathematics. Out of the 29 PTMs in the data base, 7 were classified into incorrect mathematics. It could be student's incorrect mathematical thinking or an incorrect solution. For

example, when a teacher asked a group of students “what is zero divided by zero”, that group of students immediately answered “zero” without any hesitation. The students’ incorrect mathematical understanding provided an opportunity for the teacher to clarify why zero can’t be a divisor.

Table 1: Summary of PTMs Identified in the Data (modified from Stockero and Van Zoest, 2012, p. 10)

Pivotal Teaching Moment	Teacher Decision
Incorrect mathematics (7)	Extends/connections
	Pursue student thinking (2)
	Emphasize mathematical meaning (2)
	Wait to allow student to explore (3)
Sense-making (12)	Extends/connections (2)
	Pursue student thinking (2)
	Emphasize mathematical meaning (7)
	Wait to allow student to explore (1)
Confusion (10)	Extends/connections (2)
	Pursue student thinking (3)
	Emphasize mathematical meaning (1)
	Wait to allow student to explore (4)

Sense-making. Aligned with the findings of Stockero and Van Zoest (2012)**,** PTMs occur most when students are trying to make sense of the mathematics. There were twelve sense-making PTMs in our data. For example, when a teacher asked students how to graph the function $y = -(x-3)^2 + 4$ from $y = x^2$, they talked about how -1, -3, and 4 affected the graphs. The teacher sketched the graph, but the graph passed through the origin, which was incorrect. One student commented “So it doesn’t matter how wide the parabola is?” This comment showed that the student was trying to make sense of graphing quadratic equations from parent functions. It provides a chance for the teacher to highlight the critical aspects of the mathematics at hand.

Mathematical confusion. Ten PTMs occurred when students expressed that they were confused about a mathematical idea. For example, when a teacher wrote down the general form of a parabola $y = a(x - h)^2 + k$, he asked students what they noticed when they graphed quadratic equations on their stations. One student said that when h is positive, the graph shifted to the left; when h is negative, the graph shifted to the right. Another student put up his hand and said, “I know it is right, but I don’t understand why it is right? Because, like, if it is positive, shouldn’t it go to the right? If it is negative, shouldn’t it go to the left? That’s what we’ve been taught.” This gives the teacher an opportunity to revisit vertex form, the graph and other important mathematical ideas related to it.

Teacher Decisions in Response to PTMs

There are four types of teacher decisions made by the experienced teachers: pursue student thinking, emphasize mathematical meaning, wait to allow student to explore first, and extend/make connections.

Pursue student thinking. One of the decisions made by the teachers in the data base is to pursue student thinking. The teacher tried to understand the meaning of what a student had said by asking the student to provide more information about their thinking (Stockero & Van Zoest, 2012). This occurred seven times in the data base. One example happened when the teacher

asked how many tiles she would need if she wanted to tile the border of a 9 x 4 pool. One student answered 26 tiles. By asking the student to explain how she got the answer 26 tiles, the teacher helped students to clarify that while 26 tiles was the perimeter of the 9 x 4 pool, it was an insufficient number of tiles to completely surround the pool, leaving the corners untiled.

Emphasize mathematical meaning. Another response to a PTM is to emphasize the mathematical meaning behind the subjects. There are ten such instances in the study. One of them happened when the teacher asked students to provide some real-world examples that show indirect (inverse) proportion. One student answered, "The farther you drive, the less gas you have. So you get more miles, but you have less gas." The teacher grasped this opportunity to emphasize that "In a true indirect proportion, you never could touch the x-axis", but in this student's case, it is likely that the car would completely run out of gas. This led the class to seek a better illustration of an indirect proportion, coming up with the relationship between the speed driven and the time taken to drive a set distance.

Wait to allow student to explore first. This type of response happened eight times in the study. When a PTM occurred, the teacher didn't immediately offer an answer to the student. Instead, he/she waited for a few seconds to allow students to explore first. During this process, two possibilities might happen. One is teacher-teach-student and the other is student-teach-student. The teachers in this study chose to use this method nine times. One instance happened when students got "error" on their calculators while operating zero divided by zero, the teacher didn't immediately pointed out that 0/0 is undefined. Instead, from the point she posed the question "why do you think that is?" at 15:25, she waited for 26 seconds before she actually explained it. During these 26 seconds, she allowed students to struggle with this question. Schoenfeld (2011) claimed that the simple act of waiting after asking a question made it clear that the questions are not rhetorical but are meant to provoke student responses.

Another instance happened during a group of students' presentation. These students were trying to demonstrate how they solved the question, "how many hot dogs must be sold in order to raise at least $250?" They set up an inequality $250 \geq .50h - 450$. The teacher waited patiently until this group finished presenting their method. Then she asked the whole class, "Are there any questions for this group?" This gave observing students some time to make sense of the group's method, and one pointed out the mistake – choosing the wrong inequality symbol. Schoenfeld (2011) wrote, "Giving "the answer" prematurely can deprive students of the opportunity to do sense making on their own, and perhaps even of the confidence that they can do it" (p. 138). This instance shows how students can take advantage of classroom discussions to make sense of complex concepts.

Extend/make connections. This type of response happens when teachers encourage students to reflect on their ideas and further make connections between their prior knowledge and their claims (Stockero & Van Zoest, 2012). The teachers in this study chose to use this method four times. In one instance, the class had been working on using their equations to predict what the area would be for an oil spill of one liter. One group came up with the answer 150,000 square centimeters. When they were asked to convert that to square meters, they responded "divide by 100" and "divide by 1,000". The teacher asked them to think about how many square centimeters are in one square meters and encourage them to draw a picture to help them. In this way, the teacher provided students with an opportunity to develop connections among mathematical concepts and to move beyond their existing mathematical knowledge.

Discussion and Limitations

Pivotal Teaching Moments can occur in any classroom if teachers make an effort to make student thinking public. They offer teachers intense opportunities to deepen students' knowledge, positively impacting their achievement in mathematics. As mathematics teacher educators, it is our feeling that helping teachers improve their abilities to recognize and effectively respond to pivotal teaching moments will offer great rewards in the classroom. Aligned with Stockero and Van Zoest (2012), the findings of this study display a better understanding of PTMs and teacher decisions in response to them. These findings can be used by teacher educators to help teachers capitalize on PTMs during classroom instruction, especially in the case of novice teachers who have less experience and knowledge to rely on as they encounter such moments (Stockero & Zoest, 2012).

One limitation of this study was that the videos that were used were professionally edited videos. The purpose of those videos was to display effective instructional strategies; therefore, some parts of the classroom practice may be ignored or deleted. Future studies should be designed to observe real-time classroom practice, in order to determine a clearer picture of what all of the students in a classroom are grappling with, and what factors influence the teacher's decisions that are observed. Including interviews with the teachers about his/her decisions when planning and executing lessons would also add depth to the study and inform the practices of teacher educators.

Continued research on Pivotal Teaching Moments is warranted as long as mathematics teacher educators are concerned with developing teachers' abilities to take advantage of 'teachable moments' in the classroom. By examining teaching practice through the lens of PTMs, pre-service and inservice teachers can learn to recognize and act on these moments with their own students, with the aim of increasing their abilities to positively impact students' academic achievement.

Acknowledgements

The authors would like to thank Dr. Jennifer Chauvot for her valuable feedback at various stages of writing this manuscript.

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[i] The student participants in this research are Latino and African American, located in an urban center, and of lower socioeconomic status (SES), all of these terms will be used to reference them. Additionally, since Latinos and African Americans are considered non-white populations, the term "students of color" is also used to refer to both groups simultaneously. While this depicts whites as not a "color" (and therefore at times cultureless) and could also refer to Asians, Asian Americans, as well Arabs for instance, the term is specifically focused on Latinos and African Americans in this paper. There are problems with this term as with all groupings of diverse groups of people, but currently students of color represents a common reference for these student groups.

THE EMERGING PRACTICE OF TEACHING MATHEMATICS AS AGAPE

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Mathematics is a prerequisite for accessing academic and economic opportunities that are equated with having access to society in general. The question is not if to teach mathematics but how to teach mathematics? Within this paper is an attempt to organize equitable teaching practice through the lens of agape (pronounced äˈgäˌpā), or unconditional love. Teaching mathematics as agape, is then the target of a self-study of teaching practice in a secondary mathematics classroom.

Keywords: Equity and Diversity, High School Education

Purpose of the Study

Mathematics education has a few postulates most would not dispute: 1) mathematics is necessary for access to academic and economic opportunities and 2) not all students are given an equal chance at learning mathematics (Diversity in Mathematics Education, 2007; Gutiérrez, 2007; Moses & Cobb, 2001). What if the response to these postulates flows out of agape (pronounced ägäpā), or unconditional love? Given the importance of providing students with access to mathematical practices, the influential role mathematics educators can have in promoting or denying access (Berry, 2008), and agape as the “highest guiding principle” (Wivestad, 2008, p. 308), this paper addresses the research question: What would teaching mathematics as an act of unconditional love look like? In other words, what does *teaching mathematics as agape* look like? What follows is a description of the target of *teaching mathematics as agape,* an idealized relationship between students and mathematics, and a description of a central practice of teaching mathematics as agape that emerged from a self-study attempting to enact this pedagogical ideal within a secondary mathematics classroom.

Theoretical Framework

The classroom is a relational space with the key players being the students, the teacher, and mathematics. A comparable perspective is what Palmer (1998) describes as a “subject-centered” classroom, and what Lave & Wenger (1991) describe as a “community of practice”. Learning can be thought of as strengthening the relationship between the subject and the knower (Lampert, 2001; Palmer, 1998). This model suggests that participants are constantly interacting with mathematics as an entity to relate to and understand, and not just a set of examples and/or procedures to memorize and regurgitate. Teaching mathematics as agape is offered as the means to promote an ideal relationship between students and mathematics.

Agape is a love that is “given irrespective of merit” and “seeks to give” (Morris, 1981, p. 128) and is used to organize equitable approaches to teaching (e.g. culturally relevant pedagogy (Ladson-Billings, 1995)), inclusive pedagogy (Udvari-Solner, Villa, & Thousand, 2005) and teaching mathematics (e.g. teaching mathematics for social justice (Gutstein, 2006)) within a classroom community of practice (Lave & Wenger, 1991). Mathematics pedagogy and equity pedagogy were examined through the lens of agape to theorize an ideal relationship between students and mathematics. Thus, the target of teaching mathematics as agape, an ideal

relationship between students and mathematics, is partitioned into four facets that are defined and necessitated through the literature.

The first facet of the relationship between students and mathematics implied by teaching mathematics as agape is designated as functional, and describes when a student can work with mathematics to achieve success as defined by society (Frankenstein, 1990; Gutiérrez, 2007; Gutstein, 2006; Ladson-Billings, 1995; Moses & Cobb, 2001; North, 2009). In the age of high-stakes testing that is tied to the evaluation of both students and teachers, "…students must achieve. No theory of pedagogy can escape this reality" (Ladson-Billings, 1995, p. 475). But given the inequities identified at the beginning of this article, instruction needs to begin by looking at those who are to receive it.

The second facet of the relationship between students and mathematics implied by teaching mathematics as agape is designated as communal, and describes when a student can work with mathematics in the contexts and through the practices of the student and the student's community (Gonzales, Moll, & Amanti, 2005; Gutstein, 2006; Horton, 1990; Ladson-Billings, 1995; Moses & Cobb, 2001; North, 2009; Udvari-Solner et al., 2005). By explicitly incorporating the tenets of inclusive education, this facet expands on Ladson-Billings' "cultural competence" and Gutstein's "community knowledge" in that it seeks to accommodate the needs, abilities and interests of all who are to engage with the classroom instruction. But if the system is already producing inequities, then providing access to the system could be equated with perpetuating the system, and perpetuating the inequities that are being produced (Apple, 1992).

The third facet of the relationship between students and mathematics implied by teaching mathematics as agape is designated as critical, and describes when a student can work with mathematics to analyze and question the world (Frankenstein, 1990; Gutiérrez, 2007; Gutstein, 2006; Ladson-Billings, 1995; North, 2009). If students are going to learn mathematics within a context, why cannot learning happen within a context that emerges from their daily reality and work toward answering a question of interest about that reality. But if engaging with mathematics only reveals problems in the world, students may not continue to choose to work with mathematics.

The final facet of the relationship between students and mathematics implied by teaching mathematics as agape is designated as inspirational, and describes when a student can work with mathematics to envision and work toward a better world (Gutiérrez, 2007; Gutstein, 2006; North, 2009). If working with mathematics allows one to see inequities that emerge from the local community, than that same work would likely provide insight into how those inequities can be rectified or what can be done to improve the reality of the students and their community. Given this ideal relationship between students and mathematics that is the target of the practices of teaching mathematics as agape, then the subject of this study is to identify the specific teaching practices that bring out this ideal relationship between students and mathematics.

Methods

Teaching mathematics as agape was used as a pedagogical goal in a "self study" (Zeichner & Noffke, 2001) of the author's teaching practice in "an attempt to bring together theory and book knowledge with real-world situations, issues, and experiences" (Berg, 2007, p. 232). The boundaries for this case were tied to documenting how the relationship between students and mathematics was promoted within the classroom.

This study was situated at a high school in a lower middle-class, rural community in the Midwest. More specifically, the setting was one section of the first course of the school's

mathematics sequence. The participants were twenty students and the author, as teacher-researcher. The data generated during the study was a teacher journal, student journals, and student work. The teacher journal (Cochran-Smith & Lytle, 1993) was the primary source of data and has been shown to be a useful tool in generating data for practitioner inquiry (Gutstein, 2006; Lampert, 2001). Each day's journal entry included a lesson plan, a pre-lesson reflection, and a post-lesson reflection, which included memorable quotes and interactions from the day's events. In addition, notes on interactions with "critical friends" (Anderson, Herr, & Nihlen, 2007) were also collected within the teacher journal.

The data was analyzed through a series of "coding cycles" (Saldaña, 2009). The first cycle used Richard's (2009) process of "taking off from the data" (p. 77) in order to capture initial impressions of the data. The next cycle involved "provisional" coding (Miles & Huberman, 1994; Saldaña, 2009), which utilized a set of codes generated from the four facets of teaching mathematics as agape. The last cycle of analysis was a merging of the first two cycles.

Results

In response to the research question, "What does teaching mathematics as agape look like?" is the emergent practice of "turning". Repeated disruptions in teaching practice were found that were manipulated into opportunities to facilitate the relationship between students and mathematics. It became apparent that "turning" was a means for identifying where the teacher-researcher responded with teaching practice to "promote well-being", or to promote the relationship between students and mathematics, when "confronted by that which generates ill-being", or disruptions in teaching practice. This goes beyond being a reflective practitioner (Zeichner & Liston, 1996) in that disruptions in teaching practice were identified as assets to improve practice, rather than problems to solve.

Specific examples of "turning" were identified because they demonstrated facilitation of all the facets of the desired relationship between students and mathematics that is suggested by teaching mathematics as agape. The scenarios represent struggles in practice that many in-service teachers of mathematics would experience within their own classrooms. The examples emerge from a general perception of students' ways of participation in the classroom, the creation of a lesson structure, and the creation of a mathematics investigation.

Taking into account all examples, "turning" is about having an asset-based perspective of the classroom space. Ladson-Billings (1997) suggests conceiving of classrooms as "complex systems" where understanding a classroom is not only about knowing who is in the class, but how those in the classroom interact with each other. Thus, "turning" is beyond identifying the strengths that individual students bring to the classroom, but about identifying the assets of the classroom as a complex system.

Conclusion

The description of the four facets of an ideal relationship between students and mathematics represents an attempt at organizing pedagogy through the lens of agape. The study represented progress toward attempting to facilitate this ideal relationship by examining what *teaching mathematics as agape* may look like, with "turning" identified as a central practice. "Turning" was aligned with having an asset-based perspective of students, a cornerstone of inclusive education (Udvari-Solner et al., 2005). Given this conclusion, some questions arise for further investigation. For example, how can teachers learn to notice assets, for opportunities to "turn", in order to develop relationships with mathematics? Additional research enacting and refining

teaching mathematics as agape can also reveal more practices, that can accompany this notion of "turning". In conclusion, the "highest guiding principle" (Wivestad, 2008, p. 308) of agape, has been shown as a productive guide toward addressing the postulates that began this paper.

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TEACHERS' IMPLEMENTATION OF PRE-CONSTRUCTED DYNAMIC GEOMETRY TASKS IN TECHNOLOGY-INTENSIVE ALGEBRA 1 CLASSROOMS

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This study examined the combined influence of mathematical discourse, pedagogical strategies, and dynamic geometry software use on level of cognitive demand within three 1:1 laptop Algebra 1 classrooms. Findings indicated that teachers predominantly utilized whole class discussion, with questions being used slightly more often than statements. Higher-level questions were posed more often than lower level questions, but teachers' pedagogical practices varied. When using technology, teachers most often discussed the screen or explained the screen. Two themes emerged; use of higher-level questions/statements while discussing the screen corresponded to high cognitive demand and use of lower level questions/statements to explain the screen corresponded to decreased cognitive demand. Implications for mathematics teachers, mathematics teacher educators, professional development, and researchers will be discussed.

Keywords: Algebra and Algebraic Thinking, Classroom Discourse, High School Education, Technology

Introduction

Technology use with a focus on 21st century skills, coupled with recent adoption of Common Core State Standards for Mathematics (2010), mark a new challenge for mathematics teachers. Communication, discourse, and tools for enhancing discourse (NCTM, 1991, 2000) play an integral role in successful implementation of technology and mathematics standards. With regard to communication and discourse, Stein, Engle, Smith and Hughes (2008) described five practices focused on helping teachers incorporate student thinking into classroom discussions. Dynamic geometry environments represent a tool that has the potential to enhance mathematical discussions, and one context that has great potential to examine the intersection of classroom discourse, pedagogical practices, and technology use is a 1:1 laptop learning environment.

The overarching question that guided this study was, "*How do teachers implement pre-constructed dynamic tasks in technology-intensive Algebra 1 classrooms?*" The purpose of this study was to 1) characterize the nature of mathematical discourse, 2) describe the extent to which teachers use the five practices for orchestrating productive mathematical discussions, 3) examine the design and technology features of pre-constructed dynamic geometry tasks and how teachers use them, and 4) explore the combined influence of 1-3 on level of cognitive demand when teachers implement pre-constructed dynamic geometry tasks.

Context and Participants

The study took place within the context of a larger professional development project. Participating teachers attended a week-long institute and completed twelve hours of online professional development throughout the 2011-2012 school year; however, only one school system from this project was selected for the study. The school system chosen was a small, rural school system in the eastern part of a southeastern state. The school system was participating in a 1-1 laptop learning initiative, and at the time of the study, was classified as one of the twelve lowest performing school systems in the state based on student performance on state mandated tests. Three teachers within this school system, located at two high schools, agreed to participate in the study. Each school

operated on a block schedule, meaning students enroll in 4 ninety-minute classes per semester. All teachers had five or less years of experience, and one teacher became a mathematics teacher via lateral entry.

Related Literature

Mathematics education research on discourse, pedagogical practices to support discourse, technology design and use, and level of cognitive demand informed the conceptual framework and data analysis for the study. Cayton (2012) synthesized definitions of discourse to characterize *mathematical classroom discourse* as "verbal interactions that focus on mathematics and serve to clarify, challenge, expand, investigate, or justify content, solution strategies, or contributions from self and others" (p. 23). The five practices for orchestrating productive mathematical discussions (Stein et al., 2008) served as specific pedagogical practices.

The focus of the study was teachers' use of pre-constructed dynamic geometry sketches created with *The Geometer's Sketchpad*. Sinclair (2003) reported five important design principles related to dynamic geometry sketches and learning materials. Specifically, pre-constructed dynamic geometry sketches should match the intent of student learning materials by: 1) drawing attention through the use of color, motion, and markings, 2) provide affordances for student actions, 3) provide alternate paths, 4) support experimentation, and 5) provide a shared image for the class. Drijvers, Doorman, Reed, & Gravemeijer (2010) characterized six teacher technology orchestration types that provided a lens to examine the way technology was utilized during implementation. Tech demo, explain the screen, link screen and board, discuss the screen, spot and show, and Sherpa at work describe ways teachers may orchestrate technology during a lesson.

With regard to cognitive demand, the Task Analysis Guide and Mathematical Task Framework (Smith & Stein, 1998) characterize high level and low level tasks, demonstrate how tasks unfold at each phase of implementation, and emphasize that teacher actions at each phase may alter the level of cognitive demand of the task. The next section will describe the methods utilized to conduct the study and details how the literature informed the conceptual framework and data analysis.

Methods

Research Design and Conceptual Framework

A qualitative, observational multi-case study served as the specific research design. Data sources included video-taped interviews, classroom observations, and planning sessions, as well as artifacts from classroom observations, planning sessions, and online professional development. The conceptual framework combined the Mathematical Task Framework (Smith & Stein, 1998) and the five practices for orchestrating productive mathematical discussions (Stein et al., 2008) into a dynamic model characterizing task implementation and use of the five practices in 1:1 computing classrooms.

Data Analysis

Participants were observed for three teaching sets (Simon, Tzur, Heinz, Kinzel, and Smith, 2000), and excerpts of the video that included use of pre-constructed dynamic geometry tasks served as episodes for analysis. The Task Analysis Guide (Smith & Stein, 1998), Sinclair's (2003) design principles, and the Mathematical Rubric for Potential of the Task (Boston & Wolf, 2006) were used to evaluate the potential level of cognitive demand of pre-constructed dynamic geometry tasks. Discourse was analyzed using a modified version of the Oregon Mathematics Leadership Institute (OMLI) Classroom Observation Protocol (RMC Research Corporation, 2005) and Boaler and Brodie's (2004) question types. Teacher technology orchestration types were utilized to code how

teachers used pre-constructed dynamic tasks during implementation, and the five practices served as codes for interview, group planning, and classroom observation data to characterize the extent of teachers' use of the five practices. The triangulation of discourse, technology use, and use of five practices was compared to the Mathematical Rubric for Implementation of the Task (Boston & Wolf, 2006) to evaluate the implemented level of cognitive demand of the pre-constructed dynamic geometry tasks.

Results

Discourse

Teacher to whole class was the predominant mode of mathematical discourse, followed by student to teacher and teacher to student. Questions were used slightly more often than statements. Questions were used to gather information, explore mathematical relationships and/or meanings, probe student thinking, generate discussion, and orient/focus student attention on important aspects of the problem solving situation. Overall, questions characterized as higher level by Boaler & Brodie (2004) were posed more often than lower level questions.

Five Practices

Teachers' use of the five practices for orchestrating productive mathematical discussions varied greatly throughout the semester. Only one teacher demonstrated progressively visible use of the five practices over the course of the semester.

Technology and Use

Teachers' Orchestration Types (Drijvers et al., 2010) included *discuss the screen* and *explain the screen* most often during implementation. Also, *link screen and board* facilitated connections between representations in the pre-constructed tasks and more traditional mathematical representations. *Tech demo*, s*pot and show*, and *Sherpa at work* were used sparingly.

Implemented Level of Cognitive Demand

Trends for level of implemented cognitive demand were based on teachers' implementation of common pre-constructed dynamic geometry tasks. Two predominant themes for level of cognitive demand stemmed from the results reported above. Given that pre-constructed dynamic geometry tasks adhered to design principles.

- When teachers employed higher-level questions/statements in conjunction with *discuss the screen* for their orchestration type, the implemented level of cognitive demand increased.
- When teachers employed lower-level questions/statements in conjunction with *explain the screen* for their technology orchestration type, the level of cognitive demand decreased.

Discussion

These results are consistent with Sherman (2011), where the decline or maintenance of high cognitive demand depended more on teachers' classroom practice than the role of technology during implementation. Student preparedness to engage in high level thinking tasks in general, teachers' anticipation of students' needs while using technology to engage with the task, and teachers' responses to student questions and difficulties were noted as three critical areas of classroom practice. The findings from this study speak to the latter two classroom practices.

All participants effectively anticipated students' mathematical and technological needs prior to task implementation. This helped teachers handle mathematical and technological difficulties with minimal interruptions to the learning environment. Also, teachers' use of questions and statements in conjunction with technology orchestration types speaks to effective responses to student inquiries and difficulties. The use of *discuss the screen*, with high-level questions and statements, illustrates a

student-centered response to questions and difficulties that places the cognitive demand on the students and emphasizes student thinking.

Implications

Results from this study provide implications for classroom mathematics teachers, teacher educators, providers of professional development, and future researchers. For classroom mathematics teachers, technology orchestration types and high level question/statement types represent very efficient and accessible ways to prepare for and reflect upon classroom instruction. Teacher educators could also utilize the conceptual framework and results from this study in methods courses for pre-service teachers. Such a course has the potential to assist pre-service teachers in 1) identifying mathematical content, process and/or technological goals, 2) selecting/designing high level tasks aligned with identified goals, and 3) using the five practices in conjunction with technology orchestration types and high level questions/statements to implement the task with students. A critical part of the course would be reflecting upon implementation of the task to consider factors that contributed to the decline or maintenance of level of cognitive demand.

One avenue for future research may involve professional development. For example, strategies within the professional development could involve participants 1) reading the *Five Practices for Orchestrating Productive Mathematics Discussions*, 2) learning the definitions for mode and type of discourse,
3) learning technology design and orchestration strategies, and 4) analyzing video cases in reference to the five practices, question/statement types, and technology design and usage. These learning activities encompass the implications noted for pre-service teachers, while delving deeper into ways to evaluate ones' teaching practice using research tools.

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A FRAMEWORK FOR EXAMINING HOW TEACHERS SUPPORT COLLECTIVE ARGUMENTATION

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Our teacher support for argumentation framework is based on data collected in a study that explored how prospective secondary mathematics teachers supported collective argumentation during their student-teaching experiences. Our participants used three types of support for collective argumentation: directly contributing argument components, asking questions that elicited parts of arguments, and using other supportive actions. This framework goes beyond frameworks currently in use to allow examination of how teachers impact the development of mathematics in classrooms and how teachers support the reasoning of students. It facilitates examination of who contributes the reasoning in the class and what the teacher does to prompt and respond to the reasoning that is contributed.

Keywords: Classroom Discourse, Secondary Mathematics

As suggested by recent policy documents and recommendations for teachers (e.g., Council of Chief State School Officers & National Governors Association Center for Best Practices, 2010), to facilitate productive mathematical discussions, teachers must engage in behavior that helps students build from their own understandings toward appropriate understandings of mathematical ideas. The purpose of this paper is to describe the ways in which teachers provide support for the collective argumentation in their classes and propose a framework for thinking about how teachers and students construct mathematical arguments.

Background

Researchers agree that the teacher plays a pivotal role in orchestrating mathematical discussions, even, or especially, when he or she is not acting as an arbiter of mathematical truth. Staples (2007), in her description of an experienced teacher establishing inquiry practices as normative in her classroom, foregrounded important aspects of the teacher's role in a collaborative classroom: whether and how a student's idea is taken up by the collective. Boaler and Brodie (2004) and Stein, Engle, Smith, and Hughes (2008) argued that a teacher's actions and questions influence the nature of a classroom discussion.

Understanding and constructing mathematical arguments are important parts of disciplinary practices of mathematics. Reasoning and proof are accepted to be foundational to the discipline of mathematics, but engaging students in formal deductive proof at an early age may not be developmentally appropriate. Argumentation, as a precursor to proof, is fundamental to the establishment of mathematical knowledge. We believe focusing on collective argumentation allows mathematics educators to think about what makes a discussion distinctively mathematical.

Recent work in mathematics education has highlighted collective argumentation as an important part of classroom discourse. Building on Toulmin's (1958/2003) model of

argumentation, mathematics educators, following Krummheuer (1995), have examined collective argumentation in classroom settings. We define *collective argumentation* very broadly to include any instance where students and teachers make a mathematical claim and provide evidence to support it. An argument, as described by Toulmin, involves some combination of *claims*, *data*, *warrants*, *rebuttals*, *qualifiers*, and *backings* (see Table 1 for definitions). Toulmin conceptualized an argument as occurring with a specific structure in which these parts of arguments relate to one another in specific ways. In practice, arguments are often complicated: statements offered as data may function as both data in an argument and claim in a subargument.

The Teacher Support For Argumentation Framework

The *teacher support for argumentation* framework is based on data collected in a study that explored prospective secondary mathematics teachers' beliefs about mathematics, teaching, and proof and how they supported collective argumentation during their student-teaching experiences. Our analysis of 277 arguments supported by our two participants during their student-teaching experience led to the development of this conceptual framework.

These teachers used three types of support for collective argumentation: directly contributing argument components, asking questions that elicited parts of arguments, and using other supportive actions. The research group analyzed each type of support separately. We used Toulmin's (1958/2003) model to determine the classifications for the direct contributions (e.g., claim, data, or warrant). We developed codes for questions and other supportive actions and then collapsed the codes into meaningful categories. The three types of support, along with the categories of each, constitute the teacher support for argumentation framework (see Table 1).

Table 1: Teacher Support for Argumentation Framework

Direct Contributions		**Questions**		**Other Supportive Actions**	
Claims	Statements whose validity is being established	*Requesting a Factual Answer*	Asks a question that can be answered quickly by providing a mathematical fact	*Focusing*	Actions that serve to focus the students' attention and/or the argument
Data	Statements provided as support for the claims	*Requesting a Method*	Asks students to demonstrate or describe how they did something	*Promoting*	Actions that serve to promote mathematical exploration
Warrants	Statements that connect data with claims	*Requesting an Idea*	Asks students to compare, coordinate, or generate mathematical ideas	*Evaluating*	Actions that center on the correctness of the mathematics
Rebuttals	Statements describing circumstances under which the warrants would not be valid	*Requesting Elaboration*	Asks for an elaboration of some idea, statement, diagram	*Informing*	Actions that provide information for the argument
Qualifiers	Statements describing the certainty with which a claim is made	*Requesting Evaluation*	Asks students to evaluate a mathematical idea	*Repeating*	Actions that repeat what has been or is being stated
Backings	Usually unstated, dealing with the field of the argument				

As an example, consider the following classroom conversation. This conversation occurred at the end of a class in which the students had been working in groups to develop a formula for the

sum of the measures of the interior angles of any polygon. Groups of students presented their formulas at the end of class. The following transcript and diagram give the first presentation.

Ms. Bell: So, Travis, tell us what you did.
Travis: All right. [Writing: 360 540 720 900.]
Ms. Bell: So what is this you're writing on the board?
Travis: What do you mean? That was the
Ms. Bell: Sum of the?
Travis: Sum of the interior angles.
Ms. Bell: Interior angles, okay.
Travis: So to get these, it would be--you're just--
Ms. Bell: What are the side lengths that correspond with that, Travis?
Travis: n minus 2--n would be the number of sides, minus two, times 180. So this 360, was 4-sided, so 4-2 is 2, times 180 is 360. This was 5-sided, so you would do 5 minus 3--or 5 minus 2 is 3, times 180 is 540.
Ms. Bell: Okay.
Travis: So it would just be the number of sides minus two--
Ms. Bell: So did you find that by trial and error? You were just trying different--?
Travis: Yeah
Ms. Bell: Okay, cool. Go ahead and write your formula you got up there for it, a function in terms of n.
Travis: All right. I'm going to write both of them. [Travis writes $(n\text{-}2)180$ and $F(n) = 180n - 360$]
Ms. Bell: So that's what Travis originally had up top, and then Keith said, 'hey, what about the distributive property? We can just multiply that through.' Good deal. Thanks, Travis.

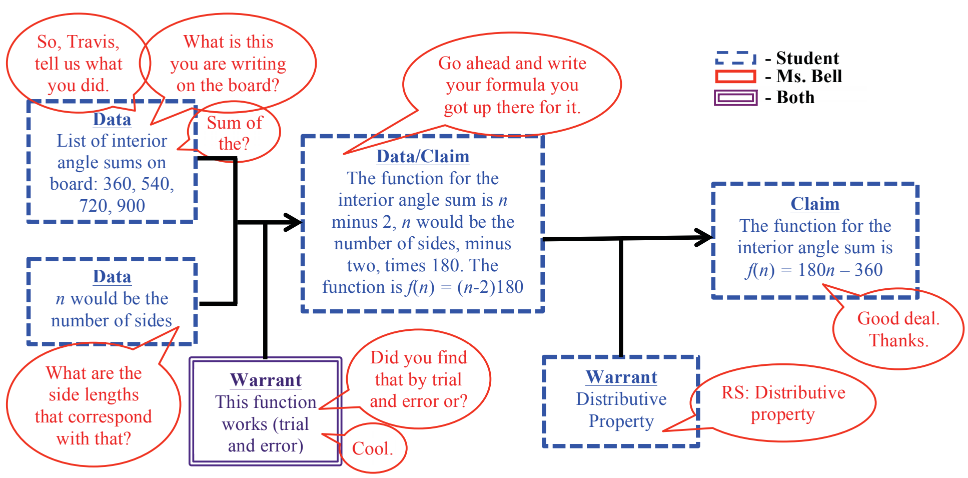

Figure 2: Diagram of Classroom Conversation

In this episode of argumentation, Ms. Bell asked several questions at the beginning to help Travis express his data to the class in a way that they would understand: “So what is this you’re writing on the board?” and “Sum of the?” were questions *requesting elaboration*. The questions required him to clarify that the list of numbers he was writing on the board was the list of interior angle sums that he would use for the rest of his argument. She *requested a factual answer* when she asked, “What are the side lengths that correspond with that?” He identified those side lengths as well as giving the variable that he would use for the generalized function. He stated his

function verbally, but she supported the collective argumentation by asking him to display it on the board, *repeating* it for the benefit of the other students. Ms. Bell's question, "So did you find that by trial and error?" *requested a method*, suggesting what that method might have been. As the conversation progressed, Ms. Bell also *repeated* Keith's warrant of the distributive property and *evaluated* Travis's claim by saying, "Good deal. Thanks, Travis."

Ms. Bell's support for the collective argumentation in this conversation took several forms. Most of her questions served to clarify Travis's explanation to the rest of the students. In particular, she made sure each part of the argument was made explicit for the class. Without intervention by Ms. Bell, much of Travis's group's thinking might have been opaque to their classmates. One important part of the argument, the warrant connecting the group's data to the function in their claim, might have been left implicit by Travis. Our framework draws attention to the importance of Ms. Bell's question ("So did you find that by trial and error?") by pointing out its location as related to a warrant in the argument and shows that Ms. Bell had some input into the content of that warrant by showing that the warrant was co-constructed by the teacher and student. We found similar questions and other supportive actions across Ms. Bell's teaching.

Conclusion

We propose this framework as a device that allows one to examine classroom conversations in a particularly mathematical way. It goes beyond current frameworks to allow examination of how teachers impact the development of mathematics in classrooms and how teachers support the reasoning of students. This framework allows one to examine who contributes the reasoning in the class and what the teacher does to prompt and respond to the reasoning that is contributed. Although we believe that it is useful to look at questions and other supportive actions without diagramming the argumentation that occurs in a classroom, it is when the questions and other supportive actions are examined in the context of the claims, data, and warrants of an argument that the most mathematical aspects of the classroom conversations come to light.

Acknowledgments

This paper is based on work supported by the University of Georgia Research Foundation under Grant No. FRG772 and the National Science Foundation through the Center for Proficiency in Teaching Mathematics under Grant No. 0227586. Opinions, findings, and conclusions in this paper do not necessarily reflect the views of the funding agencies.

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EXAMINING TEACHERS' EXPECTATIONS FOR ENGLISH LANGUAGE LEARNERS' CONSTRUCTION AND CRITIQUE OF VIABLE ARGUMENTS

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Although extant research related to ELLs encourages increased classroom communication as a means of building students' mathematical understandings and academic language, many teachers remain unsure of how to facilitate communication around viable mathematical arguments due to the academic language demands of such endeavors. I examined teachers' expectations for the construction and critique of mathematical arguments in three ninth grade mathematics classrooms comprised entirely of ELLs and the influence these expectations had on the cognitive demand (Henningsen & Smith, 1997) of mathematics tasks. The findings suggest teachers routinely deemphasized viable arguments as a means to assuage language demands for ELLs. This practice led to the enactment of a high proportion of low cognitive demand tasks.

Keywords: Secondary Mathematics, Equity and Diversity, Instructional Activities and Practices

The Common Core State Standards' (2010) third mathematical practice refers to students' abilities to construct viable mathematical arguments and critique the reasoning of others. This mathematical practice involves classroom communication along all four of the Teachers of English to Speakers of Other Language's (TESOL, 2006) language domains—listening, reading, speaking, and writing. For example, students must be able to read or listen to mathematical arguments in order to critique them. Similarly, one must write or speak to formulate a viable mathematical argument or critique.

In light of the Common Core's (2010) inclusion of the mathematical practice related to the construction of mathematical arguments, a position in alignment with the National Council of Teachers of Mathematics' (NCTM) Process Standards (2000) and Reasoning and Sense Making (2009) document, teachers of English language learners (ELLs) face the challenge of encouraging this sort of communication with students who are not yet proficient in English. This situation is further complicated by the unprecedented growth in ELLs over the past decade (National Clearinghouse for English Language Acquisition, 2011). Although extant mathematics education research related to ELLs encourages increased classroom communication as a means of building students' mathematical understandings and academic language, many teachers remain unsure of how to facilitate communication around viable mathematical arguments due to the academic language demands of such endeavors. In this study I examined teachers' expectations for the construction and critique of mathematical arguments in three ninth grade mathematics classrooms comprised entirely of ELLs and the impact these expectations had on the cognitive demand (Henningsen & Smith, 1997) of mathematics tasks.

Theoretical Framework

In looking for evidence of classroom expectations related to the mathematical practice of viable arguments, I employed Henningsen and Smith's (1997) notion of cognitive demand as a means of describing the impact these expectations had on the mathematics tasks. Cognitive demand refers to the type of thinking required to complete a mathematical task. Using this

construct to examine the tasks allowed me to discuss the ways in which the classroom expectations for viable arguments impacted the mathematical thinking around tasks. Henningsen and Smith discussed the requirement for mathematical explanations as a characteristic of higher cognitive demand tasks. In contrast, tasks focused on numerical solutions and excluding requirements for justification are classified as having a low cognitive demand. The recommendation for the inclusion of high cognitive demand tasks requiring ELLs to create and critique mathematical arguments is supported by mathematics education literature (e.g. Moschkovich, 2002), however, there has not been an examination of teacher practice in this area.

Methods

In conducting this study, I employed a qualitative case study methodology (Stake, 1995). I examined classroom data from three secondary mathematics teachers' classrooms— Ms. Thomas, Ms. Hunter, and Mr. Dubois. Each teacher was in his or her sixth year teaching and had earned an ESOL certification from the state. Each of the classrooms was a sheltered mathematics classroom, meaning all of the students were classified as ELLs, in which English was the sole language of instruction.I purposefully selected (Patton, 2002) these teachers because of their role as sheltered mathematics teachers.

This study is part of a larger study in which I examined teachers' selection and enactment of mathematics tasks for ELL students. The primary data sources for the present study are surveys, a series of interviews, two weeks of classroom observations, and the teachers' curriculum materials. I analyzed the data using the constant comparison method decoupled from grounded theory. This involved many rounds of inductive coding. I used the notion of the construction of viable arguments proposed by the Common Core State Standards (2010) defined as students "understand[ing] and us[ing] stated assumptions, definitions, and previously established results" (p.7) to guide the analysis as I looked for instances requiring students to formulate such arguments in curriculum materials and classroom interactions. I first analyzed data from each teacher individually and identified emerging themes using analytic memos. I then collapsed these themes into codes as I analyzed each of the data sources for each teacher. Finally, I performed a cross case analysis looking across the teachers to identify relevant themes.

Findings

Each of the three teachers in the study created similar expectations around their students' construction and critique of convincing arguments. All three teachers conflated students' mathematical and English proficiencies. For example, Ms. Thomas stated that her accommodations for ELL students included selecting problems with integer solutions so that her students would not get "bogged down by the math" resulting from rational solutions. This conflation was evident in the ways in which the teachers discussed their expectations for what constituted an acceptable mathematical justification or proof from their ELL students.

There were a number of instances in which the teachers described or enacted modifications to mathematics tasks that eliminated the need for mathematical arguments or justification. One such instance occurred during the curriculum interview with each of the teachers when Mr. Dubois stated, "I think I would skip the proof part." His reluctance to bring in proofs to his class was evident during my observations. During an interview he stated that it would be nice for his students to know why things work but that it was not necessary for the end of course test. Ms. Hunter and Ms. Thomas also stated their reluctance to have students complete mathematical proofs, both formal and informal,in their sheltered courses. The teachers discussed these

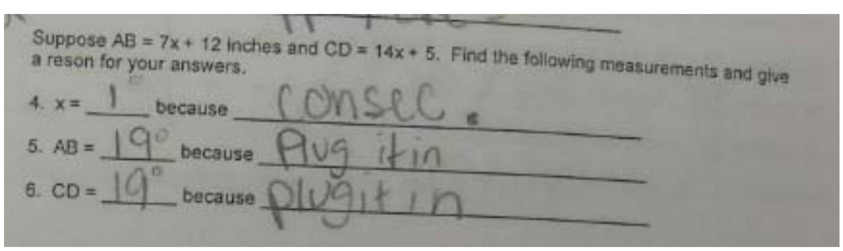
expectations regarding proof for their sheltered students as different from their expectations for native English speaking students. Ms. Thomas stated that she allowed ELLs to have their own way of doing proofs. This “ELL version” was meant to simplify the process but was also less precise and rigorous than formal proof. For example, she considered fill in the blank type activities a modified version of proof. The teachers’ satisfaction with imprecise and unclear mathematical explanations routinely contributed to a decline in the cognitive demand of tasks.

The explanation of mathematical thinking was seldom student generated with students more frequently explaining their thinking only when prompted by the teacher. This was the case in Mr. Dubois’ classroom where he often used the question “why” during whole class discussions after students gave single word responses. It is important to note, however, that the students’ responses to his question of why were often simplistic. For example, the following exchange occurred when the class was discussing the Triangle Inequality Theorem and whether or not a triangle could exist with the given side lengths,

Mr. Dubois: Gabriella, next one (referring to side lengths 8,3, and 1)
Gabriella: No.
Mr. Dubois: Because it is smaller.
Mr. Dubois: Yes, what is smaller?
Gabriella: When you add.
Mr. Dubois: Yes (moves on to the next student)

Mr. Dubois –asked the previous students to explain his or her thinking, but the students still did not provide explanations until prompted by Mr. Dubois. Exchanges such as this were common throughout my time in the teachers’ classrooms. Students rarely explained why or how they arrived at a solution without the teacher’s prompting. There was no expectation for students constructing viable mathematical arguments. Furthermore, many of their responses lacked depth. This may result from the lack of such expectations in the tasks presented to students and the classroom norms developed over time by the teachers. In either case, the lack of student explanation typically aligned with the maintenance of a tasks’ already low cognitive demand.

Several opportunities arose in which the teachers’ prodding for student explanations of mathematical thinking could have raised cognitive demand; however, the teachers’ acceptance of brief, often superficial responses, prevented this from occurring. One such example comes from Ms. Thomas’ classroom. In this task Ms. Thomas asked students to solve for missing quantities such as angle or side measures in a given quadrilateral. Next to each answer blank Ms. Thomas also included the word “because” and another blank in which students were to write a property that justified their solution. There seemed to be no expectation for precise mathematical reasoning in the student responses. Ms. Thomas was satisfied by student responses such as those shown in Figure 1. Ms. Thomas presented one of the only tasks that included problems asking students to justify their responses, though the justifications were not mathematically rigorous.


Suppose AB = 7x + 12 inches and CD = 14x + 5. Find the following measurements and give a reson for your answers.

4. x = 1 because consec.
5. AB = 19° because Plug it in
6. CD = 19° because plugit in

Figure 1. Sample Student Response Deemed Acceptable

Discussion

In this study I examined teachers' expectations related to viable arguments from ELLs in a ninth grade mathematics classroom. Each of the teachers discussed his or her reluctance to engage students in both formal and informal proof and seldom required students to provide mathematical arguments or critique the reasoning of other students. This omission was consciously decided in order to assuage the language demands of such activities, a position contrary to mathematics education research advocating for the inclusion of such activities as a means of developing students' English language capabilities in mathematics (e.g. Khisty & Chval, 2002). Moschkovich (2002) specifically asserted the need for ELLs to engage in the work of mathematicians, work that includes "engage[ing] in mathematical arguments about these problems, paralleling the sorts of arguments academic mathematicians might make" (p.9). The omission of expectations for convincing arguments decreased student opportunities to engage with high cognitive demand tasks. Each of the teachers had earned ESOL certification implying they had achieved some level of expertise in teaching ELLs, however, each of the teachers also stated that they would benefit from additional, math-specific preparation in which they learned how to implement rich mathematics tasks with ELLs.

Perhaps due to the exploratory nature of this study, the results led to more questions than answers. What accommodations can teachers make to instruction to allow ELLs access to the mathematics without lessening requirements for justification? How can teachers foster communication that promotes the construction and critique of viable arguments with ELLs? How might mathematics educators prepare teachers to foster this type of communication?

Further research is required to better understand how to support teachers in facilitating classroom communication around mathematical arguments with ELL students. The need for this work is particularly urgent with the forthcoming shift to the Common Core State Standards alongside a drastic increase in the number of school aged ELLs in the U.S.It is my hope that this study may encourage mathematics educators to undertake this challenge and address this issue with both preservice and inservice teachers.

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RELATING PERSONAL MATHEMATICS AND CLASSROOM MATHEMATICS: MATHEMATICAL PROCESSES AS PROBLEM-SOLVING STRATEGIES AND AS PEDAGOGICAL STRATEGIES

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This case study examined the relationship between an early career secondary teacher's personal mathematics and her classroom mathematics. We examined processes (generalizing and justifying) and actions on products of those processes. We found that the teacher's focus on generality and use of generic examples (Mason & Pimm, 1984) in her personal mathematics were adapted as strategies for justifying in the classroom. Implications address the need for teacher educators to attend to the beliefs and values that preservice teachers have regarding mathematics that together influence the teacher's ways of drawing upon personal mathematical processes in the service of teaching.

Keywords: Instructional Activities and Practices; Mathematical Knowledge for Teaching; Teacher Education-Preservice; Teacher Knowledge

Introduction and Framework

Recommendations for mathematical coursework in the preparation of teachers tend to assume that a teacher's mathematical knowledge is an important influence in the classroom (e.g., Conference Board of the Mathematical Sciences, 2012). However, the nature of that knowledge and its relationship to effective teaching are not clear. Although others have explicated the specialized form of mathematical knowledge that teachers use (e.g., Ball, Lubienski, & Mewborn, 2001), we argue that by also examining the ways that teachers engage in mathematics we can better understand the relationship between teachers' personal mathematics and the nature of the mathematics in their classroom. A better understanding of this relationship may help teacher educators prepare preservice teachers to craft instructional environments within which students are engaged in doing mathematics.

We characterize *doing mathematics* as engagement in mathematical processes and in actions on mathematical products (Zbiek, Heid, & Blume, 2012; c.f. Farrell & Farmer, 1988). For this paper we focus on the processes of generalizing and justifying in one teacher's personal mathematics and how those relate to justifying and generalizing in her classroom. We define generalizing as "the constructive act of extending the domain to which a set of properties applies from multiple instances of a class or from a subclass to a larger class of mathematical entities" and justifying as "the constructive act of explaining how one knows a mathematical claim is true or producing a rationale for belief of a mathematical claim" (Zbiek et al., 2012, p. 3).

With each process we associate a type of product (generalization or justification) and actions thereon, such as evaluating a justification or stating a generalization. Using this approach, we explored how a preservice, then first-year, teacher engaged in mathematical processes and actions on products in personal mathematics, how she engaged students in processes and actions on products, and the nature of the relationship between her own processes and actions and those in her classroom.

Method

Our data come from the case of a secondary mathematics teacher, Olivia, whom we followed from her preservice coursework into her first year of full-time teaching. Data sources included five task-based interviews, 12 observations of Olivia’s teaching with accompanying pre- and postinterviews, and her textbooks and lesson plans and classroom materials. Combinations of video recordings, audio recordings, and field notes were used to capture each observation and interview. Verbatim transcripts, annotated by members of the project team, were examined for evidence of processes and actions on products. Codes were discussed among the research team until consensus was reached regarding the clarity of descriptions and agreement on coding. Simultaneously, the team identified patterns, themes, and relationships within the data, reviewed transcripts to search for disconfirming evidence, and revised themes and claims accordingly.

Findings

Processes in Olivia's Classroom

The mathematical activity that occurs in Olivia's classroom is more accurately characterized by actions on generalizations than by engagement in generalizing or justifying. However, we noticed two episodes in which Olivia engaged students with generalizing as a means of justifying. In student teaching, Olivia attempted to justify the claim that $a^0 = 1$ by first showing that $a^{1-1} = 1$, and then claiming that “if we generalize it” by replacing a^{1-1} with a^{n-n}, the result still holds. In first year teaching, she engaged students in justifying the claim that the sum of the exterior angles of a convex polygon will be 360 degrees by first working with students to show that the claim would hold for a pentagon (see left side of Figure 2) and then asking them to “generalize this and use it for any size polygon,” to which students responded by suggesting the proof shown on the right in Figure 2.

6) Zero power property $a^0 = 1$

Proof: $a^0 = a^{1-1} = \dfrac{a^1}{a^1} = 1$

$$a^0 = a^{m-m} = \frac{a^m}{a^m} = 1$$

Figure 1: Olivia's Complete Proof of the “Zero Power Property”

We believe that Olivia uses generalizing as a teaching strategy to help students understand or develop justifications. This strategy depends upon treating a specific justification as a generic example (Mason & Pimm, 1984), which is a specific example that is interpreted and used as a proxy for any and all elements of a class. Olivia's request to generalize the proof from the pentagon to the n-gon requires that students be able to recognize that the computational sequence does not depend upon the number of vertices involved and could be executed on any arbitrarily chosen example.

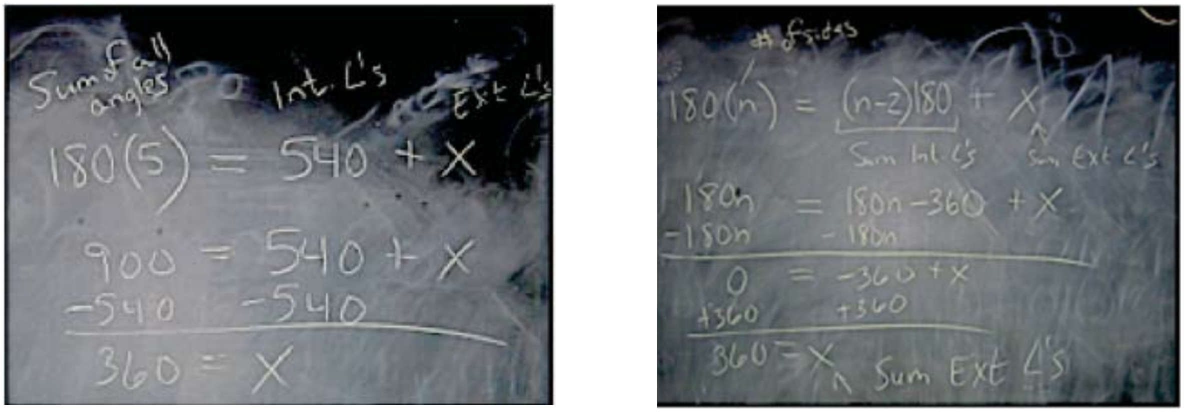

Figure 2: Proof That Exterior Angle Measures of a Convex Polygon Sum to 360 Degrees

Generalizing in Olivia’s Personal Mathematics

The use of specific cases as generic examples was also observed in Olivia’s personal generalizing and can be illustrated by two examples from the second task-based interview. We presented the images of the four Arrays shown in the left-hand side of Figure 3 to Olivia and asked whether there would ever be an array for which the number of interior circles exceeded the number of exterior circles. As she searched for ways to relate the interior and exterior circle counts (INT and EXT), Olivia noted that Array 6 had “only an additional three [exterior circles]” compared to Array 5 (See the right-hand side of Figure 3). She then treated this “additional three” as a general relationship between consecutive arrays, creating an additive recursion to compute values of EXT. Subsequently, Olivia used Array 6 and Array 7 in a similarly general way, noting that adding a seventh row to Array 6 would increase INT by “seven minus one . . . minus two,” which she recorded as “7 – 3” and, then, as “n – 3,” the latter of which she used recursively to compute additional values of INT.

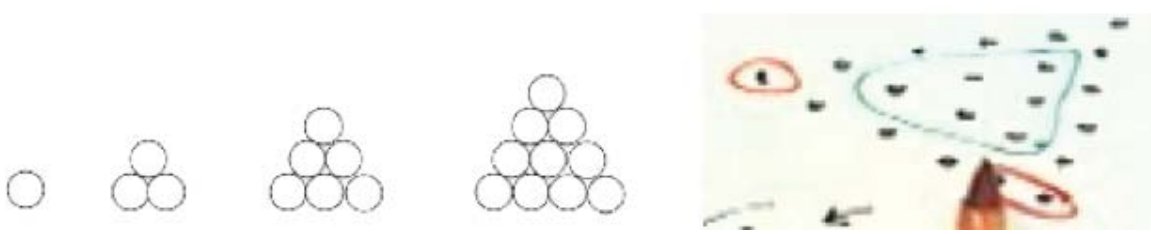

Figure 3: The Sequence of Triangular Arrays With Olivia's Drawing of Array 6

Conclusion

Olivia’s case suggests that a preservice teacher’s ways of doing mathematics is not, by itself, a direct indication of how that teacher will engage students in mathematical processes in the classroom. In spite of Olivia’s capacity to engage in mathematical processes within problem solving, her classroom is primarily characterized by attention to acquainting students with mathematical products (in particular, symbolic representations of generalizations) and with engaging them in activities that emphasize applying those products in the completion of exercises.

The preceding characterization suggests that there are influences that make the completion of exercises a more pressing concern for Olivia than engaging students in mathematical processes.

However, when circumstances called for justifying in her classroom, we can see ways in which she leveraged the generalizing that she used to solve mathematical problems as a strategy for solving a teaching problem. In interviews associated with her classroom teaching, Olivia stated that a justification should "prove it for everything" and an argument doesn't prove if "it doesn't really show you is that going to be the case for every one," but also that students will be convinced by empirical examples ("I don't think that they would need [many] . . . maybe five"). The conflict between empirical argumentation and argumentation focused on generality creates a pedagogical problem for Olivia. To solve this problem, she seems to have drawn upon her mathematical strategy of generalizing through generic example and adapted it as a pedagogical strategy for bridging the two types of argumentation in the classroom.

Implications for Teacher Education

Olivia illustrates that simply ensuring that prospective teachers are able to engage in processes is not the same as preparing them to address mathematical processes in their classrooms. In Olivia's case, the adaptation of her personal processes to the classroom was necessary given the importance that she assigns to generality as a quality of justifications. It seems that one aspect of preparing teachers to engage students in mathematical processes may involve paying attention to their conceptions regarding the essential qualities of mathematical products (e.g., generality of a justification), because this may contribute to the teacher's decisions about whether and how to engage students in processes in the classroom.

Additionally, teachers who are prepared to engage students in processes must be able to identify and acknowledge classroom situations that afford opportunities to leverage mathematical processes in the service of solving mathematical and pedagogical problems. We might speculate a host of influences that lead teachers to emphasize actions on products to the exclusion of mathematical processes. However, from Olivia's case we can see that an emphasis on products and actions in a classroom does not eliminate opportunities to engage students in mathematical processes, provided that the teacher has identified circumstances within which those processes are deemed relevant. If we accept the premise that processes (the doing of mathematics) should be an integrated part of mathematics education, then it is important to prepare teachers to seek, find, and seize opportunities to include such processes in their classrooms.

Acknowledgments

This research was supported by a grant from the National Science Foundation (Award No. ESI-0426253). Any opinions, findings, conclusions, or recommendations expressed herein are those of the authors and do not necessarily reflect the views of the National Science Foundation.

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TEACHER PERCEPTIONS AND PRACTICES: INCREASING STUDENT ENGAGEMENT IN THE GRADE 9 APPLIED MATHEMATICS CLASSROOM

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This study investigates the teaching practices that three Grade 9 Applied Mathematics teachers use to increase student engagement and enhance student learning. Qualitative data were collected in the form of teacher interviews, classroom observations and teacher journals. Findings show that these teachers consider aspects of both social (e.g., creating a classroom community and developing a teacher-student relationship) and academic (e.g., using technology, manipulatives, group work and student-centered activities) domains of student engagement in their teaching, but to varying degrees and with different emphases. All teachers noted that these strategies also appeal to their students' based on their characteristics as early adolescents.

Keywords: High School Education, Instructional Activities and Practices, Teacher Beliefs, Teacher Education-Inservice/Professional Development

Less than 4% of students who fail Ontario's Grade 9 Applied level English or Mathematics have completed secondary school after four years (King, Warren, Boyer, & Chin, 2005). The literature emphasizes the importance of student engagement and its direct implication on student retention and academic success (e.g., Lowe et al., 2010). Indeed, much research posits that students are at risk of dropping out if they do not form positive relationships with their peers (Janosz, Le Blanc, Boulerice, & Tremblay, 1997) or if they lose interest in schooling and do not feel as though there is a personal benefit from staying in school (Ensminger, Lamkin, & Jacobson, 1996). This study investigates the teaching practices that three Grade 9 Applied Mathematics teachers use to increase student engagement and enhance student learning. I draw from Willms, Friesen, and Milton (2009) and Janosz, Archambault, Morizot, and Pagani's (2008) notions of engagement to frame my research. Specifically, I use two domains of student engagement, social and academic, to examine the factors that teachers used to increase student engagement.

Perspectives

Adolescence is a developmental period in which the individual undergoes many changes and as such, is at greater risk for potentially negative repercussions (Archambault, Janosz, Morizot, Pagani, 2009). During adolescence, individuals often require strong emotional and social support to reach success. Students need to be engaged socially to be successful (Janosz et al., 2008). Social engagement encompasses factors including: social isolation/rejection, quality of student-teacher relationships, and participation in extracurricular activities. Students at this level need to feel like they belong. A sense of comfort with their peers will increase the chance that an adolescent will be successful (Osterman, 2000). Schools need to be an environment in which students feel invited and supported. Specifically, teachers can create a classroom-learning environment that fosters a sense of community (Hargreaves, Earl, & Ryan, 1996).

Adolescent learners appreciate the chance to take ownership of their learning (Chapman, Skinner, & Baltes, 1990). Hand (2010) suggests that teachers allow students ample opportunity to construct knowledge through exploratory, student-centered activities. Teachers should also use

tasks and activities that are matched to their students' interests. By making mathematics learning relevant to the students' lives, the students will become more invested in their learning. They will see learning as something worthwhile to engage in rather than solely a task to be accomplished (Davis, 2006). Another way that teachers can make mathematics relevant to the early adolescent learner is through the use of technology (Gee, 2003).

Many of the above considerations parallel the Ten Dimensions of Mathematics Education (McDougall, 2004). This framework for effective teaching allows teachers to focus on the areas of teaching practice that generate higher levels of student achievement.

Method and Data Sources

This study investigates the ways in which Grade 9 Applied level mathematics teachers attempt to engage their students. To this end, I endeavoured to capture how teachers engage their students and the rationale for these decisions. The method is exploratory case study (Yin, 2009) using qualitative methods (teacher interview, classroom observations, and teacher journals). The participants are three Grade 9 Applied Mathematics teachers who teach at public secondary schools within a large urban centre in Central Canada. All three teachers were part of the Collaborative Teacher Inquiry Project (McDougall, Jao, Kwan, & Yan, 2011) that focused on improvement of instructional strategies in Grade 9 Applied Mathematics.

Data were collected throughout the first semester of the 2011-2012 academic year. Data were generated through three sources: semi-structured teacher interviews, field notes from classroom observations, and teacher journals. An initial interview determined the background of the teacher and asked about impressions of student engagement. Teachers were asked to keep a journal for the duration of the study. The structure of the journals was left open-ended to allow teachers to comfortably articulate their reflective and emotional impressions about their class. I visited each teacher bi-weekly and took field notes about the teaching strategies that were used, how the teacher used them and how the students responded to these strategies. Following each classroom observation, I conducted interviews with the teachers. Each interview started by asking the teacher to reflect on his/her lesson, followed by questions specific to the teacher based on emerging themes from journal entries, classroom observations and earlier interviews.

The data analysis includes an initial exploratory review of the data and a constant comparison analysis (Miles & Huberman, 1994) of interview transcripts, classroom observation notes, and teacher journals. Computer qualitative research software, nVivo9, is being used to assist in the analysis of the data. The initial coding scheme was based on themes from literature about student engagement and is being elaborated based on the emerging themes.

Findings

I use the cases of Benjamin, Mathieu, and Nadia to describe ways in which Grade 9 Applied Mathematics teaching increase student engagement. Consistent across all three cases is the use of similar instructional strategies including the Ontario Ministry of Education's TIPS4RM (OME, 2005) resource and the use of technology. Findings show that these teachers consider aspects of both social and academic domains of student engagement in their teaching, but to varying degrees and with different emphases. I describe this in more detail within each case.

The Case of Benjamin

Benjamin teaches physical education and mathematics, and coaches extracurricular sport teams. He is also the school's Numeracy Lead, which entails attending meetings to discuss numeracy initiatives at the school and district level.

To increase student engagement in his Grade 9 Applied Mathematics classroom, Benjamin

integrates many factors into his teaching. Specifically, these factors fall under both the social and academic domains for student engagement. In the following quote, Benjamin describes one student whom he described as going from not-engaged at the beginning of the semester to being fully engaged by the end.

> I have one student who, at the start of this semester, I thought would be a non-attender, because that is what he is in most of his other classes. But he is engaged by a number of the strategies that we have used and he does not miss class ever. [T]here is certainly a proudness [sic] when I see him in the hallway and he says, “How is it going?”. I think that in providing engagement and just giving him some sort of validity in the classroom has been huge.

Benjamin’s description of this student demonstrates that he considers factors in both the social and academic domains for student engagement. Within the social domain, Benjamin mentions that there has been an increase in this student’s self-confidence. In the academic domain, Benjamin notes that the inclusion of a variety of teaching strategies had positive repercussions on this particular student. As evidenced by an improved attendance record, Benjamin believes that the strategies that he uses in class are compelling enough for this student to make the decision to attend class more often.

The Case of Mathieu

Mathieu has been teaching for over 35 years. Mathieu is qualified to teach Grades 7 to 12 mathematics, science, and computer science, and has his Honours Specialist in mathematics. Mathieu has recently taken on a leadership role at his school, as an Assistant Curriculum Leader.

Formerly a self-confessed “traditional teacher”, Mathieu changed his teaching approach as a result of the Collaborative Teacher Inquiry Project. Since then, he has noticed a change in the engagement of his students, something that he directly attributes to his reformed teaching methods. Thus, Mathieu continues to focus on the academic domain for student engagement.

> In my class, we use the new methods, which are working in pairs, student interaction and group work. The kids are always engaged and do the richer type problems where they have to collaborate, work together, try to come up with a solution so it is mostly student driven. So the teacher is more or less just a facilitator who gives directions. We try to get the kids engaged every day. So they are doing stuff. Hands-on. So they are not sitting the being bored.

Through Mathieu’s description, we can see his focus on factors within the academic domain. These teaching strategies allow students to become interested in learning the material and support their developing mathematics understanding.

The Case of Nadia

Nadia has been teaching for 13 years and has qualifications to teach mathematics and French for Grades 7 to 12. Nadia also has an Honours Specialist qualification in mathematics.

For Nadia, student engagement extends beyond students being interested in the mathematics content. Nadia explains that, especially for Applied level students, students show their engagement on a personal level. If students are engaged, they will connect with their peers and their teacher on a personal level. Even after students have completed the mathematics work for the class, Nadia says that students who are engaged will linger. Engaged students will socialize with their peers and develop a stronger relationship with the teacher. Nadia says that she is still relatively new to the Grade 9 Applied Mathematics course and the instructional strategies shown to increase student engagement. She says:

> Everything is new. I cannot do everything. I need my own time to do well. I need to be

comfortable and to be comfortable, I need to be able to put in my own time. As a teacher, you need to be comfortable first. And it has to be your style.

In addition to Nadia's belief that social engagement is the key priority for student engagement for the Applied level student, it is not surprising that Nadia's tentativeness with certain instructional strategies further emphasizes Nadia's focus in the social domain.

Conclusion

The evidence from the study shows that all three teachers were cognizant of attributes of their early adolescent learners as the teachers sought to increase student engagement in their Grade 9 Applied Mathematics class. Six major findings from the cases of Benjamin, Mathieu and Nadia can be summarized as follows: (1) developing student self-confidence is a critical component of increasing student engagement for early adolescent learners; (2) teachers may focus on one domain more than the other as a result of their personal comfort with that domain; (3) domains for student engagement and the factors found within these domains are not independent; (4) the Ontario Ministry of Education's TIPS4RM resource (OME, 2005) is an effective way to increase student engagement; (5) technology is also an effective and relevant way to increase student engagement; and (6) the use of a framework for student achievement (e.g., McDougall, 2004) may support teachers efforts to increase student engagement.

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TEACHER QUESTIONING: DOES MKT AND SUPPORTING STUDENT AUTONOMY PREDICT IT?

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The present study examined elementary teachers' choice of questions and how their level of MKT and dispositions for supporting students' autonomy predicted such choices. Results suggest no statistically significant relationship between MKT and questioning choice. However, autonomy support was found to be a consistent predictor across scenarios. Potential interactions between student responses and the effect of autonomy were also observed.

Keywords: Mathematical Communication; Teacher Knowledge; Teacher Questioning.

Background and Objectives

Teachers' questioning strategies strongly influence the manner in which students engage in mathematical discussions, as well as their success in developing mathematical understandings (e.g., Franke et al., 2009; Hufferd-Ackles et al., 2004; Kazemi & Stipek). A predominate factor inherent in more successful questioning approaches is the ceding of control from the teacher to their students (e.g., Hufferd-Ackles et al., 2004). Yet, encouraging student autonomy in association with one's questioning strategies may have limitations in its effectiveness. Rather, too strong a focus on supporting student autonomy may remove necessary focus on providing appropriate mathematical scaffolds in discussions (Chazan & Ball, 1999), and may result in use of inappropriate questions. Initial observations from qualitative studies suggest teachers' mathematical knowledge for teaching (MKT) may facilitate teachers' questioning strategies (Cengiz et al., 2011). Yet, such studies are few in number, and use relatively small samples. Therefore, the present study sought to examine this potential relationship with a larger sample to provide more breadth to the literature. Additionally, teachers' disposition towards facilitating student autonomy and its influence on teacher questioning was also included in this study. Thus, the purpose of the present study is to examine preliminary data to assess the extent to which primary grade teachers' MKT and disposition to facilitate student autonomy predict their choice of question in hypothetical mathematics teaching scenarios.

Teacher Questioning and Potentially Related Factors

Boaler and Brodie (2004) examined lessons from grade school teachers and identified 9 basic classifications: *gathering information* (requiring immediate answers for known facts/procedures); *inserting terminology*; *exploring mathematical meanings and/or relationships* amongst concepts; *probing* for students to explain their thinking; *generating discussion* from students in the classroom; *linking and applying* mathematics to other areas of life or study; *extending thinking* to situations where the math could be similarly used; *orienting and focusing* on key elements or aspects of the situation; *establishing context* by making connections outside of mathematics. Research using these classifications suggests *gathering information* and *generating discussion* questions may be less reflective of meaningful questions than, for example, *probing* and *orienting and focusing* questions (Boaler & Brodie, 2004; Kosko et al., 2013). Given the specificity that this classification scheme allows, it was used in the present study. While not using Boaler and Brodie's (2004) classification scheme, other research suggests that a key feature in effective questioning, and questions, is support for students' mathematical autonomy (Hufferd-Ackles et al., 2004; Kazemi & Stipek, 2001). While autonomy appears to be an essential component of effective questioning, and general facilitation of mathematical discussion, there are arguments suggesting too much autonomy can limit the

effectiveness of discussion (Chazan & Ball, 1999; Sfard, 2007). Specifically, such research has argued that while a certain ceding of control by the teacher to students should be incorporated in teachers' questioning, a certain degree of scaffolding must also be inherent. Yet, this skill may be an inherent aspect of teachers' MKT. If so, teachers with higher levels of MKT may possess the specific skills necessary to allow for students' mathematical autonomy while including scaffolds to limit such autonomy in ways that support the appropriate development of mathematics.

Studies examining the connection between MKT and teachers' facilitation of mathematical discussion are limited, but informative. Cengiz et al. (2011) observed six teachers in 1st through 4th grade and found that teachers' level of MKT interacted with the way they facilitated discussions. However, Grassetti (2010) observations of 3 novice teachers suggested their levels of MKT did not correspond with their use of effective questioning strategies. These 3 studies provide a mixed view of whether and how MKT influences teacher questioning. This may simply be a characteristic of the limited sample sizes in and prevalence of such studies. However, the varying findings may indicate the influence of other, confounding factors that may regulate the effect MKT has on teachers' effective use of questioning. It is with these considerations in mind that the present study asks the following research question: *To what degree do teachers' support of student autonomy and level of MKT predict their understanding of appropriate questions in facilitating mathematical discussion.* To investigate this research question, 3 hypothetical scenarios of mathematical discussion were illustrated through use of comic-based classroom representations (Herbst & Chieu, 2011). Scenarios stopped at a key point where participants were asked to select the most appropriate question to ask to facilitate students' mathematical understanding. These measures provide a preliminary examination of teachers' understanding of more appropriate questioning strategies in relation to aforementioned factors.

Methods & Analysis

Data were collected from 41 elementary teachers (K-4) in a Midwestern U.S. state across rural and suburban settings. Teachers were asked to complete a survey packet that focused on teachers' perceptions, knowledge, and habits regarding facilitation of mathematical discussion. Three dependent variables were used in the present study. Each dependent variable represented a set of categorical responses to a cartoon-based depiction of an elementary classroom (1st, 2nd, and 3rd grade) in which a teacher asked a question and a student provided a response. Students depicted in 1st and 3rd grade vignettes provided mathematically valid responses, while the student in the 2nd grade vignette provided an incorrect response. The participant was then asked to select one of four potential questions that would best facilitate a "solid understanding of the math involved." The four potential questions were structured following Boaler and Brodie's (2004) scheme. Specifically, *Gathering Information*, *Probing*, *Generating Discussion*, and *Orienting & Focusing* were selected as the four forms of questioning to evaluate, due to their prevalence in previous research (Boaler & Broadie (2004); Kosko et al., 2013). In order to ensure comparability across items, each item was embedded in the context of early algebra patterns. Two independent variables were included for analysis. Teachers' relative autonomy support for students (*Autonomy*) was assessed with the Problems in School survey. This survey presents a series of written vignettes to teachers and asks them to rate four potential actions in response to the vignette. The calculated outcome measure provides an indicator as to the teachers' disposition to how controlling versus autonomy-supportive they are in their instruction (M = 4.80, SD = 1.74, $Range$ = -.25 to 9.13). For the second independent variable, a 2006 revised version of Hill, Schilling, and Ball's (2004) MKT assessment for 'patterns, functions, and algebra' was used (α=.78, M= .00, SD = .83, $Range$ = -1.49 to 1.72). This particular MKT assessment, thus, aligned with content in the dependent variables.

Multinomial logistic regression (MLR: see Hosmer & Lemeshow, 2000) was used to examine participants' choice of question for each mathematics teaching scenario. Specifically, participants

were asked to select one of four potential questions following a depicted teaching scenario. MLR is a form of logistic regression which uses one category (one of the choices available) as a reference outcome, and creates separate logistic regression comparisons between the reference outcome and each other classification. For the present analysis, *Probing* was used as the reference outcome due to its representativeness of more generative questioning forms. So, I examined the degree to which each independent variable contributes to participants choosing an *Orienting & Focusing* (question rather than a *Probing* question); *Generating Discussion* (instead of a *Probing* question); and *Gathering Information* (instead of a *Probing* question).

Results from the MLR analysis suggest several intriguing results both within and between each scenario. *MKT* was found to not have a statistically significant effect on question choice across all scenarios. Results also suggest that participants with dispositions more supportive of student autonomy were less likely to select a *Generating Discussion* question over a *Probing* question.

Table 1. Results from Multinomial Logistic Regression.

Comparison		1st Grade β (logits) / S.E.	2nd Grade β (logits) / S.E.	3rd Grade β (logits) / S.E.
Orienting & Focusing | *Probing*	Intercept	-2.67* / 1.60	6.35* / 2.68	-.14 / 1.38
	Autonomy	.23 / .29	-1.02* / .45	-.15 / .27
	MKT	-.05 / .55	.18 / .78	-.41 / .48
Generating Discussion | *Probing*	Intercept	1.14 / 1.85	5.77* / 2.82	3.02 / 2.14
	Autonomy	-.86[a] / .46	-1.07* / .49	-1.24* / .58
	MKT	-1.14 / .87	.06 / .88	-.56 / .82
Gathering Information | *Probing*	Intercept	-	5.60* / 2.54	-1.54 / 2.60
	Autonomy	-	-.73[a] / .41	-.25 / .51
	MKT	-	.51 / .73	-1.24 / 1.04

$^{a}p < .10$, $^{*}p < .05$, $^{**}p < .01$

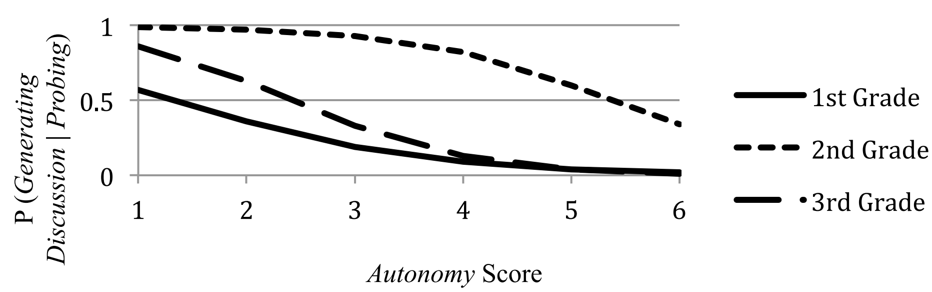

Figure 1. The Effect of *Autonomy* on the P (*Generating Discussion* | *Probing*).

While autonomy was generally found to influence choice of *probing* questions over *generating discussion* ones, an effect related to context was observed (see Figure 1). Specifically, the observed effect of autonomy scores was less for the 2nd grade scenario as opposed to the 1st and 3rd grade scenarios. In the scenarios where students provided a correct response (1st & 3rd grade), participants'

autonomy support appears to have interacted more with the decline in probability than in the scenario where the student provided an incorrect response (2nd grade).

Discussion and Conclusion

Findings from this study are preliminary in nature, but suggest potentially interesting patterns in why elementary teachers elect to ask certain questions over others in mathematical discussions. First, the findings from this study suggest that participants' level of MKT as currently being assessed, does not predict their choice of question in the depicted scenarios. This finding should be interpreted with caution, given participants were asked only to select one question. Therefore, the items used here did not assess sequences of questioning, which may be a better demonstrator of MKT. Alternatively, it may be that the present MKT assessments may adequately assess certain teaching practices, but not others. Therefore, further research investigating how MKT influences teachers' questioning strategies is much needed. Second, regardless of context, participants with higher *Autonomy* scores were less likely to choose a *Gathering Information* question over a *Probing* question. *Gathering Information* questions on the surface may seem to support student autonomy. However, *Probing* questions provide a certain scaffold for reasoning that *Gathering Information* does not. Finally, the interaction regarding a context where students provide an incorrect response may suggest a context where autonomy supportiveness is less effective, and thus questioning types may also vary accordingly. Future research is needed to further investigate these trends in additional scenarios and contexts.

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A PRACTITIONER'S INSTRUMENT FOR MEASURING SECONDARY MATHEMATICS TEACHERS' BELIEFS SURROUNDING LEARNER-CENTERED CLASSROOM PRACTICE

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In this paper we present the Mathematics Teaching Pedagogical and Discourse Beliefs Instrument (MTPDBI), a 20 item multiple-choice survey designed and analyzed using Rasch Measurement Theory. Items on the MTPDBI span the topics of beliefs about the nature of mathematics, teaching and learning mathematics, and beliefs about classroom discourse practices. The MTPDBI is shown to be a reliable measure of secondary mathematics teachers' beliefs specific to reform-oriented pedagogical practices and classroom discourse.

Keywords: Teacher Beliefs, Measurement, Teacher Education-Preservice

Purpose

In this paper we present the development and initial validation study results for the Mathematics Teaching Pedagogical and Discourse Beliefs Instrument (MTPDBI), a multiple-choice survey designed and analyzed using Rasch Measurement Theory. Although other instruments measuring teacher beliefs exist (e.g., Andrews and Hatch, 1999; Schmidt et al., 2007) these measures are often long and very general. The MTPDBI is composed of twenty multiple choice questions designed to identify teachers' beliefs about pedagogical practices related to reform-oriented teaching and discourse practices

Perspective

Following more than two decades of research examining the connection between mathematics teachers' beliefs and practice, there are a variety of theories in this area: beliefs are consistent indicators of practice (Cross, 2009); beliefs change as a consequence of change in practice (Guskey, 2002); or, change of belief and change of practice occur for a variety of reasons with one the precursor of the other or vice versa (Franke, Carpenter, Levi, & Fennema, 2001; Nathan & Knuth, 2003). With research supporting the idea that teacher beliefs are related to practice in some way, it then becomes relevant to investigate teachers' beliefs related to areas of practice such as classroom discourse.

Beliefs specific to mathematics teachers are usually separated into three categories of beliefs: nature of mathematics, teaching mathematics, and learning mathematics (Ernest, 1991). Mathematics teachers' beliefs in these categories can be described along a continuum from absolutist beliefs to fallibilist beliefs. Absolutists believe that the body of mathematical knowledge is static and already existing, and therefore the teaching and learning of mathematics is concerned with fluency with existing mathematical procedures. Fallibilists believe that the body of mathematical knowledge is growing and changing and thus, teaching and learning mathematics is about creating understanding and discovering new ideas.

Central to investigation of teachers' classroom practices and their relation to reform-oriented pedagogy is the concept of classroom discourse. The National Council of Teachers of Mathematics [NCTM] (2000) clearly defines the role of discourse in the mathematics classroom as a pivotal element in the learning process and later describes discourse as "the ways of representing, thinking, talking, agreeing, and disagreeing" (NCTM, 2007). Describing levels of discourse in a mathematics

classroom should include analysis of a variety of aspects of classroom communication: the initiator of discourse, the closer of a conversation, the questioner(s), the ownership of ideas, and the changing of ideas (Hufferd-Ackles, Fuson, & Sherin, 2004; NCTM, 2007). Teachers must create the opportunities for discourse to occur through selecting appropriate tasks and inviting students to participate (Smith & Stein, 2011).

With literature on mathematics teachers' beliefs and mathematics classroom discourse as a foundation, Wilson's (2005) model for the creation of an instrument was employed to create a map of the construct for mathematics teachers' beliefs about the nature of mathematics, learning and teaching mathematics, and classroom discourse all ranging from teacher-centered (absolutist) to learner-centered (fallibilist) beliefs. This map served to define the construct being measured and as a map for item construction.

Methods and Data

In order to support the content validity of the instrument (Wilson, 2005), research reported by Raymond (1997), Davis (1997), and Cooney, Shealy, and Arvold (1998), and others was reviewed for the formation of survey items. Initial survey items were reviewed for content and wording by an expert panel of mathematics education researchers and by a group of graduate students in mathematics education.

The MTPDBI is composed of 20 questions that represent four domains within teacher beliefs: the nature of mathematics, teaching mathematics, learning mathematics, and classroom discourse practices. Each question in the survey has four responses, each of which is selected to represent a different level on the construct: teacher-centered, emerging (from teacher-centered), approaching (learner-centered), and learner-centered. For the purpose of model estimation and participant scoring, the responses were coded with values 1 through 4, with 1 representing teacher-centered and 4 representing learner-centered. Two sample questions are provided in Figure 1. Question 1 is in the domain of classroom discourse practices and Question 2 is in the domain of beliefs about student learning.

1. I believe it is most important to ask questions during classes:
    a. To assess whether or not students are paying attention.
    b. To evaluate student knowledge.
    c. To better understand my students' thinking.
    d. To encourage further student exploration and, if necessary, change direction of a lesson.
2. I believe it is most important for students to learn to:
    a. Generate and explore their own mathematical questions.
    b. Explain reasoning for processes and explore connections between problems.
    c. Solve problems and explain reasoning for processes.
    d. Solve specific problems accurately.

Figure 1: Sample MTPDBI Questions.

A pilot study was completed for the MTPDBI during 2011. The survey was administered to 142 participants, including 77 pre-service and 65 practicing teachers. All participants in the pilot study completed the same version of the MTPDBI. The pre-service teachers completed a paper version of the survey and the practicing teachers completed an online version of the survey.

Results

The data were analyzed using WINSTEPS software and a Rasch Partial Credit Model (Masters, 1982). The advantage to using Rasch Measurement Theory is that an item difficulty parameter for

each item and a person measure for each person are provided, and these are located on the same continuum in units called logits (the logarithm of the ratio expressing the odds of success for a person on an item). The empirical continuum can be compared with the expected continuum used to develop the items. Each item difficulty measure and person measure has associated standard errors and fit statistics. In addition, an item measure reliability is reported separate from a person measure reliability.

The participants in the study exhibited measures all along the continuum, with items indicating a teacher-centered focus low on the continuum and items indicating a learner-centered focus high on the continuum. In addition, participants from all groups (pre-service and practicing teachers) were measured along the continuum rather than clustered in a particular range of logits. This indicates that the survey can be a valid measure for both groups. However, the items did not span the entire continuum. This finding indicates that the items on the survey do not accurately measure persons who are higher on the continuum (closer to the level of learner-centered). Creation of additional items or revision of existing items may correct this deficiency.

Item reliability was .96 with infit and outfit statistics within a reasonable range of .6 – 1.4 (Wilson, 2005; Wu & Adams, 2007). The test reliability, Cronbach's Alpha (KR-20), was .75.

Closer examination of individual questions indicated that more work is needed in some areas of the instrument. Figure 2 shows category probability curves for items 18 and 3. These curves show the probability of scoring a 1, 2, 3, or 4 as a function of where the person scores on the teacher-centered to learner-centered continuum. The higher the score, the more learner-centered would be the teacher's beliefs. Figure 3 shows that the people lowest on the continuum are most likely to select 1 on item 18, as expected. The people higher on the continuum are most likely to select 2, 3, and 4 according to their position on the continuum as expected.

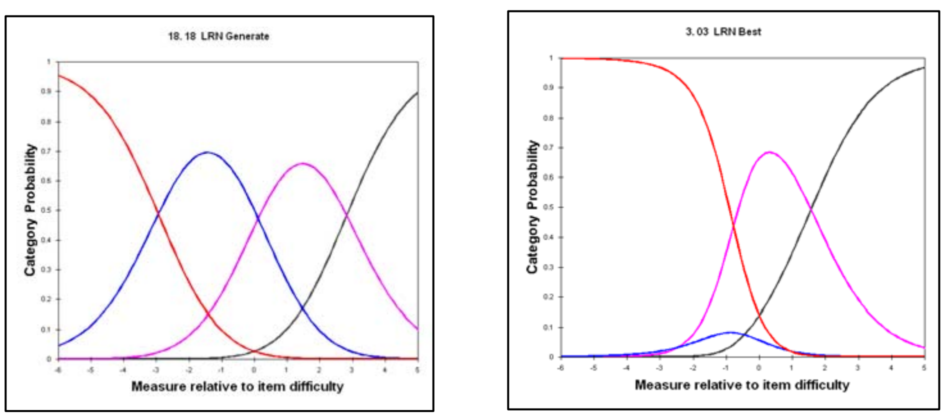

Figure 2: Category Probability Curve for Items 18 (left) and 3 (right).

However, other items did not produce the expected results. For example, item three in the domain of beliefs about the learning of mathematics, produced the second set of category probability curves shown in Figure 2. These curves indicate that the response related to the emerging level of the construct was not chosen by many participants. The model for this question, and a few others like it, may indicate that the continuum may need to be sorted into three levels instead of four levels.

Significance

The first author employed the MTPDBI as one component of a qualitative study on teacher change within a school-based methods course; the findings of that study supported the measures generated by the MTPDBI. The teacher who was observed to be more traditional in his teaching measured closer to the teacher-centered end of the continuum and the teacher who was observed to be more reform-oriented measured closer to the learner-centered end of the continuum. Comparison

of the pre- and post- MTPDBI scores indicated a shift in beliefs among the participants that was supported by the qualitative observation data from that study.

The validation study results for the MTPDBI indicate that this instrument, with revisions, may be useful to mathematics educators in a variety of ways: as an indicator of pre-service teachers' beliefs during a methods course, as an indicator of practicing teachers' beliefs during a professional learning experience, and as an indicator of change in teachers' beliefs in a longitudinal study. An instrument, such as the MTPDBI, that is brief and measures teachers' beliefs about reform-oriented classroom practice is important to support further study of the interplay between beliefs and practice of secondary mathematics teachers.

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SECONDARY MATHEMATICS TEACHERS' RELATIONSHIPS WITH HIGH-STAKES EXAMINATIONS

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High-stakes examinations have become a constant part of teachers' lives over the past 10 years. This study illustrates the relationship that two particular secondary mathematics teachers have developed with student written high-stakes examinations. This study was conducted in Alberta where students write provincially mandated exit exams worth 50% of their final grade. The research was undertaken using philosophical hermeneutics and unstructured conversations with the participants over the course of a school year. Conversations with teachers illustrate the varied relationships they have developed with the high-stakes examinations.

Keywords: Assessment and Evaluation; Classroom Discourse; High School Education

This paper reports early findings from a study focused on the lives of secondary mathematics teachers within a context of high-stakes student examinations. The research was conducted in Alberta where students write a provincially mandated exit examination, called a diploma examination, worth 50% of their final grade. These examinations are considered high-stakes for students because they significantly impact final grades, graduation, and post-secondary admission (Agrey, 2004; Barksdale-Ladd & Thomas, 2000; Dager Wilson, 2007; Webber, Aitken, Lupart, & Scott, 2009). These examinations are not considered high-stakes for teachers, as there is no formal consequence for teachers based on student results (Abrams, Pedulla, & Madaus, 2003). Three teachers were involved in the study but this paper focuses on one of those teachers and the relationship she has developed with the examination.

Purpose of the Study

The purpose of the study is to broaden understanding of secondary mathematics teachers' teaching lives within a context of high-stakes examinations; specifically, how secondary mathematics teachers make sense of high-stakes student examinations. One way that teachers make sense of high-stakes examinations is illustrated by relationships that are formed with the examination and how they incorporate the examination into their teaching practices. Previous research has reported on teacher identity (see Rex & Nelson, 2004; Upadhyay, 2009; Walls, 2008), teacher emotion (see Brady, 2008; Steinberg, 2008), and teacher practice (see Abrams, et al., 2003; Agrey, 2004; Yeager & Hover, 2006) within a context of high-stakes examinations; however none of this work fully attends to relationships teachers have developed with the high-stakes examination themselves.

Theoretical Framework

The methodology for this study is based in Gadamer's philosophical hermeneutics (2004, 2008). Philosophical hermeneutics is concerned with developing understandings of texts and takes a broad definition of what texts might be. Essential to philosophical hermeneutics, the researcher exposes existing prejudices or pre-understandings regarding the topic of the study (Fleming, Gaidys, & Robb, 2003; Paterson & Higgs, 2005). As such, within this study, I have a particular orientation to and view of high-stakes examinations based on my own experiences

with them as a secondary mathematics teacher and as a developer of such examinations. I have a trusting relationship with and deep knowledge of the secondary mathematics high-stakes student examinations in Alberta that stems from my continued engagement with the examination.

From my interactions and conversations with other secondary mathematics teachers in Alberta, I became aware that not every teacher had the same, or even similar, understanding of the high-stakes examinations that I had developed. This research is based on my desire to understand how other secondary mathematics teachers make sense of high-stakes examinations and to broaden my perspective of thinking around high-stakes examinations. What I noticed through this work is that teachers' relationships with the examination influenced how those teachers talked about and incorporated aspects of the examination into their practice. This paper presents the meaningful understandings (Prasad, 2005) that I have reached regarding secondary mathematics teacher relationships with high-stakes examinations.

Method

This paper presents data and understandings of the relationship that Wanda, a secondary mathematics teacher, has developed with high-stakes examinations. At the time of this study, Wanda had been teaching for approximately eight years and was teaching mathematics at a mid-sized rural school. Students in the school are from an affluent community and the school has a tradition of academic excellence on the high-stakes mathematics examination. Wanda had taught high-stakes examination courses previously and would be teaching a high-stakes examination course during the time period of our conversations. An important factor in selecting participants for this study was that the teacher had not participated in the marking of or the developing of any high-stakes examinations. I had the opportunity to engage with the high-stakes examination in those ways and I wondered if, by becoming familiar with the development and marking process, my relationship with the examinations was positively influenced.

Conversation was used as the basis for interaction with the participants (Carson, 1986, p. 83). Carson describes conversational research as a way to open up spaces for thoughtful reflection by all parties engaged in the research. The conversations were unstructured and free-flowing allowing for the participant to explore thoughts and to make connections as they arose. Questions that were "open and reflexive" (Vandermause, 2008, p. 72) were asked during the conversations for clarification purposes and to further explore ideas that participants mentioned. I had three, one to two-hour conversations with Wanda between November 2011 and April 2012. Whitehead (2004) notes that "[t]heoretical saturation is not sought in hermeneutic studies" (p. 514). What is sought are "field-based, thick descriptions" (Rex & Nelson, 2004, p. 1293) that inform our understanding of an experience or set of experiences.

Both the audio and transcription of the conversation were used as texts for data sources. I found that the feel of the conversation could not be adequately captured by the transcript thus I would listen to our conversations to reconnect to the time and place of the conversation and the participant herself (Fleming, et al., 2003). Thoughts, comments, and questions were written in a research journal after the conversation and were also used as texts for analysis. Once the conversation was transcribed and checked, transcripts were sent to Wanda to check and edit as she wished. When the transcript was returned, passages that highlighted, comments were made on why that passage was highlighted, and early understandings that were being developed were recorded. Throughout the analysis phase, I engaged in a dialogue with the texts (Fleming, et al., 2003) that served as a way to clarify my developing understandings and as my way of engaging in the hermeneutic circle in my search for "an interpretation as coherent, comprehensive, and comprehensible as possible" (Ellis, 1998, p. 27). It was through engaging in the hermeneutic

circle that I began to understand that how Wanda talked about the high-stakes student examination was reflective of the relationship she developed with it. I continued with the process of reading, writing, reflecting, until I reached an understanding of Wanda's relationship with the high-stakes examination.

Results

The data that illustrates Wanda's relationship with the high-stakes student examination are being represented by a pastiche of her words. Rex and Nelson (2004) use pastiche as a way "to represent the teacher's professional position in his or her own voice" where segments of different transcripts are "linked together to produce coherent texts" (p. 1297) called pastiches. These rich descriptions are not meant to be comprehensive of the teacher's perspective but are purposefully created to maintain the teacher's voice and to illustrate the understanding that has been developed of the relationship that the teacher has with the high-stakes examination. Analysis of Wanda's words is presented in the discussion section that follows.

> *Wanda:* I'm not afraid of the diploma in any way because I know our program is solid and I know my kids and I know where they are. I don't have a problem with testing the curriculum. I don't have a problem with how the questions are asked. The diploma exams don't bug me. I have no problem. I cover the curriculum; my kids come out above the provincial average. I have no problem as long as we're bigger than the provincial average. Big worry if I was less or something, like if the average comes out at 76 or something and I'm at 74, they'll haul me in and I'll have to justify it somehow, but I've done it like ten times. I just felt validated that I wasn't way off right 'cause you get that examination booklet and you're thinking please don't let there be something on here that I forgot. Ever thought that maybe you forgot a unit (chuckle) and you're sitting there and you're like I had an awful lot of time review this year. Did we not do all the perms and combs, what did I miss? But I was really happy that I was able to tell them the types of questions that were going to be on the diploma right. I can kind of predict and I know what's going to be on there.

Discussion

Wanda's comments illustrate a complex and contradictory relationship with the high-stakes examination. Wanda claims that she has 'no problem' with the examination, yet she states that she worries about being 'below the provincial average.' Because students in her school have traditionally performed well on the examination, Wanda worries that she will have to defend herself if her students score less than the average. She feels "vulnerable against the judgment" (Kelchtermans, 1993, p. 453) of the school administration. She is confident in her knowledge of the examination as she feels she can 'predict' the kinds of questions that are going to be on the examination, yet she worries that 'maybe she forgot a unit.' Wanda, like the teachers in Jeffrey and Woods (1996), doubts her "competence and adequacy" (p. 329) when being faced with an external measure of her teaching. From her comments, I believe that Wanda sees this examination as being high-stakes for herself as well as for her students. What is at stake for Wanda is her image of herself as a professional, her professional identity (Beijaard, 1995; Rex & Nelson, 2004; Upadhyay, 2009; Walls, 2008).

This study illustrates one teacher's relationship with a high-stakes student examination. There are many teachers across the globe living and working within a context of high-stakes examinations that have developed relationships with high-stakes examinations. Further study with respect to teacher relationships with high-stakes examinations and how those relationships

are developed would play an important role in informing us as a community of educators working within a context of high-stakes examinations. Further research would serve to broaden our perspectives on how teachers relate to high-stakes examinations can help us better understand how to best support teachers in their work and how to best prepare pre-service teachers to work within a high-stakes context.

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MUTABILITY AND RESILIENCY OF TEACHER BELIEFS AND PRACTICES: A CASE STUDY

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This pilot study examines the shifts in beliefs and practices for a new teacher in terms of the influences of her high school math classes, teacher training, cooperating teacher's beliefs and practices, and the middle school where she now teaches.

Keywords: Beliefs, Learning Theories, Teacher Education-Preservice, Standards

Lessons from the Math Wars

The "Math Wars" are, in part, a cautionary tale of mismatch between what teachers believe and what standards imply about teaching and learning, and even about what constitutes mathematics (Schoenfeld, 2004). Teachers trained in particular theories of education tend to develop resilient beliefs and practices that are resistant to change. Preservice teachers may cling to the implicit beliefs of their own high school experiences (Labaree, 2010; Stuart & Thurlow, 2000). Faced with standards and curricula that assume a different set of beliefs about teaching and learning, teachers may graft new curricula onto old practices. This can result in "lethal mutation," where the potential power of a lesson is diluted by leaching it of cognitive demand (Brown & Campione, 1996). High cognitive demand is not common in traditional mathematics classrooms, which feature a steady rhythm of lecture, seatwork, and memorization (Boaler, 1998).

The lessons of the Math Wars bear heeding, because the impending implementation of the new Common Core State Standards for Mathematics (CCSS-M)—and particularly the Common Core Mathematical Practice Standards (CCMPS)—places teachers in the same constructivist role as did the National Council of Teachers of Mathematics (NCTM) standards (CCSI, 2010). Teachers who have experienced only traditional mathematics classrooms will need to build a new understanding of teaching and learning if they are to lead classes that foster the CCMPS. Additionally, many teachers hold beliefs that the higher order reasoning reflected and reinforced in the CCMPS is for "advanced students," who are often singled out for speed and accuracy in completing mathematical tasks (Horn, 2007).

This is not the first rollout of standards for teaching mathematics. But this time, the stakes are higher. Tied to Race to the Top funding, the CCSS-M has a high adoption rate (45 states). Also, high stakes examinations are designed to measure fidelity of implementation, with very real material outcomes for students and teachers linked to test results. Let us assume that the standards imply better learning, and that an impediment is misalignment of teacher beliefs with the CCSS-M. What might motivate changes in beliefs? How are the tensions between competing beliefs resolved? As good mathematicians, let us first tackle the problem as a simpler case—let us examine the shifting beliefs of a new teacher.

Study Participant, Contexts, and Methods

Cherie is a European American, middle-class woman in her early twenties. She was singled out for advanced classes at a diverse, public high school. She completed her undergraduate

degree and master's teacher training at the same prestigious university. Her field placement was at a large, racially and socio-economically diverse high school. Within that setting, she taught Honors Geometry; the students were predominantly European American with middle to upper socioeconomic status (SES). She now teaches at a charter middle school with a 99% Latino/a population, and nearly all students are eligible for free or reduced-price school lunch. The school actively recruits students who are below proficient levels on the state's achievement exams with the goal of closing academic achievement gaps.

For her final master's program exhibition, Cherie presented two drawings that depict very different classrooms (Fig. 1 & 2), drawn at the start and end of her one-year program. They were so strikingly different from one other that I wanted to learn more about how her beliefs had changed. Interviewing Cherie gave me insight into what she intended them to mean, and provided a code-able data set. The prompt for the drawing "The Quintessence of Learning," was created and refined by J. Lythcott. I modified an interview protocol created by H. Borko (1999) to elicit teacher beliefs about learning and the roles of students and teachers. The semi-structured interview took place after Cherie's first semester as an inservice teacher. I coded using Charmaz's (1996) open coding and grounded theory, and triangulated the data with classroom observations and fieldnotes.

Findings

In describing her own experience as a high school mathematics student, Cherie stated that "[T]he skills you needed to get by in…class were the ability to sit, and listen, and take notes, and follow example problems, and then do them on your own. So it was that pretty traditional pattern of learning." When asked to contrast her experience with the one she offered her students, she added that "[My students] worked in groups…which I never did in high school math, and they were asked to figure things out before we got to formulas. They [were] asked to think for themselves a lot more…but, mostly what I was asked to do, as a math student in high school, was memorize the information that was given to me, and produce algorithms." When asked for a specific example of her mathematical teaching practices, she talked about inquiry-based groupwork, and in particular, lessons that became common referents for students. She recalled an Honors Geometry lesson, designed to build conceptual understanding of the formula for the surface area of a sphere, in which students tessellated orange peels (a proxy for surface area) onto four circles with the same diameter of the orange (proxies for a great circle and sphere). Students later talked about that lesson and the formula it generated as a linked pair. Cherie also referenced the class's co-constructed definition for cylinder: "…rather than 'here's the definition of a cylinder,' having them look at cylinders and decide what would be the most precise, inescapable definition of a cylinder you could possibly find, and they came up with awesome, awesome stuff."

As a preservice teacher, Cherie's classroom practices differed significantly from those of her cooperating teacher. The key pedagogical difference was that the cooperating teacher was traditional in her methods, whereas Cherie actively adopted an inquiry-based model that frequently employed group work. Cherie reported that the cooperating teacher "always had great things to say about the group work and everything, but didn't like doing it herself."

Cherie explained the differences in her vision of teaching from the beginning to the end of her master's program. Figure 1 shows what she remembered of "the image of a classroom…a teacher giving instruction from the board, students sitting in their desks… pretty static in their seats, and the teacher up at the front of the room, and gesturing to what she's teaching on the

board, which is an equation." She added that "the tools for learning, or the modes for learning, the kids, the only thing you can see on their desks are pencils and paper, and the teacher has a whiteboard". She compared this to Figure 2 where "there's way more varied things that they're using… and they're down there where everybody has access to them, and is using them". She also differentiated the speaking roles, pointing out that "in [Figure 1] the teacher is up and away from the students and talking to them, and they're sitting, you can't even see their faces." This was notably different from the Figure 2, in which "everyone is facing each other, and having a conversation," and that they were "on the same level." She added that "the students…are talking more than the teacher; the teacher's just kind of smiling, and listening, and I would imagine she's thinking of a question she's about to ask them."

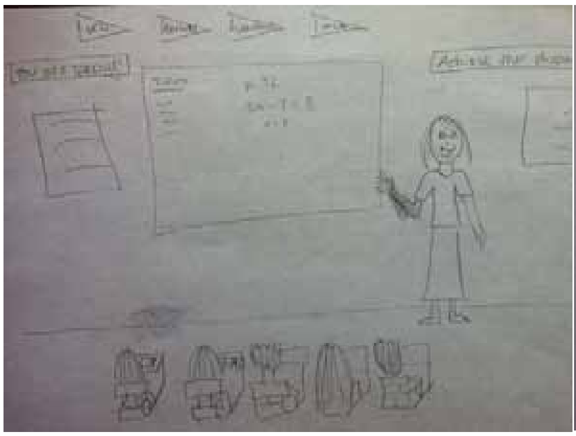

Figure 1: Cherie's "Quintessence of Teaching" Beginning of teacher program, June 2011

Figure 2: Cherie's "Quintessence of Teaching" End of teacher program, June 2012

When Cherie was a math student, she was able to understand her teachers' two-dimensional representations of three-dimensional objects. As a student teacher, she discovered that many of her students struggled with that approach, and changed her lessons to include physical modeling. Her teaching experiences, and her own coursework on dialogic discourse and meaning-making, led her to believe that students needed "to actually be able to touch what they're learning about."

Cherie describes her current view of the work of teaching as "orchestrating" curriculum, norms, management, and assessment. This includes the need to occasionally "frustrate" students in the work of pressing them to think "honestly" and develop "perseverance" in solving problems, while "noticing" their accomplishments and letting them know that she would not "abandon" them. Cherie sees the work of the students as entering a mutual "trust" with a teacher, one in which they are "willing to try" what the teacher asks of them, even if it pushes them beyond their comfort zone.

As Cherie moved from her student teaching placement to her first year teaching middle school, she carried with her a conviction that *all* students should be held to higher order mathematical practices, such as those envisioned in the CCMPS: "My kids really need those experiences more than kids with different life experiences and backgrounds." Cherie is working in direct opposition to the "pedagogy of poverty" (Haberman, 1991).

Discussion and Conclusion

Cherie's story is intriguing. Her vision of teaching first shifted from teacher-centric to student-centric, then remained intact as she navigated from teaching high-track high school

students to low-SES middle school students often written off as "incapable" of inquiry-based, student-centric teaching and learning (Horn, 2007). Figure 1 depicts Cherie's experiences as a student of mathematics in a traditional mathematics class. Figure 2 depicts a vision of teaching that is constructivist and inquiry-based, one that literally levels the teacher with students—the teacher as an inquirer alongside her students, seeking to better understand their thinking, seeking to scaffold the next set of questions, pressing for deeper understanding and stronger explanations. I argue that this vision is in better alignment with the CCMPS, one with higher cognitive demand. Cherie believed that the sorts of practices that most benefitted her middle/high-SES freshman Geometry Honors students, who were largely European American, were of even greater benefit to her low-SES Latino/a middle school math students, evidence of a belief in teaching for equity. Cherie's teacher beliefs and practices mark a significant departure from those she seems to have absorbed as a student of mathematics and those of her cooperating teacher. Further, they were *resilient* as she moved to a different teaching environment. There is evidence that Cherie's experiences as a master's student had a large impact on her beliefs in the power of experiencing mathematics through model building and dialogic discourse. In a tug-of-war of competing beliefs, which influence "wins?" Is Cherie a remarkable outlier, or is there something that can be learned from her experience that might provoke other teachers to re-examine their own beliefs about teaching and learning? This case study raises intriguing questions about the alignment of influences and resulting beliefs and practices.

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Acknowledgements: Thanks go to J. Boaler, A. Cain, C. Eilbert, E. Fortune, D. Labaree's writing class, R. Lotan, J. Lythcott, S.K.Selling, the Stanford Math Education group, and three anonymous reviewers. Any errors are my own.

UNDERSTANDING STUDENTS' ATTITUDES TOWARDS AND PERCEPTIONS OF HOMEWORK JOURNALS

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Writing in mathematics has become increasingly more common in mathematics classes of all age levels. The purpose of this study is to explore and describe students' attitudes towards and perceptions of homework journals, a specific type of writing in mathematics. Students enrolled in a college mathematics course were interviewed about their attitudes towards and perceptions of writing in mathematics. Based on the analysis of this data, students were found to have both positive and negative opinions of homework journals, such as liking instructor feedback from the homework journals and disliking the open-ended nature of the homework journals. In addition, homework journals were found to be a valid mathematics writing task consistent with the current research literature on designing meaningful writing tasks.

Keywords: Affect, Emotion, Beliefs, and Attitudes; Instructional Activities and Practices; Post-Secondary Education

Purpose of the Study

Mathematical writing can challenge students to purposefully reflect on their learning (Borasi & Rose, 1989), to think critically about mathematical ideas (Emig, 1977), and to deepen their mathematical understandings (Hamdan, 2005). While writing in mathematics can serve as a powerful tool, students' perspectives on mathematical thinking and learning through writing in mathematics are not well documented. The purpose of this study is to fill this gap in the research literature by exploring and describing students' attitudes towards and perceptions of homework journals, a specific type of writing in mathematics. Specifically, we will investigate the following research question: What are students' likes and dislikes associated with homework journals?

Theoretical Framework

Writing in Mathematics

Based on the current literature related to journal writing in mathematics, there are three important characteristics of journal writing assignments in mathematics: 1) prompts (Borasi & Rose, 1989), 2) clear expectations (Hamdan, 2005; Powell & Ramnauth, 1992), and 3) responses of the teacher (Borasi & Rose, 1989).

The first important characteristic of journal writing tasks, which is common among almost all of the examples in the current literature, is the prompt. The prompt can be relatively generic (Borasi & Rose, 1989) or extremely specific (Jurdak & Zein, 1998). The prompt can also be affective or cognitive (Jurdak & Zein, 1998). Regardless of their format, prompts are a critical element of journal writing.

Another important characteristic of journal writing tasks is well-communicated expectations. Teachers may communicate expectations in a variety of ways (Hamdan, 2005; Powell & Ramnauth, 1992). However expectations are communicated, it is valuable for both teachers and students to know what the teacher is anticipating from the journal responses in order for the journal writing to be meaningful to both teachers and students.

A third important characteristic of journal writing tasks is the response of the teacher; receiving feedback from their teacher is extremely important for students (Borasi & Rose, 1989; Jurdak & Zein, 1998; Powell & Ramnauth, 1992). Teachers' responses, comments, affirmation, and challenges based on students' journals are all important aspects of journal writing.

Homework Journals

Based on the literature related to journal writing in mathematics and mathematics education at the postsecondary level, we developed a journal writing assignment, called homework journals. Homework journals, which are based on the structure of Powell and Ramnauth's (1992) description of a "multiple-entry log," are a particular type of writing in mathematics in which students select one interesting or challenging problem from their assigned homework problems. On the left-hand column of a piece of paper, students solve this one problem in detail, clearly communicating their problem-solving process. On the right-hand column of the same piece of paper, students then write a short reflection about this problem and their problem-solving process. This reflection is prompted by a number of suggested topics and questions that students choose from. Teachers read and respond to each student's homework journal with comments, feedback, and questions.

Methods

Context and Participants

The context of this study was a college calculus course, *Elementary Calculus with Trigonometry*. Throughout the semester, students were assigned weekly homework assignments. The homework assignments consisted of 8-10 homework problems. Students were also required to submit a weekly homework journal along with their weekly homework problems. Nine students volunteered to participate in an interview outside of class.

Data Collection and Analysis

The nine students were interviewed using a general interview guide (Patton, 1990). Using a general interview guide allowed the interview to be systematic and comprehensive, but also afforded flexibility with important issues that came up during the interview. One of the main purposes of the interviews was to discuss students' attitudes towards and perceptions of writing in mathematics, specifically related to the homework journals.

A phenomenological design (Rossman & Rallis, 2003) was implemented to explore and describe students' attitudes towards and perceptions of writing in mathematics. The nine student interviews were videotaped and transcribed. The transcriptions were then coded in order to organize, analyze, and interpret the interview data.

Results

For the purposes of this study, we only focus on some of the results concerning what the nine students liked and disliked about the homework journals. The students' likes and dislikes about the homework journals demonstrate whether or not they hold a favorable or unfavorable attitude towards the homework journals and whether or not their perceptions of the homework journals are positive or negative.

What Students Liked About Homework Journals

The students described a number of different aspects of the homework journals that they liked. Students liked 1) the consistency and the format of the homework journals, 2) the feedback they received on the homework journals, and 3) the ability to make choices concerning the

homework journals. Below we elaborate on each of these "likes" and provide descriptions to help understand student's attitudes and perceptions.

Consistency and format. Three students, Mia, Sophie, and Kelly, liked the consistency of the homework journals. In particular, Sophie liked the consistency of the homework journals in that it was something that she did every week and that all of the homework journals were kept together in a single location. Kelly appreciated the consistency between the expectations her instructor communicated and the way her instructor graded the homework journals. Kelly believed that the communicated expectations and the grading practices matched and her instructor followed through with those expectations. Kelly contrasted this with a previous experience with writing in her high school calculus class: "When he [calculus teacher] graded them, he would tell us one thing and then when he graded it, he would be extremely harsh and grade from things he hadn't told us. So it kind of frustrated me more than anything."

Feedback from the instructor. Two students, Lydia and Chris, liked the feedback they received on the homework journals. They both felt that this feedback was an important and meaningful component of the homework journals. Lydia said, "I really like that she always wrote feedback." Lydia went on to describe the feedback she received on her homework journals and how helpful that feedback was, both for gaining a better understanding of the mathematical content and for preparing for tests. Chris also really appreciated the feedback he received on his homework journals. He said, "I like the personal feedback from our instructor, I really like that." The feedback they received from their instructor was very important to both Lydia and Chris and was one aspect of the homework journals that they liked the most.

Making choices. Adele liked that the homework journals gave her the freedom to choose which prompt to answer. She said, "I liked the choice to write about whatever you wanted."

What Students Disliked About Homework Journals

The students described a number of different aspects of the homework journals that they disliked. Students disliked: 1) the way that the expectations about the homework journals were communicated to them and how their homework journals were graded, and 2) the open-ended nature of the homework journals. Below we elaborate on each of these "dislikes" and provide descriptions to help understand students' attitudes and perceptions.

Communicated expectations from the instructor. Mia and Chris disliked the way expectations were communicated and the way their instructor graded their homework journals. Mia, for example, felt very strongly about the fact that her instructor had not communicated expectations because, more than halfway through the semester, she still did not understand what was expected of her on the homework journals. She lamented, "I didn't like all the points I missed on them! I feel like that was where I lost most of my homework points because I wasn't able to fulfill expectations in the writing section." Mia did not know what the expectations were and therefore was frustrated because she felt like she could not fulfill the expectations. Chris also expressed frustrations about his grades on the homework journals. Both Mia and Chris disliked the way the homework journals were graded.

Not enough structure or guidance. Adele and Melissa disliked the open-ended nature of the homework journals; they did not think there was enough structure or guidance to the homework journals. Melissa did not like the freedom she was given to choose which reflection prompt to answer. She explained, "I didn't like that it is kind of open-ended. I'm someone who likes to have very specific directions. So I'll be like, 'What do you want me to write? I'll write what you want me to write if you just tell me what you want me to write!'" Melissa wanted to be given specific instructions about what she was expected to write in her homework journal; she did not

like multiple choices for the reflection prompt. Both Adele and Melissa disliked the freedom of choices they were given with the homework journal.

Discussion and Conclusions

The students' likes and dislikes of the homework journals help describe their attitudes towards and perceptions of homework journals. Most students in this study liked some aspects of the homework journals and disliked other aspects of the homework journals. Therefore, it is difficult to determine whether or not a student entirely held a favorable or unfavorable attitude towards the homework journals or whether or not a student's perception of the homework journal was entirely positive or negative. However, what the students expressed as their likes and dislikes of the homework journals does provide some insight into their attitudes towards and perceptions of homework journals.

Students' likes and dislikes of the homework journals also confirm the strength of the homework journals as a valid and valuable journal writing task in mathematics, based on the three important characteristics of journal writing in mathematics. First of all, some students in the study, like Adele, expressed how much they liked being able to choose which prompt they wanted to respond to and reflect on each week. Although other students in the study, like Melissa, did not like the choice or variety of the prompts, this seemed to be a personal preference for those students. However, both students indicate that they were away of the multiple prompts were presented to students. Secondly, some students in the study, like Kelly, thought there was consistency between what her instructor communicated about the homework journals and the way the homework journals were graded. This provides evidence that expectations were communicated to the students. However, there were other students in the study, like Mia, who did not know what they were expected to write about in their homework journals, confirming the point in the research literature that communicating clear expectations is extremely important. Finally, some students in the study, like Lydia and Chris, valued their instructor's feedback, indicating that there was an opportunity for the instructor to provide meaningful feedback. Homework journals satisfy the three important characteristics of journal writing in mathematics.

While the students' attitudes towards and perceptions of homework journals were varied, it appears that homework journals provided a useful tool for actively engaging students in doing mathematics and reflecting on mathematics and/or mathematics learning by combining problem solving and reflection in a simple two-column format. Similar to the Powell and Ramnauth (1992) study, the students in this study were both actively engaged in doing mathematics and reflecting on that mathematics through the homework journals.

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CHALLENGES OF LEARNING MATHEMATICS IN A SECOND LANGUAGE

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This paper presents a self-study that explores challenges of learning mathematics when the primary language of the learner is not the language of instruction. An English-speaking mathematics educator (with limited Spanish-language proficiency) observed elementary school mathematics classrooms taught in Spanish in Guatemala and in dual language classrooms in the western United States. Cogenerative dialogue with a bilingual mathematics education colleague supported this self-study. Issues of equity and implications for teaching mathematics to students whose first language is not the language of instruction are discussed.

Keywords: Equity and Diversity, Classroom Discourse, Teacher Knowledge

Objectives/Purposes

U.S. schools are becoming increasingly linguistically diverse (National Clearinghouse for English Language Acquisition, 2011). Being able to speak more than one language is a strength, not a deficit; however, it is also true that linguistic diversity can impact teaching and learning in complex ways (Moschkovich, 2007). For example, measures such as the National Assessment of Educational Progress (NAEP) demonstrate significantly lower performance in mathematics for English language learners (ELLs) than for other students (National Center for Educational Statistics, 2012). This disparity suggests that U.S. schools, overall, are not supporting ELLs adequately. Research to learn more about teaching and learning of mathematics to students whose first language is not the language of instruction needs to be a priority.

As part of a larger study that investigates discourse in linguistically diverse mathematics classrooms, this paper presents a self-study at the intersection of mathematics and language, focusing on a mathematics educator's learning. An essentially monolingual (English-speaking) mathematics educator (with limited Spanish-language skills), sought to investigate her own learning in elementary mathematics classrooms where Spanish was the language of instruction.

Perspectives/Theoretical Framework

Language is critical to teaching and learning of mathematics for all students (Moschkovich, 2002, 2007; NCTM, 2000) and for the growing population of English language learners (ELLs) in particular (NCELA, 2011). Sociocultural theory, with its contention that higher mental functions derive from social interaction, provides a theoretical framework for analysis and discussion of language as a mediating tool in the teaching-learning process. Language is a primary means for how we communicate about ideas and come to understand something new (Vygotsky, 2002). Sociocultural theory provides a framework for exploring math/language perspectives such as vocabulary acquisition, use of linguistic registers, and participation in meaningful mathematical discourse (Moshchkovich, 2002).

Although there is evidence to suggest cognitive advantages related to speaking more than one language (e.g., cognitive flexibility, better problem solving, and higher order thinking skills, Hakuta, 1986), there are challenges involved in switching between languages (e.g., response time for arithmetic operations may be slower when using a second language, Moshchkovich, 2007). Further, language use and comprehension (or lack of) can impact students' attitudes and

"appreciation of mathematics" (NCTM, 2000) and, in turn, their self-efficacy. Levels of efficacy affect effort, persistence and resilience (Bandura, 2001). Monolingual educators can sympathize with challenges involved with learning mathematics in a second language and can employ strategies to support student learning – for example, SIOP (an instructional approach that offers a framework for planning and implementing high quality instruction for ELLs) (Echevarría, Vogt, & Short, 2010). Sympathy and strategies provide beginnings, but there is more to be learned. Therefore, the research question follows: *What can a monolingual, English-speaking mathematics educator learn about relationships of language to mathematics education through an immersion experience where Spanish is the language of mathematics instruction?*

Modes of Inquiry

Participant 1 (P1) is a monolingual (English-speaking) mathematics educator who participated in an immersion experience in Guatemala that involved language instruction and formal observations of mathematics classes in public elementary schools. P1 followed this experience with observations in dual language classrooms in the western U.S. where mathematics classes were taught in Spanish. Participant 2 (P2) is a bilingual mathematics educator (first language, Spanish) who participated in cogenerative dialogue with P1 (Tobin & Roth, 2005). P1 and P2 served as both participants and researchers.

For the self-study, classrooms where math was taught in Spanish were observed. In Guatemala, one second-grade classroom was observed 3 times. In the U.S., one kindergarten and one first-grade classroom were observed 3 times each; two second-grade classrooms were observed 2 times each. The larger study included additional data from classrooms where strategies (e.g., SIOP) were employed to support ELLs, but where math was taught in English.

Data sources for the self-study include: field notes of classroom observations; audio and/or video recordings of classroom dialogue; journals documenting and reflecting on classroom observations and experiences; cogenerative dialogue (via email) related to journals and field notes between P1 and P2. Cogenerative dialogue involves reflection where members refer to the same set of events and explanations are cogenerated, thus supporting reflection on experiences and co-generation of perspectives (Tobin & Roth, 2005).

Qualitative research techniques were employed that draw from self-study methodology (Loughran, 2007). Open coding was used to identify prominent themes in the data. Constant comparison methods facilitated axial coding to further assemble and identify the themes (Straus & Corbin, 1990).

Results

Thematic analysis revealed that language is critical to understanding mathematics. P1's immersion in Spanish-language classrooms helped her to gain greater appreciation of challenges inherent in learning math in a second language. To illustrate key themes and issues, an observation of a math lesson in a second grade classroom in Guatemala is highlighted.

The teacher displayed representations showing circles, lines, and numbers on a white board (see Figure 1). The teacher asked students to come to the board to complete parts of the representation and to explain their work (in Spanish). Verbal exchanges followed a triadic structure with the teacher initiating, the student responding, and the teacher evaluating (Cazden, 2001). It seemed to P1 that the class was reviewing previously learned skills. P1 sat in the back of the room (with notebook and audio recorder), trying to make sense of the language, the representations, and the mathematics. Following are excerpts from P1's journal:

- *Even with some Spanish language skills and, hopefully, math skills (for 2nd-grade math!) … I did not know what was happening most of the class. It became a puzzle for me as I copied every example, as best I could, to see if I could figure out what was happening*
- *My initial thoughts as I copied example after example from the board was that they were doing some form of decomposition of numbers.*

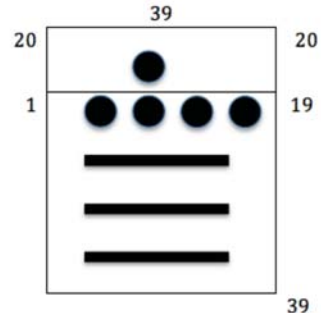

Figure 1: Representation From 2nd-grade Classroom in Guatemala

After reviewing on the white board, the students were directed to cut out paper circles and strips. Also, they each folded a paper in two parts to create a "mat" for placing the circles and strips. Then, the teacher asked students to form numbers – for example, "Forma el numero 21." The students represented the numbers on their mats using the circles and strips. "Practicantes" (high school students training to be teachers) walked around correcting and helping the students. For example, P1 heard a "practicante" count by 20s with a student (providing a hint that 20s were important). The following excerpt is from P1's journal (*in italics*):

I kept thinking that I had figured it out, trying to guess the correct picture before looking at student work. Sometimes I'd get it right and sometimes wrong. One thing that I didn't figure out until the end was that the lines represented 5 times the place value. My scribbly notes include things like, "Boy am I lost … I don't understand… some kind of decomposition… I don't understand why they are using 20s instead of 10s."

It wasn't until the end of the class when homework was written on the board that P1 was able to puzzle out what they had been doing. The teacher wrote, "*Tarea: formar los siguientes números utilizando la numeración maya en su cuaderno …"* [Homework: form the following numbers using the Mayan numeration system in your notebook …]

"Aha!" I said to myself. "Mayan numbers!" … suddenly there was a context. There was a potential reason for using 20 as a place value – a different number system.

Having a context made a difference, but P1 still needed time to think (*in English – her primary language*), drawing on notes taken during class. Eventually, P1 was able to figure out the number system *[the numbers to the left show the place value. A circle represents 1 times the place value. A line represents 5 times the place value]*, but not before recognizing the impact of language, representation, and context on her ability to learn and perform mathematically:

I had to ask myself, if I, a university mathematics educator, was confused in a second-grade math class, how would second grade student in similar circumstances feel?

Cogenerative dialogue with P2 helped to further unpack the experiences. P2 suggested that "live" experiences help to provide some sense of the challenges involved with learning and doing mathematics in a second language and/or within an unfamiliar culture. P2 asked P1 to think about how experiences trying to learn mathematics in such a context might impact students' self efficacy. P2 asked P1 if she had been confronted with having to answer a math question publicly

in Spanish. P1 had not had to do this, but could imagine herself as a student, trying to shrink down so as not to be have to publicly participate. Self-esteem would have been an issue for P1.

Learning a second language is a worthwhile endeavor; however, there are challenges when using the second language while trying to learn mathematics – especially for a beginning speaker. Some issues that emerged from cogenerative dialogue between P1 and P2 follow:

- Academic language is much more challenging than conversational language.
- Working to understand even basic instructions in a second language can be exhausting.
- Asking meaningful questions in a second language can be difficult (and intimidating).
- Lack of opportunity to reason in one's primary language can hinder sense making.
- One may choose not to participate publicly.
- Visual representations can help, but are not sufficient.
- Unfamiliar representations and contexts may present additional challenges.
- One is likely to appear (and feel) less intelligent than one really is.
- Personal experience as a second-language learner can be enlightening.

The issues presented are consistent with existing literature (e.g., Echevarría, et al., 2010; Moschkovich, 2002, 2007). However, they are more poignant when personally experienced.

Discussion/Conclusions

There are potential implications for mathematics teaching and learning in a second language. For example, language comprehension may impact students' attitudes toward mathematics and, as a result, their self-efficacy. Because self-efficacy is associated with effort, persistence and resilience, this may, in turn, impact academic performance (Bandura, 2001). Teachers need to better understand challenges inherent in trying to make sense of mathematics when language and/or representations and/or cultural contexts are unfamiliar. Issues of equity are at stake. Awareness is an important first step, but more needs to be done to figure recommendations for supporting students whose first language is not the language of instruction.

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EXAMINING AN INTEGRATED RELATIONAL APPROACH TO ELEMENTARY MATHEMATICS PRACTICE

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There have been recent calls from within the field of teacher education to identify "the components of building and maintaining productive relationships with students" (Grossman & McDonald, 2008, p. 188). Although research in mathematics education illuminates the efficaciousness of developing strong relationships with students, how mathematics teachers establish caring relationships with their students has been underspecified. In this study, we describe an integrated relational approach to elementary mathematics practice and examine how a group of 4th grade students respond. Knowing more about the relational demands of mathematics teaching may provide insight into how pedagogical relationships influence students, particularly among students from diverse racial, ethnic, socioeconomic, and linguistic backgrounds.

Keywords: teacher practice, elementary mathematics, pedagogical relationships

The field of K-12 mathematics education is concerned with developing students who have an understanding of content *and* possess productive mathematical dispositions (e.g.. NRC, 2001). Work in mathematics education has illuminated that particular features of classroom practice influence student learning and participation (Franke, Kazemi, & Battey, 2007; Hiebert & Grouws, 2007). A growing body of work indicates that the ways in which teachers and students interact around mathematical activities shapes how students engage in learning, which may subsequently influence their levels of mathematical understanding (Franke et al., 2007). Chiefly, teacher-student interactions mediate student engagement with tasks, guide students in attending to key mathematical ideas (Hiebert & Grouws, 2007), and support the development of positive mathematical identities (Martin, 2007).

A related line of inquiry suggests that strong interpersonal relationships between teachers and students can enhance learning outcomes (Noddings, 1984), and research illuminates the key role teachers play in cultivating and facilitating students' emerging relationships with mathematics (Bartell, 2011; Martin, 2007). That is, emotionally positive teacher-student relationships contribute to increased student effort, motivation, and persistence in learning mathematics (Midgley, Feldlaufer & Eccles, 1989). Of particular significance, quality relationships may be particularly central in the experiences of students who have been historically underserved in mathematics classrooms (Martin, 2007; Murrell, 1994).

Scholars acknowledge, however, that one of the fundamental complexities practitioners negotiate in the classroom is maintaining a relationship with students that is both supportive *and* productive (Labaree, 2000; Rosiek, 2003). Kennedy's (2005) study of elementary teachers illustrated how practitioners struggled with "competing ideals of nurturing students who are still young and emotionally immature…and of helping them learn important academic content" (p. 25). Teachers indicated that developing positive relationships with students and fostering their self-esteem was a fundamental aspect of their work but suggested this goal was related, but distinctly separate from attending to academic goals. Specifically, they theorized that supporting

the development of student self-esteem was a “prerequisite to learning” (p. 49) and affirming students’ competencies would lead to stronger student outcomes. As such, addressing students’ academic misconceptions was particularly “troublesome” for teachers, and they “abhorred” the idea of telling students they were wrong. Yet, as Kennedy notes, “since students are novices at the subjects they are learning, they are likely to often be wrong, thus placing teachers on the horns of an agonizing dilemma” (p. 50). Seen this way, establishing a productive pedagogical relationship “requires a remarkable capacity for preserving a creative tension between…two opposites, never losing sight of either teaching’s relational means or its curricular goal” (Labaree, 2000, p. 230).

The overarching goal of this research is to investigate how teachers build and maintain strong pedagogical relationships with their students and to examine how this relationship influences mathematics practice and student learning. The study is guided by the following research questions:

1) How does an integrated caring approach shape one teacher’s mathematics practice in a fourth grade classroom?
2) How does an integrated caring approach shape the mathematical experiences of students in a fourth grade classroom?
   - How does this approach shape students’ conceptions of the discipline?
   - How does this approach influence student engagement, motivation, and persistence in learning mathematics?

Theoretical Perspective

The theoretical assumption driving this study holds that mathematics teaching and learning is relational, the nature and quality of the social relations between teacher and student shapes the formation of a pedagogical relationship, and the quality of this relationship influences the ways in which students engage, learn, and come to identify with the discipline. This study takes on the theoretical lens of care (Bartell, 2011; Hackenberg, 2010; Noddings 1984). Perhaps most widely recognized is Noddings’ (1984) work on *care theory*, however, in recent years, scholars have built upon and refined this description of care in order to explicitly address mathematical cognition (Hackenberg, 2010) and issues of equity (Bartell, 2011). Here, we suggest the use of an integrated caring perspective, particularly one that more explicitly addresses issues of disciplinary knowledge and equity, is appropriate for the purposes of this study.

Drawing from Tarlow’s (1996) work, we conceptualize caring as a nuanced and negotiated process that has multiple dimensions, or, as she puts it, “a phenomenon with a past, present, and future” (p. 57). We join together three central aspects of the caring process in this framework: (1) personal caring, (2) mathematical caring, and (3) political caring. The space where these dimensions overlap and interact is what we refer to as an *integrated caring approach,* which we consider to be an attempt on the part of the teacher to build and sustain a pedagogical relationship along these dimensions. This framework seeks to “enable all students to participate substantially in classroom mathematical practices and to develop competence and a sense of affiliation with and empowerment towards the discipline” (Gresalfi & Cobb, 2006, p. 54). Furthermore, an integrated perspective of care allows for the concurrent examination of what teacher-student

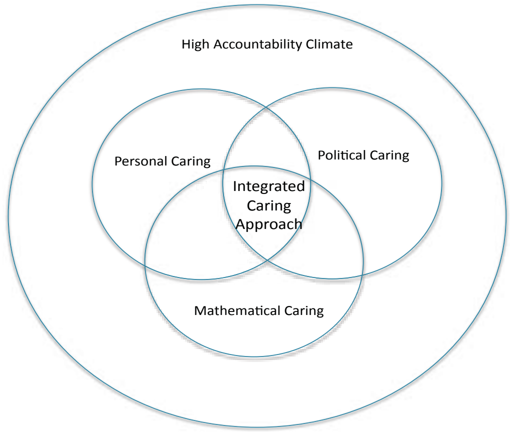

Figure 1: Conceptual Cramework for an Integrated Caring Approach

relationships look like in diverse classrooms, how they form and sustain, and how particular teacher moves open up or constrain students' opportunities to learn.

Data Methods

The design of this study incorporates the use of practitioner inquiry (Cochran-Smith & Donnell, 2006) and is influenced by design experiment methodology (Cobb et al., 2003). The first author assumed the role of teacher-researcher in a 15-week teaching experiment study in a culturally, racially, and linguistically diverse fourth grade classroom at a Title I school in California. Data collection methods were designed to capture how an integrated caring approach to mathematics practice was enacted in the classroom and how students appeared to respond. Data sources include 30-35 audio-recorded lessons, a daily teacher journal, lesson plans, interviews with focus students, and student work (e.g., surveys, classwork, assessments, journals).

Instructional design and research are interrelated processes in design experiments, thus on-going conceptual analysis of *how* students are responding to the experiment is conducted in addition to retrospective analysis (Cobb et al., 2003). Ongoing analysis included daily cycles of implementation, reflection, and revision. Retrospective analysis of data attends to the presentation of mathematics content, classroom participation structures (particularly in relation to students' engagement); and teachers' positioning of students' mathematical ideas.

Preliminary Results

An integrated caring approach appeared to be beneficial in facilitating students' engagement in mathematical activity and influential on students' conceptions of the discipline and their developing mathematical identities. Developing personal relationships with students and leveraging personal and mathematical knowledge of individual students was a significant resource in supporting student mathematics learning. Creating a supportive classroom environment that encouraged risk-taking and problem solving also shaped the nature of student participation and learning.

Significance

Although existing studies point to the salience of strong teacher-student relationships, as a collective field, we know little about how this relationship develops or how this relationship

potentially shapes teacher practice. Similarly unclear are the kinds of classroom relationships that make a difference for student learning, particularly in schools with culturally and racially diverse student populations. Emerging work related to caring relations (e.g., Bartell, 2011; Hackenberg, 2010) illustrates the complexity of care and the challenges that arise as teachers seek to attend to both relational and academic aspects of practice when attempting to provide rigorous learning opportunities for students. There are recent calls from the field of teacher education to focus on the relational aspects of practice and identify "the components of building and maintaining productive relationships with students" (Grossman & McDonald, 2008, p. 188). This work seeks not only to contribute to practical understandings of how elementary mathematics teachers demonstrate care in but also to examine the interactive relationship between teachers, students, and content and provide theoretical insight into the complexity of care in the context of mathematics teaching and learning. Gaining a deeper understanding of the relational demands of mathematics teaching may give insight into how to better prepare teachers to work effectively with a wide variety of students from different racial, ethnic, socioeconomic and linguistic backgrounds.

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CONSOLIDATION ACTIVITIES IN TECHNOLOGY-BASED ENVIRONMENTS: DIFFERENT TEACHER APPROACHES

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In this paper we describe how teachers are implementing the "consolidation activities" in a technology-based unit designed to help students understand key aspects of geometric similarity. We have reason to believe that differences in implementation of these activities may impact affect student learning. In this preliminary paper, we analyze two types of implementation, one that we call "time-based management" and one we call "activity-based management."

Keywords: Technology; Geometry; Middle School Education; Curriculum

Introduction: Dynamic Representations and Curricular Activity Systems

The classroom activities discussed here are part of the SunBay Digital Mathematics Project (SunBay Math), a multiyear effort to raise the quality of middle grades mathematics instruction in a large and diverse urban district. The project is based on a *curricular activity system* approach (Vahey, Knudsen, Rafanan, & Lara-Meloy, 2013) that integrates learning progressions, teacher professional development (PD), curriculum materials, and software, recognizing that these are all situated in a larger educational context. In this paper we illustrate how the teacher mediation of students' interaction with the technology-based and non-technology-based content can lead to dramatically different classroom environments, and we posit that these differences may influence the range of student learning outcomes observed across different teachers.

Core to our materials is the use of dynamic representations. Dynamic representations embed mathematical relationships in (typically digital) objects that the student can manipulate via specific hot spots carefully chosen by the environment designers (Moreno-Armella, Hegedus, & Kaput, 2008). Manipulating a single hot spot varies one specific mathematical relationship while other mathematical relationships remain invariant. As students manipulate the environment, the environment constrains the actions allowed while providing feedback on the embedded mathematical relationships (Ares, Stroup, & Schademan, 2009). For instance, dragging a hot spot that uniformly scales a geometric object while simultaneously updating a set of measures can enable a student to see that while the absolute lengths of sides change, the multiplicative relationship between the sides remains the same.

Description of the Geometric Similarity Unit and Consolidation Activities

In this analysis we focus on materials designed to help students develop qualitative and quantitative definitions of the similarity of geometric figures (e.g., triangles, rectangles, and parallelograms). Using a set of workbook materials and corresponding GeoGebra documents created by the team, students investigate key aspects of geometric similarity, including the use of *within ratios* (in which the side-to-side ratio within one polygon is compared with the corresponding side-to-side ratio within another polygon), *between ratios* (in which the side-to-

side ratios across corresponding sides of polygons are compared), and the use of scale factor.

Students investigate similarity in the context of working for a design firm. Early in the unit students establish that for a copy of an object to be considered good (e.g., not distorted when made a different size), it must be mathematically similar to the original. Key to the narrative flow is Terry the Technician, a character who provides the students with different methods of object manipulation, some of which result in similar objects and others do not. By using Terry's copy machines, which use embedded dynamic representations to enable students to perform actions such as grow rectangular objects and measure their side lengths, the students investigate different aspects of similarity and engage with key misconceptions (such as the additive misconception).

The unit is designed for students to work in pairs to explore aspects of mathematical similarity. At key points in the unit students are asked to complete letters to Terry describing in detail what he has to do to make sure his copy machines create only mathematically similar objects. These letters are examples of what we call *consolidation activities*. In consolidation activities students reflect on what they have learned (they are neither expected to engage with the dynamic representations nor exposed to any new concepts or ideas) and compose written mathematical explanations.

Methods and Data Collection

The investigation presented in this paper was conducted according to the principles of design-based research (e.g., Penuel, Fishman, Cheng, & Sabelli, 2011), using quantitative and qualitative methods to construct a shared empirically based perspective that allows us to improve the design while also advancing theory building.

Participants were 15 middle school teachers and their 788 students in one of the 25 most populous urban school districts in the United States. Teachers volunteered and subsequently were invited by school and district administrators to participate in the research project. Sources of data collected included identical pre- and post-unit assessments designed to test students' understanding of the target mathematics, observation notes collected for at least one lesson of each participating teacher, and two sources of video from the same lesson that was observed (one focused on the teacher, the other using Camtasia software to capture the screen of and audio from a randomly selected set of students).

To analyze student learning we calculated a student gain score by subtracting the pretest score from the corresponding posttest score. To analyze the observation and video data we used the observation notes and videos to create a list of key episodes for each observed classroom. These episodes were indexed to learning activities and provided a summary of the class activity during that episode. We took a comparative perspective, which required that we find all teachers for whom we collected observation and video data for the same consolidating activity. We chose a task in which software-based explorations prepare students for the consolidating activity of writing a letter to Terry. We reviewed the observation logs to find key events that described each teacher's classroom as he or she implemented the consolidating activity. Once these descriptions were in place, we looked for common themes in the data and also compared student learning for teachers with different styles of implementation.

Results

Analysis of the student learning gains indicated a significant difference in the mean pre- ($\bar{x}$ = 6.5723, s = 4.07410) and post-unit test scores ($\bar{x}$ = 9.132, s = 4.30954). $t(787) = -21.015$, $p <$ .001), with variance among classes. We analyzed the video and observation data to investigate any patterns of activity that could explain the variance among classes. We found that teachers

engaged in two primary forms of implementation of the consolidating activity, one we call *time-based* management and the other *activity-based* management, as described below.

Time-based Classroom Management

Teachers using time-based management tended to base their lessons on the timing of their class periods and allowed students to work through the technology-based materials at their own pace. There was a common format to their class lessons:

- An *introduction*, in which teachers may have provided warm-up problems and then oriented the students to the day's activity.
- The *main body* of class time, in which teachers allowed students to engage with the material as individuals, pairs, or small groups generally going at their own pace. Teachers often walked around the class during this time, providing students individualized help and occasionally providing whole-class guidance for common difficulties.
- A *wrap-up*, in which teachers reviewed the day's activity, sometimes provided reflections on learning, and assigned any homework.

This implementation appears to be based on an understanding that all activities are equally well served by pairs or individuals using the materials at their own pace and that the most effective time to allow for student reflection is at the end of class, once all the students have had a chance to experience the day's activities. Of the five teachers for whom we have video of the letter-writing activity, two were categorized as using a time-based management technique.

Activity-based Classroom Management

The format of activity-based implementation had much in common with the time-based implementation, but the teachers punctuated the main body of class time with whole-class discussions that were designed to leverage the reflection and consolidation opportunities found in the letters to Terry. These discussions lessened the opportunity for students to work at their own pace. A typical class proceeded as follows:

- An *introduction*, in which teachers may have provided warm-up problems and then oriented the students to the day's activity.
- A *series of activities*, in which teachers allowed students to engage with the material as individuals, pairs, or in small groups, supporting students as described above.
- Occasional *consolidation discussions* that punctuated the students' individual activities and had the class reflect on core activities. The letters to Terry often served this role.
- A *wrap-up*, in which teachers reviewed the day's activity, sometimes provided reflections on learning, and assigned any homework.

Teachers who used this form of implementation limited students in exploring the materials at their own pace. This implementation appears to be based on an understanding that certain activities are more suited to student reflection and consolidation and that these activities may be less effective when addressed by individuals or pairs. Of the five teachers for whom we have video of this activity, three were categorized as using an activity-based management technique.

Student Learning: Initial Results

An initial descriptive comparison showed that the students of the three teachers who used activity-based management generally had higher learning gains than those of the two teachers who use time-based management. Although one of the teachers who used the activity-based management may be considered an exception because of relatively low gain scores, his class had the highest overall post-test score, so his students may have experienced a ceiling effect. Given

the relationship between these two implementation styles and student learning is preliminary, we will examine it as we move forward in studying different implementation models of our, and other, technology-based materials.

Discussion and Conclusion

Although preliminary, these case studies shed light on the ways teachers enacted the consolidation activities and provide fodder for helping teachers and researchers consider the role of such activities in technology-based environments. If further research bears out our analysis, several implications are of importance to the field and to teaching practices. One is that the notion of students working at their own pace, at least in the type of environment discussed here, may have significant drawbacks. Many technology-based learning environments individualize instruction, either by letting students work at their own pace or by adapting instruction for each student. In contrast, the SunBay materials provide students with meaningful and collaborative mathematical experiences, and consolidation activities are core to ensuring that students connect their experiences to the formal mathematics. When students work at their own pace in SunBay-like materials, they may not be aware of the purpose or value of consolidation activities and so may not take the time to reflect on what they have learned thus far. It is then up to the teacher to individually identify and remediate those students who did not engage in reflection. In contrast, when the teacher engages in whole-class discussions about consolidation activities, although students are constrained to working at the pace of the rest of the students, they have the opportunity to reflect on what they have learned, hear how other students are thinking about the content, and can be identified by the teacher if they need additional help.

A second implication is in how we conduct teacher professional development. Our PD focuses on supporting teachers in implementing the materials in a way that they will find practical, while ensuring that core aspects of the program are implemented effectively. This research may identify consolidation activities as a core aspect that should be identified for teachers as key to effective implementation. Conversely, additional analysis may uncover some teacher strategies that successfully allow student reflection while still engaging in a time-based implementation, and such strategies would be valuable to include in ongoing PD.

Acknowledgments

This material is based on work supported by EDUCAUSE through the Next Generation Learning Challenge, the Pinellas Education Foundation, the Helios Education Foundation, and Pinellas County Schools. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the funders. The authors thank the teachers and students who were willing to use our materials and provide feedback and the teams at SRI International and the University of South Florida St. Petersburg.

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REFLECTIONS ABOUT QUESTIONING: A CONTINUUM OF DEVELOPMENT

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This paper reports on the development of the Reflections About Questioning Continuum, which was designed to characterize teachers' reflections on video recordings of lessons they taught during a 3-year professional development project.

Keywords: Teacher Education-Inservice/Professional Development, Classroom Discourse

A wealth of literature underscores the value of teacher professional development aimed at enhancing classroom discourse, especially teacher questioning (e.g., Bay-Williams & Karp, 2008). In this paper, we report on a professional development program in which teachers identified questioning as a meaningful lens through which to examine and improve their instructional practice.[1] We focus on the ways in which teachers reflected on their questioning when examining video recordings of their teaching and present a continuum that characterizes the nature of teachers' reflections about their questioning.

Theoretical Perspectives

Our research resonates with the growing body of literature on teacher noticing (e.g., Sherin, Jacobs, & Philipp, 2011) and research documenting the benefits of video as a tool for teachers to examine and interpret their instructional practice (e.g., van Es & Sherin, 2009). We relate our work specifically to the van Es and Sherin (2002) study in which prospective teachers were "learning to notice" (p. 572) by being prompted to attend to noteworthy events in video recordings of their teaching and write essays about what they noticed. The researchers developed a trajectory for analyzing the prospective teachers' essays, which included four levels: (a) prospective teachers described events they observed literally and made judgments about those events; (b) they described and judged aspects of practice as they happened but were somewhat more interpretive of the events they observed, although they did not necessarily provide specific evidence to support their argument; (c) they interpreted classroom situations and related specific events to particular principles of teaching and learning; and (d) they connected their noticing to principles of teaching and learning and provided "pedagogical solutions based on their interpretations" (p. 581). The van Es and Sherin (2002) *Trajectory of Development in Learning to Notice* provided a lens through which we interpreted data in this study.

Context of the Study

The data for this study were collected within the context of a mathematics-science partnership project conducted as a 3-year Master's degree program in which teachers earned a degree in either mathematics or science education. The 22 teachers participating in the program were from a public school district in a mid-sized city in the Midwest. Throughout the program, the project team[2] collected data from a variety of sources to document changes in teachers' content knowledge and instructional practice. The most robust source of teaching data we collected came from a three-phase *video reflection cycle*. At the beginning of the program, each teacher planned, taught, and video recorded a lesson that became the basis for reflection and

revision throughout the program. About halfway through the program, teachers reviewed their initial lesson plan, viewed the recorded lesson, and reflected (in writing) on their teaching in light of the coursework, readings, and program activities in which they had subsequently been engaged. They modified the lesson plan, wrote a rationale for the modifications, re-taught and recorded the modified lesson, and reflected on the differences between the two lessons as taught. This process was repeated a third time, at the end of the program.

Analyses of the recorded lessons indicated changes in the teachers' instruction and we have evidence that the video reflection cycle was an effective means of prompting the teachers to reflect critically and meaningfully on their instructional practice (Langrall, Morey, & Mooney, 2012). In this paper, we report on the development of an analytical tool to examine how teachers reflect on their questioning when analyzing video recordings of their teaching and to characterize the nature of the teachers' reflections.

Methods

We identified ten teachers who produced substantive reflections for all three phases of the video reflection cycle. Using a constant comparative approach (Strauss & Corbin, 1998), the first author analyzed each teacher's written artifacts to characterize the nature of the reflections. For the analysis reported in this paper, the teachers' written reflections were parsed into units of data that contained direct or indirect references to questioning. Typically, reflections were comprised of about five units of data. For each unit of data within a reflection, the first author wrote descriptive phrases that characterized the teacher's comments related to questioning. She then examined all phrases (across teachers and reflections), clustered similar phrases, and generated descriptors for each cluster. The phrases describing units of data suggested differences and similarities in the focus of teachers' reflections on their questioning. For example, phrases such as "links questioning to multiple strategies," "links questioning to student explanations," and "links questioning to accessing student thinking" collapsed into the descriptor "utilizes questioning as a pedagogical tool." Three descriptors were generated and ordered as points along a continuum that represented a progression in the depth and level of sophistication of the teachers' reflections. At this point, the second author became involved in the analysis and two end points were developed for the continuum. At one end of the continuum, we created a point to indicate that none of the teachers' reflections during the first phase of the video reflection cycle addressed questioning. At the other end of the continuum, we created a point to accommodate characterization of a teachers' reflection that was more sophisticated than the fourth descriptor and allowed us to avoid the ceiling effect with the continuum.

Reflections About Questioning Continuum

The Reflections About Questioning Continuum (Figure 1) is defined by five anchor points that we believe represent a progression in the development of teachers' reflections. This continuum shares important attributes with the *Trajectory of Development in Learning to Notice* developed by van Es and Sherin (2002). In the discussion that follows, we describe each point on the continuum and relate it, as appropriate, to a respective level on the van Es and Sherin trajectory.

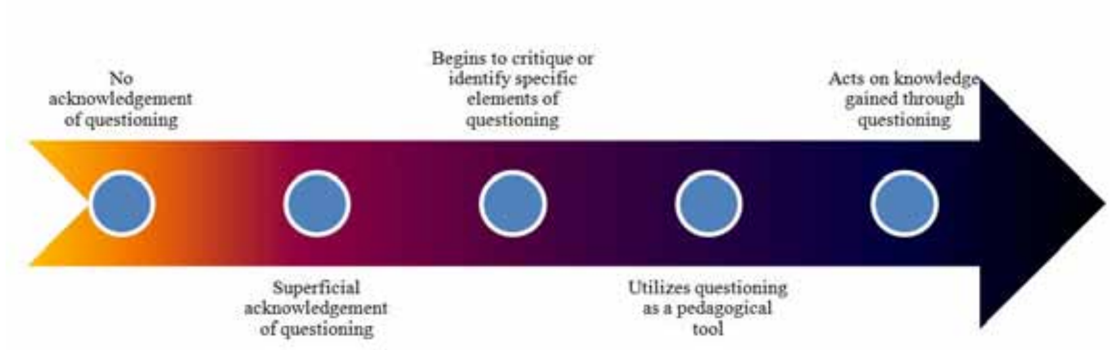

Figure 1: Teacher Reflections about Questioning Continuum

No Acknowledgement of Questioning

The continuum begins with the category "no acknowledgement of questioning." We included this point on the continuum because none of the teachers in this study reflected on their questioning in the first phase of the video reflection cycle.

Superficial Acknowledgement of Questioning

A reflection was characterized as acknowledging questioning in a superficial manner if it merely indicated that questions were asked or referred simply to the quantity of questions posed. For example, teachers made statements like "I need to ask more questions" or "In this lesson I asked a lot of questions," but did not go further to critique the nature of their questioning. We consider this characteristic similar to the first level of the van Es and Sherin (2002) trajectory in which teachers provided literal descriptions of the events they noticed and made judgments of the pedagogy they observed.

Begins to Critique or Identify Specific Elements of Questioning

Reflections at this point on the continuum made reference to different types of questions posed or provided a critique of the questioning teachers' observed in the current or previous video recording. Teachers would often identify the questions they asked as being high level, low level, procedural, conceptual, guiding, or probing, but not provide examples of the types of questions they referred to. When teachers critiqued their questioning, they referred to questions they perceived as important to ask but did not explain *why* it was important to do so. For example, a teacher claimed: "I need to ask more conceptual than procedural questions." This point on our continuum relates to the second point on the van Es and Sherin (2002) trajectory in the sense that the prospective teachers in their study were becoming more interpretative in their essays, but generally failed to provide specific support for the events they described.

Utilizes Questioning as a Pedagogical Tool

The fourth point on the continuum characterizes reflections that contain evidence of questioning being used as a pedagogical tool. That is, teachers acknowledge the role of their questioning in encouraging discourse in the classroom, requiring students to share multiple strategies, and providing insights into students' thinking. Furthermore, the connections between these pedagogical practices are clearly stated in teachers' reflections. The following reflection illustrates the connection between questioning and student thinking: "After watching my first lesson I realized I answered more of my own questions than my students did, which is not a good thing. Students cannot learn if I am doing the thinking, they need to be able to communicate their own ideas." This point relates to the third level of the van Es and Sherin (2002) trajectory, where

prospective teachers' interpreted classroom events with principles pertaining to teaching and learning.

Acts on Knowledge Gained Through Questioning

The upper bound of the continuum characterizes reflections in which teachers comment on *how* they can use knowledge gained through questioning to inform instruction. Although this point was not fully developed in any of the reflections we analyzed, one teacher did indicate that she could adapt instruction based on what she learned about her students' thinking through questioning. We see this as similar to the fourth level of the van Es and Sherin (2002) trajectory, where prospective teachers provided pedagogical solutions to the instructional challenges they noticed.

Discussion

The Reflections About Questioning Continuum was sufficiently robust to characterize the data collected from our professional development work with teachers. The potential of the continuum to be used more broadly is strengthened by the characteristics it shares with the work of van Es and Sherin (2002) and we believe that other teacher educators could use the continuum to examine teachers' reflections on their questioning practice. The research literature identifies different types of questions that can be posed in the classroom (e.g., Boaler & Brodie, 2004) and this continuum extends such work by providing a way to describe how teachers reflect on the questions they pose and the questioning techniques they employ. Future research needs to examine the connection between teachers' actual practice and reflections on their teaching. The Reflections About Questioning Continuum will be a useful tool in conducting this kind of research.

Endnotes

[1] The work reported in this paper was supported by the Illinois Mathematics-Science Partnership Grant No. 4936-72-064-5450-51.

[2] Members of the project team included Cynthia Langrall, Marilyn Morey, Josh Hertel, Elif Safak, and Edward Mooney. Nicole Wessman-Enzinger joined the team for the analysis of teacher reflection data.

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INQUIRY IN MOTION: AN INVESTIGATION OF A PROFESSIONAL DEVELOPMENT PROGRAM TO CHANGE TEACHERS' BELIEFS

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Educational reform and improving teachers' classroom practices has been a longstanding concern. It is often claimed that professional development programs should focus on changing teachers' beliefs in order to enable teachers to reform their instruction. This study examines whether and how middle school math and science teachers changed their beliefs toward inquiry-based instruction within one-year of a professional development program. Regarding the statistical preliminary findings from the belief survey, it was evident that both math and science teachers' beliefs about the use of inquiry during instruction increased, but for different factors.

Keywords: Inquiry-Based Instruction, Teacher Beliefs, Teacher Change

Purposes

Educational reform in math and science has been an issue since the 1950s. Ever since, reform efforts have struggled to make lasting and influential impact on teacher practice. The current implementation of the Common Core State Standards for Mathematics (CCSSI, 2010) and the development of the Next Generation Science Standards (Achieve, 2012) reinforce the need to find ways of effectively transforming teacher practice to a more inquiry-based model (Keys & Bryan, 2001). Transforming teacher practice has been and continues to be a major research focus in educational inquiry, with teachers' personal beliefs and their implications for transforming teacher practice playing a major role. While there were inconsistencies among the data that tie teachers' beliefs to their practice, data has been collected that provides evidence that implies connection between the two (Mansour, 2009). This paper seeks to provide additional evidence and implications of the transformation of math and science teachers' beliefs after one year of professional development on inquiry-based instruction.

Theoretical Framework

This study is situated within the framework of teacher beliefs and teacher change through professional development programs. The reason for choosing this framework was because of the fact that the survey that the teachers filled out focused solely on the beliefs that the teachers had concerning their teaching practices.

Belief as a Cognitive Construct

Belief is an admittedly ambiguous construct which researchers have used in a variety of different ways (Pajares, 1992). Therefore, serious attention has been given to clarify this construct and its consistent use in research. Since the focus of our research centers around teacher beliefs and their transformation, we are choosing Pajares' generalized definition which states that "Belief is based on evaluation and judgment" (1992, p. 313).

Teacher Beliefs and Their Change

In order for teachers to make substantial changes in their teaching practices, teacher beliefs must be accessed (Ernest, 1989). Without such a change in the beliefs of teachers, reform in teacher practices is not likely. Guerra and Nelson (2009) agree that "research reveals that for lasting changes in behavior to occur, beliefs and assumptions must be brought to consciousness and the deep structures supporting behaviors must be addressed" (p. 354). According to Hunzicker (2004), "changing a teacher's beliefs requires new information to be presented repeatedly over time to the point that the person begins to feel disequilibrium between current beliefs and new information" (p. 45). In other words, longitudinal teacher professional development is the key for a change in teacher beliefs and therefore practice. Regarding this idea, this research study was initiated to investigate whether math and science teachers' beliefs about STEM education change due to a longitudinal professional development initiative emphasizing inquiry-based instruction.

Professional Development Programs

Guskey (2002) describes professional development programs as an attempt to change classroom practices of teachers, their beliefs and attitudes towards different teaching approaches, and improve students' learning outcomes. As an alternative to the traditional teacher change model which initiates a change teacher classroom practice in order to change their beliefs and attitudes, Guskey proposes a process beginning with Professional Development Programs (PDP), which can result in Classroom Practice Change (CPC), Students' Learning Outcomes (SLO), and Belief and Attitude Change (BAC) respectively. This model explicitly underlines the necessity of positive improvement in students' learning outcomes as evidence that teachers changed their beliefs and attitudes. Guskey's model of teacher change informed the design, enactment and examination of this study. We focus our results on the link between PDP, CPC and BAC.

Methods

This study tracks the degree to which teachers transformed their beliefs related to inquiry-based instruction in middle grades classrooms as a result of a professional development program that focused on providing support and training to improve content-embedded inquiry-based learning. Teachers were provided two weeks of teacher training in the summer (64 hours), four group follow up sessions during the academic year (10 hours total), four or more full class observations with debriefing after (at least 4 hours), and numerous individual support sessions (about 5 hours). This totaled over 80 hours of involvement for each participant each year.

The summer primarily involved modeling examples of practice, debriefing practice, and developing new inquiry-based lessons in teams that were based on a two dimensional instructional model (Marshall, Horton, & Smart, 2009). This Model is predicated on engaging students in rigorous inquiry learning and includes formative assessment and teacher reflective practice at each step of the inquiry learning process. We define inquiry-based instruction as the opportunity for students to explore math and science content prior to a formal explanation being developed or provided. Support during the academic year included co-planning, co-teaching, observations and debriefing observed classroom instruction.

This is a five-year long professional development project, allowing some project teachers to participate over multiple years. However, this study reports only on the experiences of first year project teachers. Our research question is, "To what extent do teacher beliefs in regards to STEM education change within the first year of professional development in inquiry-based instruction.

Three sources of data were collected during the study: 1) An electronic observation protocol to evaluate teacher classroom practices (Marshall, Smart, & Horton, 2010), 2) Measures of Academic Progress (MAP) to measure student growth in science content knowledge and process knowledge (Cronin, Kingsbury, Dahlin, Adkins, & Bowe, 2007), and 3) Belief Survey to measure teacher beliefs regarding inquiry-based instruction. The internal consistency value (Cronbach's Alpha) was .75 for the overall survey. The survey data are the focus of this paper. Teachers completed a pre-survey prior to beginning the year-long professional development. At the conclusion of their year post-surveys were completed.

Survey data was analyzed using SPSS 20.0®. An exploratory factor analysis of the 17 survey questions was completed in order to group the questions into separate constructs for analysis. Once these were determined, we ran a dependent t-test and multivariate analysis in order to make statistical inferences based on the transformation of teacher beliefs.

Results

We first conducted an exploratory factor analysis on the overall survey data collected for first year participants in each of the four years of the project using the SPSS dimension reduction feature. Data was collected from 35 math and 76 science teachers. 7 items were excluded from the factor analysis because of low reliability coefficients. Regarding the scree plot for the dimension reduction and the cut-off point of an eigenvalue greater than 1.0 on the rotated factor loadings from 10 items, four factors were retained with Verimax rotation:

Table 1: Factors and Their Reliabilities

Factors	Factor Labels	Reliabilities
1	Instructional Effectiveness	0.773
2	Support by Colleagues	0.764
3	Motivation	0.600
4	Content Standards	0.670

The main reason to conduct the factor analysis was to reduce the amount of factors measured within the survey, and to analyze the change in terms of these factors for first-year and second-year teachers in the project. In order to make this comparison, a composite score was determined for each factor and each teacher, and a dependent t-test was applied for each factor. In addition, we also conducted paired t-test analysis for each item which could not accommodate in any factor. Surveys that were collected at the beginning of the first year and at the end of the each year of the project were paired for each teacher.

Our t-test analyses indicated a significant change on the Instructional Effectiveness factor ($p < .01$) for all first-year participants. We also conducted dependent t-test analysis for science and math teachers separately. When calculated separately, first-year participant math teachers demonstrated no significant difference for any factor and survey items. First-year participant science teachers demonstrated significant difference for: Instructional Gains ($p < .05$), and Support ($p < .05$) factors. For the second-year participants, math teachers showed significant difference on the Instructional Effectiveness factor ($p < .01$). After two years in the program, math teachers ($n=5$) agreed more that they know how to make their instruction effective while using inquiry. However, this finding was not reliable as much as the findings for science teachers because of low sample size for math teachers in the second year.

Discussion

Regarding the preliminary findings from surveys, it was evident that the professional development program to foster inquiry-based instruction support science teachers to change their beliefs about the use of inquiry during instruction more than math teachers. The first year of this program was not successful in changing math teachers' beliefs about how to use inquiry effectively for the instruction. The first year was much more impactful for science teachers, as it influenced science teachers' beliefs about their knowledge of how to change instruction, how to manage classroom, how to orchestrate discussions, and how to assess students' learning while reforming their practices towards inquiry-based instruction. Working with science teachers throughout the project might have helped math teachers to find ways to integrate math with other subjects. However, one year in the project was not sufficient for math teachers to change their beliefs in the same manner as the science teachers. One possible reason for this difference might be math teachers' conservative beliefs about the nature of math.

The difference on beliefs between science and math teachers might also stem from the initial state of their classroom practice. Regarding the classroom observations, science teachers showed they knew more about how to create an inquiry-based instruction than math teachers at the beginning of the program. The initial shift for math teachers through one year might have been a bigger move and therefore resulted in less change in their beliefs compared to science teachers.

Returning back to Guskey's (2002) teacher change model, these preliminary findings showed that CPC for one year might be influential for teachers' BAC towards reform. In such a change, evidence of SLO might have been a confounding variable. Further analyses including students' achievement need to be included so as to test Guskey's model of teacher change process for the context of inquiry-based instruction.

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SECONDARY MATHEMATICS TEACHERS' BELIEFS REGARDING CONTENT AREA READING

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Reading supports learning in all disciplines, yet is seldom used in mathematics classes. With the advent of the Common Core State Standards, new emphasis is being placed on students' ability to engage in mathematical practices such as understand word problems, read and critique arguments, make explicit use of definitions, read and comprehend technical grade level texts, (CCSSI, 2010). Mathematics text is difficult to read, presenting demands in higher-level thinking and comprehension skills. The development of academic literacy at higher grades requires the integration of explicit strategy instruction with sense-making activities involving disciplinary texts (Greenleaf, Schoenbach, Cziko, & Mueller, 2001).There is good reason for mathematics teachers, who are more familiar with the demands of mathematics text than literacy teachers, to accept the responsibility to develop these reading skills in their students (Reehm & Long, 1996). This study investigated secondary mathematics teachers' beliefs about the role of reading in mathematics learning and examined shifts in their beliefs after participation in professional development focused on integrating literacy and mathematics.

Twenty-six teachers were interviewed before and after their participation in the project. Interviews probed teachers' uses of reading strategies and the purposes behind integrating reading into their instruction. Interviews were first coded for emergent themes and then the teachers' overall purposes for incorporating reading were characterized according to a continuum from traditional to constructivist. Following the initial coding, the researchers identified shifts that occurred regarding individual themes or teachers' overall orientations for reading use.

Initial analysis suggests that these teachers held primarily traditional views regarding the use of reading strategies in mathematics classes. Most teachers described the purpose of reading mathematics as a way for students to get information, learn vocabulary, increase reading skills, and prepare students for college. A few teachers described the purpose of reading as supporting conceptual understanding and connecting to real life. While most teachers maintained their traditional views, about one third of the teachers shifted their views in the constructivist direction following participation in the project. The results suggest that through engaging in professional development designed to address the purposes of and strategies for teaching reading in mathematics classes, it is possible to begin to shift teachers' beliefs and practices. To support mathematics teachers in addressing reading in mathematics, we need to better understand their beliefs and further examine how to support teachers in shifting these beliefs towards better alignment with current standards.

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THE EXTENT OF MATHEMATICAL CREATIVITY AMONG GIFTED STUDENTS

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This pilot study investigates the extent of mathematical creativity among 57 eighth-grade talented students in the Mathematically Talented Youth Program. The reasoning of these students is examined when solving a problem, as is the degree of mathematical creativity and aesthetic in their approach in solving a non-routine mathematical problem. The analysis explored whether the students' mathematical thinking was dependent solely upon previous mathematical knowledge and skills. The majority of the students relied on technical algorithm to solve the problem. Although talented students coped well with the thinking challenge, most of them operated at a very basic level of creativity. There is a need to broaden and develop mathematical-logical thinking as an integral part of instructional programs in mathematics.

This study examined the mathematical aesthetic in students' problem solving solutions. The aim of the study was to undertake a preliminary investigation of the extent of mathematical creativity and aesthetics in solving non-routine problems, among students in the Mathematically Talented Youth Program. The study was conducted to answer the following research questions: (1) What kinds of reasoning did the talented students use in solving a non-routine problem? (2) What was the degree of creativity and aesthetics in problem solving among students who attended the Mathematically Talented Youth Program? (3) According to the degree of creativity, can we conclude that mathematical thinking among the students of the Mathematically Talented Youth Program is dependent solely upon previous mathematical knowledge and skills?

The level of difficulty of the thinking challenge presented to the students in this study was appropriate, as shown by the majority of the students answering the question correctly. However, the type of reasoning, the level of creativity, and the degree of mathematical aesthetics varied greatly among the students. Three types of reasoning emerged among the talented students in the program: analytical reasoning, practical reasoning, and creative reasoning. The majority of the students in the program (about 40%) were at the most basic level (Level 1) of mathematical creativity. Although students can be categorized as talented, the level of mathematical creativity may not necessarily. The students in this study did not seem to develop a high level of mathematical aesthetics when solving the problem. Students relied heavily on prerequisite knowledge and although presented their solutions with high clarity, they failed to express simplicity, structure, and cleverness in their solutions. Furthermore, there lacked elements of surprise in their solutions. Although the research literature on talented students support the notion that talented students are more creative, no specific level of creativity was found in this segment of the population. Understanding the development of mathematical creativity and aesthetics among gifted and talented students is crucial to initiate and support their growth.

Future studies should examine instructional support and classroom practices that afford the development of mathematical creativity and aesthetics over a period of time during their program. Teachers need to emphasize not only creative ways to solve problems, but also the elegance of the solutions because that is what mathematicians do.

TASKS AND QUESTIONING WITH AND WITHOUT TECHNOLOGY

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The way technology is implemented in the classroom influences the effect it has on mathematics learning. Using technology to explore mathematical ideas engages students differently than using technology to check calculations (Zbiek, Heid, Blume, & Dick, 2007). The emphasis teachers place on different learning goals (e.g., skills or conceptual understanding), the kind of tasks they pose, and the kinds of questions they ask are aspects of teaching that influence the opportunities students have to learn (Hiebert & Grows, 2007). This poster will describe and distinguish the tasks, and questioning of two teachers, Edward and Rosa, in Algebra 1 lessons during which dynamic technology was and was not used as a mediator of instructional activities.

We modified the Mathematical Task Framework (Stein, Grover, & Henningsen, 1996) to determine the level of cognitive demand of each task in three stages: 1) as presented in curriculum materials, 2) as set up by the teacher, and 3) as discussed. Also we used a modified version of Boaler and Brodie's (2004) questioning scheme to explore how questions influence task discussion. We sought to answer: 1) How is the use of dynamic technology related to the fidelity of tasks (consistency of a task within the three stages)? 2) How is the use of dynamic technology related to level of questioning?

In the absence of technology, Edward lacked fidelity to the task, by increasing the cognitive demand of the tasks during set up or discussion. In the presence of technology, Edward was able to maintain the level of cognitive demand for high demand tasks. Edward maintained an emphasis on the meaning and explanations of procedures and content during discussion of the tasks. In the absence of technology, Rosa had fidelity to the tasks she used, tasks that were inherently low level. Most of the questions Rosa used in this context were related to gathering information, rather than engaging students in mathematical thinking. In the presence of technology, Rosa could not maintain fidelity to a high level task. In fact, she had difficulty engaging the students in the task at all, even when she attempted to reduce it to a set of procedures to follow. Further research is needed to establish a more robust connection between the use of various question types and other pedagogical choices in technology-rich settings.

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A COMPARISON OF FOUR PEDAGOGICAL STRATEGIES IN CALCULUS

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Keywords: Post-Secondary Education; Affect, Emotion, Beliefs, and Attitudes; Data Analysis and Statistics

The quality of education in science, technology, engineering, and mathematics (STEM) fields is an issue of particular importance. The President's Council of Advisors on Science and Technology (PCAST, 2012) recently reported that fewer than 40% of students who intend to major in a STEM field actually complete a STEM degree; many of the reasons students give for switching implicate uninspiring, dull, or unimaginative introductory courses. The MAA is currently conducting an NSF funded national study examining the characteristics of successful programs in college calculus. Using the same surveys as the MAA study, my work builds on this project by examining the effects of different instructional strategies on student outcomes.

My work uses emergent perspective (Cobb & Yackel, 1996), which holds that students' learning is enabled and constrained by their participation in classroom activities. Therefore, classes that present more opportunities for student engagement and participation are conjectured to offer more opportunities for robust student learning and improved beliefs and dispositions.

In the Fall 2012 semester at a large public university in the southwestern United States, Calculus I was taught by four different instructors using four different instructional techniques: traditional lecture, interactive lecture, an inverted model, and a technology-intensive lecture. I identified the following research question: How do students in the four classes compare in their: a) persistence in STEM major tracks? b) attitudes, dispositions, and beliefs about mathematics? and c) conceptual and procedural achievement in calculus?

Students completed start- and end-of-term surveys, which included questions on demographics; preparation; beliefs and attitudes about mathematics; and college and career plans. I also collected scores on the Calculus Concept Inventory (Epstein, 2006), the Calculus Concept Readiness assessment (Carlson, Madison, & West, 2010), and a common final exam. These data were analyzed using quantitative methods. My results were in three main categories: persistence, achievement, and beliefs and attitudes. Briefly, there were no significant differences in persistence; several measures of beliefs and attitudes differed significantly across classes; differences in final exam scores were attributable to differences in preparation; and there were large differences in grade distribution and fail rates.

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MATH TALK IN ELEMENTARY SCHOOL CLASSROOMS: WHAT FEATURES ARE COMMON?

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Keywords: Classroom Discourse; Elementary School Education

Participating in mathematical discussion has been heralded as a key component to students' developing a robust understanding of mathematical concepts (NCTM, 2000). The Common Core Standards and the assessment systems that will accompany them demand that students explain their thinking. While many studies have documented particular classrooms where children have engaged in rich mathematical discussions, this practice is still relatively rare in elementary school classrooms (Walshaw & Anthony, 2008). We have been working to promote mathematical discussion within one small elementary school district that serves a diverse population of economically disadvantaged students. We propose to share the results of our observations of 19 third and fourth grade classrooms to describe the variety of approaches to discussion being enacted in the classrooms. We believe that our analysis will be useful as we consider what support teachers might need to advance the level of discussion in their classrooms.

Two researchers visited all 19 of the third and fourth grade classrooms at the five schools in the district between January and March, 2013. Teachers were asked to teach a typical lesson and the observers took field notes during the lesson to document the various elements of discussion. The researchers independently scored the lesson immediately afterwards using a rubric constructed by the research team, and then discussed their scores to determine inter-rater reliability and to resolve differences between their scores. The rubric includes the following five categories, each rated on a 4-point-scale: (a) variety of approaches to the lesson tasks used by students (b) quality of mathematical explanations by the teacher and by students (c) opportunities for students to actively participate in the discussion (d) equitable participation, and (e) connections between mathematical ideas.

At the writing of this proposal, we have observations from 17 of the 19 classrooms. The data indicate that 3 of the 5 categories were common among the classrooms we observed, with the most common feature being equitable participation. We also frequently observed students using a variety of approaches to problems and experiencing frequent opportunities to participate. The quantity of student talk and participation during math lessons was high. However, connections between ideas and in-depth explanations, both indicators of the quality of class discussion, were observed with much less frequency. Explanations consistently scored lower on our rubric than all other features of discussion, and were often lacking important elements of a concept or focused exclusively on procedure. This suggests that although student participation in mathematical discussions is relatively high, there is still work to be done to increase the quality of the more complex features of mathematical discussions.

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OBSERVING MATHEMATICAL PLAYFULNESS IN THE CLASSROOM

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I define play as activity in which participants imagine and provisionally accept ideas that imply a structure of rules. This notion of play is influenced by Bruner's (1976) view of the functions of play as: a) minimizing the consequences of action, and b) providing opportunities to try combinations of behaviors. I draw on Vygotsky's (1978) assertion that play involves imaginary situations and the rules that necessarily follow. Therefore, mathematical playfulness involves play where the activity itself is mathematical. The relationship between imagination and rules in mathematical play depends on the mathematics of the activity.

Suggestions of mathematical play can be seen in the Standards for Mathematical Practice in the Common Core State Standards for Mathematics (CCSSI, 2010, pp. 6-8). When Standard MP3 says students "make conjectures and build a logical progression of statements to explore the truth of their conjectures" (p. 6), students must entertain ideas and the mathematical consequences of those ideas. These are hallmarks of mathematical play.

To understand the role of mathematical play in classrooms, I have developed a framework to observe interactions identified as mathematically playful practices. The framework is organized into three categories of activity. Each of these categories corresponds to mathematical practices in research, and satisfies the definition of mathematical play. *Mathematically playful exploration* is activity in which participants are engaging with a problem situation in ways that allow them to explore the affordances and constraints of the environment. *Mathematically playful explanation* is the experimental effort to communicate, clarify, and justify mathematical ideas. This includes argumentation, in which ideas (such as conjectures) are described in a way that exposes them to challenge and the possibility that they be found incorrect. This subconstruct of the framework is intended to capture activity that incorporates aspects of mathematical playfulness to share ideas by convincing others to consider and accept them. *Playful mathematization* is the activity of using models in a flexible, speculative way to capture some aspect of experience in a mathematical structure.

The aim of my current research is to apply this framework to observations in classrooms using a grounded theory approach to analyze how these practices unfold. The refinement of this framework will allow researchers to look for patterns in how students and teachers engage in and support mathematically playful practices, enabling greater understanding of the educational role of playfulness in the mathematics classroom.

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MIDDLE SCHOOL MATHEMATICS TEACHERS' PERSPECTIVES OF IMPLEMENTING HIGH COGNTIVE DEMAND TASKS

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Keywords: Middle School Education, Instructional Activities, and Practices, Teacher Education-Inservice/Professional Development

Stein, Grover, and Henningsen (1996) argue that student engagement in mathematics increases when the level of cognitive demand of tasks in the classroom is increased. By engaging students in high cognitive demand tasks, it can increase students' problem solving skills and provide them with opportunities to engage in problems that are non-routine and require complex mathematical thinking. Henningsen and Stein (1997) claim that teachers need to be able to select tasks appropriately, as well as implement them at a high level of cognitive demand in order to support student thinking. I will examine teaching practice over time in middle school mathematics classrooms with teachers who are implementing more high cognitive demand tasks after providing the teachers with specific interventions aimed to help maintain the level of demand, as well as gain teachers' perspectives on how the implementation process is progressing in the classroom.

In this case study, I will examine mathematics teachers at a diverse middle school in the Southeastern United States who have expressed interest in implementing more high cognitive demand tasks during the 2013-2014 school year. I will provide interventions focused on teachers' enactment of high cognitive demand tasks and help teachers create classroom environments that will foster student engagement with the tasks. Data collection will include classroom observations to determine the level of cognitive demand during implementation, teacher interviews after each observation period to gain teachers' conceptions on task implementation, and video-stimulated recall to reflect with teachers on their instructional moves. I will work with the teachers in setting up classroom norms before the school year begins and perform classroom observations daily the first two weeks of school to see how the teachers implement tasks in the beginning of the school year and observe the classroom for three days each subsequent month through the year.

I will be seeking to understand task implementation from the teachers' point of view while comparing the teachers' perspectives to my own from classroom observations. I intend to add the body of knowledge on task implementation and help inform professional development aimed at supporting teachers implement high cognitive demand tasks in the mathematics classroom. By identifying the roadblocks and successes as identified by the teachers when implementing high cognitive demand tasks, it can inform next steps in helping other mathematics teachers successfully implement high cognitive demand tasks in classrooms.

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WITH A LITTLE HELP FROM MY FRIENDS: USING SCAFFOLDING TECHNIQUES IN PROBLEM SOLVING

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The purpose of this study was to explore students' use and effectiveness of scaffolding to solve non-routine mathematical problems in middle school mathematics classrooms containing students of diverse abilities. Students were given two different types of scaffolds to support learning within a sixth grade geometry unit. First, students solved a math problem by using a four square method—identification of critical components, strategy selection, computation, and analysis of answers. The second type of scaffolding occurred with different types of grouping, alternating among teacher led instruction, working in groups of four, in pairs, and finally individually. Measures included students' mathematical knowledge (state tests), daily non-routine problems and student satisfaction of scaffolding methods.

Although the definition of scaffolding is not consistent among research studies, Saye and Brush (2002) note that most scaffolding techniques can be classified as either soft or hard. Soft scaffolds are teacher actions that are performed in support of students at the moment of instruction—when a learner has specific needs (Roehler & Cantlon, 1997). Examples of soft scaffolding with this study include students working in small groups or partner situations. Hard scaffolds are external supports that are developed in advance of the instruction. Examples of hard scaffolding in the current study include the four square approach to problem solving. Some studies have demonstrated effectiveness of scaffolds in helping students to manage information (Cho & Jonassen, 2002), reflection (Davis & Linn, 2000), metacognition (Holton & Clark, 2006), and knowledge acquisition (Roehler & Cantlon, 1997). To analyze data gathered on the graphic organizers, an eight-point scale was used. A control group solved the problems individually, while two classes used the soft scaffolding. The control group was then compared against the experimental. Z-scores and effect size tests were used to document significant differences from current to previous administrations on the Ohio Achievement Assessments. Implications of this study go beyond standardized test results. Results indicated a significant improvement in problem solving at the $p < .001$ level, whereas post-test scores showed no significant differences between the participant groups. Results of student satisfaction surveys (of scaffolding types) are consistent with literature on students with special needs.

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TEACHERS' LEARNING FROM PROFESSIONAL DEVELOPMENT FOCUSED ON MATHEMATICS CLASSROOM DISCOURSE

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Research has shown that providing students with opportunities to engage in mathematical argumentation and conceptual explanations can improve students' learning (Chapin & O'Connor, 2004). Unfortunately, mathematical conversations in which students engage deeply with one another's ideas are not common practice in mathematics classrooms (Cazden, 2001). Effective professional development is recognized as a critical component to improving mathematics classroom practices (Ball, 1995; Wilson & Berne, 1999). Thus, experiences that develop teachers' understanding of ways to foster mathematics discourse in classrooms are an important step to help support students' learning of mathematics. This study investigated what teachers took away from one such practice-based professional development focused on mathematics classroom discourse. This presentation focuses on features of classroom mathematics discourse that were evidenced in the ways teachers talked about and reflected upon their classroom practice.

A review of the literature on mathematics classroom discourse revealed four categories of practices that have been shown to influence students' learning. These categories are: (a) making student thinking visible, (b) acknowledging the teacher's role in shaping discourse, (c) promoting a focus on mathematics during classroom discussions, and (d) shaping social norms of the classroom. These categories serve as an analytical framework for this study.

Four middle school mathematics teachers engaged in a professional development pilot program focused on mathematics discourse in secondary classrooms. There were seven sessions across the school year during which the group of teachers met with the university facilitator. Data collected includes video recordings of the professional development sessions, three classroom observations and three semi-structured interviews. The study used a grounded theory (Strauss & Corbin, 1990) approach to explore the concepts the participants learned from the professional development. This poster presentation will present the preliminary analysis of this study. It will share the issues of practice that served as teachers' initial focal points, the ways teachers' reflected on the development of their classroom discourse practices, and the challenges teachers' encountered as they worked to implement the practices they learned from the professional development sessions.

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TEACHER TENSION: WHEN SHARING STUDENT STRATEGIES CONFLICTS WITH THE LEARNING TRAJECTORY

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Mathematics teachers often encourage students to share their solution strategies with the whole classroom. This teacher move can unveil how teachers listen to their students mathematical thinking and sequence students' strategies to frame their conception of a lesson's learning trajectory (Empson, 2011; Kazemi & Franke, 2004). But teachers can also use this teacher move in a way that confuses students by sharing strategies in a haphazard fashion (Hiebert et al., 2005). In this study, we examine how and why mathematics teachers have students share mathematical strategies. What beliefs do mathematics teachers hold about students' sharing strategies in the context of a lesson's learning trajectory? What are ways this sharing aligns or conflicts with the teacher's conception of the learning trajectory?

We worked with nine mathematics teachers from 5^{th} to 8^{th} grade classrooms from three different schools, all within the same suburban district in the Mid-Atlantic United States. Each teacher taught the same lesson. Students explored combinations to build trains of various lengths and justified how they knew they had exhausted every combination. The teachers taught the lesson over two-days and attended a one-day professional development workshop emphasizing this learning trajectory.

We analyzed video of each teacher's lesson and post-teaching interviews. We also looked at MKT scores and pre/post-belief surveys for each teacher.

We found every teacher spent at least 18 minutes having students share strategies, meaning they prioritized it as an important part of their lesson, which often lasted less than fifty minutes. Within this teaching move, we found a wide variety in how teachers had students share strategies. Two teachers deliberately sequenced students' organizational strategies from simple-to-complex. Four teachers sequenced strategies based on surface-level features, such as whether the student used numbers or drew pictures. And three teachers randomly chose strategies based upon their proximity to students. In these latter two cases, this teaching move conflicts with the learning trajectory of this lesson; it does not build up to a way for students to justify how they know they found every combination.

Our teachers latched onto this teaching move of having students share strategies. But few of them did it in a way that supported the mathematical goal and learning trajectory of the lesson.

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RESEARCH ON PROFESSIONAL DEVELOPMENT AND TEACHER INSTRUCTION: SUSTAINABLE CHANGES IN TEACHER PRACTICES

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Teachers shy away teaching mathematics in the way called for in the national and state reform documents (e.g., National Council of Teachers of Mathematics, 2000). Professional development programs may help teachers change their instructional practices. Given substantial federal, state, and local investment in professional development each year, studies are needed to investigate what elements of professional development are most effective and how teaching practices change over time because of participation in these activities. In this study, we investigated the extent to which teachers' instructional practices changed after they attended professional development activities and the extent to which teachers sustained those practices over time.

To answer our research questions, we used the data conducted as part of evaluation of a professional development program which was designed to prepare high school teachers to improve the quality of mathematics education. The program targeted teachers in two urban districts that mainly served low-income students or students of color. For two consecutive summers, teachers attended 4-week summer programs to increase their knowledge in important mathematical content areas. During each academic year, teachers met with program developers on a monthly basis. After teachers had completed their first summer in the program, external evaluators conducted classroom observations beginning in fall 2005 and continuing through spring 2010. Each observed lessons were scored by using the classroom observation protocol developed by the research team of the mathematics education center at Rice University. Factor analyses were then used to identify the key features of the lessons. Classroom observation data were analyzed by using a three-level multivariate model approach, with time points nested within teachers, and teachers nested within cohorts.

The results of this study indicated that teachers changed several aspects of their instruction after they attended the program. Teachers made prominent and permanent changes in their practices so that students shared their knowledge, justified their conclusions, and talked more about mathematics ($p \leq .0001$). Similarly, students actively engaged in thought-provoking activities, and teachers created an environment for students to be members of a learning community in which intellectual rigor and constructive criticism were valued ($p \leq .0001$). Teachers also became more explicit regarding what they expected from their students during group activities and whole-class discussions ($p < .02$). However, teachers did not show a steady growth in practices that allowed students time to discuss and share what they understood with their peers and work in groups, or to explain their understandings to their peers. Finally, neither the teachers nor students began to use a variety of means to present concepts after the program was initiated.

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THE USE OF REPRESENTATION BY TWO CALCULUS TEACHERS

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Keywords: Instructional activities and practices, Post-Secondary Education

Students' difficulties with the conceptual understanding of the derivative are well documented in mathematics education (e.g. William, 1991). There is however no such agreement on the use of representations by teachers. While Duval (1993) pointed out that using and articulating different types of representations when learning a concept leads to a deeper understanding, Selden, Mason and Selden (1999) observed that students have great difficulties coordinating different types of representation. Eisenberg and Dreyfus (1991), among others, even describe reluctance on the part of students to use different types of representations. The use of representations by teachers in calculus classes could help understand this behaviour. This poster focuses on a study that aims to better understand the way teachers use representations in their teaching of calculus.

To describe the way calculus teachers use representations in their teaching, we need to define what we mean by representations. To do so, we first use the theory of representations (Duval, 1993) which includes different types of representations (verbal, algebraic, graphical, etc.) and actions done with these representations such as treatment, production, conversion, and coordination. Because the representations described in Duval's theory are conventional representations, we also consider the more intuitive functional representations (Hitt, 2006). For the study, two calculus teachers were observed during 8 teaching sessions. These sessions were videotaped and, the three of them that focused on the introduction to the derivative were analyzed.

Analysis shows common elements in the way the two teachers used representations. Firstly, we observe that most of the representations used by both teachers are algebraic representations. Moreover, even when they produced other types of representations, such as graphs, the teachers rapidly referred back to the algebraic representations. Secondly, the teachers have a preference for formal representations. For example, when they produce a graph, it is often a quadratic curve open upward, in the first quadrant. Finally, while the teachers coordinated and articulated different representations adequately, these manipulations were often implicit. For example, they used the information contained in a graph to produce an algebraic representation without explicitly relating it to the graph. Overall, the elements observed in the practice of two teachers bring new interrogations on how students can understand and interpret implicit manipulations of representations. This leads us to a new study, the premises of which will be presented on the poster.

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RESPONSES TO STUDENT ERRORS IN CHINESE AND U.S. MATHEMATICS CLASSROOMS

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Theoretical Framework and Method

Treatment of students' errors or misconceptions has been a longstanding issue in mathematics education. Specifically, Borasi (1994) made a strong argument that the errors should be used as "springboards for inquiry" to promote students' reasoning in mathematics classroom. In their cross-cultural study, Schleppenback and her colleagues (2007) depicted a rich picture of Chinese and U.S. elementary teachers' responses to student errors. U.S. teachers were more likely to respond to students' errors with a statement about the error, while Chinese teachers were more likely to respond to errors with a follow-up question to solicit student explanations about why they made the errors and how they may be corrected. The current study is to extend these concepts to middle school algebra classrooms in China and the U.S.

The lessons examined in this study were taken from Chinese and U.S. data sets of an international comparative study (Clarke et al., 2006). A fine-grained analysis of one Chinese lesson and one U.S. lesson was conduced to identify the patterns of dealing with students' errors.

Results and Discussion

In the U.S. lesson, the teacher usually encouraged students to present their solutions as much as possible. However, the teacher never asked how they got to the solution. Following general comments on students' work, the teacher then presented the correct solution to the class without exploring the reasons that may have caused the students' errors. In the Chinese lesson the teacher adopted different strategies to illuminate students' errors. Strategies utilized include: student discussion, double-computation, transforming students' errors through deliberate questioning, and clarification of missing concepts.

In summary, students in the Chinese classroom did not feel embarrassed for making errors and discussing their errors, and the teacher purposely made use of errors to promote student understanding. Examining the Chinese practice of dealing with students' errors and misconceptions may provide valuable insight for U.S. mathematics educators in how to view student errors and how to use them to promote student understanding.

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IDENTIFYING RELATIONAL MATHEMATICAL INSTRUCTIONAL MOVES

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Keywords: Affect, Emotions, Beliefs, and Attitudes, Instructional Activities and Practices, Middle School Education

The purpose of this poster is to identify and conceptualize instructional moves that can be leveraged to concurrently support middle school students' learning of mathematics, positive dispositions toward mathematics, and development of social relationships among and between students and teachers in mathematics classrooms. The work of teaching is complex. Identifying instructional moves that can be leveraged to meet multiple goals simultaneously will afford the likelihood that they will be implemented by teachers and that students can benefit. This poster serves as a first step in a research endeavor to identify such instructional moves, support teachers in making sense of and enacting them, and investigate students' experiences of them.

We intend to define various types of *relationships* that play out in mathematics classrooms and how instructional moves support their development: relational understanding of mathematics (Skemp, 1976), positive interpersonal relationships, and positive dispositions toward mathematics, including students' awarenesses of growth in their understanding and interpersonal relationships. Productive instructional moves that promote relational understanding, such as pressing students to elaborate their thinking (Kazemi & Stipek, 2001) are not usually discussed in terms of social or dispositional benefits, but such benefits are possible.

Below, in Table 1, we provide examples of two instructional moves and conjectures about how they could meet multiple outcomes concurrently.

Table 1: Two Examples of Conjectured Relational Mathematical Instructional Moves

Instructional Move	Supports Learning	Supports Social Relationships	Supports Positive Dispositions
Pressing students to elaborate upon their thinking during classroom discourse.	Students engage in sense-making and develop as communicators.	Students connect with others through dialogue and feel valued by being heard by their teacher.	Increases students' self-efficacy through public validation of their ideas.
Posing tasks that are at the boundaries of students' zones of potential construction.	Allows students to engage in vertical learning (major reorganizations of students' schemes and operations).	Through experiencing success with a challenging task, students develop feelings of trust toward their teacher if challenges are bearable.	Increases students' self-efficacy through experiencing significant mathematical progress.

In our poster, we will elaborate upon these instructional moves and identify others through a review of research literature. Future directions for this work will be shared as well.

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TEACHING MOVES TO SUPPORT ARGUMENTATION IN TWO DIFFERENT CLASSROOMS

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The Common Core mathematics standards require that we understand what is required of teachers to help students learn the practice of mathematical argumentation. For this poster, we examine teaching moves that two teachers used to support students in making arguments, as well as other moves that could inhibit argumentation. This poster adds to the body of literature on classroom argumentation and proof, which analyzes classrooms, teachers and students (e.g. Yackel & Cobb, 1996; Bieda, 2010). We view mathematical argumentation as both a social and a mathematical activity in which teachers engage in flexible, improvisational (Sawyer, 2011) use of moves, practices and materials to support students in making arguments for themselves—arguments about mathematical ideas using methods particular to mathematics. We follow in the tradition of Nathan and Knuth (2003) and Williams and Baxter (1996) in distinguishing and analyzing the tension between social and analytic scaffolds. We reframe the tension for our teachers as being between supporting students' content learning and engaging in argumentation.

We analyzed two middle school teachers' implementation of one lesson. One teacher used whole group discussion to provide support for argumentation, including setting norms, defining stages of argumentation, and creating a discourse community. Much of the mathematical content arose in students' small groups, each of which the teacher visited only briefly. In the lesson examined, argumentation was at the fore. A second teacher focused on efficiently working through the mathematics content of the unit, sprinkled with short arguments that she made herself or that she asked students she perceived to be high-achieving to make. She also skipped one part of the activity that explicitly asked students to make a conjecture. In the lesson we examined, this teacher emphasized content over argumentation.

In the poster session, we will share video from these teachers' classrooms and offer the opportunity to discuss how the tension between content and argumentation plays out.

Acknowledgments

This material is based on work supported by the National Science Foundation. Any opinions, findings, and conclusions expressed in this material are those of the authors and not necessarily those of the NSF.

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DEVELOPING POSITIVE STUDENT-TEACHER RELATIONSHIPS IN STEM CLASSES

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Secondary school is full of marked changes for American students. Students experience school social environments at odds with their developmental needs, shuttling through multiple subject-specific classes threatening the quality of relationships with teachers and students' sense of support and relatedness (Eccles et al., 1993). Caring student-teacher relationships, where the teacher holds students to high expectations may be particularly important for youth of color in urban schools (e.g. Duncan-Andrade, 2009). However, student-teacher relationships are often most tenuous in public schools in urban settings (e.g. Fine, 1986). Perception of the level of teacher support is correlated to students' intrinsic value of mathematics (Midgley, Feldlaufer & Eccles, 1989). Because student-teacher relationships may impact student perceptions of teacher support, this research explores the nature of these relationships. Three STEM teachers from City Performing Arts Academy,[1] a public school in Boston, and their students were collaborators in this research. City Performing Arts Academy uses an Advisory system where every teacher serves as an Advisor for students' four-year high school careers. Advisory is a small class with the goal of promoting students' sense of belonging by attending to personalized academic and social emotional needs to improve education outcomes. Because Advisory intends to build a trusting student-teacher relationship to increase student engagement, this study investigated the nature of student-teacher relationships in Advisory and how they may differ and/or impact (positively or negatively) the relationship in STEM classes. Interviews with students included those who have their STEM teacher for both their STEM class plus Advisory as well as those who have their teacher just for STEM class (and not Advisory). Student interviews and participant observation revealed that students' relationship with their teacher in Advisory differed from and supplemented their relationship with their teacher in STEM class. Similarly, teachers' viewed their responsibilities as an Advisor as greater and more personal than that of teaching in STEM class. Both student and teacher perspectives suggest ways for STEM teachers to build positive relationships with students with or without Advisory class (e.g. by taking time after school to help with homework and supporting students with challenges outside of school.)

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THE ROLE OF MATHEMATICAL AESTHETIC IN IMPLEMENTING A CLASSROOM MULTI-USER INTERFACE: A CASE STUDY

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Keywords: Affect, Emotions, Beliefs and Attitudes, Classroom Discourse, Technology

This paper briefly outlines the conceptual framework, methodology, and findings of a case study. The setting of the study was the implementation of Multi-User Interface (MUI) technology in a mathematics classroom, e.g., participatory simulations. This investigation revealed connections between aesthetic perceptions of mathematics and teaching practices. Activities, taught using MUIs, invited student participation at various levels of understanding. As each student engaged in the MUI, he/she produced artifacts of his/her mathematical thinking, which were viewable on both a private display screen and an overhead projection. All of the students' contributions were aggregated as a whole group-constructed artifact, an object of the teacher's mathematical aesthetic evaluation (Sinclair, 2004).

The case study was conducted within the broader context of a design experiment, which investigated refinements in the design of MUIs for the classroom. Together, one teacher, her class, and the technological innovation comprise the case. The researcher conducted several videotaped teacher interviews during lesson planning sessions. The data also consisted of videotaped observations of the classes in which she implemented the activities. Data were analyzed within the conceptual framework of network-supported generativity (Stroup, Ares, Lesch, and Hurford, 2007).

The central focus of the investigation was to describe a particular phenomenology—subjective experiences—of mathematical aesthetic from the teacher's perspective. Findings indicate that aesthetic considerations emerged as part of the teacher's design constraints and guidelines for classroom implementation of the MUIs. Findings also suggest that the projections of student-generated artifacts made mathematical aesthetic visible and a substantive topic of classroom discourse.

The teacher's mathematical aesthetic sensibilities showed in her directing students' attention to salient characteristics of student solutions (Brinkmann, 2009), projected overhead. Her comments revealed her preference for more "interesting" student contributions, as opposed to the "boring ones," i. e., trivial cases. Moreover, some of the class discussion concerned the noted "interesting design" made by the aggregate contributions of each student's work. Her highlighting of particular features in artifacts of the MUIs was taken to infer that she aesthetically perceived the mathematics within generative activity.

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EXPLORING PENTOMINOES IN 7 DIVERSE PRE-K/K CLASSROOMS

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In this poster we describe how a lesson concept began in a single kindergarten classroom and was progressively adapted and taught in 7 very different environments using a variety of techniques and modifications to suit each unique classroom culture, guided by the Japanese lesson study framework (Perry, Lewis, & Hurd, 2009). The lesson was focused on solving the pentominoes challenge, which involved discovering the twelve unique shapes that can be composed with 5 unit squares lined up so that the sides are either perfectly aligned or do not touch at all (see Figure 1). This involves both grappling with, and coming to understand the concept of congruence, and also visualizing and conceptualizing the transformations of rotation and translation. The solution to this problem requires time, patience and effort and has not been explored and documented with students in this age range.

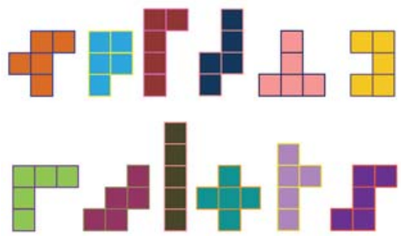

Figure 1: Pentominoes Shapes

The process of the design and uptake spanned two school years and emanated from an ongoing professional development research project based in Ontario, Canada in which teams of teachers, math coaches, early years specialists, and university researchers who collaborate to explore issues surrounding the learning and teaching of math in early years classrooms. We describe how a teacher-researcher team first conceived the lesson concept, and how it was designed and taught as a "public research lesson" in a single kindergarten classroom. From this point, the initial lesson underwent seven adaptations that, although maintaining the structure of the initial lesson, included a number of variations. These include incorporating technology, varying levels of difficulty, and in one case extending the lesson by using the pentomino shapes to include identifying 3D nets of open cubes from the configurations found.

The analyses we present in this poster will have two foci. One on student reasoning in the context of the pentomino lesson and the second on how this process supported teachers to gain new content knowledge in the area of geometry and to broaden their understanding of children's reasoning in geometry.

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MATHEMATICS TEACHER PERCEPTIONS ABOUT EXISTING DISPARITIES AMONG PUBLIC AND PRIVATE SCHOOLS IN JOHANNESBURG, SOUTH AFRICA

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Keywords: Teacher Beliefs, Mathematics Curriculum Disparities, Equity and Diversity

Beginning in 1948 with the rise of the control by the National Party and lasting some 40 years later until a negotiated settlement, the South African Apartheid was an era of racial inequality in all matters within the country including education. South African schools serving white students were overflowing with resources "while schools serving the black majority were systematically deprived of qualified teachers, physical resources and teacher aids such as textbooks and stationary" (Fiske & Ladd, 2005, p. 4). The government that took charge in 1994 post-apartheid mandated racial equality in education. In light of these expected changes, the purpose of this study is to explore disparities in mathematics teaching and learning in public and private schools. These disparities are viewed through the perceptions of mathematics teachers who navigate these two distinct spaces. This investigation was conducted during a cultural immersion component of a study abroad experience in Johannesburg, South Africa which targeted the examination of a township school in juxtaposition with a university sponsored magnet school post South African apartheid. This cultural immersion experience consisted of 6 mathematics researchers and educators being fully engrossed and embedded in the South African culture in an effort to uncover the implicit and explicit application of mathematics. The study poses the question "what are the perceptions of South African mathematics teachers regarding existing differences among resources, student population and curriculum availability among neighboring school systems." Using critical theory as a framework to promote critical consciousness and to break down the institutional structures that continue to persist long after the apartheid, we employ the use of an exploratory case study to investigate these insights. Data was triangulated using three methods. As a result of our analysis, major themes in both settings emerged encompassing teachers' issues with time management, segregation of English as second language learners, and pressures felt by unreasonable expectations for mathematics curricula. This research will contribute to the field as implications for future investigation rest in the exploration of miscellany of mathematics curriculum among schools in South Africa. Activists must organize and advocate for comparable mathematics education in the wake of desegregation and the South African apartheid. Our goal is to bring awareness to the disparities that exist within the post-apartheid South African school system and to encourage mathematics educators to advocate for equity and a balanced allocation of resources for both public and private schools. This advocacy will not only facilitate the requisite education that all students are entitled to, but it will allow all students to be competitive in the globalized marketplace. This study will also add to the theoretical base of the literature by highlighting and addressing the disparities that continue to exist in mathematics education and to remind us as educators that we must use our platform to speak on behalf of those who are not in a position to speak for themselves.

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DEFINING AND IDENTIFYING EXEMPLARY MATHEMATICS TEACHERS OF ENGLISH LEARNERS

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Keywords: Equity and Diversity, Middle School Education, Teacher Beliefs

Purpose of Study

This study attempted to identify and observe exemplary mathematics teachers of English learners (ELs) – to see what they were doing to support ELs. This poster explores some of the challenges with identifying exemplary teachers of ELs.

Methods

A district-level math coach, who selected the pool from which we recruited teachers, served as the focal participant. We drew on a single two-hour interview with the participant focus focused our coding on: exemplary teaching, supporting ELs, choosing exemplary teachers and obstacles to obtaining exemplary participants.

Findings and Discussion

There were a number of factors that the district math coach used to determine which teachers he selected. He began by looking at data at schools, then individual teachers' data, and finally the data of individual classes. The coach looked at the growth and proficiency of each teacher's students, but he also concentrated more specifically on the growth of ELs in the teachers' classes. The coach also knew or was familiar with many of the math teachers in the district because of this time in the district and his role at the district. He felt that he was able to look at the quantitative data he had available, but he was also able to identify teachers who he knew had in place a lot of the exemplary practices he had identified.

There are a number of issues in finding and selecting exemplary teachers. One of the first issues is how we define exemplary teaching. Notions of exemplary teaching involve empirical and interpretive evidence. While the district math coach cited the availability of test scores and classroom tools (calculators, word walls, prior work posted, models) as evidence of exemplary teaching, he also argued that less visible factors are also evidence of exemplary teaching, such as high expectations and understanding "where students are at." How do we identify exemplary teaching based on such invisible evidence? Are there ways to make these seemingly interpretations of teachers' "mindsets" more concrete?

While an administrator might drew on schools' and teachers' data, he could likely also make his decisions based on prior interactions with teachers, classroom observations, hearsay, and other myriad unknown characteristics, all somewhat subjective criteria. Exemplary mathematics teachers might be able to model practices for their peers. However, finding them and determining how exemplary they are can be a nebulous process.

Acknowledgements

This work is supported by a grant from the Spencer Foundation #20120086.

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EXAMINING DISCOURSE IN AN EIGHTH GRADE FACE-TO-FACE AND VIDEO CONFERENCING CLASSROOM

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Keywords: Classroom Discourse, Middle School Education, Technology

There is an established body of mathematics education research on classroom interactions (Bauersfeld, 1995; Cobb, Yackel, & Wood, 1995; Voigt 1995). A productive venue for observing growth and development of discourse in classrooms is to examine question types posed by the teacher and the evaluation and feedback students receive (Yackel & Cobb, 1996). Evaluation and feedback can also occur in distance education classrooms that use video conferencing (VC) to reach students that would not be possible otherwise due to location while providing highly qualified teachers and resources to support learning. The Initiation-Reply-Evaluation sequence is one such structure that can be used to identify discourse in the F2F and VC classrooms (Mehan, 1985).

The purpose of this study was to observe inquiry-based instructional practices of one teacher with one cohort of sixteen eighth grade students in two settings, F2F and VC, as a way of tracing similarities or differences in the types of questions and feedback/evaluation provided to students. Data collection and analysis aimed to explore the similarities and differences that might exist in the types of evaluations/feedback the teacher provided to students in these two distinct environments.

One 120-minute session from each of the F2F and VC classrooms were video recorded and transcribed. Grounded theorizing lead to emergent categorizes which included affirmations, probing questions to scaffold student thinking, asking students to explain their thinking, and asking students to either explain vocabulary, a phrase or question relevant to the given task.

Results indicated that the classroom environment affected the evaluation and feedback provided to students. Probing questions and asking students to explain their thinking were more customary in the F2F lesson whereas affirmations were used more often by the teacher in the VC lesson. Furthermore, probing questions in the F2F lesson was directed towards an individual whereas in VC, although the teacher spoke to one student, the teacher spoke to all students because the individual the teacher was scaffolding was located at a distance site.

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THE IMPACT OF PROFESSIONAL DEVELOPMENT ON INSTRUCTION AND BELIEFS: ONE TEACHER'S JOURNEY

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Keywords: Teacher Education-Inservice/Professional Development, Equity and Diversity, Teacher Beliefs, Instructional Activities and Practices

The Centering Teaching of Mathematics on Urban Youth (CTMUY) professional development program strives to inform teaching within the framework of culturally relevant mathematics pedagogy (CRP) (Rubel, 2010). CRP focuses on three distinct goals of mathematics teaching, namely: 1) developing conceptual understanding, 2) preparing students for participation in society, and 3) centering teaching on students' experiences (Ladson-Billings, 1995). This presentation will focus on a case study of one teacher's journey as she participates in CTMUY. The main focus of this research will be on the progression of her instructional practices, specifically in terms of culturally relevant mathematics pedagogy during her involvement in this group and also recognize any impact such participation may have on her beliefs in teaching mathematics and/or her beliefs in her students. Lastly, it is important to recognize and honor her experience as a Latina teacher, particularly through the lens of both race and gender.

Professional Development

This teacher participated in a 2-week summer institute in 2012, as well as two-hour meetings that occur once a month throughout the school year. All of these meetings are video recorded and audio taped. This teacher's lessons were observed 7 times, from the spring of 2012 through the fall of 2013. During the observations, careful notes were taken on instructional activities, which were then coded into total minutes for each separate type of activity conducted during the class. Visual displays of these activity codes help highlight changes in instruction. A lesson environmental rating rubric helped quantify aspects of the environment, such as engagement, support, analysis, knowledge, and discourse. Follow-up interviews were held after each observation to discuss the lesson. In-depth interviews were conducted to gain the perspective of the teacher herself, specifically in regards to her beliefs about mathematics, her students, and inquiry into her own experiences that impact her view of teaching and her teaching practices.

Changing Instruction

During the course of this professional development, teachers were provided with new tools to adapt their teaching practices, with the intent of providing their students with more challenging material, that connects to them in a more realistic and applicable way. Of particular interest in this study will be to see how these tools influenced this particular teacher's practices, as observed in the classroom, and changes in her beliefs in mathematics, and her beliefs in her students, as recognized by the way in which she speaks of her lessons, her students, and her view of teaching.

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INSTRUCTOR GESTURES IN PROOF-BASED MATHEMATICS LECTURES

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Keywords: Post-Secondary Education, Classroom Discourse, Instructional Activities and Practices

There is a growing body of research demonstrating that teachers regularly make gestures along with their speech during instruction (e.g., Alibali & Nathan, 2007). Yet, there is little research describing how non-verbal communication might influence students' opportunities for learning mathematics. The goal of the study described here is to begin to address this by describing the ways instructors' gestures are used to communicate mathematical ideas.

Using Arzaello's (2006) idea of a "semiotic bundle," and McNeill's (1992) classification of gestures, we describe each semiotic set of gestures as part of a larger *gesture space*. Our framework is focused on classifying the *use* of the gesture within the context of the gesture space, the semiotic bundle, and the entire lecture itself. We broadly classify gestures as: Indexical (used to indicate the position of an object or idea), Social (used for emphasis or to increase the instructor's immediacy), and Representational (used to represent a mathematical or non-mathematical object or idea). Each category is further described by sub-categories that align with communicational aspects of the lecture.

Our data corpus consists of six videotaped lectures of a standard junior-level abstract algebra class at a large university. Our findings include: (1) the meaning of a gesture depends on the semiotic bundle; (2) the components of gesture spaces take on a wide range of meanings; (3) indexical gestures can have complex referents; (4) temporal-spatial aspects can contribute to the complexity of referents of indexical gestures; (5) making indexical gestures from far away introduces ambiguity; (6) it can be difficult to distinguish between mathematical and non-mathematical gestures; and (7) gestures can be ambiguous, even to expert observers.

Our work adapts and extends McNeill's (1992) framework to be appropriate for advanced mathematics classes; it expands the range of indexical gestures, and focuses on communicational use rather than individual cognition. The analysis of the data reveals that even apparently simple gestures may require complex interpretative acts in order to decipher their meaning. In addition, many gestures are difficult to interpret, even for expert observers. These results add to our understanding of students' opportunities for learning from mathematics lectures.

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TEACHER JUDGEMENTS IN THE CLASSROOM: WHAT IS IT WE ATTEND TO?

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When meeting a group of students for the first time teachers can often make judgements, wittingly or not, about the students' ability. In this poster I examine clues which may enable teachers to make this judgement. Using tools of Conversation Analysis and an analysis of gesture, I show that certain features of the dialogue, along with the body language of the students, can offer clues to the understanding the students are developing. I suggest that the experienced teacher may develop a subconscious recognition of such traits, and outline some dangers that might arise in doing so.

Keywords: Understanding, Conversation, Gesture, Teacher attention

To Wittgenstein (1967), the possibility of a student understanding something will depend on whether the student can go on to write independently (143). If we relate Wittgenstein's ideas to a conversation, then understanding could manifest itself as the ability of a student to 'go on' with an exchange. By conversation, I use the definition of Davis (2001) as meaning an exchange of ideas of which the interlocutors are willing to change. To Harvey Sacks, through examining the intent of callers to a suicide hotline, the structure of a conversation was seen to be revealing, leading to the development of Conversation Analysis. In a similar way, McNeil (1992) has described gesture as 'a window to our thoughts'. It seems that there is a natural link to be found as both areas are about trying to make sense of the subconscious thinking that is occurring beneath the level of the spoken word.

Two grade 5 classes were videoed over the period of a school year while problem solving in groups. It was found that the way the group initiated their collaboration, and the structure of the discourse which ensued, had significant impact on both the group's success and the development of understanding indicated by the group members. In a similar way, the confidence of the student in their utterance was often revealed by the dynamic gesturing s/he engaged in. Such signals were revealed as potentially misleading if viewed in isolation, while a deeper analysis gave the potential for a richer analysis of the students' understanding.

As confidence grows, gestures become more dymanic and embodied.

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EXPANDING TEACHERS' EXAMPLE-SPACE: THE APPROACHES TO SELECTING AND GENERATING EXAMPLES IN HIGH SCHOOL MATHEMATICS

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Examples are not only as an integral part of mathematics, but also can be a pedagogically powerful tool for facilitating students' mathematical understanding. Studies show that examples offer insight into the nature of mathematics through their use in complex tasks to demonstrate methods, in concept development to indicate relationships, and in explanations and proofs (Bills & Bills, 2005). With respect to the pedagogical component, Watson & Mason (2005) argue, a key feature of examples is that they are chosen from a range of possibilities and it is vital that learners appreciate that range. Skemp's (1971) notion of "*noise*" – a teacher may use examples for illustrating certain ideas through his or her lens while a student may focus on its irrelevant features – suggests that teachers ought to develop awareness of choosing examples to minimize the "*noise*". In response, I propose to investigate teachers' selection and generation of examples for teaching high school mathematics in particular, considering that many high school teachers still struggle to choose appropriate examples from a wide range of resources to meet students' diverse needs.

In spite of the critical roles examples play in learning and teaching mathematics, there are only a small number of studies focusing on teachers' choice and treatment of examples. To address significant gaps in multiple research literatures that revolve around a lack of careful attention to teachers' selection and generation of example, the purpose of this study will discover the characteristics of teachers' selection and generation of examples for teaching high school mathematics. This inquiry will also seek the strategies for expanding teachers' example-space in the knowledge base of teaching. My overarching research questions are: What are characteristics of teachers' selection and generation of examples? What decisions do teachers make to select and generate examples? What strategies do teachers use to select and generate examples? And how do teachers expand their example-space?

My research will be conducted in a multiple case studies approach. By incorporating teacher interviews and classroom observations, this study will consist of a series of observational case studies where participant observation is the primary data-gathering technique (Bogdan & Biklen, 1992). The participants of the study will be five experienced secondary teachers who taught classes of different grade level and achievement level. I will meet the participants once a week throughout a semester, observing both algebra and geometry classes.

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Chapter 10: Technology

Research Reports

Brief Research Reports

Poster Presentations

TEACHER CANDIDATES' PERCEPTIONS OF MATHEMATICAL, COGNITIVE AND PEDAGOGICAL FIDELITY OF THE "FILL&POUR" VIRTUAL MANIPULATIVE

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Virtual manipulatives as cognitive tools, dynamic/interactive, Web-based representations and/or technology-based renditions, allow users to engage in mathematical meaning making. This research investigated teacher candidates' perceptions of the mathematical, cognitive, and pedagogical fidelity of Fill and Pour virtual manipulative. Findings suggest that the degree to which the mathematical entity is faithful to the essential mathematical properties of that item in the virtual environment has been granted by teacher candidates to virtual manipulative designers automatically, without closer examining of the mathematical, cognitive, and/or pedagogical fidelity. Further qualitative probing was carried through to better understand the nature of such assumptions.

Keywords: Mathematical Knowledge for Teaching, Teacher Education-Preservice, Technology, Problem Solving,

The ongoing challenge of reshaping mathematics education with integration of technology tools (NCTM, 2000; Ball, 2003; Alagic, 2003, 2004) has led to consideration of pedagogical, mathematical and cognitive fidelity of virtual tools as well as concern about teacher preparation to utilize quality math based technology (e.g., Dick, 2008; Bos, 2009). This study examined teacher candidates' perceptions of the mathematical, cognitive, and pedagogical fidelity of a certain problem solving virtual manipulative tool. The study is an illustration of challenges that we face in teacher preparation programs related to use of technology in mathematics classrooms.

Virtual Manipulatives: Mathematical, Cognitive and Pedagogical Fidelity

This section provides a brief review of the research literature related to representational and cognitive characteristics of virtual manipulatives and their mathematical, pedagogical and cognitive fidelity as they relate to quality teaching and learning mathematics.

Virtual Manipulatives as Cognitive and Representational Tools

Virtual manipulatives (VM) are typically designed as Java or Flash applets. They are often modeled after existing manipulatives such as geoboards, tangrams, base ten blocks, fraction bars,…They allow learners to relate concrete models to abstract mathematical concepts. Dorward (2002) defined virtual manipulatives as "computer based renditions of common mathematics manipulatives and tools" (p. 329) while Moyer, Bolyard, & Spikell (2002) defined them as "... an interactive, Web-based visual representation of a dynamic object that presents opportunities for constructing mathematical knowledge" (p. 373). Therefore virtual manipulatives must be web-based and they must afford users to interact/manipulate these dynamic objects which shows potential for learning and teaching mathematics interactively (Moyer-Packenham, Salkind, & Bolyard, 2008). Other names for virtual manipulatives include interactive math applets and Mathlets (JOMA Web Site 2006).

As cognitive tools, virtual manipulatives support, guide, and extend the thinking processes of their users. They are based on the principle that learners need to make their own meaning of new

concepts. Jonassen (1992) defined cognitive tools as "generalizable tools that can facilitate cognitive processing" (p.2) and "make effective use of the mental efforts of the learner" (Jonassen, 1996, p.10). Virtual manipulatives have properties that go beyond their counterparts, physical manipulatives. They provide learners a way - often both visual and verbal including hints, feedback with pop-ups and help features - of representing their understanding of a new concept/phenomena and how it relates to their existing understanding of the same idea (Derry, 1990; Alagic & Palenz, 2004; Moyer, Niezgoda, & Stanley, 2005; Zbiek, Heid, Blume, & Dick, 2007).

When learning complex new mathematical ideas it helps to interact with multiple representations (Cox and Brna, 1995). Virtual manipulatives as *external representations* may help learners' ability to transfer among multiple representations, to extend what has been learned in one context to new contexts, developing representational fluency (NRC, 2000; Zbiek, Heid, Blume, & Dick, 2007). Virtual manipulatives help develop *representational fluency* by linking symbolic, pictorial and concrete representations. Not making connection between different representations may even inhibit learning (Ainsworth, Bibby & Wood, 2002). *Representation standard* (NCTM, 2000) articulates representations as crucial components in facilitating learners' conceptual understanding of concepts and relationships. The term representation applies to both processes and products. Furthermore, the same term is used for product/process that are observable externally (external representations) as well as to those that ensue internally (mental models). In this context, virtual manipulatives contribute to richness of representations in a unique way, as they often comprise multiplicity of representations – visual, dynamically visual, symbolic and verbal (Goldin & Shteingold, 2001; Alagic, 2003).

The literature review of the existing research indicated that learners using virtual manipulatives demonstrated improvements in mathematics understanding and achievement (e.g., Bolyard, 2006; Moyer et al., 2005; Lee, Silverman, & Montoya, 2002; Lee & Jung, 2004). A study described by Reimer & Moyer (2005) about third graders using virtual manipulatives during a 2-week long unit on fractions revealed a statistically significant improvement in students' conceptual knowledge. Some research suggested that use of virtual manipulatives might have a positive effect on student engagement and developing procedural and conceptual understandings (e.g., Moyer, Niezgoda, & Stanley, 2005; Raphael & Wahlstrom, 1989). However, mathematical meaning is not necessarily explicit in use of manipulatives and VMs cannot be expected to improve learners' understanding; to be effective, virtual manipulatives use must involve active cognitive processing by learners (Ball, 1992; McNeil, 2007; Roberts, 2007; Smith, 2009). Many believe that virtual manipulatives can be particularly helpful to students with language difficulties, including English language learners (Moyer, Niezgoda, & Stanley, 2005).

As illustrated, there are a number of studies about teachers' and pupil's use of virtual manipulatives demonstrating the unique characteristics of these tools for developing conceptual understandings and teaching mathematics. However, there is no much research about teacher candidates' understanding of both potential and pitfalls in using virtual manipulatives.

Mathematical, Cognitive and Pedagogical Fidelity of Virtual Manipulatives

The mathematical fidelity of a virtual manipulative refers to faithfulness to the defining properties of a mathematical concept or a phenomenon that manipulative is attempting to represent. Simply, we ask, is a representation provided via virtual manipulative true to the mathematical concept that it is trying to represent? "In order to function effectively as a representation of a mathematical "object," the characteristic of a technology-generated external

representation must be faithful to the underlying mathematical properties of that object" (Zbiek, Heid, Blume, & Dick, 2007, p. 1174). For an example, the answer to a division problem using the calculator is truncated, but the calculator is giving you the most feasible answer for the place values it allows. "Technology's limitations are a constant concern for mathematical fidelity …" (Bos, 2011, p. 4404).

The cognitive fidelity of virtual manipulatives is related to user's cognitive engagement while developing patterns and making connections that were only possible in one's mind. A familiar examples are those applets that allow change of a parameter over time (using a slider) resulting in changes of a graph, allowing a mental process recognizing a resulting series of graphs. In other words, cognitive fidelity leads to question, Is the concept better understood due to the user's capability of acting on the related virtual object? (Bos, 2008). Cognitive Information Processing Theory as well as Dual Coding Theory (DCT) is based on the premise that two interconnected systems and their sets of codes (visual and verbal) are the base of information processes and storage. These collections of codes include both visual and verbal codes, which can represent letters, numbers, or words. According to these theories, facilitating learning with functionally independent both visual and verbal codes has cumulative effects on their recall (Clark & Paivio, 199; Moyer-Packenham, Salkind, & Bolyard, 2008).

Table 1: Pedagogical, Mathematical, and Cognitive Fidelity Chart to Determine Degree of Fidelity (Adapted for VM from Bos, 2009, p. 526.)

	Low Fidelity	Medium Fidelity	High Fidelity
Pedagogical	VM interactivity is not obvious; not intuitive, confusing to use. Not appropriate for the concept being represented. VM hard to access.	Manipulation/interactivity is not intuitive, but after reading the directions it is doable. May be easier to do without the technology.	VM is appropriate for activity. Mathematical manipulation is doable, encourages active involvement, and requires little or no training.
Mathematical	Math concepts behind VM either too simplistic or too complicated. Patterns are not revealed. Not real-life related. Leads to rote memorizing rather than conceptual understanding.	VM patterns lack predictability. Mathematical significance is minimal. Application of mathematics unclear.	VM is mathematically correct. Maximizes the use of patterns. Believable and livable use of mathematics.
Cognitive	VM static with no opportunities to formulate and test conjectures. Patterns do not make sense. Difficult to relate to prior knowledge; confusing and unyielding.	VM provides limited opportunity to explore and test patterns. Patterns require either minimal or too much manipulation to make sense of the mathematical concept behind it.	VM can be used to construct and deconstruct, test, and revise to understand the patterns and structure of concepts. Interacting leads to the patterns and greater depth of understanding.

The pedagogical fidelity of virtual manipulatives refers to the degree to which a learner believes that a virtual manipulative affords her to act mathematically in ways that correspond to

the nature of mathematical learning via discovery. Zbiek, Heid, Blume, & Dick (2007) define *pedagogical fidelity* as "the extent to which teachers (as well as students) believe that a tool allows students to act mathematically in ways that correspond to the nature of mathematical learning that underlies a teachers practice…" (p. 1187). It is about allowing learners to learn mathematics by doing – facilitating the creation of objects, acting on objects, explicating evidence - without being distracted by low quality of the applet or other technology limitations (Bos, 2009; Dick, 2008).

Degree of mathematical, cognitive and pedagogical fidelity may vary for virtual manipulatives used in mathematics. The Table 1 is a VM adaptation of a chart for determining degree of fidelity that Bos (2009) used in the study of mathematical, cognitive and pedagogical fidelity for mathematics related websites. Furthermore, Bos (2008) described a study that suggested technology high in mathematical and cognitive fidelity lead to greater student mathematical achievement. Selecting a virtual manipulative for instructional purposes requires careful consideration of the mathematical, cognitive, and pedagogical fidelity of the virtual manipulative as well as its externalized representation will affect mathematics learning and teaching (e.g., Zbiek, Heid, Blume, & Dick, 2007).

Threshold Concept and a Critical Incident: *Fill and Pour* Virtual Manipulative

The course *Mathematics Investigations* (Alagic, 2006) is designed to investigate demands of digital technologies integration and inquiry-based approaches to teaching and learning of mathematics, while bridging the gap between two traditional courses: Mathematics for Elementary Teachers and Instructional Strategies in Elementary Mathematics. The main thrust of the course are three Problem Sets assignments, each focusing on one big mathematical idea/concept developed around an open-ended, real-life related and challenging problem. The problem set consists of 6-7 additional problems scaffolding "down" the main concept. Each student is designing a unique collection of problems and submitting their work individually. However, students are encouraged to discuss collaboratively their work. Each problem set utilizes technology tools in an essential way. At the end of the problem set, a required metacognitive reflection reports about students' thinking during the process of the problem set design.

A *threshold concept* refers to realizing a new and previously unreachable way of thinking about a certain concept or phenomena. It represents a transformed way of meaning making or interpreting something relevant to learner's progress in understanding (Land, Meyer, &Smith, 2008). In the context of virtual manipulatives it refers to the fact that "Students do not necessarily see on the screen what is "evident" [to the software designer and maybe the teacher] (Dreyfus, 2002, p. 23). The following is an example of what I consider threshold concept for the students in Mathematics Investigations class because it lead to deeper understanding of the need to consider fidelity of virtual manipulatives in general.

Critical incident. During class activities teacher candidates were asked to investigate virtual manipulative *Fill and Pour* (Figure 1), with problems of the following type:

> You have a soda fountain but only two unmarked containers (one 5 ounces and one 9 ounces) that can be filled or emptied or poured back and forth as needed. Your goal is to get precisely the target amount (7 ounces) in one of the containers.
> (http://matti.usu.edu/nlvm/nav/frames_asid_273_g_3_t_4.html).

In a matter of minutes, every teacher candidate in the class was "filling" and "pouring". Some candidates quickly got answers, others struggled. As some candidates were losing their patience, a candidate offered a hint, "Just fill and pour in the same direction, eventually you will

get an answer". Very quickly, the idea spread around, everyone was showing off their cherries and ducklings (Alagic, 2006).

Teacher candidates were asked to record the process in some way. Most of them described in long paragraphs the process they followed. Some used some kind of algorithmic representation to capture the process. One of the students neatly captured the following:

Let's mark containers with L (left) and R (right). To solve this problem we can
Fill L => (transfer to) R
Fill L => R
Empty R
Transfer L (1oz) =>R
Fill L => R
Fill L => R
Empty R
Transfer L (2oz) =>R
Fill L => R
Fill L => R
L has 3oz

The same candidate provided a description of the opposite process in a similar manner.

So, correct solution is available through a sequence of automatic "fill" and "pour" steps without a deeper consideration how the problem should be solved and what kind of mathematical reasoning needs to be involved. This example was an inspiration for the study of mathematical, cognitive and pedagogical fidelity of *Fill and Pour* virtual manipulative.

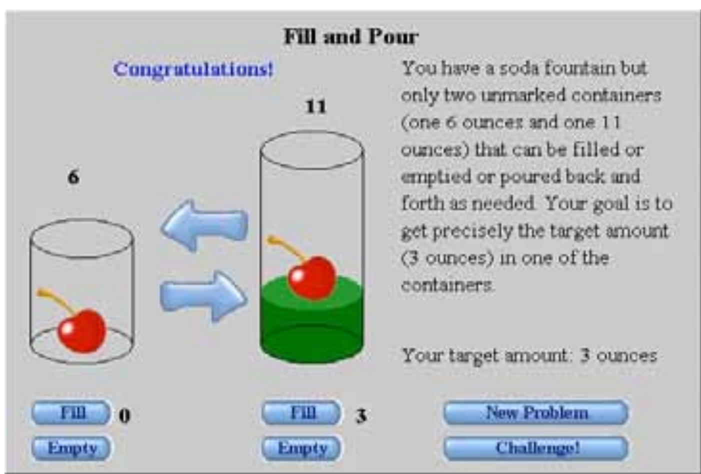

Figure 1: Virtual Manipulative Fill and Pour applet from NLVM (2006)

Mathematical, Cognitive and Pedagogical Fidelity of the *Fill and Pour* Virtual Manipulative: Teacher candidates' Perceptions

Mode of Inquiry and Analysis

Across 6 semesters, total of 224 teacher candidates voluntarily participated in this case study. Participants were elementary teacher candidates enrolled in the Mathematics Investigations class. Background information on the participants included their prior use of manipulatives and technology. All participants had prior experience with physical manipulatives and some experience with basic virtual manipulatives, such as virtual base-10 blocks, geoboards, pattern

blocks, and tangrams as these were utilized in the required class preceding Mathematics Investigations. The critical incident described above was an inspiration for investigating students discovery and understanding of the fidelity of VMs in the following 6 semesters.

Participants were asked to solve problems posed in *Fill and Pour* in two ways, using (a) paper-pencil method, and (b) virtual manipulatives. Half of the participants would first complete paper-pencil method and follow by using virtual manipulatives. Other half would work first with virtual manipulatives. This assignment was followed by metacognitive reflections required in order to better understand how students make meaning of fidelity based on this experience. Those metacognitive reflections represented data analysed for this study. Furthermore, students were given the chart (Table 1) without fidelity terminology and asked to identify one cell in each row that corresponds to their understanding of how virtual manipulative *Fill and Pour* can be characterized.

Results. 75% of participants discovered that there are three types of problems in the *Fill and Pour* virtual manipulative: (i) following the pattern of fill and pour works in both directions (from smaller to larger or larger to smaller container – the required amount can fit in either container); (ii) following the pattern of fill and pour works only in one direction (required amount cannot fit in the smaller container); and (iii) impossible problems (required amount odd number and containers hold even number of ounces). Qualitative probing of those participants that did not reach this conclusion showed that either they did not know they needed to figure that out or that all the problems they tried were of the similar nature.

Out of these 168 (75%) participants, 120 were in the groups interacting with VMs first, before attempting paper and pencil approach. The following table illustrates percentages of students identifying mathematical, cognitive and pedagogical fidelity based on only descriptors provided in the Table 1 (no fidelity terminology used, yet).

Table 2: Fidelity Chart with Number of Participants Selecting Pedagogical, Mathematical, and Cognitive Fidelity Descriptors Based on Their Interaction with *Fill and Pour* Virtual Manipulative

Related to	Low Fidelity	Medium Fidelity	High Fidelity
Teaching using *Fill and Pour*	14	10	200
Understanding Mathematics behind *Fill and Pour*	5	39	180
Recognizing problem solving patterns in *Fill and Pour*	40	27	157

Limitations. More attention is necessary to understand challenges that student have about paper-pencil approach and bridging the two representations– paper-pencil approach vs. virtual manipulative *Fill and Pour* approach to problem solving of this type of problems.

Discussion and Conclusions

This study provides an initial examination of participants' perceptions of mathematical, cognitive and pedagogical fidelity based on provided descriptors. The findings illustrate that for significant majority of participants, perceptions that using virtual manipulatives to solve problems is advantageous regardless of mathematical, cognitive and pedagogical fidelity. For 80% of participants using virtual manipulative to solve problems of the type *Fill and Pour* is a better choice (high fidelity) regardless of the fact that process is a akin to rote memorizing rather than to conceptual understanding. 90 % of participants would teach such problems using virtual

manipulatives rather than paper-pencil method. Some participants commented in their reflections that "paper-pencil" is too complicated when they know to follow the pattern "fill and pour in the same direction, eventually you will get an answer". In terms of cognitive fidelity, reflections and qualitative probing seem to identify patterns as types of problems based on (i) the required amount can fit in either container; (ii) required amount can fit only in the larger container; and (iii) impossible problems. However, it is important to notice that this reasoning is coming from somewhat automatic play with virtual manipulative and not from patterns recognized in paper-pencil problem solving.

Follow up classroom discussions helped clarify some of the misconceptions inherent in mathematics thinking and learning related to problem solving utilized via *Fill and Pour* virtual manipulative. This provides further confirmation that simply using manipulatives without follow-up conversations may lead to deepening some misconceptions; not making connection between different representations may even inhibit learning (Ainsworth, Bibby & Wood, 2002).

Implications and Further Research. This study, although focused on only one VM, illustrated potential challenges in using dynamic/interactive web-based tools in terms of mathematical, cognitive and pedagogical fidelity for three types of learners– students, teachers and instructional designers.

There are many follow up questions to be studied, some of which the author is already pursuing. This study can be considered as a pretest for understanding students conceptualization of of mathematical, cognitive and pedagogical fidelity for a specific virtual manipulative. How this threshold concept might be used to sharpen teacher candidates' inquiry into fidelity of virtual manipulatives in general? What is the effect of teacher candidates' knowledge of problem solving on their ability to conceptualize mathematical, cognitive and pedagogical fidelity for virtual manipulatives?

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SMALL GROUP TO WHOLE GROUP SHARING OF IDEAS IN AN ELEMENTARY GEOMETRY CLASS UTILIZING TECHNOLOGY

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This research reports on data and analysis investigating the sharing of ideas and strategies from the level of a small group to the level of the whole class (and vice versa) in a fourth grade mathematics classroom utilizing digital technology. This research takes a sociocultural perspective positioning meaning and learning as shaped by mediational means which can drive or suppress the transition of ideas and strategies from the small group to the whole class and vice versa. The three potential mediational forces are: social, technological, and pedagogical. The digital technology of focus is a research version of Sketchpad® Explorer on the iPad. Using this specific technology, in conjunction with connectivity, students can interact with and manipulate dynamic constructions in a small group that can then be sent to a shared classroom space to be analyzed and discussed at the level of the whole class.

Keywords: Technology, Classroom Discourse, Elementary School Education

Introduction

This research paper seeks to investigate the transition of ideas and strategies across levels (from small group to whole class to small group) in an elementary mathematics classroom utilizing a research version of Sketchpad® Explorer for the iPad that supports collaboration with connectivity. Sketchpad Explorer is based on The Geometer's Sketchpad® (Jackiw, 1991, 2009) and allows the user to drag and manipulate mathematical sketches on the iPad. Using this specific technology, in conjunction with connectivity, students can interact with and manipulate dynamic configurations in a small group that can then be aggregated into a shared classroom space to analyze and discuss at the level of the whole class.

In the transition from small group work to whole class sharing and discussion, where the small group collaborative activity transitions to a whole class level, there exists the potential to share ideas and strategies that can potentially influence the future work of small groups. For this research, we investigate the role of mediation in this transition by focusing on mediational forces influencing what is shared or suppressed at the whole class level. This perspective on mediation assumes tools, peers, and the teacher are not neutral, rather the tools, interactions between students, and between student and teacher shape the mathematical meaning within the interactions and from the interactions. Three possible mediational forces of focus are: social, technological, and pedagogical.

Theoretical Framework/Background

From a sociocultural perspective, in a classroom, meaning and learning are products of social activity. In a small group collaborating on a task utilizing digital technology to create a mathematical object, the social interaction amongst the group members is fundamental to individual learning because "human action typically employs 'mediational means' such as tools and language, and that these mediational means shape the action in essential ways" (Wertsch, 1991, p. 12). Wertsch continues referring to the student(s) as 'individual(s)-acting-with-mediational-means' such as language and other psychological tools such as words, gestures, and

diagrams, opposed to speaking only of the 'individual(s)'. Digital technologies used in the classroom can support students' exploration of mathematical ideas and can mediate their social interactions with both teachers and peers. A common pedagogical practice for small group work includes bringing the small groups back together for a whole class discussion of the task and artifacts created. In a connected classroom, this can take place quickly and with ease allowing multiple cycles of small group to whole class to small group work within a single class period.

Classroom technologies incorporating connectivity have shown to have great impact on learning and opportunities to learn in the classroom (Brady, White, Davis, and Hegedus, 2013; Hegedus and Kaput, 2004; Stroup, Ares, and Hurford, 2005). Classroom connectivity offers opportunities for both group and individual learning experiences in the classroom structured around participation. Embedded in the design and intention of software technologies such as NetLogo (Wilensky, 1999) and SimCalc MathWorlds® is the recognition that a student in a classroom is not learning or building knowledge in isolation, rather alongside other students with varying mediators at play. As such, the more recent designs and uses of classroom connectivity include pairs or small groups of students working together on a task or creating a mathematical object within a technological environment.

Goos, Galbraith, Renshaw, and Geiger (2003) found that the technology used in the classroom, graphing calculators, could facilitate communication and sharing of knowledge in both private and public settings highlighting that "learning becomes a process of appropriating tools that change the ways in which individuals formulate and solve problems" (p. 75).

Enyedy (2003) focused research around student small group work (pair) and the relations between private work on a computer and public work during whole class discussion time. This research positioned mathematical activity and mathematical understanding as a function of participation and called the individual computer simulation work and the whole-class discussions two social configurations of the classroom. Within the intersection of these two social configurations, Enyedy examined the ways in which individual and social processes are mutually constitutive. The analysis included two case studies in which examples were found tracing a student's reasoning back to the interactions with other students, the teacher, and the software used. Additionally, the author found that agreements within the pair setting potentially caused conflict with other pairs' work when presented at the whole class level.

This finding is consistent with findings from the SimCalc MathWorlds work in which small group work is aggregated at the whole class level via connectivity (Hegedus and Roschelle, 2013) and agreement is not always achieved at the level of the whole class. This resonates with work of Pea (1994) who notes that communications using multimedia-learning environments are open to multiple interpretations in how they express representations of phenomena and so there will be a need for interaction, negotiation, and repair amongst participants.

A difference between these research investigations exists in how and what student small group constructions are made public. For Enyedy, student verbalizations of strategies or findings are made public, whereas with connectivity, verbalizations of ideas or strategies along with artifacts of student constructions are made public. These public interactions have implications for everyone in the class as a learning opportunity for future mathematical investigations or revisions of strategies.

White, Wallace, and Lai (2012) developed collaborative activities for classroom networks to investigate the intersection of individual student engagement and collectively constructed artifacts that are made public via connectivity. The authors position their research within a largely social view of learning which takes words, gestures, and inscriptions of the small group

as different modalities working together for social sense-making. Students in three ninth-grade Algebra 1 classes worked in pairs, each controlling a point on a grid space, and were asked to create specific linear functions. Analyses of the video data of the pairs focused on whether the strategies were implemented by a single student in the pair, or jointly by the pair of students. Two important findings from this work includes: 1) the authors' finding that the students shifted emphasis from asynchronous actions to synchronous actions in the shared graphical space, and 2) not all pairs in the class displayed the same degree of interaction and negotiation with their partner (White, Wallace, and Lai, 2012). This is somewhat due to the relations between the mathematical objects and social relations; each shape and are shaped by each other via the mediational forces. Our broader research aims to identify the mediational forces at play for various groups and how these forces drive the interaction between students and from small group work to whole class discussion.

As Moreno-Armella and Sriraman (2010) have pointed out, research into the nature of tool mediation is a crucial goal for the further development of the field of mathematics education, explaining that over time, tool mediation has become an integral part of human intellectual activity. With the addition of connectivity, dynamic mathematics, and the affordances of the iPad device, which includes multimodal interaction, the mediational role of this technology within the classroom in which students are working collaboratively in groups, human action (student action) will shape and be shaped by these mediational means.

Methods

Descriptions

The data presented in this report was collected as part of a larger study. The participants in the main study were three fourth grade teachers and their students (n = 61) which focused on teachers' implementing multi-modal activities focused on multiplication using Sketchpad® Explorer for the iPad. The three teachers collaborated with the researcher team and together the geometric activities using Sketchpad Explorer were designed.

This paper focuses on a single class of one teacher. The teacher is State certified and has been teaching for eleven years. She had never used an iPad in the classroom before but participated in the development of the activities and training workshops focused on implementing the activities. There were twenty students in her fourth grade classroom.

The design of the activities and the teaching sequence included the following types of activities: 1) introduction to the activity, 2) small group work in which students collaborate using the iPad to manipulate/ configure a mathematical object which will get sent to the teacher, 3) individual paper work in which students respond to questions and detail their strategies for configuring the mathematical object, and 4) whole class discussion usually initiated by the teacher.

Data Collection and Analysis

Data collection. The intervention activities were implemented in each classroom once a week for three weeks during which classroom video data and student work were collected. Three video cameras captured the lesson in each classroom; one roaming camera focused on students, one roaming camera focused on the teacher, and one stationary camera with a wide angle lens in a back corner to get a visual of the whole classroom. For the purposes of this paper we focus only on the classroom video data from week 3 which introduced connectivity.

Classroom activity. The overall main goal of the activity is an exploration into partial products. There are four activity tabs within the Sketchpad sketch (see Figure 1). Each tab represents a specific element in the sequence of the activity. The goal for the first tab is for

students to create one shape from the partial products, a quadrilateral. The square and rectangular pieces in this tab are a fixed size so while the arrangement of objects can be different across groups, the total sum of the area will be the same.

In the second activity tab, the square and rectangular pieces are again a fixed size but the four pieces cannot be placed together to form a quadrilateral. Students must determine which partial product is not fitting and why. Students will write a multiplication sentence to represent the total area. While configurations across groups can be different, the total sum remains the same.

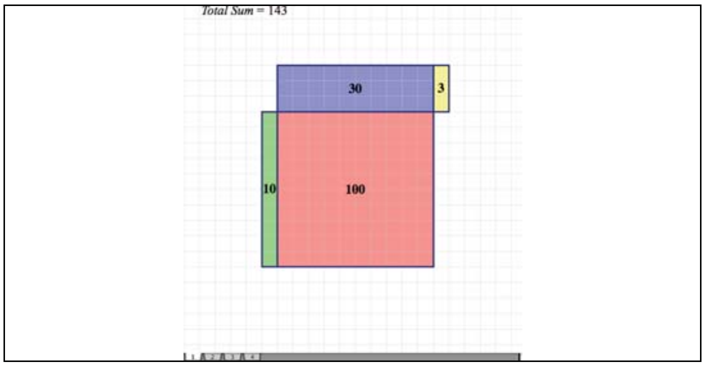

Figure 1: Screen Capture of the Activity 3 Window

In the third activity tab, students are asked to create a rectangle from the four pieces given; three fixed sized pieces, and one editable piece. The three fixed pieces are the same pieces from activity tab 2 and the one editable piece initially has the same area as in tab 2. Students will be writing a multiplication sentence to represent the total area and the total area will be the same for all students. Here, they are investigating the partial products of 11x13.

In the fourth tab, students manipulate and arrange a new set of partial products into a rectangular configuration. There are four pieces and only one is dynamic. Students are asked to write a multiplication sentence for their configuration. In this activity tab, it is possible for the total sum to be different across groups depending on how students configure the pieces; there are three possibilities. Students are asked to explain their strategy on a paper worksheet provided.

Over the four weeks the teacher of the class tried to support group work by physically rotating the position of each student in a group relative to the iPad. Students worked in groups of three (and one group of two), with one iPad per group. The iPad primarily remained on the center desk and the three students changed their seats several times during a single activity. This ensured every member of the group could reach the iPad with his or her hands/fingers. Of course the iPad did not *always* remain on the middle desks, students did pick up the device to show each other or move the device to make an edit. During this lesson incorporating connectivity the student seated at the center desk submitted the work to the teacher under his or her name.

Analytical methods. The analysis presented in this paper focused on the classroom video data from one classroom engaged in the fourth tab of the activity, described above. The aim of this analysis was to investigate the question: as student ideas and strategies are developed and shared within a small group, how do the mediational forces drive the transition to bring these ideas and strategies to the level of the whole class?

Video from both the teacher and student camera angles was transcribed. Included in the transcripts were utterances, pauses, noted gestures with descriptions, and a description of referents when known. The video of the class was watched alongside and in conjunction with the creation of the transcript and was watched during the process of coding the transcript. We coded

completed transcripts for instances of transitions of an idea or strategy, 1) where a student idea or strategy was presented within the small group, and 2) where the small group work transitioned to the whole class during the whole class discussion. A second cycle of coding identified instances where the social, technological, and pedagogical mediational forces existed.

The following conventions were adopted in the transcription: (a) conversational utterances are numbed sequentially (as a result, portions of the transcript will not begin with utterance one); (b) non-verbal information including action gathered from the video is included in brackets and italicized; and (c) [inaudible] indicates utterances that could not be heard. Due to space restrictions, ellipses (…) were used to denote a break in the transcript.

Findings

In this section we wish to present a classroom scenario that illustrates an instance of how a group strategy from one small group is taken up by another small group and shared with the class. We refer to this as borrowing since the strategy taken up by a small group can potentially be returned at the level of whole class via connectivity and the sharing of student work. With this episode we illustrate: 1) the transition from the level of the whole class to the level of the small group, and 2) the transition from the level of the small group to the level of the whole class through the lens of mediational forces, which shape the action.

Small Group Borrowing of a Strategy

The following transcript pieces are centered on one small group (Sabrina, Pablo, and Nick) and the whole class discussion of their work. All names are pseudonyms. Sabrina is seated at the center desk with her group members on either side of her. In an interaction during small group work, the teacher asks the group about their strategy. Sabrina says the strategy for their configuration comes from an idea that Fred, a student in another group, had previously done. She says,

> Um, well, when I was making this, I thought of what Fred was doing, and, um, I was thinking, like, how I was gonna, like, build it. And I wanted it to be different from everybody else's 'cause I know everyone was building it upwards and I wanted... [horizontal].

Sabrina refers to how she wanted to configure the rectangle and her own specific strategy for this creation. Her language is very personal referring to how she was going to build it. The teacher interrupts Sabrina, but from her configuration in Sketchpad Explorer and her verbalization it seems she specifically tried to do something different from her classmates because Fred and his group had done something different from his classmates during a previous activity tab earlier in the class period. While it is unclear how Sabrina determined what her classmates did for this activity tab (in order to do something different), this strategy can be seen as both mathematical and social. It is mathematical because, for Sabrina, Fred and his group have illustrated through sharing their work at the whole class level, that there can be more than one mathematical solution, or configuration in this case. It can also be thought of as mathematical because Sabrina presumably would have had to determine what some of her classmates had done and how she can complete the task differently from the other groups.

Her rationale for this strategy may have been social: wanting to stand out from others in the class, or to receive special attention from the teacher. During the previous activity tabs the teacher focused the whole class discussion on only a few arrangements specifically looking for "something different", a phrase she used several times during the class period. When previously discussing the work of Fred and his group members during the whole class discussion, the teacher identified Fred as someone who "thinks outside of the box". However, completing the

task with this strategy of finding something different from what other small groups are doing has mathematical underpinnings, even if Sabrina's configuration was mediated by social and pedagogical factors.

Small Group to Whole Class

After student work had been sent to the teacher, the teacher showed a configuration she herself had created and asked if students had the same total sum: 143. The teacher asked who has a sum smaller than her own and members from six of the seven small groups raise their hands. She asks who had a total sum larger than 143 and only Sabrina and Pablo raise their hands. Sabrina had succeeded in creating a configuration different from all her classmates with a total sum larger than the total sum of all configurations. The teacher directed attention to Sabrina's arrangement, both by referring to it as "interesting" (line 1501) and by putting the arrangement in the public space to be discussed with the whole class.

1489 *T*: Why would the total sum change a little bit depending on how you did the shapes?

1490 *James*: It's because you put that one bigger than all of...

1491 *T*: Is it bigger than all the others?

1492 *James*: No, it could be smaller and bigger. It's bigger than ours.

1493 *T*: Okay. Alright. Why else?

1494 *Haley*: Because you could move the blue, um, the purple shape anyway you want.

…

1499 *T*: How many people have a total sum smaller than that?
[Members from six of the seven small groups raise their hands.]

1500 *T*: Anyone have larger than that? *[Sabrina and Pablo raise their hands.]*

1501 *T*: Interesting. Okay. Let me bring yours up. I wanna take a look at it. Alright.

…

1508 *T*: Okay. *[Posts the work of Sabrina, Nick, and Pablo.]* This is what they have.

1509 *James*: One bigger. Are you serious? *[James turns and looks at Sabrina's group as he responds. He is referring to the total sum as being 1 value larger than the total sum of the teacher's configuration.]*

1510 *[Sabrina smiles.]*

1511 *T*: Why is it one bigger?

1512 *Pablo*: Um, because our, the blue, … is a sum of thirty-three.

1513 *T*: Okay. So, this whole thing here *[Points to blue rectangle on projected screen]* has thirty-three little squares in it?

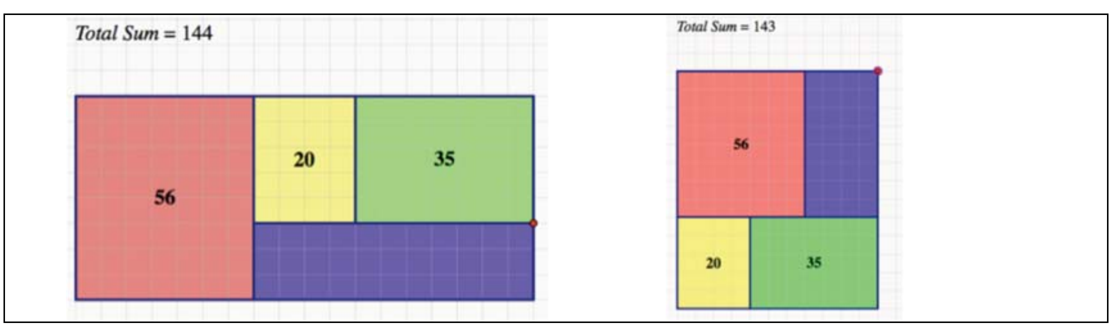

Figure 2: Sabrina, Pablo, and Nick's (Left) and Teacher's (Right) Configurations

With the affordance of connectivity and the teacher questions (lines 1493, 1499, and 1500), the technological mediational force and the pedagogical mediational force act in tandem to bring about this interesting mathematical instance. By making multiple groups' work public the class has unearthed these two configurations, which differ in total sum by 1 square unit. The teacher presses for justification by asking why the configuration is "one bigger" than the previous configuration. She presses for further explanation after a student has identified the editability of the purple piece as being responsible for the different total sums (line 1494).

The social force initially mediated Sabrina's action with this activity; she pulled on a strategy used by Fred to come up with an arrangement that was different from her classmates'. As Sabrina verbalized to the teacher, this difference originated in the shape of the configuration, to make a configuration horizontal while others were making the configuration vertical in appearance. The teacher acted with a mathematical rationale in bringing Sabrina's group configuration to the level of the whole class, this configuration was different, both in shape and in size. While it was likely not anticipated by Sabrina (or Pablo, and Nick) when creating their arrangement that it would have the largest total sum, Sabrina's borrowed strategy of trying to create something different from her peers led to an interesting mathematical learning opportunity. In the following transcription piece, Pablo is offering an explanation.

1527 *T*: Okay. So, it's one bigger because this is thirty-three. That's the only reason?
1528 *James*: Yup. Pretty much. No, wait. If you combine...
1529 *Pablo*: No, because that, the multiplication fact that goes with it is eleven times three, and that one hundred, and that for thirty-two would be, that would have a total sum of everything for one thirty, one forty-three. Because of eight times four.
1532 *T*: Okay.
1533 *Pablo*: Because if it has four...
1534 *David*: I don't get how it's like one bigger.
1535 *Sabrina*: Because it just wants to be. *[Sabrina turns her head and directs this response to David before looking back at the teacher.]*
1536 *T*: 'Cause it just wants to be? Okay. Alright. *[Sabrina laughs.]*
1537 *T*: Okay. Let's see. *[Posts a different group's work.]*

Pablo's explanation is not entirely clear although he does seem to reference his group's purple piece (with a size equal to 11 times 3), the teacher's purple piece (with a size equal to 8 times 4) and the total sum of the teacher's configuration (line 1529). It is not known whether Pablo's explanation is clear to the teacher, it is not clear to us nor is it clear for David who follows up on what Pablo says by asking why is it one [unit] bigger. Sabrina, the primary creator of this configuration, responds with "it just wants to be." This non-mathematical response does not resolve the mathematical question at hand but seemed to serve as a break in the activity as the teacher chose to move on to review the work of another small group. During this portion of class time the small group configuration remained static. The technology no longer played a mediating role in shaping the mathematical meaning; that ended when the students sent their work. The technology existed as a display space only at that point. The student interaction was mostly directed toward the teacher, and only a few students took part. At this point in the activity the interaction between the student(s) and the teacher via talk and reference to a static image was the only form of mediational means. This potentially can be limiting as could be argued in the above analysis, as the teacher moved the lesson forward by bringing up another groups' work.

Conclusions

In this classroom episode the strategy used by Sabrina, and the elevation of her group configuration was mediated by social, technological, and pedagogical means, which shaped the student action. A strategy was borrowed from another small group who manipulated their configuration to be different from others' in the class. The Sketchpad Explorer activity along with peers supported the small group exploration into configuring something different. An interaction between connectivity and the questions posed to the class by the teacher elevated the configuration of Sabrina's group to the level of the whole class. The interaction of mediational means played a critical role to get to that point which resulted in a rich learning opportunity for the class. Future work will focus on varying activities to explore the conditions under which such forms of mediation are suppressed or enhanced.

Acknowledgments

This material is based in part upon work supported by the National Science Foundation under Grant Number NSF REESE REC-0835395#. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

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EQUIVALENCE AND EQUATION SOLVING WITH MULTIPLE TOOLS: TOWARD AN INSTRUCTIONAL THEORY

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This research study was specifically concerned with the development, testing, and revision to an instructional theory for studying the mathematical concepts of equivalence and equation solving with multiple representations and multiple tools. Following a design research approach, a collaborative teaching experiment was conducted with a ninth-grade algebra teacher in which instruction was guided by a specifically designed sequence of tasks, techniques using paper-and-pencil and computer algebra systems (CAS), and theory on a hypothesized progression of learning. Retrospective analyses of data informed revisions to a resulting progression of learning and activity sequence that are being tested with pre-service secondary mathematics teachers.

Keywords: Algebra and Algebraic Thinking, Technology, Learning Trajectories (or Progressions), Design Experiments

Background and Purpose

Mathematical equivalence is a central topic in mathematics, and the domain of school algebra specifically (Chazan & Yerushalmy, 2003). Current expectations for school mathematics include justifying the equation solving process by reasoning about equivalence of equations (Common Core State Standards Initiative [CCSSI], 2010). Empirical studies on students' understanding of the equal sign have shown that students hold a diversity of understandings from operational to relational (Matthews, Rittle-Johnson, McEldoon, & Taylor, 2012), and that these understandings matter for students' abilities to solve equations (Knuth, Stephens, McNeil, & Alibali, 2006).

A specific motivation for this study was to link research and practice on the use of computer algebra systems (CAS) to support students' development of representational fluency in solving equations. CAS are a representational toolkit that facilitate the creation of, manipulation of, and movement between symbolic, graphic, numeric, and verbal representation types. For this study, representational fluency was defined as the ability to create, interpret, transpose within, translate between, and connect multiple representations in doing and communicating about mathematics. This construct helps to characterize students' conceptual understanding of mathematics because it deals with cognitive connections across representations of mathematical objects.

Some of the specific gaps in the literature that this study sought to address include the articulation of aspects of CAS-equipped classroom learning environments. For instance, more research is needed to specify the roles of representations and a balance between CAS and paper-and-pencil tools (e.g., Kieran & Saldanha, 2008). Additionally, a focus on the role of the equal sign may shed light on student thinking and understanding with respect to equivalence and equation solving (Knuth et al., 2006). This is also tied to the use of language in learning the nuances and relationships among expressions, equations, equivalence, and solutions (Kieran & Drijvers, 2006). True to the goal of linking research and practice, designing instruction based on students' learning processes pushes the field forward in defining a theory of teaching (e.g., Sztajn, Confrey, Wilson, & Edgington, 2012). On the topic of understanding of equivalence and equation solving, research studies on student learning processes that undergird a meaningful

instructional sequence are emerging (Kieran & Drijvers, 2006; Kieran & Sfard, 1999; Rittle-Johnson, Matthews, Taylor, & McEldoon, 2010).

This report focuses on a research question that guided a component of research conducted by Fonger (2012): *What means of support seem to facilitate students' development of representational fluency in a combined CAS and paper-and-pencil environment?* To address this question, following Cobb (2003), the means of support were organized around four main aspects: (1) the instructional tasks and activity sequence, (2) the tools students would use, (3) the activity structure of the classroom, and (4) classroom expectations. Inspired by Kieran and Drijvers (2006), the focus of this report is on the tested and revised sequence of tasks, techniques, and theory. The tasks specify the mathematics that students do and the techniques are the ways in which CAS and/or paper-and-pencil are used to accomplish some mathematical goal. The theory is an empirically based conceptual progression of expected tendencies that students will encounter as they formalize their understandings through engagement with tasks, tools, techniques, and other interactions (cf. Kieran & Drijvers, 2006; Sztajn et al., 2012).

Research Design and Theoretical Frameworks

A design research approach (Gravemeijer & Cobb, 2006) was followed in order to effectively design for, test, and revise an empirically based instructional theory. The three phases of this research were: preparation for the experiment, conduct of a teaching experiment and ongoing analyses, and retrospective analyses. During the first phase of the research a conjectured instructional theory was posited based on a review of relevant literature, briefly summarized above and elaborated by Fonger (2012). Conjectured and revised elements of the instructional theory are elaborated in the next section.

The second phase of the research involved a teaching experiment conducted in collaboration with a classroom teacher (Cobb, 2000). The teacher had four years of experience teaching courses in algebra and geometry and had used non-CAS graphing calculators in her instruction. The setting of the research was an algebra classroom at a large urban public high school. During the five-week teaching experiment, the teacher taught all lessons and the researcher served as a participant observer in the classroom. Consistent with the research design and the researcher's epistemological foundations, an interpretive lens on classroom interactions guided the ongoing analyses of classroom activity; the classroom practices and students' mathematical activity and cognition were seen to co-evolve over time (Cobb & Yackel, 1996).

The process of carrying out and testing the instructional sequence involved three aspects that occurred on a daily basis. First, daily cycles of classroom implementation were guided by hypothetical learning trajectories (Simon, 1995) in which learning goals, learning activities, and hypotheses of students' learning process were tested. The second component was reflective in nature with the directive to link research and practice. The teacher and researcher engaged in thought experiments or directed reflections every day after class for 45-90 minutes with student work, lesson notes, and task-technique-theory frameworks to explore the questions "is instruction meeting the set-out goals?" and "how should we improve the next lesson to account for student understandings demonstrated in the most recent lesson?" On Fridays we met for an hour or more to address the questions "how did the weekly teaching sequence support or differ from the conjectured instructional theory?" and "what revisions should be made for next week's plan?" After each thought experiment, the researcher identified critical moments that occurred during the classroom episodes and summarized those in daily and weekly summary files. Critical moments were identified as segments of a teaching episode that seemed to well-support or contradict the proposed learning goals and means of support being tested.

The third phase of the instructional experiment included creating new hypothetical learning trajectories for the next class session that took into account the daily instruction. Figure 1 illustrates the overarching instructional theory and the reflexive relation between theory and practice, which guided ongoing analysis and experimentation.

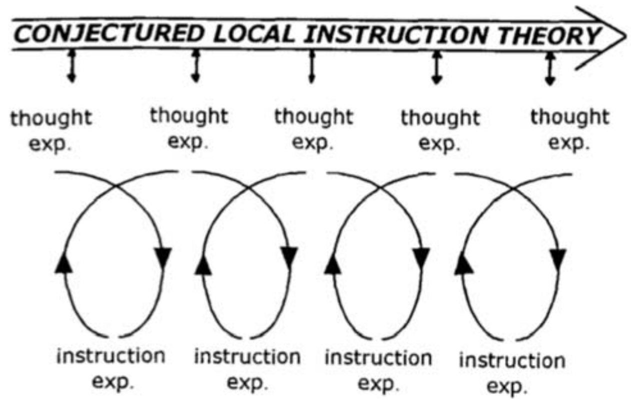

Figure 1: Daily cycles of experimentation were guided by an overarching instructional theory (Gravemeijer & Cobb, 2006).

The final phase of the research involved retrospective analyses of all data. This included: pre- and post-test data from select students of the classroom, initial and final interviews with select students (discussed by Fonger, 2012), teacher and student classwork, classroom and individual student video, observational field notes, daily and weekly class summaries and debriefing notes from collaborative and individual thought experiments about daily instructional episodes, and weekly debriefing session notes. Both Studiocode and HyperRESEARCH tools supported management and analysis of all data.

The data analysis method during the retrospective analysis stage resembled what Gravemeijer and Cobb (2006) described as a constant comparative method (Glaser & Strauss, 1967). In this process, conjectures about the instructional theory that had been identified during ongoing analyses were later confirmed or refuted based on evidence from a given classroom episode; these conjectures were then tested again against the subsequent episode. This process of confirming and refuting conjectures was repeated until all teaching episodes and critical moments were analyzed in chronological order. Critical moments were coded according to the main aspects of the instructional theory: Activity Sequence, Activity Structure, Learning Progression, and Classroom Expectations. As code names within each category were refined (e.g., clarified description, new code name, etc.), all data in that category were re-coded according to the updated code categories. By design, all instructional theory components and descriptions were revised throughout the ongoing and retrospective analyses.

Results: An Emerging Instructional Theory

An instructional theory involves two main aspects: learning processes guided by learning goals, and means of support for tasks, tools, classroom culture, and the role of the teacher (Gravemeijer & Cobb, 2006). The learning goals that defined the direction of the instructional design were to: (a) develop representational fluency with linear expressions and equations, (b) understand the equal sign as an equivalence relation, and (c) solve linear equations as a process of reasoning about equivalent equations. These three goals guided the sequence of learning activities or tasks. CAS and paper-and-pencil techniques were specified in the activity design as

a way to accomplish these goals. These techniques were summarized as an activity structure that was grounded in literature on the coordination of tool use (e.g., Kieran & Saldanha, 2008) and a multi-representational lens on doing and communicating mathematics (e.g., Kieran & Sfard, 1999). The theoretical component was intertwined with the tasks and techniques in support of the learning goals.

Tasks

To describe the overall sequence of activities, both the enacted and revised versions of the sequence of tasks are given in Table 1. Note that the changes that are made here are reflective of the retrospective analysis. One main area that was added to this sequence of tasks was the Cartesian Connection.

Table 1: Enacted and Revised Sequence of Tasks

Unit	*Enacted*	*Revised*
1	Multiple Representations of Equivalent Expressions	The "Cartesian Connection" in Graphs, Symbols, Tables, and Words Equivalent and Non-Equivalent Expressions in Graphs, Tables, Symbols, and Words
2	Equations are Equivalence Relations that are Sometimes, Always, or Never True	Equations are Equivalence Relations that are Sometimes, Always, or Never True
3	Solving Linear Equations with Multiple Representations	Identifying Solutions Sets of Linear Equations in Graphs, Tables, Symbols, and Words
4	n/a	Equivalent Equations have the Same Solution Sets

Ongoing analysis during the teaching experiment alluded to possible weaknesses in students' understanding of the Cartesian Connection (Fonger, 2012). As described by Moschovich et al (1993), this understanding is seen as an important pre-requisite skill for coming to understand the relationships between graphical, numeric, symbolic, and verbal representations of equations and solutions. The second area of the task structure that this affected is described in row 3 of Table 1; more attention needs to be given to the role of identifying solutions in multiple representations. Finally, the five-week teaching experiment was not long enough to fully test the mathematical topic of equivalent equations, thus is considered to be an additional unit to this sequence of activities (row 4 of Table 1).

Technique

Many of the techniques that were tested and revised in the experiment were process oriented with respect to the role of multiple representations (and supporting the development of representational fluency), and the role of using multiple tools. Table 2 gives a summary of these techniques; those with a double-asterisk (**) were added during the retrospective analysis of data.

Table 2: Techniques that Structured Classroom Activity with Revised Descriptions

Techniques that Support Representational Fluency	
Translate **	Create and interpret the meaning of a target representation with respect to a source representation of a different type.
Transpos	Create and interpret multiple representations within one representation type.

e**	
Interpret	Convey the meaning of the act/result of creating a tool-based representation; a basic, quick remark, thoughtful but not deep.
Connect	Give a correct interpretation of an invariant feature across multiple representations or types.
Generalize**	Make a generalization across several representations or representation types (e.g., abstract notation $a(x + b) = ax + ab$).
Justify**	Representations are used to confirm or ascertain a particular result or conclusion; "use representations as justifications for other claims" (Sandoval, Bell, Coleman, Enyedy, & Suthers, 2000; reasoning must be present); formal/rigorous explanation, objective, based on set practices.
Techniques that Support the Coordination of Multiple Tools	
Anticipate	Predict the result of creating tool-based representations.
Act	Create a representation and possibly explain the process of how one works within or moves between tool-based representations or types.
Reflect	React to or think deeply about representations/representation types with respect to equivalence and/or equations; heavy thought, detailed in response, subjective and developmentally oriented.
Reconcile	Negotiate differences between CAS and paper-and-pencil representations.
CAS Check**	Use the CAS to check or verify paper-and-pencil representations (often times within Symbolic representation type).

The notion of an action-consequence principle (see the techniques at the bottom of Table 2) is proposed an appropriate way to coordinate the use of mathematics technology in which tool-based results are first predicted with paper-and-pencil, then executed with technology, then reflected upon (cf. Dick & Hollebrands, 2011). The notion of reconciling differences between CAS and paper-and-pencil representations is elaborated by Kieran & Saldanha (2008). The role of CAS to check paper-and-pencil results was added to the activity structure during the ongoing analysis when it became evident as a common classroom practice in which results were consistent between multiple tools (no reconciling was needed).

Theory

The sequence of instructional tasks was guided by a synthesis of several conceptual progressions into a single learning progression (see Table 3). Two studies were central: (a) research with secondary students who studied a unit on equivalence and equation solving with CAS and paper-and-pencil (Kieran & Drijvers, 2006) [elements A1, B-symbolic only, C, D1, in Table 3], and (b) research with middle grades students who studied a unit with a multi-representational lens on equivalence solving with graphing calculators and paper-and-pencil (Kieran & Sfard, 1999) [elements A2, B, D2 in Table 3]. The elements that were added to the learning progression during the teaching experiment to support student understanding of the learning goals are denoted with a single asterisk * in Table 3; all other components were determined a priori to the study.

Table 3: Hypothesized Modifications to a Learning Progression

Element	Description

A1	*Connecting and generalizing the quantitative, visual, and verbal with symbols*. Symbolic expressions generalize numeric, graphic, and verbal patterns by allowing for compact, abstract notation.
A2	*Different representations/representation types can signify the same object*. Different representations/representation types of the *same* linear expressions and/or equations signify the same relationship, pattern, or function from different yet complementary perspectives.
B	*Equivalence of expressions from multiple representations.* Expressions are equivalent if they define the same relationship, pattern, or function.
C	Domain and range restrictions may arise in contextual situations and should be considered when determining equivalence.
C1*	Role of Equal Sign: "=" assigns variables rules/names for patterns.
C2*	Role of Equal Sign: "=" expresses identity between equivalent expressions.
CC 1*	If a point *P* is on the line *L*, *P* makes the equation of *L* true.
CC 2*	If a point *P* makes the equation of *L* true, *P* is on the graph of *L*.
D1	*Solutions to equations can be determined by equality of expressions*. Linear equations are relations between linear expressions that are sometimes, always, or never equal in value. Thus linear equations have one, infinitely many, or zero solutions, respectively.
D2	*Solving equations in one variable is conceptualized as a comparison of two functions.* Linear equations in one variable such as $ax + b = cx + d$ for real valued parameters a, b, c, and d, can be solved for the variable x by comparing the functions $f(x) = ax + b$ and $g(x) = cx + d$ for the value of x that makes the equation $ax + b = cx + d$ true. Graphical, tabular, or symbolic methods can be used.
E	*Equivalence of equations*. Equations are equivalent if they have the same solution set. Represented graphically, solution sets of equivalent equations are x-coordinates of the intersection points in the coordinate plane. Represented in tables, solution sets of equivalent equations are the inputs for which the outputs are the same.

The aspects of the learning progression for which there is more to explore include the relationship in language between expressions, equations, and functions. During the teaching episodes the students became accustomed to using the language of an equation as being sometimes, always, or never true. Another aspect of the learning progression that was not tested with this particular group of students was that of equivalent equations [element E in Table 3]. This element is still included in the learning progression because of the importance of this topic as specified in CCSSI (2010).

Implications and Ongoing Research

Consistent with the goal of linking research and practice, the task-technique-theory framework was used as a tool to help support research-practice links in the design of instruction. The articulation of a research-based sequence of tasks can be used to guide instructional decisions about the implementation of mathematical content as espoused in CCSSI (2010). Moreover, for classroom environments that support the coordinated use of multiple tools, a focus on techniques can be used as a lesson design principle to guide instructional moves. This aspect of the emerging instructional theory is also consistent with the mathematical practice of using appropriate tools strategically. Techniques focused on the role of representations are also well

suited for promoting a more conceptual understanding of mathematics as long as there is a strong emphasis on interpretation and connections.

The proposed instructional theory is characterized as emergent because based on the design research paradigm (Gravemeijer & Cobb, 2006), the testing of a learning progression should occur in several iterations in which the implementation of the first iteration elicits revisions and informs the next iteration of testing. The rationale for an iterative design is to build theory over time, not to just empirically tune “what works” but to elicit general design principles that can be used to inform other instructional design along a meaningful learning progression. Consistent with this research design, current research is underway that is focused on testing and refining the instructional theory. The task, technique, and theory elements are being integrated into a curriculum for pre-service secondary mathematics teachers. One way in which the ongoing research will extend the study discussed here is by the use of polynomial and rational expressions and equations for examining equivalence and equation solving, with more specific attention to the role of the equal sign with the newly emerging frameworks for understanding that knowledge (e.g., Matthews et al., 2012). Another way in which future research can extend and strengthen the current study is by investigating a broader range of mathematical standards so as to link topics from elementary grades on algebraic thinking to topics across the middle and high school that require justification and reasoning about equivalence and equation solving.

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THE RELATIONSHIPS BETWEEN STUDENTS' USES OF TECHNOLOGY, THEIR TASKS AND THE ARGUMENTS THEY CREATE

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Recent policy initiatives indicate that students should develop expertise in constructing viable arguments, engage in meaningful mathematical tasks, and strategically use appropriate tools, including technology. However, little research has been conducted that specifically examines the interplay between these three activities. The purpose of this paper is to describe the relationship between eighth grade students' uses of technology, the tasks in which they engage and how these relationships influence students' arguments. Findings indicate that specific types of tasks elicited certain uses of technology, and there seemed to be a relationship between the use of technology and the structure of the argument.

Keywords: Technology, Geometry and Geometrical and Spatial Thinking, Reasoning and Proof

The National Council of Teachers of Mathematics (NCTM) (2000) states that reasoning and proof are "fundamental aspects of mathematics" (p. 56) and "technology is essential in teaching and learning mathematics" (p.24). The Common Core State Standards for Mathematical Practices (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) agrees with NCTM indicating that students should develop expertise in constructing viable arguments and strategically use appropriate tools, including technology. Both organizations would agree that students should be given the opportunity to engage in activities in which they create and critique mathematical arguments and effectively use technology. Research has shown that the activities and tasks on which students work influences what they can learn (e.g. Stein, Engle, Smith & Hughes, 2008). Research has also indicated that the nature of a task may influence the use of technology and the structure of the argument (e.g. Hollebrands, Conner & Smith, 2010). However, research has not been conducted that specifically examines the interplay between argument, task and technology. The purpose of this paper is to describe these relationships for eighth grade students investigating properties and theorems associated with triangles while working in a dynamic geometry environment (DGE).

To briefly outline our investigation, we used Toulmin's (1958/2003) model of argumentation to examine students' mathematical arguments, which has been used by several researchers (e.g., Hollebrands, Conner & Smith, 2010; Lavy, 2006; Stephan & Rasmussen, 2002) in mathematics education. Once the arguments were diagrammed, we classified the students' uses of technology and the task on which they are working. For task, we coded each argument using the Mathematical Task Coding Instrument (MaTCI) (Heid, Bloom, Hollebrands, & Piez, 2003). To code for use of technology, we developed a list of codes based on the ways in which the students were using technology while engaged in tasks. Once coding was complete, we analyzed the data looking patterns and themes that emerged from the argumentation episodes, the types of tasks on which they were working and their use of technology.

Methods

As part of a larger study, the lead author conducted a two-week classroom teaching experiment with one eighth-grade mathematics class with the lead author serving as the

instructor. The study was conducted with students of varying ethnicities and socio-economic statuses at an urban public middle school in the southeast United States. The focus of the unit was the development of students' understanding of triangles, their properties, and associated theorems. For this class, technology played an integral role. Students used a DGE, The Geometer's Sketchpad (Jackiw, 2001), to explore and investigate concepts.

Participants

During the study, the instructor placed the students in pairs so they have the opportunity to have mathematical discussions with their partners while working on the tasks. These discussions were the primary focus of the study's analysis. Pairs were chosen rather than larger groups to maximize the opportunities for students to interact with the mathematical task while still having peer-to-peer discourse. From these pairings, four were purposefully selected to be the focus of the data collection based on recommendations made by the regular classroom teacher regarding their willingness to verbalize their thinking and ability to work well together. Of these four, three pairs (Amy and Judy; David and Erica; and, Heather and Mary) (all names are pseudonyms) were selected for analysis based on attendance.

Data Collection and Analysis

Data collection consisted of video and audio recordings, screen-capturing software that recorded students' uses of technology, and artifacts including students' written classwork, homework, quizzes, and exams. Of the eight class meetings, we analyzed the small group and whole-class discussions that centered on three topics (the triangle inequality theorem, the relationship between the magnitude of the side lengths and the measures of the opposite angles, and the properties of different types of triangles). The lead author created transcripts based on the audio and video recordings of the whole class discussions and the discussions for each of the three pairs of students during small group work. We used these transcripts, along with the artifacts, to identify reasoning episodes.

Using these data, we proceeded to make use of Toulmin's (1958/2003) model of argumentation to analyze and diagram students' arguments. Toulmin decomposed an argument into six components: claim (the conclusion whose validity is being established), data (the facts being appealed to as the foundation of the claim), warrant (the link between the data and the claim), backing (circumstances in which the warrant would otherwise be invalid), qualifier (confers the strength of the warrant), and rebuttal (circumstances in which the warrant does not hold). Figure 1 shows how these six components (data, claim, warrant, backing, qualifier, and rebuttal) fit together. Data is provided or constructed and "so" a claim is made based on this data. This claim can be made based on this data "since" the warrant. The warrant is relevant "on account" of the backing. The claim is valid "unless" the rebuttal occurs.

To identify an episode of argumentation, we looked for a student's verbal claim. After identifying each episode, we created a description of the argumentation for that claim which included the participants' words (from the transcripts) and actions including the students' gestures and uses of technology. Then, we diagrammed the argument according to the model developed by Toulmin (1958/2003). Each argument consisted of data, claim, and warrant, and, if contributed or needed, backing, qualifier, and rebuttal. Many times, students did not explicitly provide each component and we made inferences based on their previous work and use of technology. In these cases, we noted the inference in the diagrams by placing a "cloud" around the inferred component. If the component was known or spoken, we used a "box" to outline it. We attributed each component to the students, the teacher, or some combination of the teacher and students.

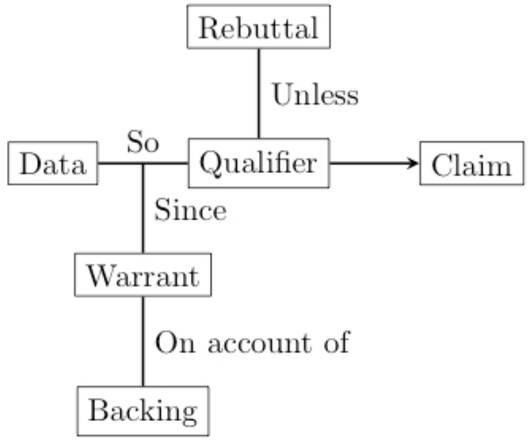

Figure 1. Toulmin's (1958/2003) Model of Argumentation

In order to determine a relationship between students' uses of technology and the task on which they are working, we first coded each argument using the Mathematical Task Coding Instrument (MaTCI) (Heid, Bloom, Hollebrands, & Piez, 2003) and then each task by the way in which the students were using technology. Heid and colleagues developed the MaTCI in order to "capture the sequence of tasks students took on as they worked with technology on conceptually oriented tasks" (p. 3). The authors designed this instrument to code tasks posed by the teacher, curriculum, and/or students. The task a student is working on may not always be the one assigned (Heid, et al., 2003). In fact, when students are working on a large task such as one posed by the curriculum, students often develop their own intermediate tasks to assist them in reaching their goal (Heid, et al., 2003). These intermediate tasks include developing and testing conjectures, producing intermediate data, describing what they see on their screen, and interpreting and justifying their results from their uses of technology. The focus of our coding was on the intermediate tasks that may or may not be posed by the curriculum. As shown in Table 1, the MaTCI framework accounts for these types of tasks by providing two levels of coding; categories and, within each category, subcategories. MaTCI was initially developed based on students' work with functions and algebraic tasks. Although the categories are general, the given explanations and examples are specific to this content. Thus, we modified the explanations to account for our students' work in geometry, specifically triangles.

For each episode of argumentation, students are developing claims based on data they collected. Thus, they are performing actions to accomplish a goal or, in other words, completing a task. Therefore, we coded the task for each argument using the Level 2 categories of the MaTCI framework. Many times, the students switched tasks within a single episode of argumentation. In these instances, we coded the additional tasks(s) using the same Level 2 categories of the MaTCI framework. For each argumentation episode, two researchers coded the same set of tasks individually and then compared their codes and resolved any conflicts by agreeing to a single code.

Table 1: Adapted MATCI (Heid, et al., 2003) Framework

Level 1 Category	Level 2 Category	Explanation
Concept	Identify	The task is to identify the name of an object when the characteristics of that object or an image of that object are presented to the student
	Describe	The task is to describe what you see (visual to perception) describing or characterizing a mathematical object.
	Elaborate	The task is to extend a previously stated idea by refining, specifying or clarifying it.
Product	Produce	The task is to create an object given a set of input values.
	Generate	The task is to create a rule or procedure for a particular problem/example/instance.
	Predict	The task is to describe what might happen under certain conditions in a novel situation. Students are asked to come up with a conjecture.
	Generalize	The task is to describe a relationship across multiple problems/examples/instances that holds for an entire class.
Reasoning	Corroborate	The task is to provide additional evidence that what is given or found is true. [Determine if something might be true or false].
	Justify	The task is to provide a logical argument for why something happens.

Once the initial and secondary tasks were coded, we proceeded to code each task in terms of the students' use of technology. For each task code, we examined the ways in which students used the technology while engaged in the task. We chose to use the task rather than the episode of argumentation as our unit of analysis because an episode may contain multiple tasks. Thus, there is potential for multiple uses of technology within a single episode as well. We developed these technology codes (see Table 2) based on our observations and revisited and refined them using a constant comparative method as described by Glaser (1965). Finally, we recorded each of task and technology codes in a spreadsheet and we used pivot tables to look for patterns and themes between mathematical tasks and technology use.

Table 2: Students Use of Technology Codes

Category	Explanation
Drag	Episodes where the use of the drag feature was used in the argument.
Appearance	Episodes where assumptions were based solely on visual aspects of the diagram
Measures	Episodes where numbers resulting from the use of commands in the "Measure" menu were used in an argument
Drag/Measures	Episodes where the use of the drag feature in conjunction with measures being displayed on screen were used in the argument
Imagined Dragging	Episodes in which the imagined movements of a figure were used in the argument
None	None of the above
Constructing	Setting up the problem situation such as dragging (adjusting) sliders

Results

The results reported in this paper are based on data taken from activities in which the goal was for students to develop an understanding of mathematical concepts and theorems related to triangles. The students used pre-constructed sketches to explore these concepts. While working on these tasks, the three pairs of students created arguments of various structures, engaged in a myriad of tasks, and used the technology in different ways. Four relationships emerged in our analysis of the students' use of technology and the tasks on which they were working.

Predict-Appearance

In over half (17/31) of the argumentation episodes with an initial task code of *predict*, the students' use of technology was the *appearance* of the figure on the screen. When students made predictions, they were not actively using the technology other than as a referent. For example, David and Erica were working on the triangle inequality task and were using the pre-constructed sketch to create segments of lengths 3, 4, and 4. Erica adjusted the sliders on the screen to create the segments, but she did not drag the segments to determine if a triangle was formed. David exclaimed, "Of course it's not going to work." This argument is illustrated in Figure 2.

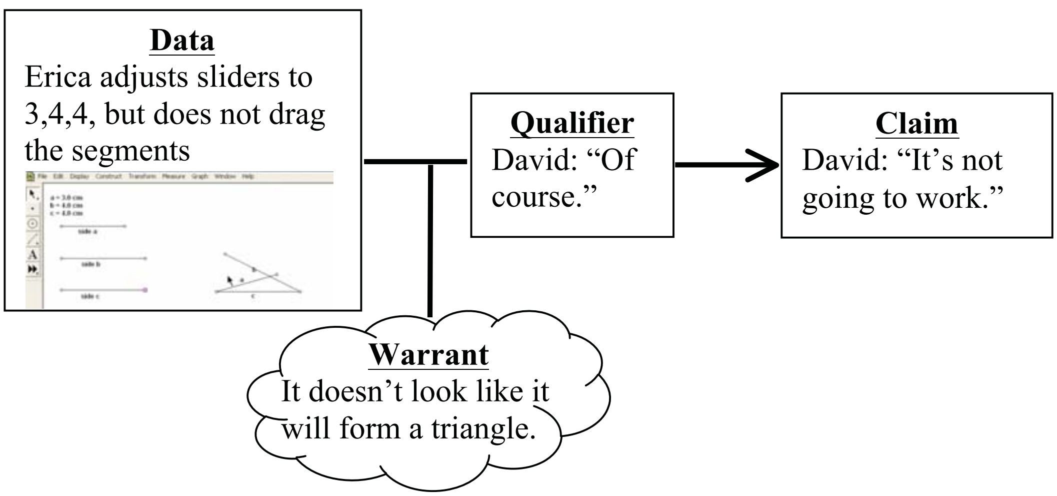

Figure 2: David and Erica's Argument with predict task and appearance

In this argument, David *predicted* whether the segments of lengths 3, 4, and 4 would form a triangle. His claim was based on the *appearance* of figure on the screen because Erica had yet to drag the segments to form a triangle prior to David's claim. David did not provide an explicit warrant for his claim, but we infer that his reasoning was based on the fact that the figure on the screen did not look like it would form a triangle.

Describe-Measure

At times, students created arguments as they provided *descriptions* of what they saw on the screen. During these episodes, the students would often use the *measurement* feature to provide a more accurate description of the figure. For example, David and Erica were determining the longest side of obtuse triangle *ABC*. On his activity sheet, he wrote that *AB* is the shortest side. Using the technology, David previously measured the side lengths to be m*AB* = 4.90 cm, m*BC* = 2.54 cm, and m*AC* = 3.01 cm. Erica displayed the obtuse triangle on the screen, which included the measures of the lengths of the sides. Erica exclaimed, "No it's not. That's wrong. That's

Martinez, M. & Castro Superfine, A (Eds.). (2013). *Proceedings of the 35th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Chicago, IL: University of Illinois at Chicago.

wrong." She told David, "Look at the measurements…*AC*." David affirmed this claim stating, "That's right." This argument is illustrated in Figure 3

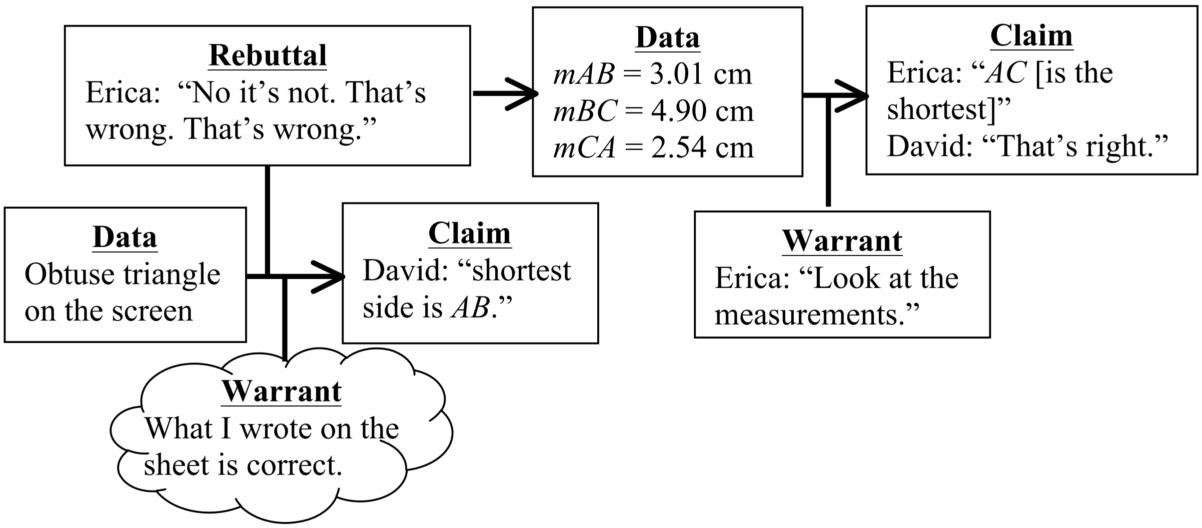

Figure 3: David and Erica's Argument with Describe Task and Measures

We coded the task for this episode as *describe*. David and Erica were not producing a triangle nor were they attempting to classify the triangle. Rather, they were providing a description of the relative lengths of the sides. To make this description, the students used measures. As students became more comfortable with the technology, they tended to stop using the measure feature and began making claims based on the appearance of the figures. Thus, when students were working on description tasks, their use of technology shifted from relying on measures to basing their claims on appearance. However, the students would use measures if prompted to do so.

Produce-Drag

Many of the tasks the students worked on when using the technology were *produce* tasks. In these tasks, students attempted to produce a figure based on certain criteria. The teacher may have given the criteria to the students via the activity sheet, general instructions to the class, or specific instructions to the pair of students. The students, at times, generated their own criteria to produce a figure. While producing the figure, the students often made use of the *drag* feature. For example, Heather and Mary were determining whether segments of lengths 3, 4, and 4 would form a triangle (similar to Erica and David in the argument illustrated in Figure 1). Heather adjusted the sliders accordingly and dragged the endpoints. As Heather dragged an endpoint of the figure and was unable to immediately form a triangle, Mary stated, "No [it will not form a triangle]." Heather replied, "No it's too long. But that's weird, wouldn't *b* be able to reach 3?" She continued to drag the endpoints and was able to form a triangle. She exclaimed, "Wait, wait, I got it." In this argument, Heather and Mary were attempting to produce a triangle with side lengths 3, 4, and 4. To accomplish this task, Heather used the drag feature of the technology.

Generalize-None and Justify-None

When students worked on tasks in which they were creating *generalizations* or *justifying* their ideas, they generally did not actively use technology. Instead, the students would often appeal to mathematical definitions, theorems, and formulas; patterns they noticed in their

recorded data; or, the authority of another individual. For example, one of the arguments common to all pairs of students in structure was in response to the question on the activity sheet, "Why was it impossible to construct a triangle with some of the given lengths?" One student, Erica, responded, "One's [segment] too long or too short." This argument is illustrated in Figure 4a. The question asked the students to generalize across the examples. The data used by the pairs of students to support their claims were their answers to the examples sets of segments on their activity sheet. To gather this data, the students used technology. However, when responding to this question, the data had been previously collected and their reasoning was not based on their active use of technology, but on the product of their previous uses. Thus, Erica did not actively use technology in this argument.

Similar to generalization tasks, students rarely used technology when engaged in justification tasks. In fact, of the seven episodes coded justify, six did not include the use of technology. For example, the teacher posed the question, "Can we have two right angles in a triangle?" The class, including Judy, responded, "No". When asked for her reasoning, she simply stated, "It would look like a square." This argument is illustrated in Figure 4b. To answer and justify her response to the teacher's question, Judy did not use technology.

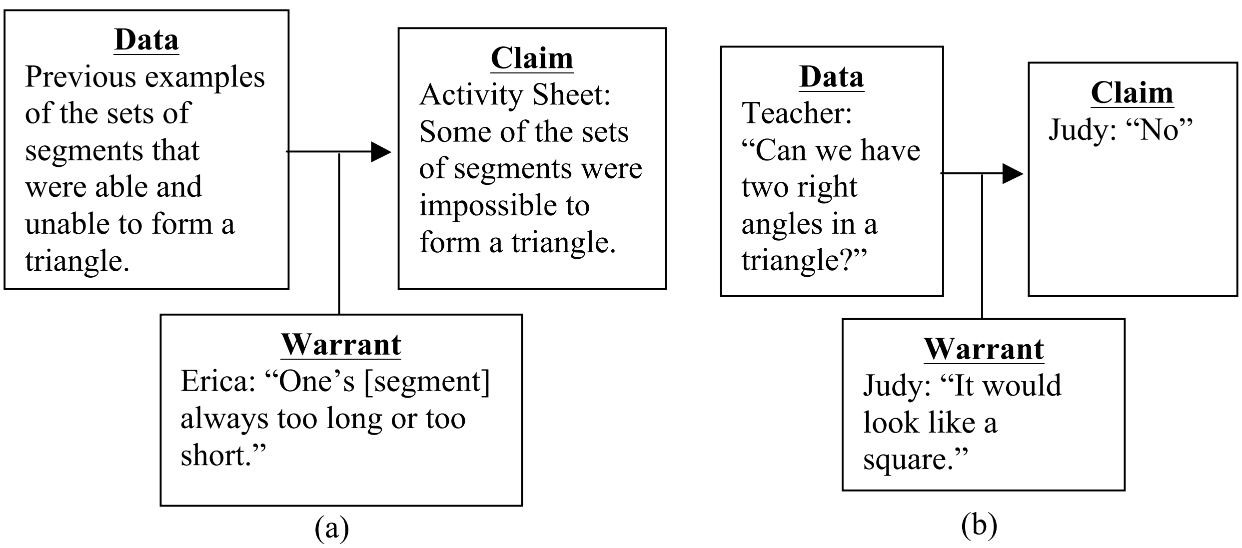

Figure 4: Arguments in which students did not use technology and engaged in (a) generalize tasks and (b) justify tasks.

Discussion

As the results of this study indicate, relationships exist between the ways in which students use technology when engaging in particular tasks. Specifically, students use the *appearance* of the figure when engaged in *predict* tasks, use the *measurement* feature when working on *describe* tasks, use the *drag* feature to *produce* a new figure, and do not use technology when *generalizing* or *justifying*. Hollebrands, Conner and Smith (2010) had a similar finding in their study of the arguments college geometry students created when working with technology. When the college geometry students provided explicit warrants for their claims, the students were generally not using technology and they were working on proofs, a particular kind of justification tasks. The authors attributed this finding to the students' prior experiences in learning mathematics at the collegiate level where the students were expected to provide formal

proofs, which did not involve the use of technology. We cannot do the same to the middle school students in this study because it is unlikely they had been exposed to formal proofs. Rather, their lack of use of technology while engaged in these types of tasks may be due to their inexperience in using technology when working on these types of tasks.

As previously mentioned, when students were not actively using technology, they were mainly working on *generalization* and *justification* tasks. However, other researchers (e.g. Healy & Hoyles, 2001) found that students will create generalizations while using technology. Some pairs of students in Healy and Hoyle's (2001) study used a DGE to investigate relationships among the angle bisectors of a quadrilateral. The pairs of students did not create the same constructions and did not arrive at the same conclusions. However, those students that were successful were able to construct and measure aspects of their diagrams and developed generalizations while using the DGE. The students in the current study did not have the option of creating their own diagram. Instead, the students in the current study used a teacher-generated pre-constructed sketch that limited the students in how they could modify and/or measure aspects of the diagram. Perhaps, the students in the current study would have been more likely to use the technology while working on *generalization* tasks if they had been given the opportunity to create their own diagrams.

The results of this study should inform the design and implementation of activities that engage students in using technology to learn mathematics. Teachers should have a better understanding of the types of tasks in which their students are engaged by examining the ways in which their students are using technology. Teachers and teacher educators should also be aware that students are unlikely to use technology while engaging in *generalize* and *justify* type tasks. Teachers and teacher educators should design activities that encourage students to engage in these types of tasks with technology.

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FEEDBACK AND LEARNING IN AN iPAD APP TARGETING NUMBER CONCPETS FOR PRESCHOOLERS

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We describe the design and research of a freely available iPad app to help Pre-K children develop flexible and robust understandings of number. Building off the theories of Robbie Case and colleagues, Critter Corral helps children integrate multiple conceptions of number, including ordinal relations, cardinality, and relative magnitude (Griffin, Case, & Siegler,1994). An in-progress study examines three models of feedback in the app. In implication feedback, learners see the implications of their mistakes (too many, too few) and make adjustments to fix them (add more, take some away). In corrective feedback, after a mistake, learners are told the correct answer and shown why it is correct. In answer until correct feedback, learners are told they are incorrect and keep trying until correct. Pre- and post-tests and in game learning measures are used to evaluate the effectiveness of the app and of the three models of feedback.

Keywords: Number Concepts and Operations, Technology, Pre-School Education, Early Childhood Education

Children's early math skills are powerful predictors of later school achievement (Duncan et al., 2007), and there has been an increasing focus on math learning in early childhood education, particularly the development of a flexible understanding of number (National Research Council, 2009). This goes beyond simply knowing the counting sequence to include cardinality, relative magnitude, estimation, numeral identification, 1:1 correspondence, and set composition/ decomposition (Pupura & Lonigan, 2013.) This paper describes a freely available iPad app called Critter Corral that aims to help Pre-K children develop a robust understanding of number through gameplay *(www.critter-corral.com)*. The learner's overarching goal in the app is to help return a Wild West town to its former glory by helping each of the five businesses in the town. Each business (game) targets number concepts in different way. For example, in some games, the quantities are discrete and countable, while in others they are continuous. Some games involve estimation, while others focus on systematicity when counting a disorganized array. In all games, the student's task is to create a 1:1 correspondence with a target amount. For example, Figure 1 shows the restaurant game. The learner's task is to count how many customers there are to tell the chef how much food to cook (Figure 1a). A simulation shows the result. If the learner is correct, there is a 1:1 correspondence between customers and food. Each customer has one piece of food and is happy.

An important element of the app is the feedback learners see if they answer incorrectly. Here, we describe one version of feedback in the game. (Later, we discuss a study that compares feedback.) In this version, if the learner is incorrect, she sees the implications of her incorrect answer (too many, too few). For example, if the learner counts too few customers, the chef does not cook enough food (Figure 1b), and some customers are left hungry (Figure 1c). Similarly, if the learner enters a number that is too large, the chef cooks too much food. The learner can fix her answer by adding or taking away food (Figure 1d). The other four games in the app follow similar play patterns. For example, in a theater game, the learner sees a chair that is missing one

leg, and chooses which size leg will fix it. If the learner chooses the correct leg size, it fits the chair (perfectly fills the empty space) and fixes it so the characters can sit and watch a show. If the leg chosen is too short, it does not fill the space and the chair tilts forward. If it is too tall, the char tilts backwards. The student can then fix her answer until the chair is fixed.

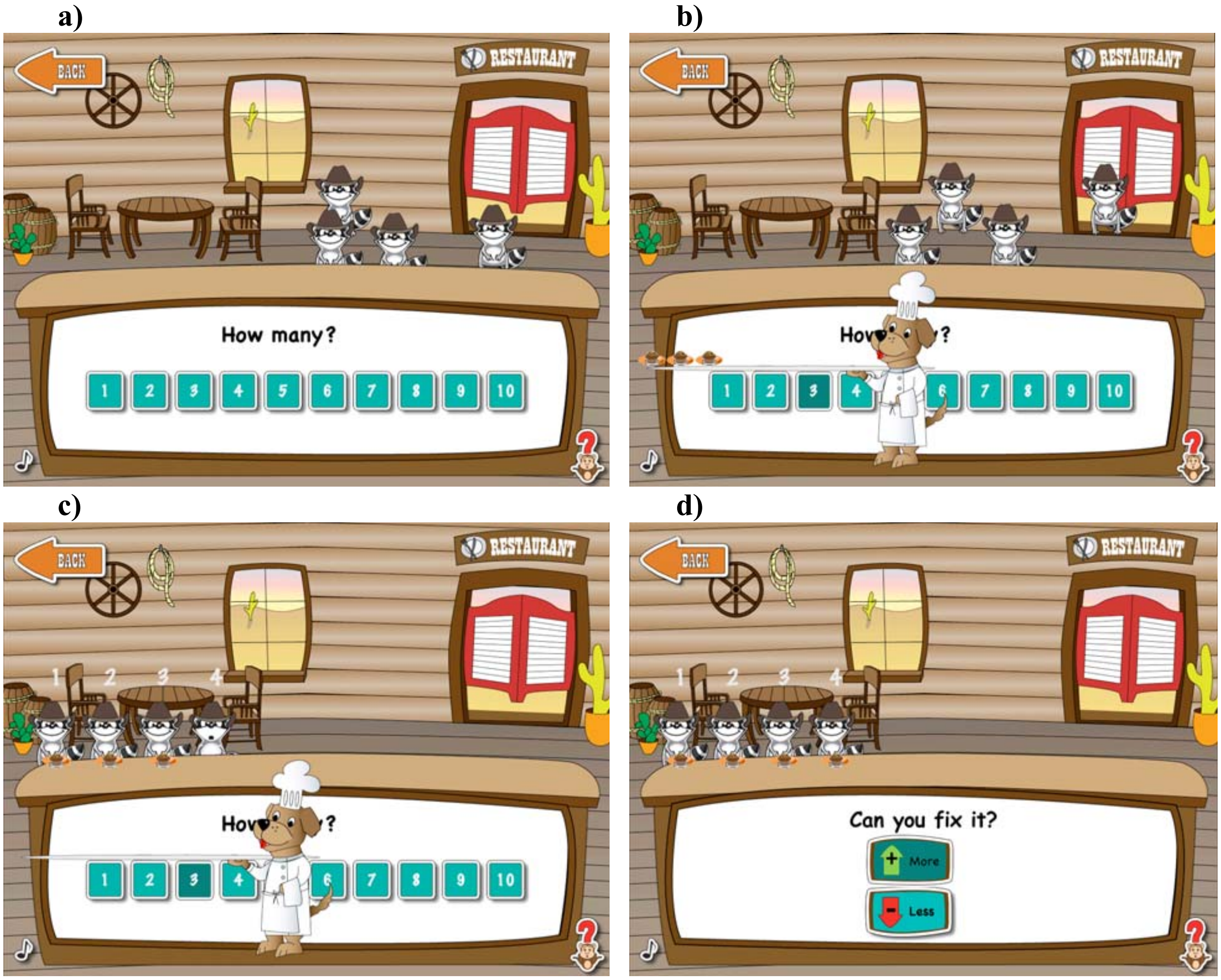

Figure 1: Restaurant sub-game. a) Tell chef how many customers so he knows how much food to cook. b) Learner enters 3. Chef cooks 3. c) Some customers are without food d) Learner corrects their error by adding one more.

The app builds off the theories of Robbie Case and colleagues, seeking to help children integrate different facets of number, including ordinal relations, cardinality, and relative magnitude, by encouraging children to flexibly move between different representations and conceptions of number, and building on the spatial structure of a number line (Griffin, Case, & Siegler, 1994; Case & Okamoto, 1996). For example, in the restaurant game described, when the child presses “3”, the digit 3 becomes associated with a set of three objects (three foods).

The chef pauses, and children can begin to compare the set of foods on chef’s plate to the set of customers (Figure 1b). As the customers and then foods are brought to the table, they are arranged from left to right, consistent with the spatial structure of the number line. This structure further helps students recognize that 3 foods is not enough for the 4 customers - 3 is less than 4,

and it spatially reinforces that relationship. In the level shown, students add or remove foods one at a time. In later levels, students decide how many foods must be added or taken away to feed all the customers (for example, how many less is 5 than 7).

We hypothesize that the kind of feedback where learners see the implications of their mistakes (too many, too few) and make adjustments to fix them (add more, take some away), will particularly help students develop a sense of relative magnitude (e.g,. 3 is less than 5; 3 is much less than 8). We call this kind of feedback implication feedback (IF) (Blair, 2009). However, for young children, it may be the case that seeing the results of incorrect answers causes unnecessary confusion and cognitive load (Kirschner, Sweller, & Clark, 2006); perhaps when they make a mistake, students should only be shown the correct answer and why it is correct, and then model the correct answer. We call this corrective feedback (CF). IF and CF have theoretical support, but are relatively uncommon in math learning apps for young children. By far the most common form of feedback is answer until correct (AUC) feedback (Blair, 2013). In this feedback, if a student answers incorrectly, the program indicates that the answer is incorrect (such as with a buzzer sound or red change of color), and the student keeps trying until they give a correct response. One potential limitation of this kind of AUC feedback is that children may focus primarily on whether they are correct or not, rather than on the quantitative dimensions of the problem and feedback space (Blair, 2009).

The current study examines which of these models of feedback is most effective for learning and engagement for young children in a technology-rich math learning environment. The existing research on feedback and learning does not sufficiently address this question. Most experimental studies in the feedback literature take place the context of learning from multiple choice tests and lists of word associations (e.g., Mory, 2004), which do not allow for IF, nor approximate the rich kinds of learning and feedback available in technological environments (e.g., de Jong, 2006; Bowers, Cobb, & McClain, 1999).

Ongoing Research on Learning and Feedback in the App

A study in progress evaluates the math learning app, and uses three conditions to investigate whether IF, CF, or AUC is more effective for young children's learning in a technology-based environment. In all conditions, presentation of problems is the same. There is no difference between conditions when the student gives a correct answer (they see the simulation run with the correct answer in all conditions). The condition differences occur when students give incorrect answers. The IF condition was previously described. In the CF condition, if the learner makes a mistake, he is told he is incorrect and shown the correct answer. For example, the restaurant game corrects a student who answered 3 instead of 4 by saying, "Oops. 3 is not correct. The correct answer is 4". Then the animated simulation runs with the correct answer, such that there is always a 1:1 correspondence between customers and food. "Let's count and see why 4 is the correct answer. 1,2,3,4 customers. Chef should cook 1,2,3,4 foods." The student then enters the correct answer and moves on to the next problem. In the AUC condition, if the learner is incorrect, he hears a noise indicating he made a wrong response and keeps trying until he selects the correct response, at which time the animated simulation runs with the correct answer.

We have recruited all 12 of the transitional kindergarten (TK) classrooms in a large urban district for the study. Transitional kindergarten is a public school program for children who just miss the cutoff for kindergarten (they turn five in October – December). The study uses a crossover design. We call 6 classrooms Cohort A and 6 classrooms Cohort B. All students (Cohorts A and B) are given a pre-assessment of math abilities. Children are asked questions

about number, relations, and operations in the form of a one-on-one interview. Some measures are drawn from standardized math tests (e.g, the TEMA-3), while others are created by the research team to specifically target concepts covered in the game. After the pre-assessment, students in Cohort A use the app for approximately 15-20 minutes per day, 3-4 times a week, for 4 weeks. Within each class, students are randomly assigned to CF, IF, or AUC condition as much as possible, within the practical constraints of the classroom, such as having pre-arranged center time groups. Students remain in their conditions for all sessions. After the first 4 weeks, all students (Cohorts A and B) do a mid-assessment, which is isomorphic to the pre-assessment. This allows us to compare students who used the app (Cohort A) to a control condition who did not (Cohort B), as well as look at effects of feedback condition within Cohort A. After the mid-assessment the 6 classes in Cohort B who did not use the app use it for approximately sixteen 15-minute sessions over 4 weeks. (Cohort A returns to normal instruction.) A final interview-based assessment allows us examine the longer-term effects of having used the app for Cohort A, and replicate the pre-post and within-condition findings for Cohort B. In addition to the interview assessments, in game learning measures, including accuracy and response time, will be analyzed to look for learning differences between the two feedback conditions. A pilot of the study has been completed. Sample sizes were too small to see condition differences, but there were significant pre- to post-gains. The larger study will be finished in the fall of 2013.

Acknowledgments

This work has been supported by a grant from the Wallenberg Foundation and is being developed within the AAALab at Stanford University.

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EXPLORING THE POTENTIAL OF TECHNOLOGY THROUGH PROFESSIONAL DEVELOPMENT FOR MATHEMATICS TEACHERS

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Technology, pedagogy, and mathematical content can be taught to elementary teachers to (a) support their understanding of students' learning and thinking about mathematical concepts with technology; (b) change their conception of how technology tools and representations support mathematical thinking; (c) provide instructional strategies for developing lessons with technology; and (d) serve to enhance teacher's abilities to critically evaluate and use curricula materials for teaching mathematical ideas with technology.

Keywords: Technology, Teacher Education-Inservice/Professional Development

Literature Review

Many suggest that the use of technology with pragmatic constructivism teaching approaches can develop students into critical thinkers and problem solvers by providing students with an opportunity to reason and collaborate using mathematical representations (Healy, Kynigos, 2010; Kaput, Noss, Hoyles, 2002; Niess, 2007). Two meta-analyses on computer technologies in mathematics education showed encouraging results and credit part of their success to pragmatic constructivist teaching approaches (i.e., problem-based, inquiry-oriented and situated cognition) and suggest that appropriately designed computer technologies can play a role in supporting and encouraging students as they learn (Hattie, 2009; Li & Ma, 2010).

Computer technology will not reach its potential in maximizing teaching and learning without pedagogical integration (Conlon & Simpson, 2003; Niess, 2007). In order to pedagogically integrate a technology, teachers must first understand the affordances of the specific technology and then relate the affordances to their classroom goals during lesson planning (Angeli & Valanides, 2009). The process begins with *cognitively* integrating these affordances with teachers' knowledge of specific mathematical tasks and instructional guidance. Technology affordances that teachers construct or activate are important for planning the use of technology in class in instructional units. An atmosphere of reasoning and critical thinking when teamed with technology can be very powerful and transformative (Donnelly, 2010).

The study described below examines the integration of technology into a graduate mathematics specialist program—the Elementary Master Mathematics Teacher (MMT), based on pragmatic constructivist and the TPACK conceptual model.

Conceptual Model

TPACK, a framework used to help teachers understand the relationship between technology, pedagogy and mathematical knowledge, assists in holistically viewing the relationships involved in integrating technology into learning and instruction. TPACK identifies the nature of knowledge used by teachers for technology integration in their teaching while addressing the complex, multifaceted and situated nature of teacher knowledge (Mishra & Kohler, 2006). The model includes technology knowledge (TK), pedagogical knowledge (PK), mathematical content knowledge (CK), and overlapping relationships of pedagogy and mathematical knowledge (PCK), technology and pedagogy (TPK), and technology and mathematical knowledge (TCK) (Mishra & Koehler, 2006). The conceptual model suggests that as pedagogical, content, and technology knowledge intersect the domains converge in the center creating the "sweet spot"

where transformative learning occurs and creative thinking is stimulated.

Theoretical Framework

According to Windschitl (1999), constructivism is based on the assertion that learners actively create, interpret, and reorganize knowledge in individual ways. "These fluid intellectual transformations," he maintains, "occur when students reconcile formal instructional experiences with their existing knowledge, with the cultural and social contexts in which ideas occur, and with a host of other influences that serve to mediate understanding" (p.752).

Pragmatic constructivism refers to a way of knowing that comes out of purposefully changing the environment and then reflecting on this change. Dewey (1988) maintained that a pragmatist approach does not imply that genuine knowledge comes neither by thinking about something abstractly nor by acting uncritically, but rather by integrating thinking and doing, by getting the mind to reflect on the act.

Methods

A quasi-experimental pretest posttest design was used based on a sample selected to represent kindergarten through sixth grade teachers with at least three years of teaching experience and students classified as *at-risk* as determined by Texas Education Agency (TEA) criteria (Public Education Information Management System [PEIMS], 2011-2012). By design the study represented a wide range of elementary teachers (age, nationality, type of school environments) who all taught at-risk students. The study addressed the following research questions:

1. What effects are noticed about teachers' attitude toward the use of technological, pedagogical, and mathematical knowledge?
2. What effects emerged in comparing teachers' lesson plans over a series of three semesters using Lyublinskaya and Tournaki's (2011) TPACK Levels Rubric?
3. What effect does the MMT program have on the practicing teachers' students' mathematical achievement as shown by Texas Assessment of Knowledge and Skills yearly exam?

Participants

The population studied includes 45 practicing teachers enrolled in a masters degree program with an emphasis on elementary mathematics. The teachers ranged in age from 22 to 60 years old and taught in five different school districts having at least 50% or more at-risk students as identified by TEA. The cohort's ethnicity was 3% Asian, 10% Black, 27% Hispanic, and 60% White; and gender 7% male, 93% female.

Treatment

The study's focus centered on three semesters where mathematics concepts were taught using technology. Teachers were engaged as learners and doers of the mathematics through exploring rich word problems using technology. Teachers, divided into assigned groups, arrived at what their students would see as a relevant problem and refined the problem to one driving question related to be resolved through a multi-disciplined approach and technology. With the driving question at the center teachers brainstormed the various cognitive avenues students might take and the information their students would need to solve the problem. A wiki was used as a platform for the teachers to collaborate on lesson plans and use as a presentation tool for their instructional unit that included their driving question, anchor video, concept map (bubblus.com), calendar, lesson plans, interactive web sites (recorded with Jing), project based assessments, and other technology-based resources.

Data Sources

The TPACK Survey by Schmidt, Baran, Thompson, Koehler, Shin, and Mishra (2009) was used to obtain data. The intent of the administered survey was to determine elementary mathematics teachers' attitudes about their use of TPACK. Though originally designed to determine use of knowledge levels as a self-reporting survey, the survey best reflects participant attitude and use of TPACK domains. The test was modified to include only mathematical knowledge content. The TPACK Survey was administered during first semester of and last semester of the 36-hour mathematics specialist program, then one year later.

Lyublinskaya and Tournaki's (2011) TPACK Levels Rubric was developed based on the TPACK framework for technology integration in the classroom where teachers progress through five progressive levels in each of four components of TPACK as identified by Niess, van Zee, & Gillow-Wilese (2010). The developers organized the rubric as a matrix where each cell represented a specific TPACK level (one of the four components of TPACK). Thus, each row of the rubric represented a specific component of TPACK and each column of the rubric represented a specific level of TPACK. A 5-point Likert scale is used. TPACK Levels Rubric was tested for reliability and validity. Correlations examined whether there was a relationship among the four components of the rubric for each expert were found significant.

Texas Assessment of Knowledge and Skill mathematics test (TAKS) was used to determine students' change in mathematical understanding. For internal consistency, the Kuder-Richardson Formula 20 was used with reliabilities in the .80s and .90s (Texas Education Agency, 2008). At the beginning of the year the previous year's TAKS test was administered for the grade level providing a pre-test. Then at the end of the year (April) a different version of the TAKS mathematics test was given as the post-test.

Results

TPACK survey

To measure attitude about knowledge in the TPACK domains the researcher administered the research instrument (Schmidt, et al., 2009) three times by way of a web-based survey. Internal consistency and reliability of the instrument when used by *inservice* teachers created Cronbach's alpha coefficients for the subscale, ranged from .61 to .95, all within an acceptable rate. The TCK and PCK were not recorded because only one question was used for each subcategory. The coefficients of reliability (or consistency) indicated good to excellent alpha scores leading the researcher to proceed to use a matched-pairs *t*-test.

Results of matched-pairs *t*-test yield a statistically significant improvement ($t(45) = 2.64$ at $p \leq .01$) with a small effect size for TK. A positive change for MK was statistically significant ($t(45) = 6.70$ at $p < .01$) with a medium effect size; PK yield a statistically significant improvement as the result of the intervention, ($t(45) = 8.16$ at $p < .01$), with a higher effect size; TPK yield a statistically significant improvement with a ($t(45) = 15.15$ at $p < .001$) with a high effect size.; and TPCK yield a statistically significant improvement ($t(45) = 12.28$ at $p < .001$) with a high effects size greater than one standard deviation.

TPACK Levels Rubric

The qualitative analysis of 155 teachers' 5E lesson plans with two coders rating the technology using Lyublinskaya and Tournaki's TPACK Levels Rubric (2011) showed growth and increasing maturity in technology use (TPACK) over the three-semester sequence (mean first semester .98, second semester 2.25, and third semester 3.74 out of a total of 5). An ANOVA was conducted yielding an F score of 21.41 and a $p < .00$. The inter rater reliability was .85.

Student Mathematical Knowledge

The results for teachers' students' Texas Assessment of Knowledge and Skills mathematics scores (2009-2010) indicated significant improvement from beginning of the year mathematics assessment to end of the year mathematics assessment ($p <. 01$) with a mean improvement of 27 score points and medium effect size.

Conclusion

Evidence supports the belief that technology, pedagogy, and mathematical content can be taught to elementary teachers in ways that (a) support their understanding of students' learning and thinking about mathematical concepts with technology; (b) change their conception of how technology tools and representations support mathematical thinking; (c) provide instructional strategies for developing lessons with technology; and (d) serve to enhance teacher's abilities to critically evaluate and use curricula materials for teaching mathematical ideas with technology.

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AN INTRA-ACTION OF AGENTS IN A GEOMETRIC ACTIVITY

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This brief research report challenges the assumption of learning theories that depend upon a rational thinker using tools to achieve an end. Adopting a post humanistic approach, granting agency to both the mathematical tool and the mathematical discipline, a different lens is used to analyse a grade 9-mathematics class. In using a tool (the Geometer's Sketchpad), students create a mathematics that depends upon their own development and experiences. The construct of agency is used to look at the engagement of both human and non-human actors. The results offer new conceptions of geometrical figures entangled with student's actions.

Keywords: Affect, Emotion, Beliefs and Attitudes, Instructional activities and practices, Geometry and Geometrical and Spatial Thinking, Technology

Introduction

In reconfiguring the perspective of math practice, post humanists are adopting a shift from looking solely at humans and acknowledging the role of non-human actors (Barad, 2007). These researchers identify that non-human components in a learning situation are often overlooked and granted minimal significance within the activity of learning. If one is to identify three key elements in an interaction consisting of a student, a tool and mathematics, one may adopt that these elements are not only joined in activity but also are defined and created in activity. Using the construct of agency argued to be an essential process of activity and using Barad's intra-action framework, I look at the classroom activity of students engaging with a geometry tool and the resulting development of all agents. The focus will not be solely on the students, but on the intra-actions (Barad, 2008) between subject, their tools and the mathematics.

Theoretical Foundation and Framework

In her work in quantum theory and her analysis of Bohr, Barad (2008) offers the term intra-action to refer to entities coming together and becoming defined within that coming together. If one views the human as being ready to learn or to be presented with curriculum, the idea of external content and how it is interiorized becomes the main focus. With this assumption a learner's agency is simplified significantly and their affects, actions, behaviours and cognition are viewed as resulting from within.

Looking specifically at the individual, Knox (2011), a developmental psychologist, specifies that self-agency is both a necessary and a developing aspect to learning. Yet, self-agency is not a stable or necessarily coherent component of one's experience, nor does it necessarily exist in a unified form (Knox, 2011). Korsgaard (2009), a philosopher who looks at the constitution of self, describes that there is no "you" prior to one's choices and/or actions because in a quite literal way one's identity is constituted by choices and actions in activity.

Self-agency, the subjective self, able to act, is essential in learning (Knox, 2011). It is an aspect of learning often not considered and yet, I suggest, it is an imperative aspect to learning, understanding, and for growth of knowledge. To further this idea, the subjective self does not stand-alone but is in intra-action with other forms of agency. When a student is acting with a tool such as *The Geometer's Sketchpad* (GSP) (Jackiw, 1988) there are contingencies that arise, as well as constraints, past experiences of computer use, or psychological experiences with mathematics. There is an abundance of considerations that affects the intra-action between a student and a tool. In this study the considerations will focus on material agency as well as the mathematical (disciplinary) agency. Mathematics and computers evoke certain ways of acting, in which case I grant them agency as Pickering has done in his analysis of how

scientists work. The material, non-human element imposes a restriction upon the user. Further, a Dynamic Geometric Software (DGS) may extend possibilities or distribute activity of the individual's activity.

Agency can be defined as synonymous with intra action. But yet is action that requires agency (Korsgaard, 2009). It is not part of a person or thing but emerges in activity. It is associated with process, a doing. Based on a performative idiom (Pickering, 1995), agency is a construct allowing us to identify who or what is doing the acting. Agency is synonymous with action and is not "part" of person or thing by emerges in action. Pickering (1995) attributes materials agency in his work in looking at scientific practices. The phrase that he has made popular is dance of agency in which agency moves back and forth among agents.

Pickering (1995) uses the words "resistance" and "accommodation" as terms to describe scientists' performative action with the world. Pickering describes agency as a dialectical interaction between a subject and an object and remains within a performative idiom. The "dance of agency" model is the back-and-forth transition between the actors' agencies. He refers to this as a repeated process of intention, resistance, and accommodation. I suggest that Pickering's model and framework can be seen as analogous to the interaction of a student with a computer. Although the context and purpose are different, I contend that the framework is analogous because it positions students as implementing an act that becomes the first of a very intricate process of resistance and accommodation.

It is also important to recognize that how the subject views their actions develops a sense of agency. When a student states that they are going to do something this is an example expressing a sense of agency. A sense of agency is when the student believes that they are under no influence and they are acting as they desire. Acting is a process of self-development (Korsgaard, 2009) and this development of agency is the sense that one is the agent of an action. If actions occur without resistance self-agency develops. One gets a sense that they can act, and in reflection construct a belief that they are the one that is doing something. However, if there is an obstacle or a resistance (Pickering, 1995), both sense of agency and self-agency are influenced. It is important to note that although self-agency might seem to be an intention, resistances can significantly affect one's action.

The question of this study looks at how self-agency and material agency co-constitute each other in a geometric activity. I suggest that this lens leads to recognizing activity as a dance between mathematics, cognition, emotion and ontologically accessible mathematical objects not as separately defined entities but as ontological entanglements of interacting agencies (Barad, 2007).

Methodology

The theoretical framing of this study demands close attention to the back and forth and integrated intra-action (Barad, 2007) of the student using tools in a mathematical activity. Attention to discourse, written or verbal, provides the means by which I identify activity. I use James' (1983) distinction of the "I" voice as expressions of self-agency and his distinction of the "me" voice as the objective self, as that which is being acted upon. I will use these distinctions of voice to identify resistance and extensions. These will be examples of material agency. Student discourse will be a major source of identifying intra action between themselves and the software. I chose a computer-based environment to study in the hopes that the intra-action would be more visible and that the self-agency more felt for the students.

Research Activity and Participants

The data for this research is drawn from a larger study looking at the role of agencies on activity and learning. The episodes presented here are based within a classroom of grade 9 (14 years old) students in a high school in Canada. The grade 9 curriculum has a large geometry component and the studies in the school have been focused on students working with *The Geometer's Sketchpad* (GSP) to construct and explore a variety of geometric relationships.

In this study, the researcher collected data from a grade 9 class in which the teacher took all the students to a computer lab, sat them in pairs and requested the students use GSP to construct a triangle and a square. During this activity, the teacher allowed students to explore the software's environment, as this was the first time the students had used the program. The construction of the square provided a

challenge in that most students "fit" four segments together, but when the teacher dragged one of the vertices of the "almost-square", the figure would morph into another shape. The next class, the teacher had all the students in a traditional classroom and requested that they write, working in pairs, how they would determine whether a given figure is a square. All written work was collected and analyzed. Data from the computer lab was collected by using SMRecorder.

Analysis

The episodes presented here are examples that highlight agency. I outline two examples from the computer lab. I then outline examples of written material from the students' responses to the question posed in the classroom. Examples were chosen because they were rich in intra-action and agency.

The following two episodes took place in the computer lab.

Justin and David constructed a triangle and then Justin translated it partly off the screen and the question "Is this a triangle?" was posed.

Justin: Is this a triangle? (Figure 1)

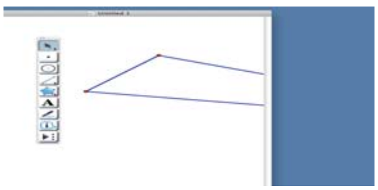

Figure 1: Justin's Triangle

This action of moving the triangle off the screen is not about a student who decided to focus on philosophical considerations ("Is this a triangle?"). It was an intra-action of student agency and material agency moving along a different trajectory than of their classmates. Since the material agency offered a new form of practice, dragging of a geometrical figure, in this case, the student participated in an activity not necessarily intended. The tool offered an opportunity. To view the student as having an intention and following up with an action is to make a very anthropocentric assumption. If we move away from the mastery and authorship assumptions and grant agency to the things we work with, we identify new engagements that develop and emerge. The Geometer's Sketchpad allowed Justin to construct and act his self-agency and specifically manipulate the tool's agency, repositioning the mathematical object.

Another example, similar in nature is presented as episode two. According to James the diagram in Figure 2 is not a triangle. James discussed, with the teacher, how GSP expected endpoints to be connected properly otherwise segments could be dragged away from each other and the shape did not retain invariant features. James challenged the idea of endpoints and intersections. A new way of categorizing intersections was introduced; intersections did not become "points" as designed in the tool but with the addition of self-agency, James adopted a new conception of a triangle.

James: The four sides must be touching but not intersecting.

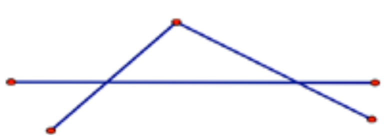

Figure 2: James' Non-Triangle

The following descriptions are from the written work in the classroom. After the students had visited the computer lab their written explanations of how to construct a square were available for analysis. Two pairs of students who had successfully constructed a square in the computer lab wrote, after their detailed steps, that they had constructed a "perfect" square. One pair wrote "aww yeah" after their claim. This comment indicates a confidence that their steps are correct. Their experience with squares that "break",

they now use, by way of experience, a claim that square can be "perfect". Their written steps use the I voice, and their reference to the perfect square may indicate that these square are different, they act different. This is exactly the finding of Sinclair & Yurita (2008), that the whole idea of what a square is changes radically since it is something that you can make (for starters), then transform and also "break."

Other students wrote such statements as:

Paul: repeat steps 3 and 4 until a square magically appears on your screen (wooh)

John: I would determine this is a square by …

The square no longer has the existence it may have had before. One needs a sense of agency to begin the enactment and a self-agency to endorse the square. Without the ability to flip, move, drag, the determination of whether it was a square might not be possible. The students used agency to pull the square apart but, as evidenced in the lab, when the square was properly constructed, the material agency, and the disciplinary agency held the square together. It is the parts of all the layers involved in acting, from the students, the tool and the math. This is an example of the square existing, as a square, only with the action of the student who in turn is identifying the invariant properties and concluding that the figure is in fact a square. The self-agency of the student is co-constituted with the agency of the constructed image in the DGE. In addition, the student reflects upon their actions and determine that by way of experience, their agency is enacted, and their sense of agency is developed as evidence by the I voice in their written work.

Discussion and Conclusion

When one speaks of mastery of a tool they speak from a very anthropocentric perspective. If we grant materials agency and acknowledge a role they play in determining the development of an emerging practice, it is possible to recognize that the tool can also master us. How the tool moves to affect our self-agency is dependent upon the experiences we have had with the tool. Providing students opportunities to act, they come to see themselves as participants, which may lead them to experience self-agency.

The mathematics experienced in these episodes is dependent upon personal experiences with the tool. The tool provides the opportunity to develop mathematics and self. This study troubles existing, humanist assumptions about the role of tools. If the tool can alter the way we develop, as well as the way we look at our own involvement in mathematical activities, both the way digital tools are designed as well as the way they are presented can have very important effects on our mathematical experiences and our own sense of self.

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MATHEMATICS THINKING AND LEARNING AS MICROWORLD-MEDIATED MATHEMATICAL ACTIVITY

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This paper draws on findings from a prior study for which I designed and developed a dynamic geometry environment called Configure and used it in the episodes of a teaching experiment to engage children's informal and intuitive topological, or at least non-metric, conceptions and support their development. I found that "Amanda" developed significant and authentic forms of geometric reasoning and have given the name "qualitative geometry" to these findings. These newly identified forms of reasoning were seen to develop as Amanda engaged in mathematical activity mediated by the microworld of Configure. These findings are used to provide a perspective on mathematics thinking and learning as engagement in microworld-mediated mathematical activity. Essential elements of this particular form of mathematical activity are provided.

Keywords: Elementary School Education, Geometry and Geometrical and Spatial Thinking

Background

Piaget's (1956) investigation into the child's representational thinking about the nature of space provides existence proofs that young children possess intuitive ideas fundamental to topology. This is a domain of mathematics that is typically conceived as advanced and therefore reserved for the few who are able to study it in college. Not surprising, these findings were met with skepticism[1] in ways that resonate with Papert's (1980) observation that "Nothing is more natural than the most advanced ideas in mathematics should be inaccessible to children" (p. 161). From a traditional perspective, there is a good argument to be made that this is indeed the case, because advanced mathematics depends on what he refers to as an "infrastructure of formal, mathematical training" (p. 161). However, *mathematical microworlds* (Balacheff & Kaput, 1996; Papert, 1980) provide the sorts of "transitional systems" (p. 122) that generate an alternate and "revolutionary" (p. 185) perspective by providing an infrastructure that democratizes access to ideas traditionally conceived as advanced upon which learners' *in*-formal and intuitive mathematical ideas are engaged.

Microworlds Provide an Environment for Mathematical Activity

Configure (Greenstein & Remmler, 2009) [Figure 1] is a microworld (Balacheff & Kaput, 1996) that was designed to engage learners' topological ideas and support their development. The essential features of this microworld are as follows. The primitive objects are an *arc*[2] and a *segment.* The arc consists of two endpoints and a "helper point." The endpoints may be dragged to change the length of the arc and the helper point may be dragged to alter its curvature. The segment consists of two endpoints, each of which may be dragged to either change the length of the segment or change its orientation. The three *transform* tools can be used to enact topological transformations of shapes in ways that resonate with the rubber sheet conception of topology.

Figure 1: Screenshot of *Configure*

These tools correspond to the axioms of topology, because they can be used to perform topological transformations on the primitive objects (either independently or in composition with other primitive objects). In addition, *Configure's* capacity for dragging affords users opportunities for "reasoning by continuity" (Sinclair & Yurita, 2005, p. 5), thus making salient the properties of shape that remain valid in all of the successive stages of a topological transformation. The design of *Configure* incorporates these essential features so as to provide the structure of a formal system that resonates with the cognitive structure of topology (Piaget, 1970b).

Theoretical Perspectives

I am proposing a conception of learning mathematics as engagement in microworld-mediated mathematical activity. The analysis of learning as the development of children's topological reasoning is guided by Piaget's framework of genetic epistemology (Piaget, 1970a) and his model of the development of cognitive structures (Bringuier, 1980; Gardner, Kornhaber, & Wake, 1996; Piaget, 1970b). That children possess a cognitive structure (Piaget, 1970b) of topological ways of thinking about shape is suggested in the results of Piaget's (1956) investigations. And an understanding of topology is typified by kinds of reversibility, a notion that is central to Piaget's characterization of operationalized thought across his entire body of work (Piaget, 1970b). One form it takes is the transformation of a shape into an equivalent image, followed by the inverse transformation of the image to produce the preimage. This perspective on learning informed the design of *Configure.* By embedding features that support the construction of mental representations of shape from perception, *Configure* provides the user with the means to construct these representations, the elements of which must be reflectively abstracted via actions on shape. Consequently, these actions characterize a mediated mathematical activity that cannot be divorced from the mathematics itself. There is no distinction between *how* the mathematics is learned and *what* mathematics is learned. This is what it means to do mathematics. This is engagement as mathematical thinking and learning.

Research Question

Given that early forms of topological reasoning have been seen to develop as a result of learners' engagement with a software environment that supports fundamental topological representations and transformations, what are the essential elements of this engagement and what is their relationship to the development of the mathematical knowledge they support?

Methodology

In the spring of 2009 I conducted a teaching experiment (Steffe & Thompson, 2000) that lasted approximately eighteen weeks. "Amanda" (age 7) is one of the children who was nominated and consented to participate. I met with her nine times for episodes lasting between thirty and fifty minutes. I collected data from a pretest, video recordings, loosely-structured interviews (cf. Spradley, 1980), screencasting software, jottings (Van Maanen, 1988), and analytical memos (Strauss & Corbin, 1998) in

the form of local reflections on observations and more global reflections on salient and emerging themes. Analysis (Corbin & Strauss, 2008; Strauss & Corbin, 1998) from this proposed perspective on learning is ongoing as new rounds of coding are being implemented for the primary purpose of crafting images of learning/construction/sense-making from salient moments that resonate with essential elements of authentic mathematical activity.

Results

In the prior study from which this data is drawn, Amanda used *Configure* in the episodes of a teaching experiment and was found to have developed significant and authentic forms of geometric reasoning (Greenstein, 2010). I refer to those findings as "qualitative geometry" so as to privilege Amanda's ideas over the more formal topological ones. The sequence of rectangles in Figure 2 describes the enacted themes that characterize of each of the episodes. They indicate elements of the experiment's teaching trajectory.

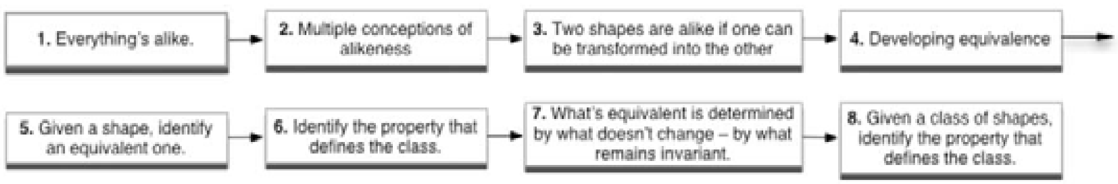

Figure 2: Themes of the Episodes of the Teaching experiment

At the onset of the teaching experiment, Amanda described how all of the shapes she was given were alike using resemblance-, attribute- and "morphing"-based conceptions of alikeness. As she progressed through the episodes of the teaching experiment, she moved further away from attribute- and resemblance-based conceptions of alikeness toward the assimilation of alikeness as determined by the structure of the environment. In Episode 4, for instance, upon completion of the transformation of a 3-sided, "open" square into a U-shaped arch, Amanda leaned back in her chair and smiled, as if to say that the equivalence is both obvious and necessarily true. This expression is evidence not only of Amanda's assimilation of a scheme of qualitative geometric equivalence, but more importantly, of a cognitive structure of qualitative ways of thinking about shape. Thus, the resonance between the structure of the microworld and Amanda's cognitive structure of qualitative ways of thinking about shape indicates that the microworld of *Configure* provided Amanda with an opportunity to engage and further develop her qualitative geometric reasoning.

By the conclusion of the experiment, Amanda had developed a conception of alikeness as "transformational." The development of this conception made it possible for her to describe fundamental properties of equivalence classes of shapes and to use a class-defining property to either identify or draw new shapes belonging to a given class. Property-based distinctions were evident in the names she assigned to those classes (e.g., "cherries," "worms," and "blocks"). These distinctions are structural in character, since they are organized in terms of properties that are meaningful in relation to possible transformations. These properties, then, are mathematically significant, which suggests that Amanda was, in fact, "doing mathematics."

Conclusion

Amanda's engagement in microworld-mediated mathematical activity gave her an authentic sense of what it means do mathematics and provided an image of the fundamental quality of that epistemological orientation, which is to preserve the synthesis of mathematical activity and the mathematical knowledge from which it emerges. Furthermore, it provides a stark contrast to traditional forms of mathematics learning that could arguably be considered inauthentic. These findings are beneficial to teachers who wish to engage their students in more authentic mathematical experiences and to the developers of mathematical microworlds who wish to design environments that support this kind of activity.

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Endnotes

[1] It must be noted that from a formal mathematical point of view there is much about Piaget's mathematics that deserve critique. Nonetheless, the findings of replicate experiments (Esty, 1971; Laurendeau & Pinard, 1970; Lovell, 1959; Martin, 1976a, 1976b) do not dispute Piaget's finding that young children possess early and intuitive topological ideas; they essentially call into question his conclusion of topological primacy.

[2] The term "arc" is used most often to refer to a portion of the circumference of a circle. I use it in a more general sense to refer to any curve joining two points.

TYPES OF QUESTIONS POSED DURING PIVOTAL TEACHING MOMENTS IN A TECHNOLOGY-INTENSIVE SECONDARY GEOMETRY CLASSROOM

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This study investigates one teacher's uses of a dynamic geometry program (The Geometer's Sketchpad) in her high school geometry classes over a two-year period. The researchers identified moments during her instruction when a disruption to the regular flow of the lesson occurred. These moments were examined to consider the action that the teacher took and the types of questions that were posed. Teacher actions such as emphasize mathematical meaning, pursue student thinking, or extend and make connections were associated with probing, exploring, and generating discussion types of questions. Teacher actions such as ignores/dismisses, acknowledges but continues, and repeats technology directions were associated with procedural/factual types of questions.

Keywords: Geometry and Geometrical and Spatial Thinking, Teacher Education-Inservice/Professional Development, Technology, High School Education

Introduction

One might hypothesize that when students are using technology as a regular part of mathematics instruction that teachers will create and implement mathematical tasks that are non-routine and provide students opportunities to develop and explore conjectures (Laborde, 2002). While students engage in such tasks, there may be moments when the student or teacher encounters an unanticipated situation or develops a novel solution (Straesser, 2001). These moments might allow students to pursue a different mathematical question than what was originally posed or offer creative mathematical explanations for what they are observing on the computer screen. Classroom disruptions such as these have been characterized as "pivotal teaching moments" (PTMs) by Stockero & Van Zoest (2012). PTMs are defined as "an instance in a classroom lesson in which an interruption in the flow of the lesson provides the teacher an opportunity to modify instruction in order to extend or change the nature of students' mathematical understanding" (p. 3). In particular they identified five types of PTMs: extending, incorrect mathematics, sense-making, mathematical contradiction, and mathematical confusion. In addition to the five PTMs identified by Stockero & Van Zoest, Hollebrands, Cayton, and Boehm (2013) identified two PTMs specific to technology-intensive secondary mathematics classrooms: technology confusion and incorrect technology use. How a teacher responds in these moments may influence how students reason about mathematics. Stockero & Van Zoest (2012) also identified five teacher actions: ignores or dismisses, acknowledges but continues as planned, emphasizes mathematical meaning, pursues student thinking, and extends/makes connections. An additional action was identified by Hollebrands et al. (2013): repeat technology directions. To more closely examine a teacher's response to a PTM, and the likely impact on student learning, this study investigates the following research questions: What types of questions are posed by teachers and students within PTMs? How are these questions related to specific teacher actions in response to a PTM?

Context and Participant

Mrs. Anderson's was one of 24 teachers in a larger professional development project that involved four public, 1:1 computing school districts (each student issued a laptop) in a southeastern state and used four interrelated interventions: mathematics software programs (e.g., The Geometer's Sketchpad), teacher professional development, STEM role models, and cloud computing (VCL). During the time of the study, she was in her fourth and fifth year of teaching, all occurring at a large suburban high school with more than 1500 students. The school operated on a block schedule (four, 90 minute classes per semester), and the student population included approximately 26% minority and 25% economically disadvantaged students during the study.

Mrs. Anderson taught geometry each semester over the two-year period and was observed at least two times per semester. The use of a particular dynamic geometry environment (DGE) – The Geometer's Sketchpad – was expected for each observation. However, Mrs. Anderson made all decisions regarding the mathematical objectives and nature of dynamic geometry task utilized during instruction (e.g., pre-constructed or student-constructed). She was observed a total of 11 times, but three observations were eliminated from the analysis because students were presenting group projects. The remaining eight videos provided longitudinal data for analysis.

Methods

PTMs and teacher actions served as a framework for coding data. The first phase of data analysis consisted of the authors collectively analyzing one video recording. Authors identified PTMs and achieved at least 75% inter-rater reliability for coding the type of PTM, teacher actions, and likely impact on student learning. Second, authors independently coded a subset of the final eight videos before grouping PTMs according to teacher actions. Third, each episode was transcribed from the video. These transcripts were then coded with a focus on students' opportunity to engage in mathematics by analyzing the question types posed by the teacher or by students. Question types were coded according to Boaler and Humphries' (2005) question types: probing, exploring mathematical meanings and relationships, generating discussion, procedural/factual, other mathematical, and non-mathematical. In our data, we focused on the first four question types. Finally, results were analyzed to identify trends in question types within specific teacher actions.

Results

Figure 1 summarizes overall results for teacher actions taken by Mrs. Anderson and highlights trends in the types of questions posed by the teacher and students. First, Mrs. Anderson asked the most questions when Pursuing Student Thinking (37; 36.6%) and Emphasizing Meaning (35; 34.7%). This finding was rather interesting because she Pursued Student Thinking only 6 (14.4%) times when responding to PTMs, compared to 18 (42.9%) times she chose to Emphasize Meaning. It might be that when teachers pursue what a student is thinking they ask more questions to better understand: what the student knows (procedural/factual), how a student is thinking about what he/she knows (probing), and how those ideas are related (exploring meanings and relationships). When emphasizing meaning, her most prevalent use of Procedural/Factual questions occurred, but collectively she utilized high-level question types (Probing, Exploring Mathematical Meaning, Generating Discussion) more frequently. Patterns were also noted among different types of questions. Probing typically followed procedural/factual questions and most often occurred within pursuing student thinking. Another interesting finding was a demarcated trend in use of high-level questions versus

Procedural/Factual questions among teacher actions. High-level questions occurred most often within teacher actions of extending math/make connections, pursues student thinking, and emphasizes mathematical meaning, whereas Procedural/Factual Questions were predominant within acknowledges but continues, ignores/dismisses, and repeat technology directions.

Question Types	**Procedural/ Factual**	**Probing**	**Exploring Meaning or Relationships**	**Generating Discussion**	
Teacher Actions					***Total***
Extends Math or Makes Connections	3 (5)	2 (0)	3 (1)	2 (0)	**10 (6)**
Pursues Student Thinking	8 (3)	22 (0)	4 (0)	3 (0)	**37 (3)**
Emphasizes Meaning	14 (4)	7 (1)	10 (6)	4 (0)	**35 (11)**
Acknowledges but Continues	7 (0)	1 (0)	4 (1)	2 (0)	**14 (1)**
Ignores or Dismisses	2 (0)	0 (0)	0 (1)	0 (1)	**2 (2)**
Repeats Technology Directions	3 (2)	0 (0)	0 (0)	0 (0)	**3 (2)**
Total	**37 (14)**	**32 (1)**	**21 (9)**	**11 (1)**	**101 (25)**

Figure 1: Mrs. Anderson – Teacher (Student) Question Types within Teacher Actions

The following excerpts provide further detail regarding teacher and student questions within Mrs. Anderson's predominant teacher actions of Emphasizes Meaning and Pursues Student Thinking.

Emphasizes Meaning

Emphasizes Mathematical Meaning requires the teacher to highlight the mathematical meaning behind the issue(s) raised by the PTM. In this particular example a student was trying to construct an angle bisector.

T: Which one are you trying to do?
S: I am doing the definition of the angle bisector.
T: Okay. So if it's an angle bisector what's the big idea?
S: Half of that is that. (*Student pointing to objects on the computer screen*.)
T: Definition. That's what you are working on.
S: So angle bisector's (inaudible) I want to do...
T: So if it's half, what do you think about the theorem? What do you know about the pieces?
S: Congruent.
T: Nice. So you want to measure those pieces.

Mrs. Anderson emphasized the "big idea" related to the definition of an angle bisector. Her emphasis on meaning could be described as "funneling" (Wood, 1998). It was evident from the questions she posed that there was a particular meaning of angle bisector she wanted the student to describe. When he responded with the word she had in mind, her questions ceased.

Pursues Student Thinking

Pursuing student thinking occurs when a teacher delves more deeply into specific student(s) thinking that prompted the PTM. One example occurred when the class was working together to construct an initial isosceles triangle for the Pythagorean spiral. The teacher posed an initial question designed to generate discussion followed by a sequence of probing questions. Her questions were intended to connect the ideas of midpoint, bisector, and perpendicular to assist students in understanding the properties of a perpendicular bisector and how it may be used to construct an isosceles triangle. This PTM likely helped students make sense of an important geometric theorem (i.e., any point on a perpendicular bisector is equidistant from the endpoints of the segment that is bisected). She also employed the drag test to check the robustness of the construction.

Discussion

Analysis of the numbers of questions posed indicated that there were approximately four times as many questions posed by the teacher than by students. However, this might have been related to the ways in which the data were collected. A microphone was placed on the teacher to capture interactions with the teacher. Questions posed when students were working in pairs or small groups were not always audible. Student questions were typically procedural/factual. In addition to a large number of procedural/factual questions, the teacher posed questions to explore and generating discussion. It is not common for students to pose questions to a class of students to generate discussion, and it may be that students were not familiar enough with the mathematical ideas to pose exploring or probing questions.

The use of technology may have influenced teacher actions and questions. When students were using pre-constructed sketches there were fewer instances of repeat technology directions. When the teacher action was coded as emphasizing meaning, a greater number of non-procedural/factual question types were posed by both students and teacher. It is notable that a majority of these codes occurred during classroom episodes where students were actively constructing geometrical objects with the technology. Whereas, Stockero & Van Zoest (2012) noted the likely impact of this teacher action on student learning to be predominantly negative or low, the variety of question types identified in this study suggest the likely impact on student learning may not be negative or low. Perhaps these differences are due to the use of technology, or they may be attributed to the manner in which these actions were implemented by the teacher, as revealed by the question type analysis.

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NUMBERS ON FINGERS: LEARNING NUMBER ON IPAD

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This brief research report describes the TouchCounts *application (designed for iPad) and its two Counting and Adding worlds. I explore how a five-year_old child (Kindergarten level) builds meaning through communicative, touch-based activity involving talk, gesture and body engagement. The main goal of this paper is to show the impact of touch-based interactions on the development of children's perception and motor understanding of ordinality and cardinality of numbers. In this case study, I found a strong value of mathematics embodiment in emergent expertise in producing and transforming numbers, which can be supported with Nemirovsky's perceptuomotor integration approach theory.*

Keywords: Technology, Number Concepts and Operations, Pre-School Education

Fingers play a vital role in developing number sense for children. They use their fingers for counting while simultaneously thinking and saying the numbers, even if they are discouraged to do so. Butterworth (2000) argues that fingers play a functional role in the development of a mature counting system. Researchers have found at least five distinct brain areas, including the somatosensory cortex that are involved in representing numbers by fingers. In addition, finger movement and finger positions are associated with numerical meaning (Butterworth, 1999). In fact, there is a relation between using fingers to count and neuro-functional system (Andres, Seron, & Olivier, 2007; Gracia-Bafalluy & Noël, 2008; Sato, Cattaneo, Rizzolatti, & Gallese, 2007). Vision also plays a very important role in counting and showing quantities while young children develop conceptual understanding of counting. Therefore, we postulate that using fingers to create numbers when it is supported by audiology and visual provision will support and augment cardinality and ordinality understanding of numbers for counting and adding (Crollen, Mahe, Collignon, & Seron, 2011).

There are many mathematics educational software that have been developed for computers, which involve interaction via mouse, keyboard or/and electronic pens. Interacting with computers via those devices requires hand-eye-coordination, which is a hard task for young children (Ladel & Kortenkamp, 2009). Moreover, they "indirectly" manipulate objects through the computer via keyboard and mouse. In comparison, the touch-sensitive interface of the iPad enables children to "directly" interact and manipulate objects via both their hands and all ten fingers. Additionally, auditory, visual, tactile senses and kinesthetic touch through gestures (flicking finger, sliding finger, taping, nudging, pinching, spreading, etc.) lets to learner to be engage bodily in learning strongly.

TouchCounts

TouchCounts was designed to help children develop one-to-one correspondence between numbers and fingers, and to enhance their number sense (Sinclair, 2012; Sinclair, N., & Sedaghat Jou, V., 2013). It is an iPad app that includes two sub-applications, one for Counting and the other for Adding in three different languages: French, English and Italian.

The initial goal of the Counting world was to enable the development of one-to-one correspondence between ordinal numbers and objects. Small circles appear on the screen when

the user taps her/his finger. Each circle is labelled with a numeral and, once pressed, the iPad also says that numeral aloud. Each successive tap produces a new circle. In the Adding world, a group of numbers are created each time that the user taps on the screen; so, for example, placing three fingers on the screen at once will produce a group of three circle along with the label ‘3’ and the spoken number “three”. Pinching these groups together enacts on of the fundamental metaphor of addition, which is gathering together (Lakoff & Núñez, 2000).

Theoretical Framework

I use Nemirovsky et al.’s (2013) perceptuomotor integration approach, which emphasizes on the role of embodied practice on learning mathematics when a learner is engaged in an activity with digital technology. In this theoretical framework, “mathematical expertise involves the systematic interpenetration of perceptual and motor aspects of playing mathematical instruments" (p.372) The perceptuomotor integration approach places on Husserl's phenomenology of temporality and adopts a non-dualistic view of human cognition and, therefore, pursues a strong version of mathematical embodiment. It assumes mathematical learning occurs through a transformation of a bodily engagement of a learner in a specific mathematical activity. It is interested in whether and how learners develop fluency in using instruments (such as *TouchCounts*) and in the associated changes in the way that learners move, pause, gesture, talk, etc. As the perceptuomotor fluency emerges, “greater interpenetration of perceptual and motor aspects will be revealed by retentions and pretentions that each includes both perceptual and motor aspects” (Nemirovsky et al., 2013, p.387).

Methodology

This brief research report is based on part of an ongoing project on Sinclair’s “Tangible Mathematics Learning” in Canada. The learner described in this paper is a five-year-old girl (Sarah) who is a kindergarten student in Northern Canada. The interview was conducted in June, in a separate room in the school, and lasted about 20 minutes. The goal of the interview was to see how young children use TouchCounts, what they are interested in doing with it and how they might learn with it. Sarah is selected because she showed higher lever of tool fluency and engagement in the interview

Results and Discussion

Here are two episodes that took place in the Adding world and could appreciate my goals with that represented a range evolving perceptuomotor expertise.

60	I	This is one you may make the groups and put them together. [Using the pinch gestures on the air]

Immediately, without direct instruction, Sarah used the pinch gesture to “gather” numbers together. The interviewer asked Sarah to make a seven. In the first attempt, Sarah maked a 7 by gathering 5 and 2. Then, the interviewer asked for a group of seven in another way than adding 5 and 2. Sarah’s try on making 7 by adding 2 and 4 failed, and she realizes that 2 and 4 make 6 and not 7. Thus, she continued:

70	S	3 and 4?
71	I	OK. Let’s try it
72	S	[Taps with index, middle and ring fingers.] (Figure 1- A).
73	iP	Three
74	S	[Taps again with same three right-hand’s fingers.] (Figure 1-B).

75 iP Three
76 S [Looks at her three fingers] Oops!

Sarah realized that she has forgotten to make a 4 with her four fingers; thus, she taped on the screen and made a 1 (Figure 1-C).

77 iP One
78 S [At the same time Sarah whispers] One.

Then she pinched 1 and 3 using her index and thumb fingers and made four following by gathering 4 with 3.

79 S [Whispering] Put together (Figure 1-D).
80 S [Smiles tilts her head back]. I made it!
81 I Good job, you made it with three different groups, a group of 3, a group of 3 and a group of 1.

Figure 1: Making 7 by Adding Three Different Groups 3, 3, and 1

This episode indicates a high level Sarah's body engagement emphasizing on numbers and corresponding finger (76). Sarah's justification and gestures in the upcoming episode reveal her emerging fluency with the instrument.

91 S [puts her five fingers on iPad]
92 iP Five
93 S [taps with all fingers]
94 iP Ten
95 I Oh, cool you made 10 really nicely there.
96 S [Pinches 5 and 10]
97 S Fifteen
98 I Cool!
99 S I did a 5 [puts her right hand fingers into left hand] (Figure 2-1), 10 [puts her left hand fingers into right hand] (Figure 2-2) and then I just made a 15 [grabs numbers on the air and putting them together, Figure 2-3], added them together.

Figure 2: Sarah's Gesture for Making 15.

Several dimensions of tool fluency were evident in this short episode, as well as associated changes in the way that Sarah interacted with the device, which was demonstrating in her gestures, talks and acts. She showed a high level of shifts in bodily engagements from using one hand to both hands and eventually gesturing on the air (Figure 2). For example, Sarah's gesture in the last episode indicates her integrated understanding on adding numbers in a sense of making or creating numbers and then "gathering" them together in the air which its initial idea arose from working on Adding world (figure 2). The finding prompts further research on how interacting with TouchCounts in classrooms may support the development of children's number sense.

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MATH ON THE MOVE: A VIDEO-BASED STUDY OF SCHOOL FIELD TRIPS TO A MATHEMATICS EXHIBITION

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In recent years the number of mathematics exhibitions in US museums has grown dramatically. Yet little research exists on how people learn in these settings. Even less is known about the relationship between learning math at exhibitions and in school. This study addresses these gaps through video-based ethnographies of school field trips to Math Moves!, an exhibition about ratio and proportion currently installed in four US science centers. Informed by theories of embodied cognition, this project applies qualitative analyses to video records of student and teacher behavior on the museum floor and during surrounding classroom activities, with the goal of characterizing how school groups (a) use and make sense of math exhibits and (b) relate those experiences to participation in the math classroom. Ongoing analyses aim to contribute to theories of embodied mathematical cognition in and across formal and informal settings.

Keywords: Informal Education, Learning Theory, Technology

Introduction: Math in Museums

Math in museums is on the rise: over the past 15 years, science centers and other informal learning institutions across the U.S. have been developing exhibitions about a wide variety of mathematical topics, including geometry, number, pattern, algebra, calculus, and ratio and proportion (Anderson, 2001; Cooper, 2011). Mathematical art exhibitions are gaining prominence in museums and professional meetings both in Europe and North America, and New York City recently became home to the Museum of Mathematics.

A striking feature of many contemporary mathematics exhibitions is that their design bears little resemblance to familiar images of school mathematics. For example, visitors to geometry exhibitions will be hard pressed to find the axioms and two-column proofs characteristic of high-school geometry. And at algebra exhibitions, references to formal symbolism and syntactic rules, often considered to be at the heart of school algebra, are equally sparse. How are visitors to museums and science centers interacting with and making sense of mathematics exhibitions? And, given the apparent contrast between many of these exhibitions and traditional classrooms, how might mathematical learning experiences in museums and schools be related?

While the opportunities for engaging with mathematics in U.S. museums are expanding, to date only a handful of research and evaluation studies have investigated learning in these environments (Anderson, 2001; Cooper, 2011; Guberman, Flexer, Flexer, & Topping, 1999; Gyllenhaal, 2006; Mokros, 2006; Nemirovsky, Kelton, & Rhodehamel, 2013, In Press). Even less is understood about how learning at these exhibitions might relate to past or present participation in school mathematics (Gyllenhaal, 2006). This project explores this unmapped empirical territory by tracing the activities of teachers and students as they move between mathematics classrooms and the exhibition *Math Moves!* in the context of a school field trip.

Math Moves!

Funded by the National Science Foundation, *Math Moves!* was recently developed and installed at four U.S. science centers. The exhibition invites users to engage multiple sensory

modalities, employ whole-body motion, and jointly collaborate in a variety of open-ended activities broadly related to the mathematics of ratio and proportion. This project addresses the research questions of how, in the context of a school field trip to *Math Moves!*, teachers and students (a) use and make sense of mathematics exhibits and (b) relate their experiences with mathematics exhibits to ongoing participation in their mathematics classroom, both during the visit itself as well as during surrounding classroom preparation and follow-up activities.

Ratio and proportion, the mathematical foci of *Math Moves!*, are key topics in school mathematics and researchers have been investigating children's developing understandings in these domains for about a century (Lamon, 2007). In recent decades, work involving formal-semantic analyses of relevant mathematical constructs, neo-Piagetian clinical interviews, and design experiments has provided a nuanced account of the mathematics of ratio and proportion from a developed disciplinary perspective as well as a deeper understanding of the intricacies of children's conceptual understandings in these areas. The present project stands to contribute to mathematics education research on ratio and proportion by investigating how these content areas might come into play in an unexplored educational context as well as by examining thinking and learning about ratio and proportion from the theoretical perspective of embodied cognition.

Theoretical Framing: Embodied Cognition in Context

Both the design of *Math Moves!* and the conceptualization of this research study are informed by recent developments in theories of embodied cognition and communication, particularly in the fields of mathematics education research (e.g. Stevens, 2012), informal science education (e.g. Rahm, 2004), and visitor studies (e.g. vom Lehn, Heath, & Hindmarsh, 2001). Embodied approaches to cognition open up particularly innovative perspectives on mathematical sense-making by offering alternatives to the culturally entrenched assumption that mathematics transcends – or is independent of – the kinds of everyday experiences provided by the body's capacities for action and perception (Lakoff & Núñez, 2000; Stevens, 2012).

The design of *Math Moves!* – like that of many current mathematics exhibitions – gives form to the theoretical commitment that engaging visitors' bodies in novel ways of sensing and interacting with complex mathematical phenomena has the potential to unlock powerful new inroads to learning and doing mathematics. This commitment in turn motivates a research approach that attends closely to how learners actually use their bodies – through active perception, expressive gesture, and exploratory movement, alone or in concert with one another – to make sense of exhibitions like *Math Moves!*.

Accordingly, this study adopts and contributes to what Stevens (2012) recently described as an interactionist perspective on embodiment that locates mathematical thinking and learning in the mutual elaboration of situated talk and action among persons and material artifacts. In mathematics education, studies adopting interactionist perspectives on embodiment have investigated a variety of phenomena related to mathematical cognition, including *generalization* (e.g. Radford, Bardini, & Sabena, 2007), *modeling* (Jurow, Hall, & Ma, 2008), and *tool and symbol use* (Nemirovsky et al., 2013). Finally, in addition to being framed by theories of embodied cognition, this work is further informed by scholarship in situated and distributed cognition that identifies a need to complement longstanding laboratory and experimental research paradigms with more naturalistic studies of cognition in and across the complex formal, informal, and everyday contexts in which it arises (e.g. Hutchins, 1995; Lave, 1988).

Methods: Data Collection and Analysis

Project methodology consists of a video-based field study that complements in-depth analyses of recorded episodes of situated social interaction with a broader ethnographic engagement with the communities under study (vom Lehn, Heath, Hindmarsh, 2002). Data collection and analysis focus on naturalistic video records of student and teacher behavior in the museum and surrounding classroom activities, and are supplemented with ethnographic techniques of participant observation and informal interviewing.

Project data derive from four classroom field trips to an installation of *Math Moves!* at a large Midwestern science museum. Data include extensive video and audio recordings of student and teacher activity in the museum and during surrounding classroom activities, the collection of physical and digital artifacts, contemporaneous and retrospective field notes, and ethnographic interviews with teachers and students. Research participants include two classroom teachers, approximately 80 pre-algebra students from both private and public schools, and 8 parent chaperones. Students represent a range of grade (5th through 7th) and achievement levels, including both gifted students and students identified as having special needs.

Analyses are currently underway and involve the interplay between two major qualitative research approaches: (a) inductive characterization of the dataset as a whole, using techniques inspired by thematic analysis (Braun & Clarke, 2006) and grounded theory (Strauss & Corbin, 1994) and (b) in-depth case analyses of selected segments of video data, using techniques from interaction analysis (Jordan & Henderson, 1995). Inductive analyses serve to provide a holistic account of the phenomena under study as well as to identify emergent patterns or themes related to the focus of the research questions. Interaction analyses are being conducted on a subset of video recordings selected on the basis of representativeness and relevance to the research questions. This analytic method is motivated by a theoretical focus on embodied cognition and entails repeated, interpretively cumulative viewings of a video record with attention to multiple communicative modalities, including talk, paralinguistic features (such as prosody), embodied actions (like gesture or postural adjustment), and manipulation of the material environment (Jordan and Henderson, 1995).

Emerging Findings

Ongoing analyses have begun to identify and investigate several themes related to learner experiences on the museum floor and how those experiences are related to participation in the mathematics classroom. Emerging themes and foci currently include: (a) the role of embodied social collaboration and physical manipulation of the *Math Moves!* exhibits in transitions from additive to multiplicative reasoning; (b) interactional work taking place on the museum floor that brings distal scenes, events, and vocabularies from school and everyday life into relation with the *Math Moves!* exhibits; and (c) the consequences of surrounding classroom activity structures and worksheet design for how participants perceive the relationship between *Math Moves!* and school mathematics.

Educational Significance and Relevance to Conference Theme

This project seeks to broaden theoretical perspectives on mathematical thinking and learning by leveraging close empirical attention to both verbal and nonverbal behavior in order to advance theories of embodied mathematical cognition. The study also addresses a global need in mathematics education to balance a long-standing emphasis on school settings with a greater understanding of out-of-school learning and its relationship with school-based learning (National Research Council, 2009; Jackson, 2011). U.S. museums and science centers constitute an especially consequential context with which to pursue this line of inquiry. Statistically, they

have grown dramatically in number in recent decades (National Research Council, 2009); organizationally, they make substantive contact with the K-12 school system through field trips, school outreach programs, and teacher professional development programs (Center for Informal Learning and Schools, 2006); and culturally, they are increasingly positioned as a key presence in the public education landscape (National Research Council, 2009).

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DOES PLAYING WITH TECHNOLOGY HELP TODDLERS LEARN THEIR 1, 2, 3?

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Increasingly, more of the toys targeted at infants and toddlers are electronic (Ritchel & Stone, 2006). Many electronic devices in the marketplace such as laptops or tablet computers are explicitly geared towards educational content in early language, literacy, and numeracy. This concurs with a study reporting that 78% of parents perceive electronic toys and devices as educational (Consumer Electronics Association, 2007). However, little is known whether young children learn from such devices. Thus, the current research investigated whether playing with a tablet with Android software applications for toddlers at home would impact their understanding of numeracy. Thirty children (15 with tablet, 15 without tablet) between the ages of 22 and 32 months old and their parents participated in this 2-month study. Our findings reveal that there was no significant learning gain in numeracy between the two groups of toddlers.

Keywords: Early Childhood Education, Number Concepts and Operation, Technology

Introduction

The use of technology by young children in play and learning has been disputable. Many electronic devices in the marketplace such as laptops or tablet computers are explicitly geared towards educational content in early language, emergent literacy and/or numeracy as asserted by the manufacturers. Such assertion is further supported by a study reporting that 78% of parents perceive electronic toys and devices as educational (Consumer Electronics Association, 2007). However, the American Academy of Pediatrics (2010, 2011) has recommended that children under two years old should be discouraged from any screen time (television and other technology) exposure and children older than three years old should be limited to screen time two hours/day. Yet, a recent study by Rideout (2011) reported that 24% of children between 2 and 4 years old played with technological devices such as computers, tablets or electronic games at least once a week and 12% of these children played with such devices daily. Despite this growing usage of electronic devices by young children, there is a dearth of research on the educational benefits that commercially available electronic devices afford to them.

Existing research on technology use by young children has primarily focused on the nature of parent-child communicative interactions during play, the nature of child manipulation and interaction with the electronic devices, and the learning benefits in language and emergent literacy. Research findings have been far less conclusive. For example, a case study by Smith (2002) reveals that a 2.5-year-old boy was able to incorporate the features such as the hypertext of his CD-ROM storybooks (also known as Talking Books) into his pretend play. However, another study conducted with 58 toddlers did not find any evidence of imaginative play using an electronic home play set (Bergen, Hutchinson, Nolan, & Weber, 2010). Moreover, a recent study conducted by Woolridge and Shapka (2012) with toddlers between 16 and 24 months old using commercially available electronic toys by V-tech and Fisher-Price reveals that parents were less responsive, encouraging, and actively engaged in supporting learning using the electronic toys compared to the traditional toys in a lab setting. In terms of educational benefits derived from playing with electronic devices, no significant language gain of toddlers using an electronic device was found (Bergen et al., 2010).

There is a pressing need to examine whether early mathematics learning could be facilitated by young children playing with electronic devices. Children already exhibit individual mathematical

differences before the age of four (Klibanoff , Levine, Huttenlocher, Vasilyeva, & Hedges,, 2006); and these differences are predictive of children's later mathematical knowledge throughout elementary school (Duncan et al., 2007). These findings underscore the fact that it is important to support early mathematical learning and development via a mathematically-enriched home and/or preschool environment to facilitate young children's subsequent mathematical competence. Furthermore, the National Research Council committee on Early Childhood Mathematics (Cross, Woods, & Schweingruber, 2009) has strongly recommended that children between three and six years old should be provided with adequate opportunities to engage in developmentally appropriate early childhood math activities in numeracy.

Thus, the present research investigated whether playing with a tablet with Android software applications for toddlers at home would impact their understanding of numeracy. The tablet has interactive educational software applications for language and literacy, mathematics, science, general knowledge, and emotional and social skills, as espoused by the manufacturer. The current study aimed to (a) examine the interactive patterns of children playing with the tablet such as duration of play and types of software applications selected using an online home journal system completed by parents, and (b) assess learning gains of children's numeracy competence over a 2-month period.

Method

Participants, Materials and Procedure

Fifteen young children (7 boys and 8 girls) in the tablet group (*mean age* = 26.87 months, *S.D.* = 3.70) and 15 young children (9 boys and 6 girls) in the no-tablet control group (*mean age* = 26.40 months, *S.D.* = 2.85) participated in this 2-month pre-test-post-test control group design study. Children in the tablet group were each given a tablet to play at home with their parents for two months during the study period while children in the control group were given a tablet each at the completion of the study. Parents in the tablet group were encouraged to curtail any deemed excessive use of the tablet by their children during the study period. They were also required to complete an online daily usage questionnaire to document their technology use with their children. Variables such as number of hours the child played with the tablet and the types of software applications chosen during each play session were included. They were asked to log onto the online questionnaire as many times a day as the need arose (e.g., the child chose to play with the tablet several times during that day). In addition, parents from both groups were asked to complete a questionnaire on their views of technology use by young children.

The family socio-economic status (SES) was measured by the mother's education level, a reliable proxy for SES (Catts et al., 2001). The highest education level attained by mothers in the tablet group was as follows: 6.7% of mothers with high school, 20.0% with college/trade, and 73.3% with university, graduate or professional education. In the control group, the composition of the highest education level attained by mothers was as follows: 20.0% with high school, 13.3% with college/trade, and 66.7% with university, graduate or professional education. All families were two-parent households.

The tablet has interactive educational software applications ranging from mathematics, language and literacy, science, general knowledge, to emotional and social skills. Children can progress through these applications at three levels of difficulty based on age: Level 1 from 18 months to 2.5 years old, Level 2 from 2.5 to 3.5 years old, and Level 3 from 3.5 to 5 years old. For this current study, given the age group of the participants, only software applications from Level 1 and 2 were loaded onto the tablets. As such, each tablet contained 25 interactive games, 12 songs, 5 stories, and a built-in camera. In terms of mathematical content, there were 3 games on numbers and counting, 1 game on quantity comparison, 4 games on visual and spatial skills.

Children in both groups were individually administered two tests to ascertain their early numeracy understanding: (i) Give-N task (Wynn, 1990; 1992), which assesses early cardinality understanding of number words by asking the child to give a puppet dog the number of balls it asked for, starting from the number '1'. Each child was given a plate of 10 balls. The puppet dog will stop at the numeral the child failed to give the correct number of balls twice; and (ii) the Quantity Comparison Task, which assesses early magnitude understanding by asking the child to help a puppet find the bigger pile of fish manipulatives presented in 7 different magnitude ratios presented in the following order: 1 vs. 6, 4 vs. 8, 5

vs. 10, 3 vs. 9, 7 vs. 9, 2 vs. 3, and 10 vs. 12) (Lee & Schell, under review). Each pile of fish manipulatives was counterbalanced in each presentation such that the larger quantity is not always on the right-hand side or left-hand side of the child. These tests were administered both at the start and the end of the 2-month period.

Results

From the questionnaire on technology use by young children completed by parents of the current study, about 70% of the parents deemed that the appropriate age to introduce electronic devices is between 1 and 2.5 years old. However, 73% of the parents indicated that they do not permit their children to play with mobile devices such as iPad without adult supervision. Moreover, only 27% of the parents would encourage their children to use electronic devices such as Vtech or Leapfrog.

Our findings reveal that parents of the tablet group logged a total of 469 entries whenever their child played with the tablet using the secured online home journaling system we have set up. Based on the parents' online home journaling data, our results indicate that the children played with the tablet for 38 days (out of the 2-month period or 60 days) and spent an average of 22 minutes (S.D. = 14.80; range was 2 to 90 minutes) playing with the tablet each time.

The most frequent feature on the tablet played was Games (66% of tablet play sessions), followed by Songs (38% of tablet play sessions) and Story (27% of tablet play sessions). Only 4% of the tablet play sessions were with the built-in camera. Parents reported about 97% of their child's tablet play interactions were enjoyable; about 58% of them were at a considerable level of enjoyment for each play session with level of enjoyment ranging from a little, some, considerable to a lot. The two main reasons attributed by the parents in terms of their child's enjoyment interacting with the tablet were the interactive nature of the games such as the ability to touch the objects to see them move around and watch the responses on the screen, and the 'relate-ability' between the games and the real world (e.g., themes involving kitchen, human anatomy and park).

In terms of whether the tablet affords learning yields to the children, about 71% of the parents' online home journal entries indicated that parents did not observe that their child has done or said something that reflected his/her learning from playing with the tablet. This finding is further supported by the lack of learning gains in the two mathematical tasks completed by the children. In terms of early cardinality understanding using the Give-N task, only 46.6% (7 out of 15) of children in both groups showed improvement in understanding the cardinal meaning of the number words from 'one' to 'two'. Thus, there was no significant increase in the number of children in the tablet group in terms of learning gains in understanding the concept of cardinality at the end of the 2-month study.

Analysis of covariance (ANCOVA) was conducted to assess learning gains in quantity comparison skills. Out of a total of score of 7, children in the tablet group had a mean score of 3.93 (S.D. = 1.75) and a mean score of 4.13 (S.D. = 1.92) during the pre- and post-testing sessions respectively. Children in the control group had a lower mean score of 2.67 (S.D. = 2.02) and a mean score of 3.27 (S.D. = 2.66) during the pre- and post-testing sessions respectively. However, our results show no significant differences between the two groups of toddlers after controlling for individual differences in pre-test scores [F (1, 27) = 1.25, *n.s*., eta^2 = .02].

Discussion

Our findings indicate that no significant learning yields were obtained between the two groups of children. As such, playing with an interactive tablet with educational games may not be the answer to acquiring early numeracy skills. Our findings are disconcerting as most parents are unsure of how to engage in mathematics activities with their children (Cannon & Ginsburg, 2008) and they may have thought playing with electronic toys could facilitate children's learning. Thus, our results serve as a caution to parents that having their toddlers play with technology does not necessary lead to educational benefits for them.

Adults must be cognizant that they have to find a way not only to direct the child's attention to mathematical use in the play activity but also to provide mathematical language such as counting words

and mathematical relationships, including one-to-one correspondence and how numbers can be used in different contexts such as ordinality (e.g., putting things in order) and cardinality (e.g., how many) to help challenge them to solve problems and encourage their persistence (Best Start Expert Panel, 2006; Ginsburg & Amit, 2008; National Council for Teachers of Mathematics/NCTM, 2000).

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A PROSPECTIVE MATHEMATICS TEACHER'S REASONING ABOUT OPTIMIZATION PROBLEMS USING CABRI 3D

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In this study, one prospective mathematics teacher was presented with a set of problems situated in a three-dimensional context and asked one to find the minimal distance between two points using Cabri 3D during a clinical interview. The results suggest that Cabri 3D facilitated our participant's reasoning particularly when she represented the problem situation using a 2D representation. Dragging, measuring, and rotating geometric objects also facilitated her problem solving.

Keywords: Geometry and Geometrical and Spatial Thinking, Technology, Teacher Education-Preservice

Introduction

Optimization problems, including minimization of a distance(s), have received a great deal of attention for a long period of time (Guven, 2008) in mathematics classrooms. However, some optimization problems require students to have calculus knowledge and this narrows the likelihood that students will have the opportunity to consider optimization problems (Cuoco, & Goldenberg, 1997). With the availability of DGS, students could develop non-calculus based strategies to solve optimization problems thus allowing more students to consider these types of problems (ibid, 1997).

Guven (2008) examined prospective mathematics teachers' problem solving processes as they considered a minimal network problem using dynamic geometry software (DGS). The results of the study suggested DGS enabled them to design multiple experiments and produce numerous conjectures to explore the location of the point that would minimize the distance. As a result of intensive interactions with DGS, students were able to verify their conjectures which led them to writing proofs, and constructing generalizations and extensions of the problem. A similar problem was used by Christou et al. (2005) in which six prospective mathematics teachers were asked to find a point (center) so that one could take the shortest distance from this point to four other points when they were connected. The result of the study indicated that DGS facilitated students' problem solving abilities while acting as a mediation tool. In both studies (Christou et al. 2005; Guven, 2008), students were asked to find a point that would result in the minimal distance in total from this point to other points when they were connected. However, these problem contexts were represented in two-dimensions, and students mostly focused on searching for the exact location of the center point and reasoning about why the location made sense.

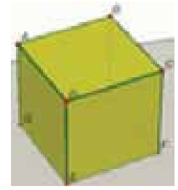
In the current study, we focused on problems situated in a three-dimensional context and asked a prospective mathematics teacher to find the minimal distance from a point to another point using Cabri 3D. We investigated how a prospective mathematics teacher's use of Cabri 3D software supported her solving optimization problems.

Methods

Participant and Tasks

Six prospective mathematics teachers were selected to participate in one clinical interview. For this study, results from one of the participants ("Kate") are reported. Kate told the interviewer that she last studied 3D geometry two years ago when she took Calculus III. She expressed her degree of comfort with three-dimensional geometry as neutral amongst very comfortable, comfortable, neutral, uncomfortable, and very uncomfortable. The clinical interview lasted approximately 90 minutes.

During the interview the following problem was posed:



> *Suppose there is an ant at point A of the cube whose edges are 4 cm and wants to go to point F using the shortest distance. The cube is closed, hence the ant will have to walk on the surfaces of the cube and the surfaces of the cube are not slippery. Can you find the shortest distance in cm that the ant supposed to walk?*

We posed questions, similar to those above, that asked to find the minimal distance between points *A* and *F* for different solids. In addition, we named vertex points of each solid exactly the same as shown above. However this study includes results only from the rectangular prism questions (we excluded one cube and one square prism questions).

Data Collection Tool and Procedure

A video camera, audio recorder, and a program that recorded computer screen (and voice of the interviewer and interviewee) were used to capture the participant's work with Cabri 3D. Because the participant was not familiar with Cabri 3D, the interviewer spent the first fifteen minutes teaching her the basic features of the program.

After the interview was conducted, the researchers transcribed the computer recording and analyzed the interviewee's descriptions, strategies, conjectures, use of technology, and outcomes/ consequences. We focused on computer screen movements because it clearly showed how the interviewee solved the problems. However, the interviewee sometimes preferred to reason about the problem using body language and provided explanations without explicitly mentioning point names like "this path is shorter than this." In such cases, video recordings were consulted.

Results

In the first problem, Kate explored that the minimal distance from points *A* to *F* of the cube could be taken by four different paths. She observed that the shortest distance would be taken if the ant walked from point *A* to the one of the midpoints of some line segments ($\overline{DE}$, $\overline{DC}$, $\overline{BC}$, and $\overline{BH}$) to point *F*. The next problem asked Kate to find the shortest path of a rectangular prism (*GE*=3 cm, *EF*=4 cm, *DE*=5 cm). For the purpose of communicating the location of the optimal point in this analysis we will use point *K* to represent the optimal point on $\overline{DE}$, point L as the optimal point on $\overline{CD}$, point *M* as the optimal point on $\overline{BC}$, and point *N* as the optimal point on $\overline{BH}$ although these points were not labeled or used during the interview.

When the interviewer asked her to describe her plan, Kate proposed to construct midpoints again because it minimized the distance in the first problem (Figure 1). However, when she represented the new path by marking a random point on $\overline{DC}$ and dragged the random point along $\overline{DC}$, she was able to see that the optimal point (point *L*) for the rectangular prism was not the midpoint $\overline{DC}$. She was very surprised, and when the interviewer asked her to reason about the optimal point on $\overline{DC}$, she said "*I don't see any reason because it's not, it's just a point it is not like the midpoint or it's not, there is not a name for it I guess*". When the interviewer asked her whether there were any other paths that might be less than or equal to 8.94 cm, she proposed to construct a point on $\overline{BC}$ and mentioned she could use this point to create a path with distance 8.94 cm. However, with this new path (Figure 2) the minimum distance was 9.49 cm. She started to compare the lengths on the top surface and side surfaces with the first path by saying "*it's not gonna work because the sides are different for like this triangle [showing triangle ADL] and this triangle [showing triangle ABM]; the sides are different so you're gonna get different hypotenuse lengths...*" Later on, she made part-to-part comparisons.

Afterwards, she proposed to construct a point on $\overline{DE}$ (Figure 3) to see if there was a shorter

distance. However, she was very unsure whether she could find a shorter distance. She made part-to-part comparison of paths considering optimal point on $\overline{DE}$ (point *K*). She mentally compared lengths of *AK* and *AL*, and *KF* and *LF*, and finally thought the last path could be longer. However, when she measured the lengths she was able to see that the shortest distance was 8.6 cm (Figure 4). She was surprised and explained this situation as follows: "*I guess because this part (AK) is longer than up here (AL) but, but the, if you use the Pythagorean theorem here (triangle EFK) it gives you a shorter c squared here to add to this (AK) and so it's short enough to where this (AK) being a little bit longer doesn't matter it's (AK+KF) still gonna be a shorter distance*". Lastly, Kate was able to show another shortest path follows point *A* to the optimal point on $\overline{BH}$ to point *F* (Figure 5).

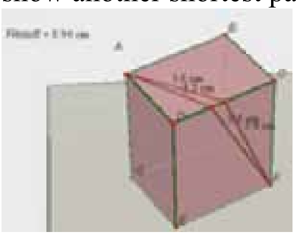


Figure 1

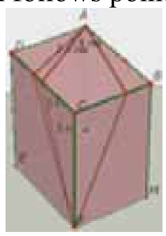


Figure 2

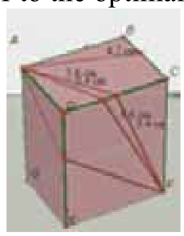


Figure 3

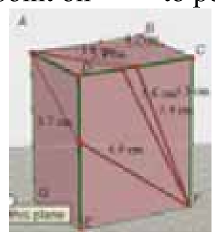


Figure 4

Figure 5

For the third problem (*GE*=5cm, *EF*=6cm, *DE*=3cm), Kate made connections with the previous problem and proposed that the shortest distance could pass through the optimal point on $\overline{DC}$. She measured the path as 10 cm and the researcher asked her whether there was a shorter distance than 10 cm. She, then, proposed to take the optimal point on $\overline{DE}$ (Figure 6) might be shorter since $\overline{DE}$ was the shortest edge. However, when she measured the new path's length and the distance was longer (11.41 cm). This may have prompted her to use her part-to-part comparisons strategy.

The interviewer asked whether she would find the location of optimal point on $\overline{DE}$ without using a trial and error method. She answered as follows: "*It wouldn't. I guess since it's not the midpoint or it is not something there is no specific way to get there, I guess. I think it would guess with trial and error.*" After this moment, the interviewer for the first time asked her to open the box dragging point *G* (Figure 7). Then, she guessed that the location of the optimal point was the intersection points of $\overline{DE}$ and $\overline{AF}$, and create a new line segment directly *A* to *F*, and said "*I guess it is the point where, the point where this edge (* $\overline{DE}$ *) of the box intersect the diagonal between A and F and that's the point where you have to go every time*".

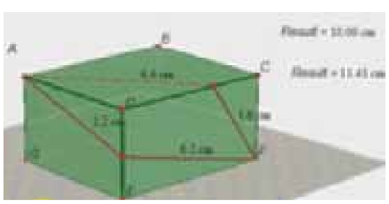


Figure 6

Figure 7

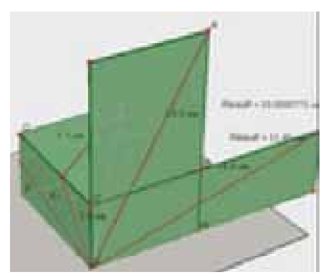


Figure 8

After Kate was satisfied with finding the location of each optimal point, a new question emerged about why each optimal point produced different total distances. She mostly explained simply why one path was shorter than another with Pythagorean theorem without making a generalization how she could identify without opening up the box. Here is a sample answer she gave: "*because the legs are when you squared and add them together, they give you the smallest number. So when you take them square that number you'll get the smallest c value for the hypotenuse*" (Figure 8).

For the last rectangular prism problem (*GE*=6 cm, *EF*=4 cm, *DE*=3 cm), she proposed to take the optimal point on $\overline{DC}$ because she misread the given edges to the problem. However, when she rotated the solid she quickly noticed that the shortest distance could be taken walking through an optimal point on $\overline{BC}$. When the interviewer asked her to compare the path lengths and explain why the second one was the shorter, she simply replied taking Pythagorean theorem into account by comparing the legs of triangles *AHF* and *AEF*. After this point, the interviewer asked her to generalize how she could find the shortest distance once more. She replied, " *I think what it is is that you're gonna take the two smallest umm segment's lengths that they give you and add them together and then the leftover length of the segment that they give you is gonna be the other leg of the triangle. And that'd give you the smallest diagonal or the smallest hypotenuse*". Afterwards she checked whether her generalization was valid for the earlier solids she studied, and she justified her generalization.

Discussion

In the current study, our participant Kate designed experiments in Cabri 3D, produced conjectures, tested their correctness by dragging objects extensively and making observations. If she could not validate her conjectures, she produced new conjectures, and validated and justified her new conjectures, which finally resulted in a generalization of the optimization problem. Similar to ours, such geometric problem solving steps for optimization problems in two-dimensional geometry using DGS were also observed in Christou et al.'s (2005) and Guven's (2008) study.

Kate had difficulty in reasoning about the locations of the optimal points, and this caused her to consider the surfaces of the solids independently, which resulted in making part-to-part comparisons of different paths on the same solids. Such difficulty reasoning about the location of the optimal point was observed in Christou et al.'s (2005) and Guven's (2008) study. Some students in these studies found the location of the optimal points as a result of measuring and dragging using DGS, but had a difficulty in explaining why these particular points worked and reasoning about the exact locations of the points. However, after Kate was directed to open the nets of the solids, she was able to reason about the specific locations of the optimal points, This may have been the critical point of her reasoning. By doing this action she was able to relate the two surfaces instead of thinking about them independently and making part-to-part comparisons. In this case, Kate was able to use a 2D representation instead of working on a 3D representation of the problem. Compared to her previous experiences with the problem before she opened up the nets of the solids, she was more goal-oriented and less discovery-oriented. She also made fewer measurements after she opened up the nets. We conclude that Cabri 3D facilitated Kate's reasoning and generalization of the optimization problem particularly after she opened up the nets of solids. Similarly, Christou et al. (2005) and Guven (2008) emphasized DGS facilitated students' generalization of geometric optimization problems, as well.

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GESTURE BLENDS IN AN IWB-MEDIATED MATHEMATICS CLASSROOM

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Given the recent proliferation of interactive whiteboard (IWB) technologies in classrooms, research is needed on their role within classroom learning dynamics. Gesture can take on a unique role in touch-based IWB environments. This study addresses the role of gesture in the learning of mathematics on an IWB through the framework of conceptual blending. Using video data of Grade 6 students and their teacher, four exemplary gesture-illustrative episodes were analyzed, and one of these episodes is presented and analyzed here. The episode reveals that gestures were used by the teacher on the IWB to relate multiple representations of an angle.

Keywords: Classroom Discourse, Technology, Cognition, Geometry and Geometrical and Spatial Thinking

Introduction and Purpose

Given the widespread adoption of interactive whiteboard (IWB) technologies in school boards across North America, this research aims to contribute to the understanding of how gestures are embodied and experienced in an IWB-mediated mathematics classroom. This relatively novel intersection is starting to warrant an increasing amount of attention (Bruce, McPherson, Sabeti, & Flynn, 2011; Eaton, 2011; Miller & Glover, 2006; Roth & Lawless, 2001). The presence and use of an IWB – with its touch input system and evolving gesture recognition – can create opportunities to engage with virtual objects and tools that could inform and alter the use of gesture in the learning of mathematics. This research considers gestures where "the hand shape or motion trajectory of the hand or arm represent[s] some object, action, concept or relation" (Alibali & Nathan, 2007, in Nathan, Eilam, & Kim, 2007, p.536). Through qualitative analysis of gesture use, this research is guided by an exploratory research question: What are the roles of gesture in the teaching of mathematics in an IWB-mediated environment?

Literature Review

The extant gesture literature connects the intentional use of gestures to improved learning outcomes. Students who were instructed or encouraged to gesture actively while learning mathematics were found to: retain knowledge better (Cook, Mitchell, & Goldin-Meadow, 2008); develop new problem solving strategies while becoming more cognitively receptive to future mathematics learning (Broaders, Cook, Mitchell, & Goldin-Meadow, 2007); and benefit from use of gesture when it is non-redundant with speech (Singer & Goldin-Meadow, 2005).

IWB technologies have been linked to enhanced whole-class discussions, more efficient presentation of visual demonstrations, improved classroom management, and as a portal to a vast range of online resources (Holmes, 2009; Bruce, McPherson, Sabeti, & Flynn, 2011). The IWB has also been characterized as an effective object of shared reference for engaging small and large groups of learners. Specifically, in a comparative study of the uses of interactive and traditional whiteboards in middle school classrooms, Fernández-Cárdenas & Silveyra-De La Garza (2010) suggest that focused classroom discourse – including gesture communication – can be negotiated by students' physical surroundings and the virtual tools and models being used on the IWB: "It is as if participants were constructing objects in the surrounding space with their hands and bodies and invoking concepts and categories with these objects to make sense of their intentions in the course of interaction" (p. 184).

Theoretical Frameworks

Cognitive Blending

In the analysis of gesture use, a variety of related factors must be considered including the nature of the mathematics problem, the context in which the mathematics is being done, and the tools that the learners have access to, such as the IWB. Thus, Fauconnier and Turner's (2002) framework of *cognitive blending* informs this analysis by relating that cognitive processing and meaning-making involves multiple conceptual spaces being 'blended' together to create a new conceptual space. Cognitive processing and thought is stimulated by this relationship since the resulting mental space "develops structure not provided by the [individual] inputs" alone, but one that emerges through the 'blending' of elements and relationships from the inputs into a meaningful conceptual space (Fauconnier & Turner, 1998, p.133).

Edwards (2009) uses Fauconnier and Turner's concept of blending as a framework to investigate pre-service teachers' use of gesture in communicating ideas about fractions. Based on the network representation model of Fauconnier and Turner (1998), Edwards (2009) provides a diagrammatic representation to present the inputs that inform a particular gesture blend. The first input, or 'Real Space', constitutes physical sources, which could include mediating tools such as an IWB. Based on prior knowledge, the second input space consists of the mental models employed by the gesturer. The third input is a generative step towards the final blend, where "whatever structure is recognized as belonging to both of the input spaces constitutes a [separate] generic space" (Fauconnier and Turner, 1998, p.143).

Methods

The principal source of data for this analysis is video footage from two Grade 6 classrooms. The data was collected as part of a study with a broader focus concerning significant learning moments with an IWB (Bruce, McPherson, Sabeti, & Flynn, 2011). Over twelve hours of video data was collected from observed mathematics lessons over an eight month period. After careful review of the video data, four gesture-rich episodes – ranging from 30 seconds to 10 minutes in length – were selected. Criteria for selection included: the clarity with which gestures were visible; instances of the use of gesture to communicate and reason mathematically; and instances where IWB use was perceived to influence gesture use. In the following analysis, the key events of one of these four episodes are presented in detail for illustrative purposes, including identification and analysis of employed gestures from the perspective of cognitive blending.

Results: Gesture Illustrative Episode

Episode Summary

In reviewing previously learned properties of parallelograms, the teacher uses the IWB and a dynamic virtual manipulative of a parallelogram to facilitate a whole-class discussion. The teacher uses two gestures to identify and represent the angles of a parallelogram. First, she identifies an angle as being created through the intersection of two line segments: in representing angle, $\angle ABC$, the teacher uses her finger to first trace the path of line segment AB and then the path of line segment BC on the IWB. Thus, the underlying focus is on the presence of intersecting line segments, and the creation of a defined space between them.

Next, in representing an equivalent angle, $\angle CDA$, she relates through gesture that an angle can be identified through three points: C, D and A. In representing $\angle CDA$, she traces the segments again but focuses attention on the points by *pressing* or tapping to highlight them on the IWB in the given order. Finally, she presses on the 'middle' point again to emphasize its importance. Thus, the teacher emphasizes an angle as being created through three points. This gesture reflects a connection to the familiar symbolic representations, namely $\angle CDA$ being equivalent to $\angle D$. In addition to these angle gestures, the teacher also employed several deictic gestures in pointing to areas of interest on the IWB. For instance, using her index finger, she points and circles the angle measurements calculated as the parallelogram's vertices are moved.

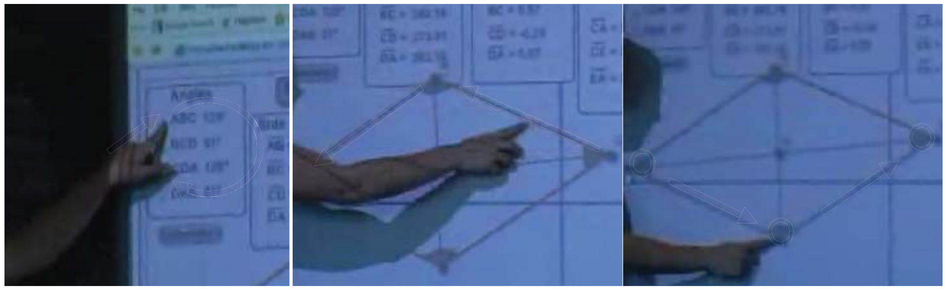

Figure 1: Gestures Used to Represent Angles on IWB.

Gesture Cognitive Blend

The gesture blend corresponding to the representation of an angle as the space between two intersecting line segments is illustrated in Figure 2, and presented to demonstrate the utility of cognitive blending in gesture analysis. In this gesture blend, the path of the finger corresponds to the act of drawing two line segments that meet at a single point.

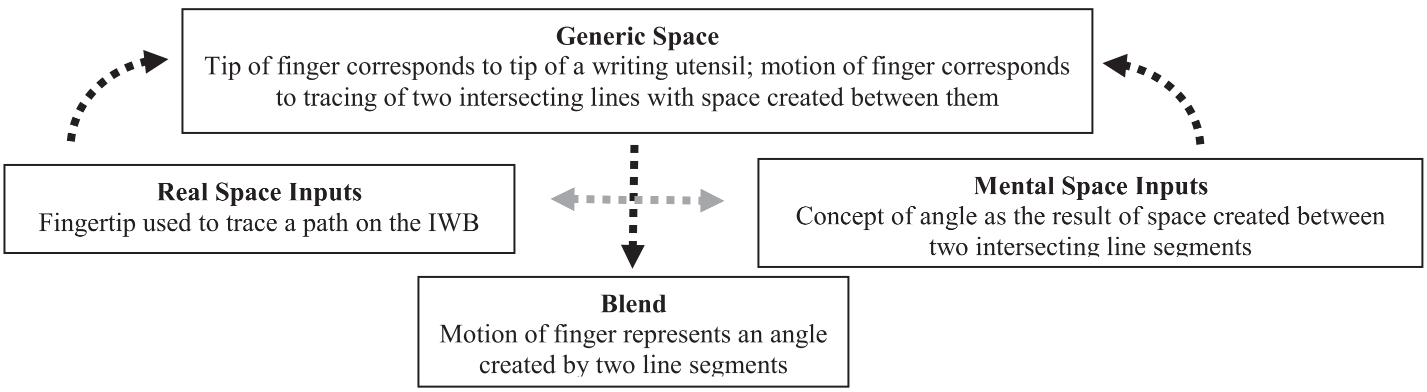

Figure 2: Blended Space of a Gesture Representing Angle Using Two Line Segments

Discussion

In this illustrative episode, gestures were used to relate multiple representations of a single concept. The teacher employed two complementary gestures representing two ways of conceptualizing an angle. Through these two gestures, the teacher highlighted the relationship between various representations (symbolic, diagram, verbal). As suggested in the literature, this use of multiple and different – albeit nuanced – gestures for the same concept may enrich students' understanding of the mathematics (Singer & Goldin-Meadow, 2005).

This episode also reveals one instance where gestures used by the teacher were related to actions on the IWB. The gesture that defined an angle through tracing line segments is closely related to the action for creating a line segment within the native IWB software. Another gesture – angle as motion through an arc of a circle, which is utilized by the teacher and her students in other portions of the video data – is very closely related to the action for rotation in the IWB software, as well as the use of the virtual protractor tool. Interestingly, the approach used by the teacher to mark an angle through the selection of three points (where the middle point identifies the angle) is related to other virtual manipulation environments. For instance, it is the method in which angles are marked in many dynamic geometry software packages, and it is conceivable that this gesture employs and reinforces previous learners' experiences engaging with this software. Thus, these gestures reinforce a multi-representational understanding of 'angle', and also model the corresponding actions that students have experienced

through past lessons.

Conclusion and Future Directions

The study of gesture in an IWB-mediated classroom is in its infancy, but there is great potential in its ability to leverage the IWB in a deliberate effort to improve student engagement with mathematics. As illustrated by the episode presented here, and the episodes in the larger research, gesture and IWB are related tools that facilitate communication and reasoning within mathematics learning. Given the emergence of more touch-based technologies, such as tablets, smartphones and ultrabooks, future research could consider how regular engagement and experience with these technologies – including gesture-based mathematics applications – can be influential in student and teacher use of gesture in learning mathematics.

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ELEMENTS OF GRAPHIC CONTRAST IN A SITUATION OF MODELING AND VARIATION

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Keywords: Design Experiments, Modeling, High School Education

There have been several studies on the use and construction of graphs (Roth & Bowen, 2001, Cen, et al, 2010). Buendía & Cordero (2005) reported theoretical understanding graphing through modeling activities and variation. The construction of these graphs is the integration of mathematical and physical categories helped us raise an activity consisting of three moments where students represent movement modeling and simulation by pencil and paper and using technology. The purpose is to interpret the student responses when graphed and variation modeling situations. Our framework takes Modeling-graphing category (Suarez y Cordero, 2008) to characterize the graphs designed by the movement from modeling. Our methodology is based on the idea of the moments of a Situation Modeling the Movement: motion modeling, designing graphics and the use of technology. We work with high school students a situation Modeling the Movement, one spring suspended on a universal holder, then a discussion for arguing among teams on graphs representing this situation and subsequently represented by using sensors, the graph of the movement. The *Moment 1* consisted of making a hand drawing to represent the motion of a spring when it undergoes the force of a weight of 50 g. The result of these interactions showed a set of meanings described the motion of the spring before and after being subjected to the movement of a mass of 50 g in one of its ends. The *Moment 2* was the drawing of a graph to represent the motion of the spring. The professor outlined the idea using technology to support, through motion sensors and software that represent the graph of position versus time; first performance had an image of what occurs when you add a weight to the spring. The contrast of the graphics, handmade and after with the use of technology, allowing the discussion focussed on the initial moment and end moment. The *Moment 3* consisted in the construction and characterization of the graphs from the data collection with the motion sensor by varying the distance point using as reference your own bodies in motion. The resulting graphs propitiated interesting and diverse contributions of ideas about the time the spring stop its motion we can rescue concepts including periodic motion, harmonic motion, balance and strength.

Acknowledgments

This paper has been made possible thanks the support received from the projects 20131847 and 20130198 approved for The Research and Graduate Studies Secretariat of the National Polytechnic Institute in Mexico.

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THE DESIGN OF A TABLET-BASED PRESCHOOL MATH PROGRAM

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In this poster we report on an innovative project to create and evaluate a cohesive set of digital and nondigital resources to promote early mathematics learning in preschools.

For the materials to be engaging, age appropriate, and supportive of young children in learning rich mathematics, we integrate research on (1) early mathematics learning trajectories, (2) developmentally appropriate joint media engagement, (3) preschool teacher professional development, and (4) evidence-centered assessment design. Integrating these approaches requires a diverse team, and we found that the mathematics education researchers, professional media and game designers, PD providers, and assessment designers have divergent perspectives, often with little commonality even in their vocabularies. To meet this challenge, we created a learning blueprint that was used as a "boundary object" across the different areas of expertise. This boundary object allowed us to integrate the perspectives of our diverse team.

We focus on subitizing and equipartitioning, two foundational aspects of mathematics not typically taught in preschool (Clements & Sarama, 2009; Confrey et al., 2009). The learning blueprint our research team created included related learning goals, example tasks, and an expected learning trajectory to inform resource design and assessment development. The design team developed digital games, nondigital activities, and a digital teacher's guide. The full team then engaged in iterative refinement processes (storyboarding, prototyping, and testing with preschoolers). The research team developed assessment tasks to evaluate children's learning.

Three preschool teachers implemented the program and pilot-tested the materials in their classes for approximately 20 classroom hours. A majority of lessons were observed using a detailed observation protocol, and students were administered pre- and post-assessments.

Students made significant learning gains ($F = 20.3$, $p < .001$). We will present the learning gains, the transformation from blueprint to learning activities and assessment items, and patterns of implementation based on qualitative analysis of our observations.

Acknowledgments

This material is based on work supported by the National Science Foundation. Any opinions, findings, and conclusions expressed in this material are those of the authors and not necessarily those of the NSF.

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TEACHER ASYNCHRONOUS NOTICING TO FOSTER STUDENTS' MATHEMATICAL THINKING

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Keywords: Design Experiments, Research Methods, Technology, and Teacher Knowledge

Emphasizing how a teacher specifically notices and responds to what a student says or does is one of the core tenants of modern mathematics education reform (Carpenter, Fennema, Franke, Levi, & Empson, 1999). Often, the very act of listening opens up space for a student to share his or her mathematical strategy, thereby positively impacting a student's mathematical growth (Jacobs, Lamb, Philipp, Schappelle, & Burke, 2007). Through a mobile app for smart phones and tablets, we explore teacher mathematical noticing with a modern twist. How do experienced K-8 mathematics teachers foster mathematical discussions with their students when these conversations happen asynchronously—not within the flow of real time?

In this design experiment, six students (three elementary and three middle school) made short videos explaining how they solved a specific mathematics problem while two teachers (one elementary and one middle school) responded using videos or text messages. All teachers attempted to use the same type of noticing, listening, and feedback as if in their classroom.

We built the mobile app using Ries's (2011) Lean Startup Model for generating rapid feedback on the usability of a prototype. Our prototype had four features: 1) students could make a video about how they solved a mathematics problem, 2) students could upload this video for their teacher, 3) the teacher could watch this video, and 4) the teacher could provide feedback/ask follow-up questions either through text messages or a new video.

While results are still preliminary, we found that the teachers enjoyed the tight focus of the short, student-generated videos. Both teachers appreciated the ability to focus in on a student's strategy through rewinding and playing each video again and again. Both teachers also felt typing out feedback through text messages forced them to be more conscious of their language and questions, which had lingering affects into their classroom teaching. Finally, we learned that teachers wanted the option to speed up/slow down the video, "tag" specific moments in the video to focus a student's attention, share/export the video, and be alerted when students received and read their feedback. Students also enjoyed experience. They were eager to create and share videos of their mathematics problem solving, felt that texting helped them better understand their own thinking, and felt the text discussions were more personal than classroom discussions. We hope this study helps the field understanding how to adapt mechanisms for asynchronous communication that values teachers' professional knowledge and continues to foster teacher/student communication.

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STUDENT USE OF PEDAGOGICAL AND TOOL FEATURES OF AN INTERACTIVE DIGITAL CURRICULUM PROTOTYPE

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Keywords: Technology, Curriculum, High School Education, Probability

There is growing need for and interest in student use of interactive digital instructional materials for school mathematics that capitalize on new advances in mathematics teaching and learning (Center for the Study of Mathematics Curriculum, 2010). Studying how students use technology tools during mathematical activity can lead to insights about strategic use of tools and student understanding of mathematics (Zbiek, Heid, Blume, & Dick, 2007).

Transition to College Mathematics and Statistics (TCMS) (Hirsch, Hart, Watkins, et al., in press) is a problem-based, CCSSM-oriented, fourth-year high school mathematics course designed for college-bound students whose intended program of study does not require calculus. The pedagogical and tool features of the TCMS digital prototype include embedded multimedia clips; mathematical and statistical software; open-ended problems linked to webbed, prompted scaffolding; synced, shared notebook; student-centered learning environment; and accessibility features such as text-to-speech software, embedded hyperlinks, scroll-over definitions and a glossary. Drawing upon tool mediation (e.g., Drijvers, et al., 2010) and work methods (e.g., Guin & Trouche, 1999), this study uses the prototype of an instructional unit designed for TCMS as a context for investigating the research question: *What are the ways students use the pedagogical and tool features of digital materials as they solve problems and construct their learning?*

The setting for this study was a twelfth-grade mathematics class at a high school in which each student had a laptop with access to the digital prototype. Eight students participated in this study. Collected during two multi-day digital investigations on binomial distribution and statistical inference, the data included video recordings of classroom practice, copies of student work, and audio recordings of semi-structured interviews. Based upon an analysis using an emergent coding scheme, this study characterizes student use of the different pedagogical and tool features of the digital prototype.

The results of this study contribute to the body of literature focusing on the student-tool relationship during mathematical activity. It provides mathematics educators with insights into the appropriateness of student use of the different features and their understanding of binomial distributions and statistical inference. The results inform curriculum developers as they design and revise pedagogical and tool features of digital learning materials.

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PhET SIMULATIONS FOR TEACHING AND LEARNING MATHEMATICS

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Technology can serve as a powerful tool for supporting students in learning mathematics (Li & Ma, 2010; NCTM, 2000). Since 2002, the PhET Interactive Simulations Project at University of Colorado Boulder has developed and studied the use of interactive computer-based simulations (sims) for teaching and learning STEM topics. While this effort began with science-based sims, the project, which includes over 125 sims to date, has initiated a significant effort in the realm of mathematics, designing new sims that align with the CCSSM. These sims are available free of charge at http://phet.colorado.edu.

In this poster, we describe PhET's goals for students, which include supporting students to: see mathematics as accessible, understandable, and enjoyable; achieve conceptual learning; make connections to everyday life; engage in mathematical exploration; and take and sense ownership of their learning experience (Hensberry, Paul, Moore, Podolefsky, & Perkins, in press). We describe PhET's design principles (e.g., use of implicit scaffolding [Podolefsky et al., in preparation], dynamic feedback, pedagogically powerful actions, and real-world connections) and how these are used to create effective sims.

We then describe ways in which PhET sims may support student learning of mathematics. Specifically, we interviewed third-grade students as they completed a sim-based activity on fraction equivalence. These interviews were video recorded. In addition, students' interaction with the sim was recorded using screen capture software. Analysis of pre- and post-test scores and video data suggests that some students' knowledge of fractions concepts improved, and their strategies for solving problems became more sophisticated after engaging with the sim. These findings suggest that PhET sims may be a powerful tool for supporting students in learning mathematics.

Acknowledgments

We thank the PhET team for their contributions, including the software development team: Sam Reid, Chris Malley, John Blanco, Jon Olson, and Mike Dubson. We also thank our teacher partners and their students for their participation in these efforts. This work was supported by grants from the O'Donnell Foundation and the Hewlett Foundation.

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ONE TEACHER'S INSTRUMENTAL ORCHESTRATIONS OF STUDENTS' SMALL-GROUP WORK WITH GRAPHING CALCULATORS

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Many researchers have explored the impact of graphing technologies in mathematics classrooms (see Heid & Blume, 2008). The implementation of such instruments often raises questions related to new complexities concerning the teacher's role in supporting rich student learning (Guin, Ruthven, & Trouche, 2005). In recent work, researchers have examined ways that teachers use whole-class discussions to guide and shape students' mathematical activity with these types of instruments (e.g. Drijvers, 2011). This present study builds on and extends related research by specifically targeting teacher actions used to support students' graphing calculator use in small group settings.

Following a case study design adapting Drijvers' (2011) *instrumental orchestrations* as an analytical framework, I explored the following research question: *What types of instrumental orchestrations does a teacher implement with students who are working within small groups?* I observed and video-recorded a single high school teacher's instruction focusing on quadratic functions over the course of four days. Coding of the video data revealed patterns in the ways that the teacher supported student activity with graphing calculators. Post-lesson interviews were used to triangulate the observed orchestrations.

This poster presents preliminary results showing that the teacher heavily favored the use of two types of orchestrations, the *Work-and-Walk-by* and the *Teacher-Demo* (Drijvers, 2011). While implementing the *Work-and-Walk-by* orchestration type, the teacher primarily verified that students were on task and making progress towards solving the assigned problems. The *Teacher-Demo* orchestration, however, was implemented in multiple ways, which led to the delineation of three subtypes of orchestrations: one where the teacher computes without showing students the keystrokes used on the graphing calculator; one where the teacher shows the keystrokes used; and one where the teacher performs some of the graphing calculator operations and pauses to prompt for student input and thinking. The latter of the three subtypes suggests an intermediate or transitional type of orchestration whereby the teacher begins to seek evidence of student thinking and student dialogue with the technology. These findings may be used to help teachers reflect on ways in which they can support student learning through the use of graphing calculators and related technologies. Further work may investigate a possible link between these types of interactions and teacher shifts from teacher-centered orchestrations to more student-centered orchestrations.

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IMPACTING ACHIEVEMENT AND SELF-EFFICACY THROUGH TECHNOLOGY

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Online education has surged into the mainstream of public and private education. "Thirty-one percent of all higher education students now take at least one course online" (Allen & Seaman, 2011). Wadsworth et al. (2007) studied 89 college freshmen who were enrolled in an online developmental mathematics course for the first time. Their results showed that self-efficacy is a significant predictor in students' final grade and that self-efficacy is statistically different between final grades in the online course (Wadsworth et al., 2007). However, the authors only studied students in an online mathematics course. The intent of my poster proposal is to provide an overview of a pilot study comparing fully online mathematics courses to traditional F2F (face-to-face) mathematics courses. *How do student learning outcomes compare for traditional (lecture-based, low technology) F2F mathematics classes, traditional (lecture-based, low technology) online mathematics classes, high technology-based F2F mathematics classes, and high technology-based online mathematics classes? To what extent does learning mathematics through an online class impact student confidence in their mathematical ability?*

In order to address the research questions, this action research pilot study will consist of a quantitative approach using a quasi-experimental design. There will be four samples of students: (a) one sample from a traditional (lecture-based, low technology) F2F class, (b) one sample from a traditional (lecture-based, low technology) online class, (c) one sample from a high technology-based F2F class, and (d) one sample from a high technology-based online class. At the beginning of the course students will be given a pre-test on mathematics content knowledge and a mathematics self-efficacy survey. This will provide a base line score in order to understand and compare students are their prior knowledge and level of mathematical confidence. At the end of the course students will be given a post-test on mathematics content knowledge and a mathematics self-efficacy survey.

A multiple regression analysis will be conducted to examine the relationship between self-efficacy, level of technology integration (high or low), mode of course delivery (online or F2F), and final grade (total points in course). The regression analysis will determine which independent variables (self-efficacy, high technology, low technology, online delivery, F2F delivery) are statistically significant in predicting the dependent variable (final grade). I will also separate final grade into groups (A, B, C, F) in order to examine the relationship between levels of performance and self-efficacy, level of technology integration (high or low), mode of course delivery (online or F2F). An ANOVA test will also be conducted using final grade (A, B, C, F) as the independent variable and self-efficacy as the dependent variable. This will indicate which groups of students are statistically different in their self-efficacy.

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CASE STUDIES OF BROADENING CLASSROOM INTERACTIONS WITH REPRESENTATIONS IN THE PRESENCE OF TECHNOLOGY

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This poster presents an investigation of the ways in which three high school algebra teachers broadened his or her classroom interactions involving representations of functions in the presence of connected classroom technology (CCT) over time using data from a larger research study stemming from national research project Classroom Connectivity in Promoting Mathematics and Science Achievement[1] (CCMS) (Irving et al., 2010). The larger research study developed a model for growth in the use and treatment of representations, technology, discourse and decisions made in the presence of CCT by analyzing the case studies of these three different teachers. Of particular interest were the manners in which two of these teachers, who originally had representationally barren classroom environments, increased the richness of representational environments. This poster restricts the focus on the larger study to the growth or lack thereof of the level of representational use and treatment.

Developing a rich representational environment for functions is thought to require multiple representations present from different sources, functions treated as both process and object, and representations need to be treated as something to reason with/about and used in explanations or justifications (Moschkovich, Schoenfeld, & Arcavi, 1993; Pape & Tchoshanov, 2001). Using these indicators a model for different levels of representational interactions was developed. Each teacher was analyzed using this model to detect if the teachers broadened their representational classroom interactions. This poster presents the how the growth or lack thereof of the teachers' treatment and use of representations in the presence of technology grew in relationship to their potentially changing orientation toward mathematics using a adapted the procedural to conceptual orientation (Thompson, Philipp, Thompson, & Boyd, 1994) to algebra classrooms.

Endnotes

[1] The research reported here was supported by the Institute of Education Sciences, U. S. Department of Education, through Grant # R305K050045 to The Ohio State University. The opinions expressed are those of the authors and do not represent views of the U. S. Department of Education.

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FACILITATING DISCOVERY LEARNING IN THE TABLET ERA: RETHINKING ACTIVITY SEQUENCES VIS-À-VIS DIGITAL PRACTICES

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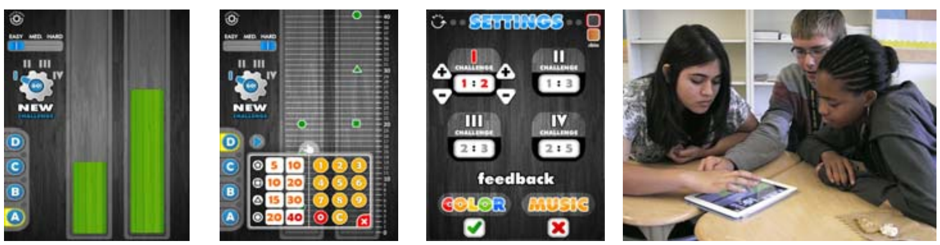
Keywords: Technology, Design Experiments, Rational Numbers

Reflecting on empirical results from a design-based research pilot intervention, we frame our study as paradigmatic of what may occur when digital devices are introduced into a classroom without anticipating students' preexisting cultural practices surrounding a technological medium. In a previous study (Reinholz et al., 2011) we evaluated a Wii-mote-based design for proportions and we used an interview protocol to guide students through a sequence of activities that implemented our design rationale. We wished to replicate this design in tablet form. However, tablets afford users free access to all interaction features.

The Mathematical Imagery Trainer for Proportion (MIT-P) is an embodied-learning device. Users solve motor-action problems and articulate their solutions prior to the introduction of formal notation. They manipulate two bars (Fig. 1a) that turn green if the bars compare in height by a "secret" ratio (e.g., 1:2). They bootstrap principles of proportional equivalence by developing strategies for making green. Various tools—grid, numerals, and table (Fig. 1b)—scaffold progressive mathematization of these strategies. Users can modify the activity's ratios, feedback, and appearance (Fig. 1c). Three 9th grade students participated in a 25 min. interview (Fig. 1d).



a. b. c. d.

Figure 1: The Mathematical Imagery Trainer for Proportion (MIT–P).
[The app is a free download at https://itunes.apple.com/us/app/mathematical-imagery-trainer/id563185943?mt=8]

The students, all fluent tablet consumers, engaged the medium in ways that inadvertently derailed our intended activity sequence and therefore undermined our design rationale. Scaling up to classrooms (Lamberg & Middleton, 2009), we modified our activity rationale so as to accommodate students' digital practices (see in Lee, 2013; Negrete, 2013).

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A SURVEY OF MATHEMATICS EDUCATION TECHNOLOGY DISSERTATION SCOPE AND QUALITY: 1968-2009

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Dissertation studies often make novel and meaningful contributions to their fields (Auerbach, 2011). These studies serve to indicate how new researchers are thinking about the field, how doctoral advisors are guiding their students, and how the baseline for research quality is considered by the field. Systematic reviews of these studies are needed to identify patterns and to determine whether and how a field is advancing. Without such reviews, the ability of doctoral advisors to effectively guide their students is severely limited, which ultimately impedes the advancement of the field. Our search for such a review of mathematics education technology dissertation studies revealed no reviews currently available.

We therefore set out to examine mathematics education technology dissertations by considering the types of technologies addressed, how outcomes have or have not changed, and how the quality of the studies has changed between 1968 and 2009 by considering two questions.

1. What is the scope of mathematics education technology dissertations?
2. What is the quality of mathematics education technology dissertation research?

We found that new researchers have consistently focused on emerging technologies for their dissertations. The number of technology studies for some types of software, such as tutorial software, was high in number but low in replication, as indicated in the Other Technology and the Miscellaneous subcategory. Such conditions limit researchers' ability to synthesize dissertation findings, thereby limiting the ability of dissertations to advance the literature base to the fullest degree.

Doctoral students as well as their mentors can play a pivotal role in such improvement by (a) attending to the QF categories as they plan, design, implement, and complete their dissertation studies and (b) demanding that dissertations represent a minimum bar for quality rather than a maximum. The mathematics education technology research community must in turn begin to demand greater clarity in its published studies, both through how researchers write about their own studies and how they review the works of others.

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PERFORMANCE ASSESSMENTS

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Technology-based learning interventions have a complex history of varied results, ranging from the surprising benefits of action games for the perceptual system (Green & Bevalier, 2007) to the depressing decrease in language skills documented in two-year-olds who watched the *Teletubbies* television show (Linebarger & Walker, 2005). Designers are intentionally aiming for certain types of learning in such interventions, but are not always able to achieve their goals. Our poster will briefly share the process of designing a performance assessment to measure the mathematical learning of pre-school children who participated in a *Kinect Sesame Street TV* Number episode designed to support counting activities.

The Kinect technology, when connected to the Xbox 360 game console and a television, essentially allows interaction between digital media and the person(s) in front of the television. This technology is quite new, and most frequently used during action-related games (such as *Dance Central 3*, which teaches players dance steps, and then evaluates the accuracy of their moves). However, a partnership between Microsoft and Sesame Street has begun developing interactive and educative episodes for the Kinect and very young children.

Both the Common Core State Standards for Mathematics (2010) and the National Council of Teachers of Mathematics standards (2000) emphasize the importance of counting for young children. Counting—and connecting the abstract to the concrete by counting objects in the world—is a crucial component to advancing in mathematical learning. Consequently, the *Kinect Sesame Street TV* episode designers developed an interactive session during which children assist Grover in picking up fallen coconuts, and are encouraged to count with him. The intervention was followed by a performance-based assessment, where the children were asked to complete various counting tasks. In our poster, we will report fully on the process of developing, implementing, and analyzing the assessment activities—and the inferences about learning we are consequently able to make.

Acknowledgments

We would like to thank Alex Games of Microsoft Studios and Rane Johnson of Microsoft Research for supporting the research that led to these findings.

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FLIPPING CALCULUS WITH TrACE: A WEB-BASED MEDIA PLAYER FOR COLLABORATION IN CALCULUS I

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The proliferation of flipped or inverted classrooms is changing the way instructors deliver course content in a variety of disciplines (e.g., Demetry, 2010; Deslauriers, Schelew, & Wieman, 2011) including mathematics. Many who implement flipped pedagogy make heavy use of web-based video lectures which students watch in preparation for class. These videos are typically hosted on YouTube, institutional web-servers or learning management systems which provide few collaborative tools to foster communication between students about the content contained in the videos. At best, these systems provide a single comment thread associated with the entire video. Unlike typical classroom settings, students are unable to ask a question at the moment it arises and engage in a discussion with classmates or the instructor about that or related issues.

Early research in computer-supported collaborative learning environments demonstrated that placing collaboration tools within the media itself increased the likelihood that students would engage in collaborative discourse and increased the quality of that discourse (Guzdial & Turns, 2000). To address the shortcomings of current web-based media environments, we have developed TrACE, a web-based video player which provides collaborative tools that can be used to link annotations to a specific physical space and time point in the video.

In TrACE, as the video plays in the left portion of the screen, markers indicating an annotation appear in the video. Simultaneously, the annotations linked to that portion of the video scroll on the right. This feature draws the viewers' attention to the annotations which are of immediate relevance. Annotations are linked to specific authors and color-coded by type for easy reference. To embed new comments, questions or reply to an annotation, students click or tap in the video itself to create the annotation marker. This feature allows students to point to a specific element on the screen which is of interest.

Currently, we are piloting TrACE in two sections of a flipped Calculus I course at University of Hartford with a total enrollment of approximately 45 students. In this poster, we will examine students' opinions of TrACE as a media player, and the quantity, the type (i.e., question, comment, response, etc.) and the quality of students' annotations.

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DEVELOPING ANGLE CONCEPTS IN A MOTION-CONTROLLED MATHEMATICS LEARNING ENVIRONMENT

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Keywords: Geometry and Geometrical and Spatial Thinking, Instructional activities and practices, Technology

Body-based experiences can play an important role in learning (Petrick & Martin, 2012); however, if students fail to draw connections between their physical movements and mathematics concepts, it is unlikely that students will learn. In this poster, we will describe the design and development of a motion-controlled learning environment that grounds angle concepts in concrete, physical experiences. Further, we share the results of a clinical interview of a fourth grade student interacting within the motion-controlled environment.

This work is based on an embodied view of cognition in which cognitive processes are developed through the interactions of our minds, bodies, and the environment (Anderson, 2003). Previously, education researchers studied students' development of angle concepts through embodied experiences, but these studies lacked a visual representation to bridge the physical to the abstract (Clements et al., 1996).

In order to more effectively ground angle concepts in physical activity, we have developed a learning environment that includes a visual representation utilizing the Kinect for Windows. Students use their arms to control two rays forming an angle projected onto a screen. As students' move their arms, the size of the angle changes, while simultaneously the background color of the screen changes depending on the size of the angle. Students are challenged to determine what makes the screen each color. The application also includes various layers of information that the researcher can adjust (such as adding a protractor overlay that allows the student to measure the angle on the screen).

In this poster, we will describe in detail the learning progression of one fourth grade student who participated in a 15 minute one-on-one semi-structured interview with a researcher. From a pre-assessment, we found that the student had little understanding of the relative size of angles, and he could not use a protractor. At first when interacting with the Kinect application, the student made judgments about the angles formed with his arms based on the height of his arms rather than the degrees between them. As layers of information were added on the screen, he began describing angle size using the amount of rotation needed to map one ray onto the other.

We found that starting with a physical, embodied representation of angle and increasingly adding information to the abstract, visual representation on the screen supported this student in developing understanding. Our future research will investigate this hypothesis.

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TECHNOLOGY AND ALGEBRA IN SECONDARY MATHEMATICS TEACHER PREPARATION PROGRAMS

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Recent standards documents from NCTM and CCSSO advocated for learning algebraic topics with technology. Research recommended that teachers make intentional use of knowledge of limitations and affordances of technology use for teaching and learning of algebra topics (Ferrara, Pratt & Robutti, 2006). CBMS (2012) recommended that pre-service teachers (PSTs) have opportunities in mathematics courses to develop this ability to choose and use technology strategically.

This study explores opportunities provided by teacher preparation programs for secondary mathematics PSTs to encounter technology in learning algebra and in learning to teach algebra. We define technology narrowly as electronic tools and software. This study is a qualitative analysis of pilot data gathered from a larger mixed-methods project. We compiled data about teacher preparation programs at three universities through a focus group of three or four graduating PSTs, interviews with mathematics and mathematics education instructors, and instructional materials for select courses at each site. We used recommendations from CBMS (2012) to develop an analytical framework focusing on: type of use, type of course, type of encounter, type of technological tool, and whether the PSTs are asked to think critically about the use or choice of technological tool for learning or learning to teach each particular algebraic topic. Our analysis includes a general overview of opportunities for PSTs to use technology in learning mathematics and in learning to teach mathematics. We then narrow our focus to examine these opportunities as they relate specifically to algebra.

We found that instructors integrated technology into mathematics education courses, with more variety of types and uses, than into mathematics courses. Our data suggest that even in mathematics courses that use technology PSTs have few opportunities to see and use a variety of technological tools. Our poster will describe general patterns of technology use in mathematics and mathematics education at each university. It will also display examples of specific algebra tasks with technology, and our analysis of those tasks according to the framework developed from recommendations in CBMS (2012).

Acknowledgement

This study comes from the *Preparing to Teach Algebra* project, a collaborative project between groups at Michigan State (PI: Sharon Senk) and Purdue (co-PIs: Yukiko Maeda and Jill Newton) Universities. This research is supported by the National Science Foundation grant DRL-1109256.

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MATHEMATICS TEACHERS' PROBLEMS AND NEEDS IN USING THE GEOMETER'S SKETCHPAD PROGRAM TEACHING MATHEMATICS

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Keywords: Affect, Emotion, Beliefs and Attitudes, Technology

Technology development is important to development of the world, including educational development. Technology, such as computer software, plays an important role in mathematics education to enhance student learning. The Geometer's Sketchpad (GSP) is one of the computer software programs used in mathematics classrooms. In Thailand, GSP has been broadly promoted to use in mathematics classrooms because it can provide visual figures representing abstract mathematical concepts.

The Institute for the Promotion of Teaching Science and Technology (IPST), Ministry of Education, Thailand, is one of the national agencies which promotes using GSP in mathematics classrooms. IPST also provides training on how to use GSP in classroom instruction. After the training, it is important to explore the results of integrating GSP in classroom instruction from the teachers who passed the training. Thus, this study was conducted to answer a research question: "What are problems and needs in using GSP teaching mathematics of the teachers who participated in IPST's training on how to use GSP in classroom instruction?"

Population in this study was 401 mathematics teachers who attended a training on how to use GSP in classroom instruction, conducted by IPST. There were 160 in-service teachers in this group who were selected by purposive sampling in order to get the research participants who passed the training and apply knowledge gained from the training to classroom teaching.

The data in the study was collected by a questionnaire on problems and needs in using GSP teaching mathematics. The questionnaire was separated into 3 parts. The first part was a set of checklist questions on personal background of the participants. The second part was a rating scale questionnaire on problems in using GSP teaching mathematics. The third part was a rating scale questionnaire on needs. The last part was an open-ended questionnaire asking for additional comments on problems and needs in using GSP teaching mathematics. The collected data was analyzed by using the Statistical Package for the Social Sciences (SPSS) for mean, mode and Pearson's correlation coefficient.

Results of the study reveal that the participants have a moderate level of problems in using GSP to produce instructional media. They have a low level of problems in teaching techniques and methods. Also, they have a low level of problems in realizing the importance of the GSP. Overall problems in using GSP teaching mathematics have 99% statistical negative correlation with teaching experience. The results show that the teachers who have more than 15 years of teaching experience have the fewest problems with using GSP teaching mathematics.

The results on the needs in using GSP teaching mathematics reveal that the teachers have the high level of needs in development of skills to use GSP and in learning on how to use GSP to produce instructional media. Also, they have the moderate level of need in teaching instrument. The teachers who have 5 – 15 years of teaching experience have 99% statistical positive correlation with needs, while the teachers who have more than 15 years of teaching experience have 99% statistical negative correlation with the needs.

Editorial Note: The author did not include references in final submission.

STUDENT-CONSTRCUTED DYNAMIC GEOMETRY TASK ANALYSIS FRAMEWORK

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Keywords: Technology, Geometry and Geometrical and Spatial Thinking, Teacher Knowledge

A number of frameworks exist that allow one to analyze the potential of mathematical tasks (Smith & Stein, 1998, and Boston et al., 2006). Sinclair (2003) offers recommendations to promote student exploration with pre-constructed dynamic geometry sketches. However, teachers often require students to create their own dynamic geometry. There is an identified need in research and practice to evaluate the potential of student-constructed dynamic geometry sketches. A student-constructed dynamic geometry sketch is a task in which the student responds to prompts as they create a dynamic geometry sketch. The purpose of this poster presentation is to introduce and discuss a framework by which such sketches can be analyzed for mathematical depth and the quality of prompts for technological action.

This work came about from analyzing a number of student-constructed dynamic geometry sketches and classroom video footage of students engaged in the sketches as part of a larger three year professional development project. The researchers used case study methodology to explore the question, "What qualities of student-constructed dynamic geometry sketches lead to pivotal teaching moments as defined by Stockero and Van Zoest (2012)?" Stockero and Van Zoest define a pivotal teaching moment as, "an instance in a classroom lesson in which an interruption in the flow of the lesson provides the teacher an opportunity to modify instruction in order to extend or change the nature of students' mathematical understanding" (p. 3). To assist researchers attempting to disclose these qualities, the Student-Constructed Dynamic Geometry Task Analysis Framework was created and employed to analyze the tasks available in the project.

The Student-Constructed Dynamic Geometry Task Analysis Framework includes two major sections: "Allowance for Mathematical Depth"; and "Quality of Prompts for Technological Action." The researchers have found benefit in using this framework to delineate qualities of student-constructed dynamic geometry tasks. The poster will include samples of student-constructed dynamic geometry sketches and the corresponding analyses with the framework.

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THE ROLES OF DYNAMIC GEOMETRY SOFTWARE IN STUDENTS AND PRE-SERVICE TEACHERS' LEARNING OF GEOMETRY

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Keywords: Geometry and Geometrical and Spatial Thinking, Teacher Education-Preservice, Technology

The introduction of dynamic geometry software (DGS) has afforded teachers and learners with new opportunities for mathematical modeling and explorations. Even though some powerful DGS packages have been available for more than 20 years, they are not widely used in the teaching and learning of geometry (Laborde, 2001). The goal of this study is to produce a comprehensive review of research evidence on the roles of DGS in pre-service teachers' (PSTs) learning of geometry to improve their education programs. Searches of three databases (ERIC, JSTOR, and PsychINFO) for peer reviewed articles published in scholarly journals between 2000 and 2012 were conducted. To get a more comprehensive understanding of the theoretical underpinnings of the roles of DGS, broad search terms were used, including 'Dynamic Geometry Software', 'DGS', and 'PSTs' using of DGS'. Only empirical studies including impacts of DGS, how DGS can be used, and the challenges faced by PSTs in using DGS were used in the analysis.

Many researchers found that DGS, specifically Geometer Sketchpad and Cabri, has positive impacts on students and PSTs' learning, conceptual understanding and motivation. For instance, Laborde (2001) indicated that Cabri facilitates learning and understanding of mathematical concepts if appropriate tools and strategies are used. Besides positive impacts of DGS, some researchers focused on how teachers use DGS in classrooms and how they could use it effectively. Wilson, Lee, and Hollebrands (2011) hypothesized that PSTs technological knowledge is a critical component in the support of students' mathematical thinking and approaches to geometry problems. Therefore, they developed a model to analyze PSTs' thinking about students' work and their interpretation of students' mathematical thinking. They proposed that the model might be used to facilitate PSTs' development of pedagogical tasks. Additionally, researchers illuminated challenges in PSTs' learning to use technological tools. Ponte, Oliveira, and Varandas (2002) stated that many PSTs were uncomfortable using DGS for their own learning and implementing the tool in teaching. Synthesis of existing literature illuminates the potential power in teaching and learning with DGS, yet obstacles continue to exist. As Hollebrands (2007) stated, DGS might provide sufficient context for teachers to gain experience about students' mathematical thinking. Therefore, this study shows that PSTs should be trained to use DGS, and they should develop confidence in using DGS and attitudes towards it.

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Chapter 11: Theory & Research Methods

Research Reports

Brief Research Reports

Poster Presentations

A STOCHASTIC MODELING OF A STUDENT'S UNITS COORDINATION ACTIVITY

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In this proposal, I introduce a method for modeling the dynamics of a sixth-grade student's accommodation of his fractions scheme to include a disembedding operation (Steffe & Olive, 2010). I will describe a three-part approach consisting of a constructivist teaching experiment, retrospective analysis, and stochastic modeling of the student's activity across ecologies within the teaching experiment (cf. Steffe & Thompson, 2000). As disembedding requires coordination of two levels of units—a disembedded part and an unadulterated whole (Steffe, 2001)—the results also serve as an initial model for describing the dynamics of an individual's units coordinating within and across contexts.

Keywords: Number Concepts and Operations, Research Methods, Learning Trajectories

Introduction

Researchers employing constructivist teaching experiment methodology (Steffe & Thompson, 2000) have developed a hypothesized hierarchy of fractions schemes; each scheme is individually defined via its associated operational necessities (see Steffe, 2001). This structure of structures has been refined and verified via subsequent teaching experiments (e.g., Hackenberg, 2007; Norton, 2008), and written assessments and clinical interviews have been used to quantitatively verify that the hypothesized sequence of scheme construction generalizes (Norton & Wilkins, 2009; 2010; 2012; Wilkins, Norton, & Boyce, 2013). However, description of the *process* of conceptual change (accommodation and assimilation) *within* the hierarchy of fraction schemes has heretofore been primarily qualitative. In this proposal, I will describe, exemplify, and discuss the utility of a mathematical model for describing the non-linear development of an important operation within the hypothesized hierarchy—the disembedding operation for fractions.

Theoretical Perspectives

Fractions Schemes, Disembedding, and Units Coordination

Building upon von Glasersfeld's (1995) tri-partite notion of scheme (recognition template, coordinated cognitive actions, and expected result), Steffe has detailed a sequence of fraction schemes in terms of learners' reorganization of their operations first constructed for whole numbers (Steffe & Olive, 2010). A pre-cursor to the first genuine fractional scheme (the partitive fraction scheme) is the development of a *disembedding* operation for fractions, in which a part can be removed from a whole without its adulteration. Per Steffe's (2001) reorganization hypothesis, the disembedding operation for fractions is a reorganization of its role in the construction of an explicitly nested number sequence, in which a number like twelve is structured as a single composite unit (twelve) consisting of twelve units (ones).

Paramount to the construction of the disembedding operation for fractions is the ability to coordinate operations with two levels of units – the disembedded part and the whole. Coordinating two levels of units *in activity* involves operating with and preserving the structural relationship of the units with reliance on sensory-motor activity. The *interioriziation* of such a

two level structure (i.e., the development of an ability for it to become part of one's assimilatory or recognition template) has been shown necessary for the construction of more advanced fraction schemes such as the reversible partitive fraction scheme (Hackenberg, 2007; Norton & Wilkins, 2010) and is also necessary for the construction of more advanced multiplicative concepts necessary for algebra (Hackenberg & Lee, 2012; Hackenberg & Tillema, 2009).

Observing Conceptual Change

Conceptual change occurs via a process of *reflective abstraction* (Piaget, 1970), which includes (not necessarily conscious) reflection on mental re-presentations of experiential activity, pattern recognition leading to abstraction, and construction of re-presentations of the activity in which to assimilate future similar experiences. As a researcher's observations are likewise constructed, modeling mathematical learning is best approached via social interaction, i.e., in the process of teaching.

> Just as students contribute mathematics to experiential situations to establish them as mathematical situations, we adults contribute our concepts and observations to what we observe students do. … So, rather than trying to make models of mathematical learning in terms of concepts embedded in the outside world, we, as constructivists, try to build models of paths of interactive mathematical communication, paths that cannot be specified a priori. (Steffe & Wiegel, 1996, p. 494).

While not realizable to the researcher before retrospective analysis, the trajectory of a student's schematic development, culminating in inference of *conceptual change,* consists of non-observation, observation in activity, observation of the use of mental records instead of physical action (internalization), and observation of immediacy without apparent conscious operational activity (interiorization). But once a scheme emerges, its successive observation again, in ostensibly the same context, is uncertain (see Tzur, 2007). Even in the most structured of clinical interview settings, describing the context of an activity necessitates reduction of the multitude of possible factors (Clement, 2000). For the purposes of this study, an activity's context is considered to have two dimensions, detailed in the methods section below: that of the task itself and that of the individual learner's environment.

Methods/Methodology

Teaching Experiment

For this proposal, I focus on Charles, who was one of four students that I taught in a six-session paired-student constructivist teaching experiment (Steffe & Thompson, 2000) over three weeks during the summer, between sixth and seventh grade. Each student's first session consisted of a clinical interview for assessing students' fraction schemes (Wilkins, Norton, & Boyce, 2013). My goals for the remaining five sessions were to (1) build and refine a model for each student's mathematics, to include their ways of operating in multiple task contexts (i.e., with circular and linear fraction models) and sequences, and (2) support (to the extent possible in the short period of time) their construction of the operations theorized as necessary for the emergence of more powerful fraction schemes. For planning purposes, the videographer/witness familiar with fraction schemes and I discussed our analysis of the progress of the teaching experiment between sessions.

Transcription Analysis

Retrospective analysis began with a rich transcription of videos. My goal was to document, from my perspective at the conclusion of the teaching experiment, every observed or inferred action by Charles, his co-participant D.J., myself as the teacher-researcher, or the witness, that might reasonably pertain to Charles' coordination of units. Context descriptions were categorized

across two independent frameworks. For social interaction, the framework suggested by Simon et al. (2010) for limited teacher-student interaction was expanded to categorize both teacher and peer actions. Categories for teacher actions included re-focusing, asking for explanation, affirming, reiterating, demonstrating, and summarizing; asking for explanation, demonstrating, and affirming are examples of categories of students' activities. For task context, Wagner's (2006) framework was slightly modified to consist of task *type* (i.e., naming a fraction versus constructing a fractional size), *aspect* (i.e., coordinate units within 8 rather than units within 2), or *setting* (i.e., linear fraction rods versus circular drawings).

Characterizations of context were used to separate Charles' observed activity into *chunks* to be analyzed. A *chunk* is a portion of time during which Charles indicated mathematical activity and the context of that activity (as coded) was unchanged. For example, if Charles responded to a task, and subsequently I asked him to explain his thinking, his second response would be in a separate chunk. Thus, the difference between two chunks is not a quantitative difference in seconds, but rather a qualitative difference in Charles' ecological environment. Since context distinctions were categorized from my perspective of Charles' environment, I attempted to bracket my retrospective hypotheses about how Charles' might have constructed his environment in my coding. However, hypothesizing indirectly affected the results of categorizing by chunk because of its role in influencing my responses and decisions during and between the teaching experiment sessions (Steffe & Thompson, 2000).

Each chunk was dichotomously coded for whether there was an opportunity for units coordination (disembedding) to be inferred. For chunks in which I was able to make an inference on Charles' units coordinating activity, I coded '1' if units coordination was inferred and '0' if I inferred that Charles was not coordinating units. I coded for these inferences based on my model of Charles' mathematics formed at the conclusion of the teaching experiment.

Stochastic Model

I created a model of the dynamics of Charles' units coordination as a derivation of a Markov model proposed by Bush & Mosteller (1953). The objects of analysis for measuring growth are an individual's propensities for changing to or maintaining a "higher level" of thinking across chunks. I use the term *propensities* instead of *probabilities* in order to emphasize that the propensities are not computed as ratios of successful outcomes to total outcomes. Rather, the first simplifying assumption is that the propensity of coordinating units (and not coordinating units) across two chunks is a function of the current inference and the propensities in the preceding chunk. This is because the context of subsequent inferences is affected by the relationships between the propensities, the current observation, and the teacher-researchers' goals, as these relationships affect the teacher-researcher's decisions. For example, upon hypothesizing that Charles seems to be "consistently" coordinating units in activity, I would likely alter the context in order to engage him in further learning, rather than continuing to engage him in a "similar" manner. I address the notion of explaining what is meant by "consistently" in the formulation below, while saving consideration of what is meant by "similar" to future work.

Define $T_i = \begin{pmatrix} p_i & 1-p_i \\ 1-q_i & q_i \end{pmatrix}$, where p_i is the propensity of *not* coordinating units on chunk i, given that units *were not* coordinated in chunk i -1; q_i is the propensity that units *are* coordinated on chunk i, given that units *were* coordinated in chunk i. One can think of p as the conditional (Bayesian) probability P(0|0) and q as P(1|1). Without a priori knowledge, there is no reason to predict that a subsequent chunk's inference will be different or the same as its

predecessor. Hence, it is appropriate to assign initial equipropensity: $p_0 = q_0 = 0.5$. Let $c_i \in \{0,1\}$ be the observation of units coordination during chunk *i*. Algorithm 1 below is my recursive method for computing p_i and q_i. By producing a plot of the values of *p* and *q* by chunk, one can communicate the trajectory of a student's ways of operating in a way amenable to a broad range of analyses.

Algorithm 1. Computing $T_i = \begin{pmatrix} p_i & 1-p_i \\ 1-q_i & q_i \end{pmatrix}, i = 1,2,3,...$

If $c_{i-1} = 0$:

a) Define $m = \min(p_{i-1}, 1-p_{i-1})$.
b) Update $p_i = p_{i-1}(1-m) + m(1-c_i)$.
c) Leave $q_i = q_{i-1}$ unchanged.

If $c_{i-1} = 1$:

a) Define $m = \min(\mathrm{q}_{i-1}, 1-q_{i-1})$.
b) Update $q_i = q_{i-1}(1-m) + m(c_i)$.
c) Leave $p_i = p_{i-1}$ unchanged.

For each chunk, one of *p* and *q* remains constant, while the other increases or decreases. Including *m* in the computation dampens the change in propensity by its likelihood of inference, so that surprising inferences have a greater effect on propensities than unsurprising inferences. Note that, other than in the pathological case of its initial state, a propensity of 1 is never possible.

Results

Analysis of data is limited to the first three sessions. Protocol 1 is an excerpt from the transcription of the second session, which was divided into four chunks (corresponding to chunks 15-18 in Table 1), each in the task setting of linear, rectangular bars. Following the transcript is a description of coding by context and chunk.

Protocol 1. Portion of Transcription from Session Two

1. *Teacher*: Okay, I'll give you a different question, and this time I'm going to give it to [Charles] first. (Puts out 1 brown bar and 8 red bars). What fraction is 1 red bar out of the brown bar? **[Note that the brown is the length of four red bars]**
2. *Charles*: (Lines 8 red bars above the brown bar). Eight-fourths? An improper fraction? Cause you said, 'the red bar of the brown bar.'
3. *Teacher*: What fraction is one (holding up one finger) red bar out of the brown bar?
4. *Charles*: (Touches brown bar). So that would be four-eighths.
5. *Teacher*: Four-eighths, can you say why (interrupted)
6. *Charles*: That would be the same as one-half.
7. *Teacher*: (To D.J.) What do you think? Do you also think its four-eighths?
8. *D.J.*: Nods.
9. *Teacher*: (Moves 6 red bars and the brown bar to D.J.s desk) What fraction is one red bar of the brown bar now?
10. *D.J.*: (Lines the red bars up, one by one, from left to right above the brown bar)
11. *Charles*: I've got the answer.
12. *D.J.*: One-third. Cause four pieces makes that (the brown) so if there were two it's close, it's like one-half, and if you take that, it's one-third, or something.

13 *Teacher*: Okay, 'cause it's close to one-half?

14. *D.J.*: Yeah, two is close to one-half, so if you take away one it's one-third.
15. *Teacher*: Okay, Charles, what do you think? You said you had the answer, what was

your answer?

16. *Charles*: (Confidently) One-fourth.
17. *Teacher*: Can you show me how you got one-fourth? Show it to me, and show it to D.J.
18. *Charles*: (Lines up all 6 red pieces). This brown bar equals up to four pieces. You said to only count one of the pieces, so that would be one-fourth.
19. *Teacher*: Did that make sense to you D.J.?
20. *D.J.*: It makes more sense than mine.
21. *Teacher*: So you really don't need these other (extra two) pieces do you, to show it?
22. *Charles*: Yeah.
23. *Teacher*: Show it to D.J. so he can see it.
24. *Charles*: See how that equals up to four pieces? So you take these (sweeps away rightmost three) and that's one-fourth.

Coding

The portion of the transcript above was divided into 4 chunks: lines 1 through 2, lines 3 through 8; lines 9 through 14, lines 15 through 24, corresponding with observations of Charles' ecological environment. In the first chunk, I didn't code any teacher actions, whereas in the second chunk, the teacher action was coded as "demonstrating" and "re-iterating," and D.J.'s action was coded as "affirming." In the third chunk, D.J.'s action was coded as "demonstrating" and "explaining" and the teacher's action was coded as "asking for explanation." In the fourth chunk, I coded the teacher's action as "asking for explanation," and "affirming", and I coded D.J.'s action as "affirming." In the first chunk, Charles' uniting of the red bars, counting by ones, and comparing size were his mathematical activity; in the second chunk, Charles' swapping the numerator and denominator and reducing were his mathematical activity. In the third chunk and fourth chunks, I inferred that Charles' was counting by ones, disembedding, and comparing size. In order to clarify my reasoning for making such inferences, I next describe my (retrospective) model of Charles' mathematics.

Hypothetical Model of Charles' Mathematics

In the first two chunks, I infer that Charles was operating with what I call a "proper-placement" scheme, first noticed during the clinical interview, in which he assimilated physical placement (top and bottom) as part of his scheme to achieve a goal of producing a *proper* fraction name. His scheme consisted of the following sequential operations: 1) Make the "top" consist of equivalent marked parts; 2) Count the number of those parts that form the same size as the bottom; 3) Swap the top and bottom if there are more pieces in the top than in the bottom to make the fraction proper; 4) Name the fraction as the number of "top" pieces out of the number of "bottom" pieces. In the latter two chunks, my interpretation is that Charles had modified his scheme to include a disembedding operation, so that even after he united the four red bars to form "four" he was able to operate on the "one" within the "four." As Table 1 indicates, such modification was temporary.

Table 1: First Three Sessions Units Coordination Inferences

Session	Inference of Units Coordination (1 = yes; 0 = no)																		
1	Chunk	1	2	3	4	5	6	7											
	Inference	0	0	0	0	0	0	1											
2	Chunk	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
	Inference	0	0	0	0	0	1	1	0	0	1	1	1	1	1	1	1	1	1
3	Chunk	26	27	28	29	30	31	32	33	34	35	36	37						
	Inference	1	1	1	0	0	1	1	1	0	0	0	0						

The Dynamics of Charles' Units Coordination

As the teacher-researcher, I was actively attempting to perturb Charles in subsequent tasks in order to encourage him to use his existing operations to develop a more robust scheme for fractions, i.e., one that would be viable and more useful than those of his "proper-placement" scheme. As seen in Table 1, there were beginnings of inferences of units coordination beginning in chunk 13, followed by successive strings of inferences of units coordination that may have appeared stable to me *during* the teaching sessions.

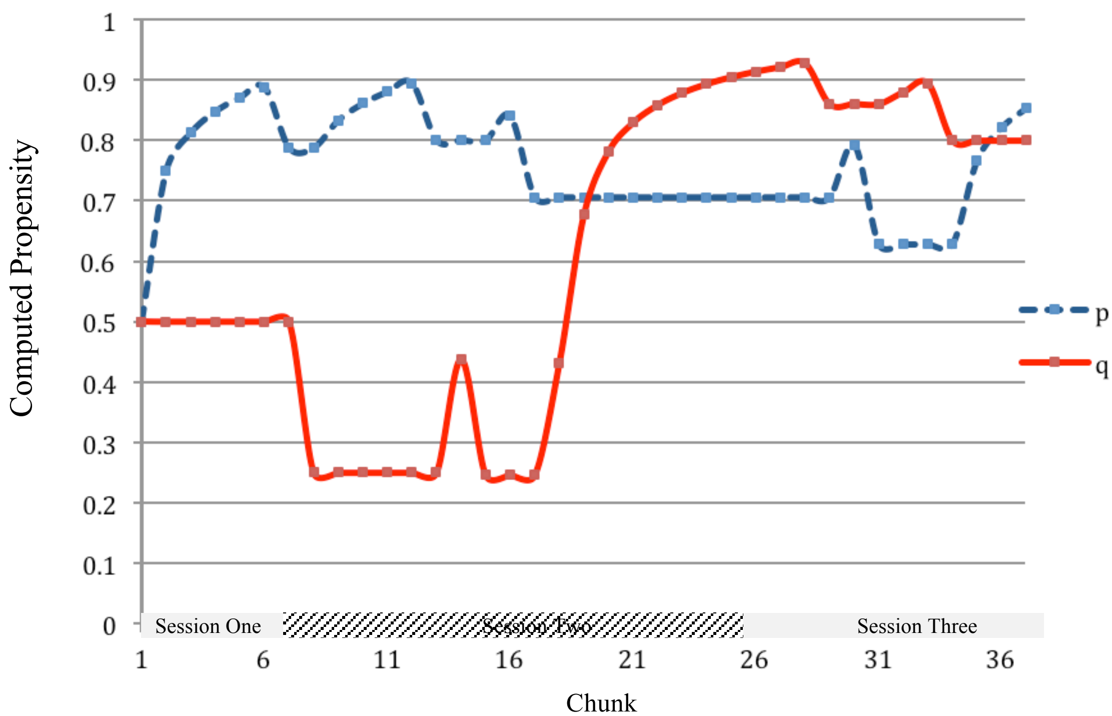

Figure 1. Results of Algorithm 1 with initial equipropensity

The graphs of p and q in Figure 1 illustrate the dynamics of my inferences of Charles' units coordination activity, influenced by my teaching goals. With those caveats, values of p or q converging to 1 indicate lack of perturbation, and a value of q converging to one indicates an accommodation of Charles' assimilatory scheme for fractions to include a disembedding operation.

Discussion

In this proposal, I have presented a third-order stochastic model of the mathematics of a student, as it can be considered a first-order model of my second-order model of Charles' mathematics (Steffe, & Thompson, 2000). It is a mathematization of my conception of the dynamics of the interactions between students' activities, teachers' inferences, and contexts that take place within a goal-oriented constructivist teaching experiment. While I make no claims that it is the *right* way, I believe that it is a useful model because of its compatibility with radical constructivist epistemology (von Glasersfeld, 1995), simplicity, and tractability. In its current form, the model has potential to explicate results stemming from teaching experiment methodology (e.g., to qualify the "emerging" or "establishing" of schemes or to compare students' growth within or across teaching experiments, cf. Simon, Tzur, Heinz, & Kinzel, 2004).

The model could also be used to further validate a theorized role of units coordination in individuals' ways of operating within the domain of fractions and across mathematics content domains.

The presented formulation could be modified to describe or predict the role of context in an individual's scheme development. For example, one might begin with analysis of the contextual factors accompanying large changes in slope in the plots of p or q. Future research might also include multiple teachers/participants and could possibly lead to a predictive or inferential model. Such formulations could eventually generate a dynamic structural model (Fischer, 2006) to simultaneously communicate both an overall structure and the expected variations and deviations from that structure in a way that could more widely influence curricular design and assessment than independent, qualitative comparisons of teaching experiment results (Kilpatrick, 2001).

There are several limitations to this formulation, however. The model only permits dichotomous, un-weighted inferences of students' ways of thinking, but within the existing hierarchy of schemes and associated operations, there are often more than two possibilities, and there are varying degrees of certainty of inference. To test for robustness, i.e., sensitivity to differences in inference or chunking, triangulation of the coding of data is necessary; such qualitative research is quite time-intensive. However, at minimum, the model provides a novel mechanism for communicating the dynamics of the non-linear assimilation/accommodation process - the trajectory and variation *within* a second-order model of a student's mathematics.

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CAPTURING MIDDLE SCHOOL STUDENTS' UNDERSTANDING OF THE CONCEPT OF AREA USING VYGOTSKY'S CONCEPT FORMATION THEORY

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Incorporating Vygotsky's concept formation theory and Berger's appropriation theory, an assessment which highlights authentic operations with the concept of area was designed and administered to 44 sixth grade students to determine what types of understanding they maintained. The results indicated the more novel the situation, the more diverse stages of understanding were exhibited by children. Inconsistent levels of reasoning across different items were revealed for most individuals. We suggest that novel assessments grounded in concept formation theories may provide greater insights on children's understanding of mathematics.

Key words: Assessment and Evaluation, Measurement, Problem Solving

Introduction

Learners' prior domain knowledge of mathematics has long been identified as a key prerequisite for development of more sophisticated mathematical thinking (Shulman & Keislar, 1966; Bauersfeld, 1995; Lesh & Doerr, 2003). This connection has most prominently been voiced within the genre of research on mathematical problem solving (Schoenfeld, 1992), and understanding the interactions between domain knowledge and problem solvers' activities including strategy use, control, and beliefs has been the source of inquiry for over three decades (Lester, 1994). The body of existing literature points at a link between what learners know and the constraints that the existing knowledge imposes on their mathematical practices. While this point certainly merits attention, we are less convinced by how children's mathematical knowledge may have been characterized based on instruments used for capturing their understanding. Conventional assessment tools often focus on whether learners can use what they had presumably learned when confronted with tasks similar to those practiced. Such practice has led to production of numerous reports indicating learners' failure at transfer of knowledge when experiencing a new situation (Niss, Blum, & Galbraith, 2007). We posit that capturing, with some degree of accuracy, what children know demands research-based assessment instruments that reveal their ways of knowing. Of particular concern is not the use of instruments that capture what conventional knowledge children may have retained, but rather the particular types of understanding they hold. Currently, such instruments are rare in mathematics education (Adams, 2012).

Objectives of the Study

The purpose of the study reported here was twofold. First, we aimed to investigate the types of understanding of the concept of area that middle school seemingly held when tackling different problems. Second, we were motivated to determine the affordances of a theory driven instrument to make visible the various types of understanding children might have of the same topic. The following research questions were used to guide the study:

1. What stages of concept development are revealed through students' interactions with the tasks in terms of Vygotsky's concept formation theory?
2. What patterns of understanding of the concept of area do middle school students exhibit?

Theoretical Framework

In this study we incorporated two theoretical perspectives as groundings for our instrument development: Vygotsky's (1962) concept formation theory and Berger's (2004) appropriation theory. Vygotsky's theory proposes a framework for an individual's concept (word or sign) development within a social environment, while Berger's theory proposes an interpretation of Vygotsky's theory in the domain of mathematics by adding and omitting certain stages. Since Berger's theory was designed based on undergraduate students' performance in calculus and linear algebra, we conjectured that the developmental framework might be different for secondary students in other content areas. In order to benefit from her work without being conceptually restricted, both theories were utilized to inform this study.

According to these two complementary theories, concept development consists of three phases: heap, complex, and concept. In the *heap* phase, the learner associates a sign with another because of physical context or circumstance instead of any inherent or mathematical property of the signs. In the *complex* phase, objects are united in an individual's mind not only by his impressions, but also by existing bonds between them. However, the bonds between objects are concrete and factual instead of abstract and logical. The complex phase further contains the sub-stages described in the following paragraph.

During the *association complex*, the learner uses one mathematical sign as a nucleus and associates other signs with some common attributes based on objective and factual justification. For *chain complex*, the learner associates one mathematical sign with another based on some similarity and then links the new sign to another by a different attribute to form a chain. With *representation complex*, the learner identifies the visual or numerical representation of a mathematical object as the object itself. Properties abstracted from such representations are considered as the properties of the object. Students with a *pseudo-concept* could use and communicate the mathematical notion as if they fully understand it, although their understanding may be based on factual connections instead of logic.

Concept is defined as a mathematical idea with consistent and logical internal links (links between different properties and attributes of the concept) and external links (links of the concept to other concepts).

The tentative formation stages for the concept of area, as we conceptualized them, are illustrated in Figure 1. Examples of students' reasoning with area in Battista's (in press) Cognition-Based Assessment (CBA) were used as the main resource in referencing students' developmental understanding of the concept. Note that in the figure, underlined terms are the stages derived from Vygotsky's and Berger's theories, while Non-Measurement and Measurement (including Unit area and Formula) are the key components in the development of measurement reasoning identified by Battista's CBA levels. Additionally, each developmental stage of each component is followed by the corresponding examples from CBA levels as well as examples from our own experiences with students' reasoning (in *Italic*).

This framework guided the design of the assessment tool as well as the analysis of participants' responses in this study.

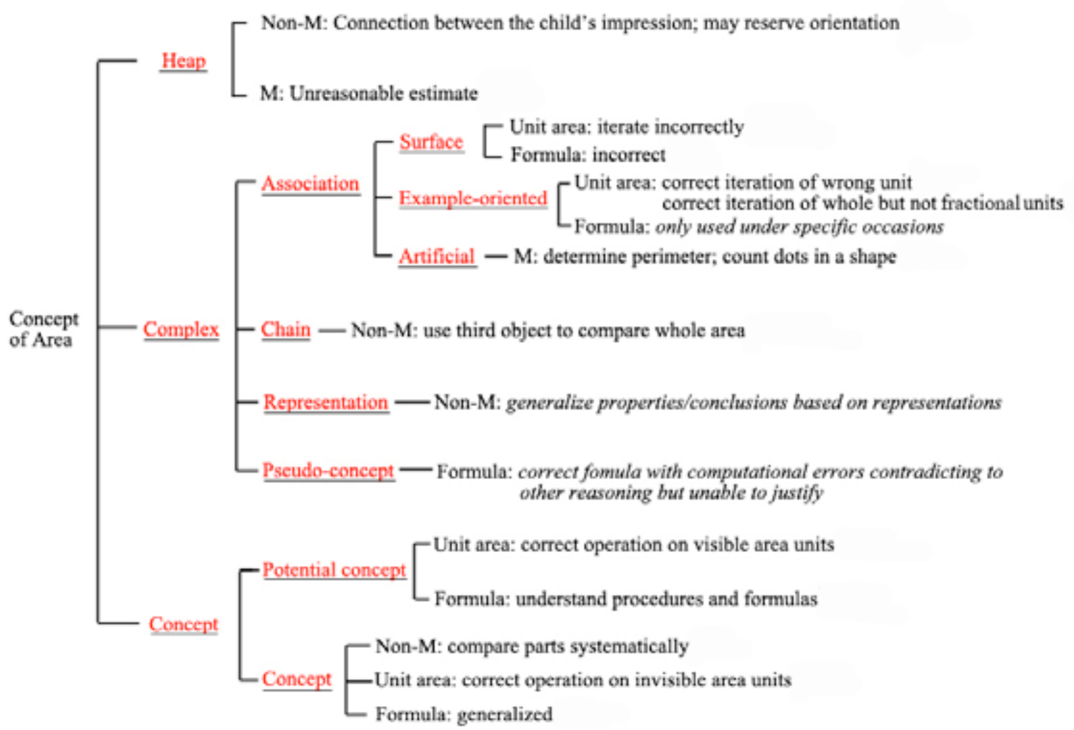


Figure 1. Tentative Developmental Stages of the Concept of Area

Methods

Participants

Participants included 44 sixth grade students from a mid-western suburban middle school. The students were from three distinct class periods of an algebra course taught by the same teacher at the time of data collection. They were observed by the lead author for 6 months prior to data collection. In the course of observations it became evident that they exhibited a range of different types and levels of understanding of the concept of area. These observations constituted the need to examine their thinking more carefully.

Contexts

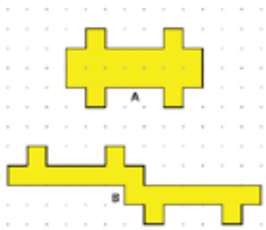
The participants were given the assessment during one class period (50 minutes). Prior to administering the assessment, students were informed that they could use calculators if they felt they were needed. They were also reassured that if they felt they needed assistance when reading the problems, the researchers would provide assistance accordingly. Lastly, they were asked not to erase their work even if they considered it wrong.

Instrumentation

Five questions were selected from existing assessments and modified to resemble novel (non-textbook-like) tasks. Resources included items from TIMSS, CBA tasks, and Problem Sets from the Math Coaching Program at the Ohio State University. Table 1 shows the difference between a conventional task and its corresponding modified novel version produced for use in the study. The third column outlines the developmental stages expected to be elicited by the novel version.

Table 1: Difference Between Conventional Task and Novel Task

Conventional task	Novel task	Developmental stage and explanatory approach elicited by the novel item

Which of these shapes has more area or room inside it, or do they have the same amount?  [CBA measurement task]	Which of these shapes has more area, or do they have the same amount? Explain how you reach your conclusion.  If you believe there's not enough information to answer the question, please state what information you would need and how you would use that information to answer it. [Item 1]	*Non-measurement reasoning:* Heap (compare based on impression) Chain complex (use third object to compare whole area) Concept (compare parts systematically)

The assessment items were aligned to associate with stages identified by the framework (with overlaps) where novel situations were created to provoke authentic interactions with the concept. Item 2 through 4 are displayed in the results section.

Analysis

Analysis of data followed three stages. First, two researchers independently reviewed all 44 sets of participants' responses to identify and document enacted approaches and coded developmental stages associated with each approach. Notes were compared for consistency in scoring. Children's approaches that were ambiguous or non-anticipated were discussed in the second step. The theoretical framework was adjusted based on the analysis of these responses; five more stages were identified and added to the original framework. Lastly, the distribution of developmental stages for each item was examined and potential patterns were abstracted.

Results

Table 2 summarizes the descriptions of different developmental stages associated with each item, and the number of times each stage appeared for each item. Stages in *italic* are those identified and added to the original framework based upon the researchers' initial analysis of students' responses. For Item 3, six students provided two types of reasoning in their responses, revealing two stages for each individual. Item 5 assessed general problem solving performance, which is not included in the table at this time. Blank answers are not included in the results.

Table 2: Number of Stages Elicited by Each Item

Stage	Description	Item 1	Item 2	Item 3	Item 4	Total
1.1	Heap - NonM: connection between the child's impression	9	3	0	0	13
1.2	Heap - M: unreasonable estimate	0	0	3	0	3
2.1.1.1	*Surface Association Complex – NonM: compare parts randomly*	2	0	0	0	2
2.1.1.2	Surface Association Complex – Unit area: iterate incorrectly	2	1	10	11*	24
2.1.1.3	Surface Association Complex – Formula: incorrect	0	1	2	0	3
2.1.2.1	Example-oriented Association Complex – Unit area: correct iteration of wrong unit or of whole but not fractional units	0	3	1	11	15
2.1.2.2	Example-oriented Association Complex – Formula: only use formula under specific occasions	0	0	0	0	0

2.1.3	Artificial Association Complex – M: determine perimeter; count dots in a shape	8	18	0	3	29
2.2.1	Chain Complex - NonM: use a third object compare whole area	1	0	0	0	1
2.2.2	*Chain Complex – M: use a third object to compare area*	0	3	0	0	3
2.3	Representation – generalize properties based on representations	0	0	0	0	0
2.4.1	*Pseudo-concept Complex – NonM: compare parts non-rigorously*	7	0	0	0	7
2.4.2	*Pseudo-concept Complex – M: empirical estimate*	6	0	1	0	6
2.4.3	*Pseudo-concept Complex – Unit area: Non-rigorous estimations on visible area units*	0	11	0	0	11
2.4.4	Pseudo-concept Complex – Formula: correct formula contradicting to other reasoning	0	1	0	0	1
3.1.1	Potential Concept – Unit area: correct operation on visible area units	0	1	0	7	8
3.1.2	Potential Concept – Formula: understand procedures and formulas	0	0	1	0	1
3.2.1	Concept – NonM: compare parts systematically	3	0	0	0	3
3.2.2	Concept – Unit area: correct operation on invisible area units	0	0	20	7*	27
3.2.3	Concept – Formula: generalized	0	0	0	0	0
Total		38	42	38	39	157

*Responses were from the copies where the grid was barely visible in item 4.

Three stages failed to be revealed in item 1-4: Example-oriented Association Complex – Formula (only use formula under specific occasions), Representation (generalize properties based on representations), and Concept – Formula (generalized). The Example-oriented Association Complex – Formula stage could be investigated by looking at both Item 3 and 5, while the latter two stages might be more likely to be elicited in a problem that allows extending and generalizing.

Figure 2 illustrates the distribution of the elicited stages for each item.

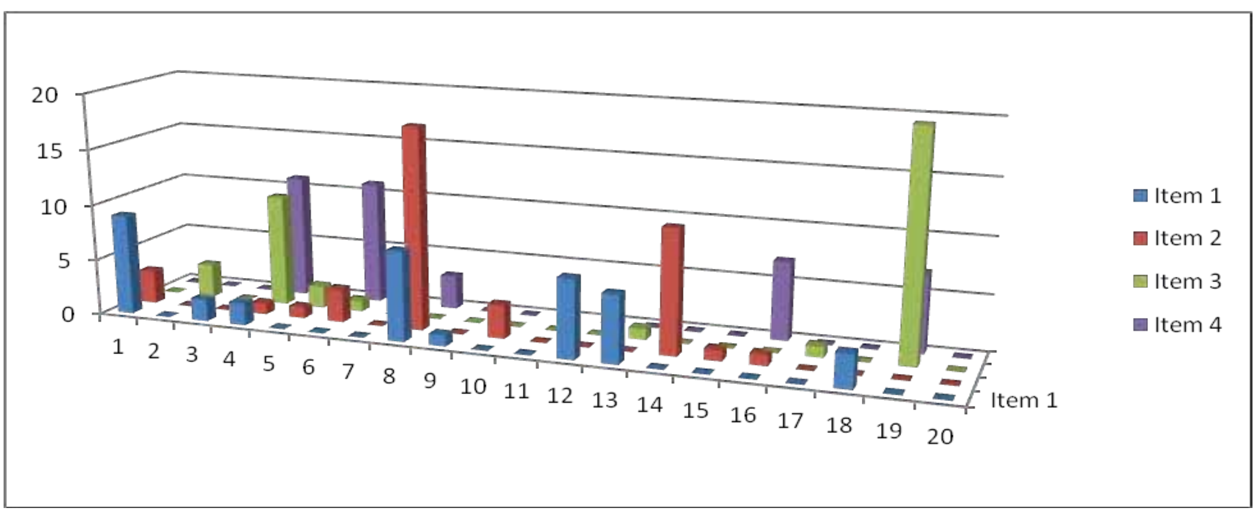


Figure 2: Distribution of Stages for Each Item

As the graph illustrates, Item 1 and Item 2 provoked more diverse stages of knowing, while Item 4 elicited the most narrowed. Item 3, arguably the most conventional mathematical task among all items, dominantly stimulated the highest developmental stage for unit area; however, in our previous study (Zhang et al., 2010), the dominant approach (37%) for this problem among 292 fifth graders was to iterate incorrect triangles (stage 2.1.1.2).

A Closer Look at Learners' Approaches and Coding Implications

To expand on the previous quantitative account of aggregate responses, in this section we use one participant's written artifacts to each of the four items to demonstrate the range of analytical schemes children utilized when examining non-conventional tasks.

Student J's response to Item 1 is illustrated in Figure 3. Note that Item 1 asks the student to compare two irregular shapes without any measurement.

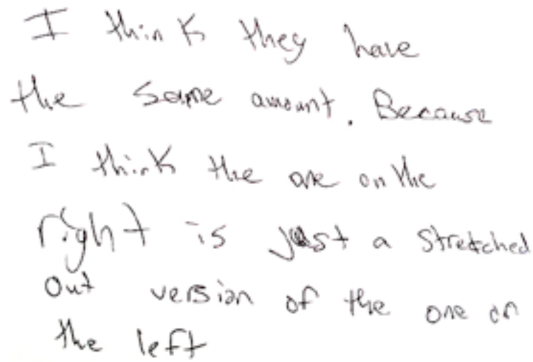


Figure 3: J's Response to Item 1

J's response to Item 1 was categorized as stage 1.1 (Heap – NonM) since the phrase "stretched out" did not provide enough concrete or logical mathematical evidence for his conclusion, despite the fact that his answer was correct. Since he did not provide any visual representation, we did not have sufficient evidence to rank his thinking beyond stage 1.1.

Item 2 along with J's response is illustrated in Figure 4.

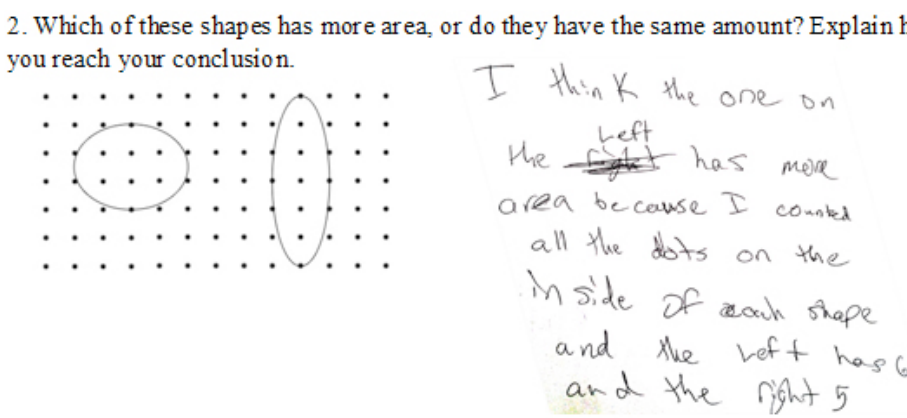


Figure 4: J's Response to Item 2

J's response to Item 2 was categorized as stage 2.1.3 (Artificial Association Complex – M) since he associated the area with the number of dots enclosed within the region. Some students considered not only the number of dots inside of each region, but also the ones "touching" the

perimeter; those responses were also categorized into this stage, although the idea might be quite different from J’s.

Item 3 along with J’s response is illstrated in Figure 5.

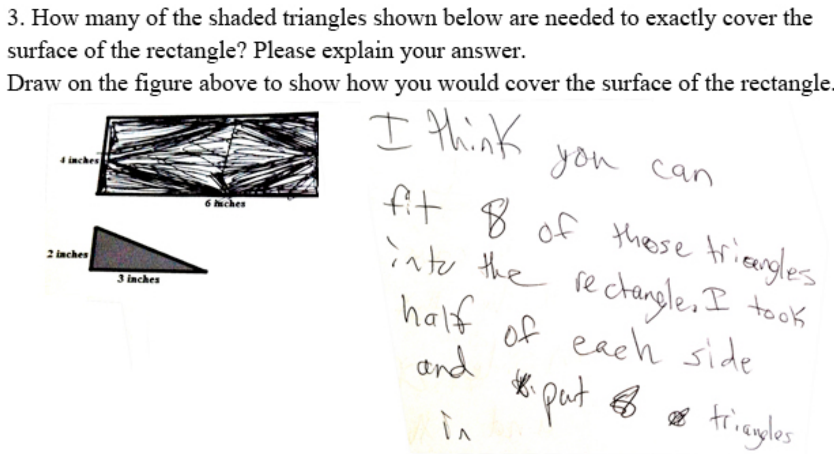


Figure 5: J’s Response to Item 3

J’s response to Item 3 was categorized as 3.2.2 (Concept – Unit area) since he correctly iterated eight shaded triangles in the rectangle. However, a part of his iteration was not very clear (the right bottom section of the rectangle). He may have iterated incorrectly during the process (stage 2.1.1.2), but the description he provided was valid.

Item 4 along with J’s response is shown in Figure 6.

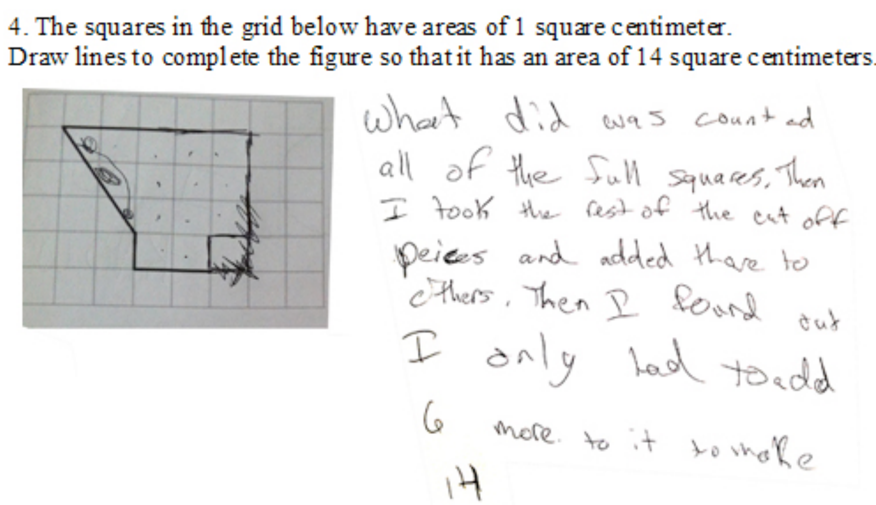


Figure 6: J’s Response to Item 4

J’s response to Item 4 was categorized as stage 3.1.1 (Potential Concept – Unit area) since he correctly paired up the four partial areas into two whole squares. Many students only paired up two partial squares while ignoring the smallest one and chose to draw a new partial area to make up the missing part (which was the part they ignored).

J’s responses to Item 3 and Item 4 placed him at concept level of reasoning, but his responses to Item 1 and Item 2 were identified as heap or lower level complex reasoning. Such inconsistency was commonly observed among the participants. 25 out of 44 students showed

concept level reasoning in Items 3 and/or Item 4 but *heap* or low *complex* level reasoning in item 1 and/or Item 2. 16 students exhibited consistent level of reasoning across all 4 items; 1 student showed *concept* level reasoning in Item 1 but low *complex* level reasoning for Item 2 to 4; 2 students showed inconsistent levels of reasoning which were different from the previous three patterns. A possible reason for such inconsistency is that individual's understanding for each component (Non-measurement, Unit area, and Formula) develops at different pace, and a higher level understanding of one component might be influenced/restricted by a lower level understanding of another component under novel situations.

Discussion and Conclusion

The major goals of the study were to examine the utility of a research based instrument grounded in theories of concept formation for revealing middle school children's conceptualization levels of area concept. Findings revealed that four of the items (1 to 4) successfully elicited 17 among the 20 developmental stages pertaining to the concept under study.

Findings suggest that stages and levels of understanding of a concept become far more visible when situations used for assessment are less familiar to what students may have experienced in textbooks. Relatively conventional situations appeared to elicit standardized approaches for solving problems, making the issue of assessing learning far simpler by categorizing them as right or wrong. Our findings suggest that novel assessments designed around concept formation theories may provide researchers greater capacity to articulate intricacies of children's understanding of mathematical concepts.

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PROMPTED AND UNPROMPTED TRANSITIONS BETWEEN REPRESENTATIONAL MODES IN CALCULUS

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This article argues for a shift in how researchers discuss and examine students' uses of representations during their calculus problem solving. An extension of Zazkis, Dubinsky, and Dautermann's (1996) Visualization/Analysis-framework to include physical modes of reasoning is proposed. An example that details how transitions between visual, analytic and physical reasoning inform students' problem solving in a calculus context is discussed.

Keywords: Reasoning and Proof, Advanced Mathematical Thinking

Background

Cartesian graphs, analytic notation and real/imagined physical scenarios are ways of representing a function. These three modes of representation provide very different opportunities for reasoning about function and translating between these modes is non-trivial (Leinhardt, Zaslavsky, & Stein, 1990).

The extent to which function is tied to physical scenarios varies depending on the particular curriculum. However, often emphasized is the connection between modes. Several researchers have explored understandings of the connection between graphic and analytic modes through tasks that prompt students to translate between these modes (e.g. Kaput, 1987; Knuth 2000). Additionally, there is a growing literature on mathematization of real and imagined scenarios and interpretation of what graphs imply about the situations they describe (Gravemeijer, & Doorman, 1999; Nemirovsky, Tierney, & Wright, 1998). Achieving representational fluency is an important part of secondary school mathematics (Knuth, 2000). This type of fluency later plays an important role in calculus.

However, very little calculus education research has focused specifically on transitions between representations. Education Researchers and curriculum developers with strong ties to the education research community have, for years, stressed the importance of including multiple representations as part of a calculus course. One major product which emerged from the Calculus reform movement of the late 1980's and early 90's is Hughes-Hallet et. al's (1994) calculus text. A guiding principle which underpins this textbook's approach is "the 'Rule of Three,' which says that whenever possible, topics should be taught graphically and numerically, as well as analytically" (p. 121). To date this text still comprises 19% of the US calculus textbook market (Bressoud, 2011). In recent years the 'Rule of Three' has been appended to include physical/kinesthetic considerations and has become a 'Rule of Four' (Kung & Speer, in press).

Many researchers have echoed the cry to emphasize more than just the symbolic aspects of calculus. Zimmerman (1991) wrote that, "visual thinking is so fundamental to the understanding of calculus that it is difficult to imagine a successful calculus course which does not emphasize the visual elements of the subject" (p. 136). However, in spite of the emphasis on multiple modes of representation, students' difficulties with implementing non-symbolic modes of reasoning have been a prevalent theme in the literature. For example, in a series of studies that explored multiple representations Presmeg and colleagues have emphasized the rareness of students who are comfortable with both analytic (verbal-logical) and graphical (visual pictorial) thinking

(Aspinwall, Shaw, & Presmeg, 1997; Presmeg, 2006; Haciomeroglu, Aspinwall, & Presmeg, 2010). They refer to such students as harmonic thinkers. In general, these studies have concluded that students who prefer graphical modes of reasoning are rare and harmonic thinkers are even rarer. Some authors have attributed students' difficulties with and resistance to visual reasoning in calculus to the over-emphasis on symbolic representations in secondary grades (Haciomeroglu, Aspinwall, & Presmeg, 2010; Vinner, 1989).

Furthermore, discussion of multiple representations has been expanded beyond analytic and graphic modes. Zandieh's (2000) derivative framework includes verbal and kinesthetic reasoning in addition to analytic and graphical. However, the work on multiple representations in calculus has tended to focus on which representations are present in student thinking on particular tasks, rather than how transitions between representational modes contribute, whether productively or not, to student problem-solving in calculus. Given the amount of emphasis translation between representation modes has received in secondary school, it seems natural to extend the investigation of this phenomena into calculus education research.

Theoretical Perspective

In this section I introduce the Visualization/Analysis-model (VA-model) and suggest an extension of it that includes physical modes of reasoning. I argue that a more detailed look at transitions among different modes of reasoning is needed and I exemplify different types of transitions in calculus tasks.

The VA-Model

My perspective extends the VA-model (Zazkis, Dubinsky, & Dautermann, 1996), which views the development of visual and analytic modes of reasoning as complementary rather than disjoint processes (see Figure 1). The modes, which may start as wholly separate entities, build on one another as reasoning develops. As students progress, their ability to translate between these modes becomes more common and the connections between the modes become stronger. In other words, the model contends that a back-and-forth relationship between the modes of reasoning does not develop overnight; it develops over time, and as it does, the transition between modes becomes progressively more natural for students to make. Figure 1 illustrates this process via a path through successive levels of visualization and analysis in which the 'distance' between visualization and analysis decreases as the levels advance.

Although the VA-model was used in prior research, these studies tend to focus on classifying individual students as visual thinkers, analytic thinkers or harmonic thinkers (e.g. Haciomeroglu, Aspinwall, & Presmeg, 2010). I see this classification as inconsistent with the VA-model since the model contends that all students make transitions between modes during their mathematical development. In other words, the model contends, implicitly, that all students are harmonic thinkers.

The tendency to shy away from examining transitions between representation modes is also present in work that does not subscribe to the VA-model. For example, Zandeih's (2000) derivative framework classifies individual students in terms of whether or not they have expressed various modes of thinking. That is, rather than focusing on transitions between modes of thinking and how one mode informs another, students were classified in terms of the modes they expressed.

Focusing specifically on transitions between representations stays true to the VA-model. Additionally, it helps illuminate how harmonic thinking develops, even in students that rarely use certain modes of thinking.

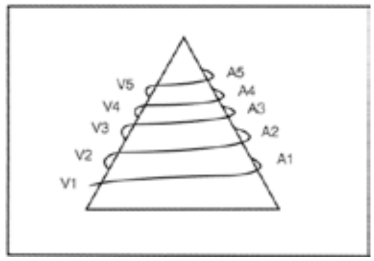

Figure 1: The VA-Model Diagram (from Zazkis et al. (1996))

The Expanded VA-Model

In line with Zandieh's work and the general trend in calculus education research to include real or imagined physical scenarios, my model adds representations that are based on real or imagined physical scenarios to the VA-model. This is consistent with a growing body of work that emphasizes the importance of physical contexts in students' understanding of calculus, such as the relationship between acceleration, velocity and position (Nemirovsky, Tierney, & Wright, 1998). In order to reflect this change the model will henceforth be referred to as the VAP-model. Mathematics is often motivated by connections to physical scenarios. So I see this addition to the model, which was originally not developed for calculus, as applicable to other areas of mathematics.

The VA-Model diagram shows levels that spiral up a triangle as reasoning advances. Visualization and analysis become closer to each other as one moves to more advance levels. The VAP-Model diagram is a tetrahedron, to accommodate the addition of a physical mode. The path between modes also spirals up with levels getting closer to each other, however, in the VA-diagram there is an orderly path that moves from visualization to analysis and back. In the VAP-diagram the path moves upward between three modes, but does so through a disorderly unpredictable path. This signifies that the transitions between visual, analytic and physical modes do not follow a specified sequence.

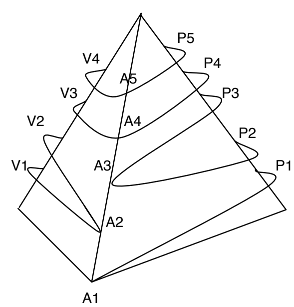

Figure 2: The VAP-Model Diagram

Students' transitions between the three modes of thinking, in the context of problem solving, are of particular interest because they inform how students use multiple modes of representation in conjunction with each other. The VAP-model contends that these modes inform each other, but what this looks like, when it happens or how such transitions can be fostered by instruction are not predicted by the model.

Instead of classifying individual students as predominantly preferring one mode of reasoning over another, I contend that the classification should be of students' claims and justifications and whether those are visual, analytic, or physical in nature. In other words I regard all students as harmonic reasoners to some extent, regardless of which representational mode is predominant in

their thinking. Within such a classification I place a special emphasis on back-and-forth transitions between modes of reasoning.

Categorizing Transitions

In the secondary grades translation tasks are often solely about the translation itself. In a calculus context, however, translation between modes is often part of the task and not the task itself. Some calculus tasks are stated in one representational mode and require an answer in another. For example, consider the following task: *"If f(0)=1, f '(0)=1, f(3)=7, f'(3)=-1 and f ''(3)=-1, sketch a possible graph of f(x)."* The task is stated in terms of the analytic mode, since the information about the function is given symbolically, and the answer is supposed to be provided in a graphical mode. Note that in order to complete the task a student is required to transition between representational modes. Solving the task requires moving from one edge of the VAP-diagram to another.

It can also be the case that translation between modes is not necessarily required in order to complete the task. However, spontaneous transitions between representational modes may occur anyway during students problem solving. For example if a student is given the following integral to solve $\int_{-3}^{3} x\sqrt{9-x^2}\,dx$, she may solve it using standard methods, such as u-substitution. The problem can, however, be solved by reasoning about the shape of the graph of $x\sqrt{9-x^2}$. The graph has a 180º rotational symmetry about the origin (odd function). Therefore every region above the x-axis has a corresponding region below the axis on the other side of the y-axis. Since the bounds of integration are symmetric with respect to the origin the integral evaluates to zero. Even though the problem is stated in symbolic/analytic terms and requires a symbolic/numerical answer the second solution makes extensive use of the graphical mode. If a student solves the task in this way, her thought process moves from one edge of the VAP-diagram to another, but this transition is not specifically required by the problem itself.

I refer to transitions between modes that are not required by the task, as *unprompted transitions*. Further, I refer to transitions that are part of the problem itself, that is, when a problem is stated in one mode and requires an answer stated in another, as *prompted transitions*. Note that prompted transitions are an attribute of a task and unprompted transitions are an attribute of a solution. So it is possible to have an unprompted transition occur within the context of a prompted transition problem.

Method

Data and Participants

This study followed a group of three average students as measured by their scores on the standardized Calculus Concept Readiness (CCR) test (Carlson, Madison &West, 2010). The group consisted of two males and one female, which were given the pseudonyms Carson, Brad and Ann. These students were observed over the course of a semester long technologically enriched calculus class taught at a large university in the southwestern United States. The class had approximately 70 students. The three students in this study worked together during in-class group work, which was recorded daily. Each of the three students also participated in three individual problem-solving interviews throughout the semester. The data in this article come from these interviews.

Analysis

All interview tasks were coded for which representation modes were prompted. Student work on these tasks was also coded for representational mode with special attention paid to when transitions occurred and how these transitions informed students' problem solving. The metonymy of many mathematical terms necessitated the use of a neutral code. The code was applied when it was unclear which mode of reasoning was being used.

Results

I begin by discussing the students' use of representations as a whole and how they relate to the representations in the task statements. Then I shift to discussing a particular student in detail.

Table 1 documents which transitions (if any) were explicitly required by each of the interview tasks and the transitions between representational modes students used in their solutions. Visual, analytic and physical are indicated with V, A and P, respectively.

Table 1: Prompted and Unprompted Transitions.

	Interview 1					**Interview 2**							**Interview 3**						
Task #	1	2	3[1]	4	5	1	2	3	4	5	6	7	1	2	3	4	5	6	7
Task Type	AP	AP	V	V	VP	AP	V	V	V	V	AP	VAP	AP	VP	AP	V	VAP	VP	VP
Ann	VAP	VAP	V	VA	VP	VAP	VP	VP	VAP	VAP	VAP	VAP	VAP	VP	VAP	VP	VAP	VP	VAP
Brad	VAP	VAP	V	V	VP	VAP	V	V	VA	VA	VAP	VAP	VAP	VP	VAP	V	VAP	VP	VP
Carson	VAP	VAP	VAP	VP	VP	VAP	VP	VP	VAP	VAP	VAP	VAP	VAP	VP	VAP	VP	VAP	VP	VP

Note that, in solutions to tasks that incorporate all three modes unprompted transitions are not possible. In table 1 two thirds of student solutions that could have shown an unprompted transition did. One important thing to notice from Table 1 is that most of the instances where students made an unprompted transition involved the addition of a visual or physical mode. The instances that incorporated unprompted transitions to the visual mode typically involved drawing a graph to aid with reasoning. Unprompted transitions to the physical mode typically involved reasoning about a graph as if it were describing the motion of some particle or vehicle, which was not part of the specified problem. A specific transition to the physical mode is explored in more detail below. The addition of an unprompted analytic component was rare but did occur in the data. This took the form of reasoning that a graph (or part of a graph) appeared similar to a known analytic function and then finding a related function analytically before translating back to the graphical mode. This behavior, which typifies analytic thinkers in Presmeg and her colleague's work, was fairly uncommon in this data set.

Table 1 documents what modes were used for particular questions but does not detail the specifics of how representational modes inform one another during the problem-solving process. The VAP-model contends that these transitions between modes are central to students' development. The next section details a particular set of transitions used to solve a graphing task.

[1] This is the task shown in Figure 3

Carson and the Derivative Sketching Task

Although each of the three interviews involved several tasks, only one task will be discussed due to space limitations. The derivative sketching task (Figure 3) is a graphing task that asks students to sketch the graph of a derivative given a particular original function. The task is similar to a task discussed by Aspinwall and Shaw (2002) that asked students to sketch the derivative of a continuous symmetric 'saw-tooth' graph that alternated between a slope of negative one and one. The graph in Figure 3, unlike Apsinwall and Shaw's graph, alternates between several different slopes. These graphs have no simple translation into analytic notation. They therefore discourage an analytic approach. Aspinwall and Shaw observed that students they classified as analytic thinkers had difficulty with the saw-tooth task. As presented, the task below does not force any transitions between modes of representation since it can be solved using only graphical reasoning.

Sketch the graph of the derivative of the following function. Think aloud as you sketch your graph.

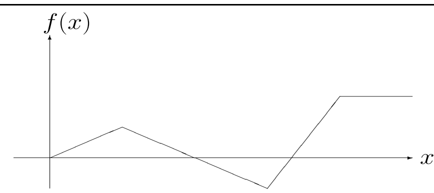

Figure 3: Derivative Sketching Task

Below is the transcript of Carson working on the derivative sketching task (Figure 3). This task is stated in graphic terms and requires a graphic solution. The transcript details a solution that does not stay solely within the confines of graphical thinking.

Carson [00:22:49]: Alright, so I know that the derivative is the slope and I took physics so I know that this is distance [writes a d under x axis] over, no that's wrong this is time [crosses out d and writes t under the x axis]. This is time over distance, which is you speed. Speed is distance over time. So this is time and this is your speed [labels axis on derivative function] and so as your distance…I'm sorry… So this is constant so you know that velocity is constant. So your velocity is something like this [draws short horizontal line segment above x-axis] and then later when it hits this tip it's at zero [marks a dot on the x-axis after previously drawn segment]. And then later when it's decelerating. Ya this is a negative speed so the graph. And it's a straight line so you know it would be something like this [draws a horizontal line under the x-axis]. And then again at this point it's zero [draws another dot on the x-axis after the second segment]. And then um accelerating. But this time it's more this is steeper. So it would be higher because your velocity would be faster. Your speed would be faster. So it would be like this [draws a third horizontal line segment above the x-axis higher than the first]. And then again right here it's zero [draws another dot on the x-axis]. And then now this one your distance isn't changing. Since your distance isn't changing. This equation [s=d/t] looks like zero over time. So the rest of the graph would look like this [draws a fourth line along the x-axis.]

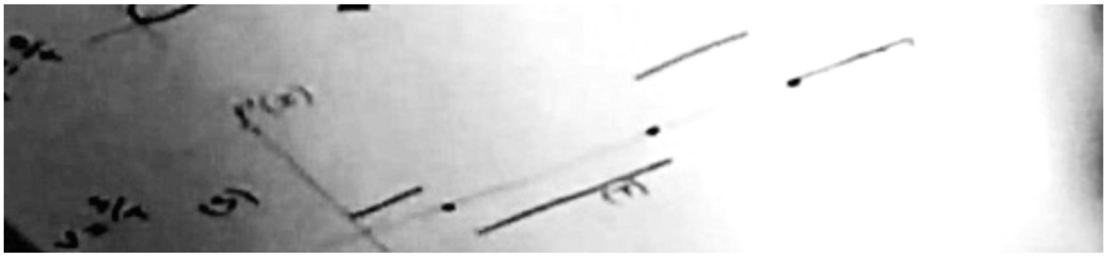

Figure 4: Carson's Solution to the Derivative Sketching Task.

In the above transcript Carson is presented with a question that makes no mention of a physical context, however, Carson attributes the function in the question to a function that describes a moving object. This transition to physical thinking is unprompted by the question. Carson does not simply shift into a physical mode and remain there. He continually moves back and forth between visual and physical modes. More specifically, he interprets a section of the given graph as corresponding to a physical motion, reasons about the velocity/speed of that motion and then translates that into a velocity graph. This cycle occurs several times throughout the transcript.

When dealing with the last segment of the function Carson switches to an analytic mode. He reasons that a non-changing position corresponds to 0/t=0, no change in distance over a non-zero change in time, before sketching the last segment of the derivative. So the above transcript shows unprompted transitions to both physical and analytic modes within a graphical problem. Relating this back to the VAP-diagram, Carson's reasoning continually alternates between the physical and analytic edges of the tetrahedron before moving to its analytic edge.

The connections that Carson makes between physical and graphical contexts led to some interesting artifacts. The given graph cannot represent the position graph of a physical object. Physical objects cannot instantaneously change directions and so it does not make sense to discuss the physical interpretation of what happens at those points. Carson, however, does not abandon the physical-graphical link. This leads him to conclude erroneously that there are zeros at the points where the graph instantaneously switches direction. This phenomenon is similar to one noted in Aspinwall, Shaw and Presmeg (1997), which they termed uncontrollable mental imagery. This is where visual images associated with students' graphical interpretations interfere with their analytic interpretations.

Further questioning revealed that Carson's translation into the physical mode only appears to affect his ability to deal with sudden transitions from increasing to decreasing or vice-versa. Consequently, his errors are limited to several discrete points. So, his use of the connection between graphical and physical modes appears to help him more than it hinders.

Discussion

In my view, calculus curricula that make non-trivial attempts to incorporate graphical and physical modes carry with them the implicit goal of fostering representational fluency. In other words, the goal of incorporating visual/graphical and kinesthetic/physical elements into a calculus course is not to expose students to separate modes of thinking, each of which targets a specific class of problems. Rather, the goal is to expose students to ways of approaching problems that can complement and elaborate each other. Simply exposing students to multiple representations does not ensure that they can translate between them. In order to better understand how to foster a rich back and forth relationship between modes of reasoning researchers need to understand what such transitions look like, how they evolve over time and

what kinds of tasks and teaching actions help foster them. This paper is a contribution to the first of these goals, illuminating what these transitions look like, both when they are prompted and unprompted. These transitions, at least in this particular data set are not uncommon. They occurred in two thirds of student solutions that could have shown an unprompted transition

If I were to stick solely to classifying general tendencies, Carson would be labeled a physical thinker because he used physical reasoning as part of his solution to every task. However, this labeling would have completely overlooked the rich unprompted transitions between modes of thinking that were integral to his problem solving processes.

These transitions between modes, in which one mode of reasoning informs another, are central to how the VAP-model views the development of analytic, visual and physical modes. More importantly, they shed light on what these transitions look like and may be used as a launching point for developing curricula and instruction that strengthens students' ability to make such transitions.

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LEVELS OF COLLEGE STUDENTS' CONCEPTION OF DUALITY IN UNDERSTANDING INFINITY

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Interpreting students' views of infinity posits a challenge for researchers due to the dynamic nature of the conception. There is diversity and variation among the students' process-object perceptions. The fluctuations between students' views however reveal an undeveloped duality conception. This paper seeks to examine college students' conception of duality in understanding and representing infinity with the intent to elucidate strategy that could guide researchers in categorizing students' views of infinity into different levels. Data for the study were collected from N=69 college pre-calculus students at one of the southwestern universities in the U.S. using self-report questionnaire and interviews. Data was triangulated using multiple measures analyzed by three independent experts using self-designed coding sheet to assess students' externalization of the duality conception of infinity.

Keywords: Advanced Mathematical Thinking

Theoretical Framework

This study is supported by APOS theory (Dubinsky, Weller, McDonald & Brown, 2005) to model the development of the duality conception via actions, processes, objects, and schema. Performing a small number of iterations constitutes an action. An action that is continuously reflected upon and interiorized indefinitely constitutes a process, and an encapsulated process constitutes an object. A coherent collection of objects and processes constitutes the schema (Dubinsky, 1991) and the schema represents the "process-object duality" (Monaghan, 2001).

We designed a framework to assess duality concept development called Action-Process-Object-Duality (APOD) (Babarinsa, Tchoshanov & McDermott, 2012) which was adapted from the APOS theory. Sfard (1991) conjectured two ways of developing a mathematical concept: structurally as an object, and operationally as a process. The structural conception refers to actual infinity, example of which is the infinity of the number of points in a segment, and the operational conception, which is the process of performing algorithms and actions, is dynamic and refers to the potential infinity (Fischbein, 2001). Fischbein claimed that seeing a function or number both as a process and as an object is fundamental for a deeper understanding of mathematics. According to Sfard (1991), the dual nature of mathematical construct can be observed verbally and through various symbolic representations. However, scholars warns that care needs to be taken in interpreting students' representations of infinity because of the danger of assumption that comes with determining students' views and due to the dynamic nature of the duality conception (Bingolbali & Monaghan, 2008; Falk, 2010, Monaghan 2001).

Majority of researches on infinity carried out in the elementary, secondary and college levels indicate that students' perception of infinity is more of a process. They define infinity as going on and on, continuing forever, endlessly etc. Monaghan (2001) draw our attention to students' usage of the word 'infinite' and 'infinity' in Luis, Moreno and Waldegg's (1991) research. Students perceived infinity to mean an object view and infinite to mean a process view. We believe that the examination of process-object conception of infinity presented by Monaghan (2001, p. 245-246) does not fully address the complex nature of infinity concept. Duality as a

fundamental hidden idea is not explicitly presented. Monaghan takes for granted the explicit representations in determining students' view of infinity by using obvious cases. We also disagree with Kolar and Cadez's (2012) interpretation of the symbol ∞ as representing the concept of actual infinity. They stated that "We believe that it represents the concept of actual infinity and indicated the awareness of the respondents about the infinite amount of numbers" (p. 404). While coding the students' responses, we realized the existence of fluctuations in students' views of infinity. This, we believe, posits challenges for researchers in interpreting students' perceptions of infinity as either a process or an object, and especially in determining the students' process-object duality conception. To overcome these challenges, the proposed study categorized the college students' duality conception into four different levels.

The Level 1 represents the isolated singular view. In this case, only one view is displayed by the student, which could either be the process view (P) or the object view (O). The strength of the conviction/view could either be dominant ('P' or 'O') or recessive ('p' or 'o'), which is insignificant in determining the Level 1. The Level 2 represents the semi-isolated dominant view. In this case, students tend to display both the process and object view, depending on the task/context. Either the strength of the students' object view is dominant and process view is recessive or the strength of the students' process view is dominant and the object view recessive ('Op' or 'Po'). Our belief is that having one view dominant over the other is an indication of strength in the dominant view, which makes us to classify both the Level 1 and Level 2 as students' singularity conception. The Level 3 represents the dual-idiosyncratic view, where both process and object views are recessive (i.e. not strong or convincing). We believe this case to be an indication of equality of views ('p' and 'o'). The level 4, which is the duality view also indicated equality of views, since both the process and object views are dominant (i.e. strong and convincing – 'P' and 'O'). We therefore classify both the Level 3 and Level 4 as students' duality conception.

Method of Inquiry

The participants for the study are N=69 college Pre-Calculus students at one of the southwestern universities in the U.S. Data for this study were collected using self-reported questionnaire administered during class time of the Pre-Calculus sections of the instructors who were willing to let their students voluntarily participate in the study. Data was triangulated using multiple measures analyzed by three independent experts using self-designed coding sheet to assess students' externalization of the duality conception of infinity. The questionnaire required that students complete the four tasks and justify their responses with explanations on: (1) draw infinity (concept-image task), (2) define infinity (concept-definition task), (3) Cookie Monster (scenario-based task), and (4) multiple choice of infinity view (self-reported task).

To analyze students' responses and determine their duality conception level, especially because of the fluctuations in students' views from process to object and vice versa based on the task and context, the students' responses were coded and organized into two major views – the dominant views and the recessive views which are further categorized into the singularity conception and duality conception, based on the strength of students' responses/views.

Results

Below we present samples of student work addressing different levels of college students' singularity/duality conception in understanding and representing infinity.

Level 1: *Singular View* (‘P’ or ‘O’, ‘p’ or ‘o’). For example, a student uses the symbol ∞ to represent infinity in the concept-image task. If we blindly follow Kolar and Cadez’s (2012) interpretation of the symbol ∞ as representing the concept of actual infinity, then we conclude that this is an object view of infinity. However, if one evaluates student’s explanation “*infinity has no ending and it continues forever and ever*”, then it is clear that the student has distinct singular process view of infinity. Level 2: *Dominating View* (‘Op’ or ‘Po’). Here, student uses descriptive context to represent infinity on concept-definition task as “*endless numbers*”. In a Level 3: *Recessive View* (‘p’ and ‘o’) example, student uses “*technical* vs. *mathematical*” context to represent infinity on the scenario-based task. Level 4: *Dual View* (‘P’ and ‘O’). Student’s dual view of infinity is illustrated by “*All of the above and everything and nothing at the same time”.*

The major outcome of the study is the fact that traditional Pre-Calculus coursework does not support the development of duality concept among college students. Contrary, it strengthens the singularity perspective on fundamental ideas of mathematics such as infinity. The Table 1 presents the data supportive of the claim above: 60% of students were at Levels 1 and 2 representing singular view of the infinity concept. The result from the table shows that of the 69 pre-calculus participants that participated in the study, only 1% actually possess the duality conception of infinity (Level 4), which limited the other 40% to the dual recessive level (Level 3). This explains that though these 29 were able to conceptualize infinity as a process and as an object, their convictions are not strong but recessive. Of the 39 participants having the singularity conception of infinity, 27 of them possess semi-isolated dominating views and the remaining 12 which is 22% of all have the singular isolated view of infinity.

We further analyzed the data to examine students’ conception within each level, Levels 1 and 2, in particular. The data presented in Table 1 below show that at Level 1, majority of the Pre-Calculus students possess the singular process view of infinity (83%). Even at Level 2, the process view is dominating: 89% of students had dominating process and recessive object view of infinity.

Table 1: Distribution of College Students’ Conception of Duality

Level	Level 1		Level 2		Level 3	Level 4
View	‘P’	‘O’	‘Po’	‘Op’	‘p’ and ‘o’	‘P’ and ‘O’
View Total	10 (83%)	2 (17%)	24 (89%)	3 (11%)		
Level Total	12 (22%)		27 (38%)		29 (40%)	1 (1%)

The study also found that coding and assessing college students’ conception of duality is a challenging and complex process due to the dynamic nature of the conception that is task-dependent and context-dependent. The results of this study could serve as a facilitating instrument to further analyze cognitive obstacles in college students’ understanding of infinity concept.

Discussion and Conclusion

Interpreting students’ views of infinity posits a challenge for researchers due to the dynamic nature of the conception. There is diversity and variation among students’ process-object perceptions. The fluctuations between students’ views however reveal an undeveloped duality conception. This study examined college students’ conception of duality in understanding and representing infinity with the intent to design strategy that could guide researchers in categorizing students’ views of infinity into different levels.

It is known that concept of duality as any other fundamental ideas of mathematics are "built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures" (Tall & Vinner, 1981, p. 151). Results of our study reveal that college students' experiences in traditional Pre-calculus course are not supportive of the development of duality conception. Therefore, it is important to provide college students with relevant experiences to build the concept of duality, which will help them to understand mathematical concepts (e.g., infinity) at a more rigorous level. Understanding the dual nature of mathematical concepts could help students become more knowledgeable and flexible in learning abstract and complex mathematical ideas. "In order to be able to deal with mathematics flexibly, students need both the process and object views of many concepts, as well as the ability to move between the two views when appropriate" (Selden, 2002). Gray and Tall (1994) describes concepts that could be viewed both as a process and an object as procept. Hence, we consider a proceptual perspective as a tool to help students at their earlier stages of learning to understand and overcome the contradictory and counterintuitive nature of infinity concept.

Practical significance of the study is that it helps to recognize misconceptions and start addressing them so students will have a more comprehensive view of fundamental mathematical ideas as they progress through Calculus coursework sequence. If pre-or-miss-conceptions are not timely recognized and addressed, then students' traditional experiences could be easily built on strong 'narrow-minded' mental scripts that could be later transferred to "immature" understanding of mathematical concepts.

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TEACHER REPORTED IMPACTS ON PRACTICE: USE OF A MODELS AND MODELING PERSPECTIVE IN MATHEMATICS TEACHER EDUCATION

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Researchers have incorporated a Models and Modeling perspective to examine teacher thinking about mathematical topics in teacher education settings. In addition to calls for additional research, questions remain about how this perspective might apply to domains beyond mathematical topics. How can a Models and Modeling perspective advance teacher thinking about what it means to teach a mathematical idea, and how that idea manifests in the classroom? In this paper the researcher will discuss findings from a study that used a Models and Modeling perspective in a teacher education setting to examine teachers' development of quantitative reasoning with a focus on the impacts in the teachers' classroom. The implications of this work contribute to researchers and teacher educators using a Models and Modeling perspective in teacher education.

Keywords: Modeling, Teacher Education-Inservice/Professional Development

Connecting teacher education to practice has long been a challenge for mathematics educators, and one approach to addressing this gap is through research examining aspects of teacher learning and systems of interpretation (Doerr & Lesh, 2003; Zawojewski et al., 2008; Potari, 2012). In response to this challenge, Lesh and colleagues have described how a Models and Modeling approach to in-service teacher education can challenge teachers to develop ways of thinking to help their students while simultaneously documenting their development for research purposes. This approach uses Model Eliciting Activities (MEAs), which are tasks that engage teachers in thinking about realistic and complex problems embedded in their practice in order to foster ways of thinking that can be used to communicate and make sense of these situations (Doerr & Lesh, 2003; Lesh & Zawojewski, 2007). MEAs have contributed to teacher development because these activities make teachers engage in applicable mathematics, consider student reasoning more deeply, and reflect on beliefs about problem solving (Chamberlin, Farmer, & Novak, 2008; Schorr & Koellner-Clark, 2003; Schorr & Lesh, 2003).

While studies have documented how this can be done, the merit of this approach for teacher education is still needed. There is also a need for additional research given the recent demands that education reform place on teacher education programs, such as the Common Core State Standards (Garfunkel, Reys, Fey, Robinson, & Mark, 2011). Teacher educators subsequently need to ensure teachers, both pre- and in-service, are pushed in productive ways to think about the mathematics and its connection to the classroom (Potari, 2012). The purpose of this study was to contribute to the body of work by focusing on the latter of these objectives, as this approach can broaden the perspective of the traditional impact of a Models and Modeling approach on teacher education. In this study I examined the development of in-service teacher models of quantitative reasoning in relation to teachers' classroom practice. The research question was: how did teacher models of how quantitative reasoning develop through Models and Modeling approach in a teacher education course, with specific focus on how teachers communicated their teaching and assessment strategies in their own classroom.

Methods

This study was grounded in a Models and Modeling Perspective, as described by Lesh and colleagues. In addition to being a powerful lens for examining teacher education, a Models and Modeling Perspective also provided guidelines for methods that can be used to provide significant findings given this study's research question. With these methods, a Models and Modeling Perspective offered a framework for understanding teachers' ways of thinking, their development, and provides a mechanism for analyzing and piecing together findings (Koellner-Clark & Lesh, 2003; Sriraman & English, 2010).

The setting for this study was within a master's program in mathematics, where teachers took a combination of mathematics and mathematics education courses over two years. I focused the study on a newly developed mathematics education course in the program, called Quantitative Reasoning in Secondary Mathematics, which was offered in the summer. This course was taught by an instructor familiar with the Models and Modeling perspective. The course content was framed to follow a models and modeling perspective, as the instructor incorporated an MEA aligned with this perspective as 50% of the course grade. In the MEA, 21 teachers worked in six groups to receive feedback about their group's task during the summer from the instructor, each other, and undergraduate students who completed the task. The final, optional, iteration occurred during the fall as teachers implemented their task in his or her classroom. Each feedback cycle prompted an updated iteration of the task and supporting documents that captured how the teachers' ways of thinking developed.

For the purpose of this report, I focus on the four teachers (Table 1) who implemented the task their group created in their own classroom. This allowed me to document the influence the course had on their view of quantitative reasoning with particular focus on how teacher models related to the classroom. Data collection consisted of the iterations of MEA documents generated by the individual teacher, who was sometimes working within a larger group. Using content analysis (Patton, 2002), I adapted Hjalmarson's (2008) analytical tool to identify themes on teachers' communication quantitative reasoning.

Table 1: Participant Description

Participant:	Joyce	Tiffany	Allie	Penny
Group	A	B	B	B
Years teaching	4	3	3	18
Grade bands	9-10	9-10	9-11	10-12
Subjects	Geometry	Algebra	Algebra	Pre-calculus

Findings

The MEA documented how the teacher models of quantitative reasoning developed, and here I summarize four themes relating to how these teachers communicated how quantitative reasoning was framed, taught, and assessed in the his or her classroom. The first theme evident in all teachers' development was an increased value placed on student thinking in the quantitative reasoning tasks. Teachers acknowledged how their thinking about quantitative reasoning developed through receiving student feedback on their task. For example, Joyce's group commented on the student feedback, saying, "did they show us enough work? How do we elicit that out of them? How do we change our questions so we get a little more verification that they know what they're doing rather than just doing it?" The other group also made comments about needing to adapt the questions in their task to better elicit student thinking. These changes were subsequently observed in how teachers in both groups communicated the task should be

framed and implemented in the classroom.

A second theme was that all teachers communicated increased attention towards promoting productive ways of student thinking about quantitative reasoning. Initially, the teachers of both groups were focused on constructing questions that examined students' quantitative reasoning. Through the MEA iterations, teachers began scaffolding questions to elicit answers, and ways of thinking, that the teachers desired. For example, group A split questions into subparts and group B added a guiding questions list for the facilitator. These changes were driven again by the student feedback as student misconceptions were identified through the task. In some cases, questions that were too difficult for students were abandoned; group A's questions on relationships between functions and group B's question about slope defined as a relationship between quantities were such cases. The answers the teachers had listed for expectations were not met, and while they identified student's lack of success, they did not address how to overcome these difficulties.

A third theme was generalization becoming an essential component in their model of quantitative reasoning. This element was manifested in two ways: the first in promoting students to generalize the mathematical ideas being explored in the tasks. For instance, Alice's final reflection said:

> My understanding of quantitative reasoning truly developed during the revision process of our MEA. By being able to take the feedback and discuss it with my group, I was able to really grasp how to incorporate the reasoning component into a given task. It was very neat to be able to take a relatively simple task, such as the plumber scenario, and turn it into an activity that reveals how students are thinking about the problem and also be able to help the students generalize what they learned from our MEA to any function representation.

The second place this theme revealed was the task revision. Group A was especially explicit about how they were generalizing successful strategies, such as pattern recognition, to incorporate in updated versions of the task in order to promote productive student thinking.

The final theme was the difficulty teachers had in acknowledging and accepting multiple answers to quantitative reasoning questions. While both Tiffany and Allie indicated quantitative reasoning and their task should incorporate multiple ways to think about and solve a problem, the actual task and assessment of student work did not reflect these goals. Group A identified this issue and altered one question in the final version in an attempt to incorporate multiple ways of thinking from students, but task as a whole did not reflect these goals. Group B did not comment on the issue, though overall this challenge was prevalent in both groups throughout the MEA.

Implications

The aim of this study was to describe how teacher models of quantitative reasoning were developed in relation to their classroom practice through a models and modeling perspective in teacher education. Common themes from the participant indicated (1) quantitative reasoning was presented more conceptually to students in comparison to teachers' original models; (2) productive student thinking was promoted through targeted decisions in the design and implementation of the task in ways that were not present originally; (3) teachers began seeing quantitative reasoning as a generalized process that appeared in the classroom as more of a habit of mind rather than tied to specific tasks; and (4) quantitative reasoning challenged teachers to consider multiple ways of thinking, one of which being its connection to their classroom.

These themes indicated teacher education that follows a models and modeling perspective could have positive impacts on classroom practice. Teachers were put in positions that challenged their prior models of quantitative reasoning and promoted development in ways that

align with teachers' practice, a characteristic of successful teacher education (Potari, 2012). Identifying areas of difficulty, such as accepting only single ways of thinking, was also an important finding, setting the stage for future work to be done investigating why these challenges occur and how to support teachers in more advanced ways of thinking about the problem. Using this information, teacher educators are better suited to structure teacher education in ways that advance teacher thinking and change teacher practice (English, 2003; Lesh, Middleton, Caylor, & Gupta, 2008).

The documentation of development reported in this study can support both teacher educators and researchers of teacher education to continue considering what it means for teachers to think productively about quantitative reasoning and other mathematical ideas. Using a Models and Modeling approach encouraged teachers to decide what productive ways of thinking are in terms of their students and classrooms. Identifying patterns of development can guide future teacher education efforts taking this perspective and support improvement to address the issues occurring in this study. Finally, this work helps set the stage for the theoretical groundwork for thinking about teacher development using a Models and Modeling perspective.

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MATHEMATICS TEACHERS USING CREATIVE INSUBORDINATION TO ADVOCATE FOR STUDENT UNDERSTANDING AND ROBUST MATHEMATICAL IDENTITIES

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Mathematics teachers need to be able to skillfully negotiate the politics of language, racism, and testing. This study examines secondary mathematics teachers who received professional development to understand teaching as a political act and who are using creative insubordination in order to support their Latin@, black, and low-income students.

Keywords: Equity and Diversity; Teacher Knowledge; Teacher Education-Pre-service

Teaching mathematics has always been a political enterprise. However, over the past decade, recent developments have intensified the need for teachers to learn about much more than mathematics, pedagogy, and students (Darling-Hammond et al., 2005). This trend is especially pronounced for educators of Latin@[i], black, American Indian, and low-income students who historically have received poor quality mathematics instruction and/or who have been the focus of back-to-basics reforms. Regardless of teachers' depth of mathematical knowledge, ability to relate to students, or propensity to develop standards-based lessons, those who are not adept at negotiating the politics of language, racism, and testing cannot adequately support their marginalized students to learn high quality mathematics and develop robust mathematical identities. Teachers need to be prepared with the kind of political knowledge that allows them to creatively side-step policies and practices that depict Latin@, black, American Indian, and low-income youth as unmotivated, inferior, or unable to learn rigorous mathematics. This study focuses on teachers who have developed such political knowledge.

Creative Insubordination and Political Conocimiento

In their ethnographic work conducted in Chicago Public Schools in the late 1970s, Crowson & Morris (1985) found "widespread rules and directives violations among site-level administrators" that they labeled partly as "creative insubordination" because these violations were benign and counter-bureaucratic and substituted the principal's values for those implicit in organizational policies directed from above. Roche summarizes various studies and notes that:

> Creative insubordination has two main purposes: to ensure that the system directives do not impinge unfairly or inappropriately on teachers and students and to avoid the possible backlash that outright defiance might incur. Crowson (1989) and Haynes and Licata (1995) argue that when principals use creative insubordination, the counterbureacratic behaviors they adopt often contain a moral element designed to balance antieducational consequences." (Roche, 1999, 257-8)

Our work (Gutiérrez, Irving, & Gerardo, in preparation) builds upon and extends the early research on creative insubordination by connecting it with teachers and showing its usefulness within the context of secondary mathematics. With respect to mathematics teaching, creative insubordination includes the following acts: creating a counter-narrative to the achievement gap; questioning the forms of mathematics presented in school; highlighting the humanity and

uncertainty of mathematics; positioning students as authors of mathematics; and challenging deficit narratives of students of color.

Elsewhere, I have described the concept of *political conocimiento for teaching mathematics* (Gutiérrez, 2012) that connects mathematical content knowledge, pedagogical knowledge, knowledge with communities, political knowledge within a community of like-minded individuals, and that takes into consideration the history of mathematics teaching and learning in a global society. See Figure 1.

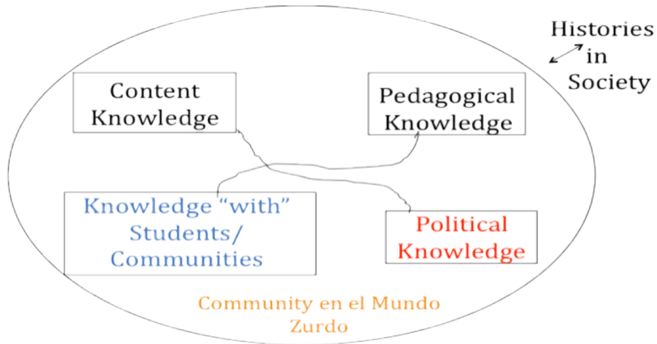

Figure 1: Political Conocimiento for Teaching Mathematics

Given this broader depiction of the knowledge needed for mathematics teaching, pre-service teachers in our program are provided with opportunities to rehearse not just mathematical lessons, but also political situations that arise in the everyday work of practicing mathematics teachers who serve marginalized youth.

Methods

This study is part of a larger longitudinal research project that seeks to understand how an equity-based teacher education program influences the knowledge bases, skills, and dispositions of 19 pre-service secondary mathematics teachers (PSMTs) who prepare for and eventually teach marginalized students (defined here as Latin@s, blacks, American Indians, and low-income students). During their two years as pre-service teachers, the PSMTs attended the regular teacher education program, including field observations, lesson planning, and portfolio development for state level credentials in grades 6-12 mathematics. They also participated in: a 3-hour biweekly seminar that focused on issues of rigorous and creative mathematics, social justice teaching, as well as strategies for supporting black and Latin@ youth, low income students, English learners, and strategies for negotiating teaching in an era of high stakes testing; conferences and movie viewings that provided deeper understandings of students in a diverse society; and met every other week with our partner teacher who modeled creative insubordination in practice. PSMTs were also required to develop activities for and volunteer in a weekly after-school mathematics club that supported black and Latin@ middle school students. Finally, each attended bi-weekly, hour-long mentoring sessions with the author and/or a doctoral student.

The PSMTs ranged in ethnic background and language proficiencies, though most were mathematics majors receiving an education minor. Participants were followed during the two years in the teacher education program and one year into their teaching. Data sources included: audio recordings of seminars and mentoring sessions; written responses to an online forum; observations of the after-school mathematics club; and extensive field notes of all of the

aforementioned sessions and meetings. Members of the research team made summaries of the mentoring sessions we conducted and provided some analyses. In addition, audio recordings of seminar and mentoring sessions were transcribed on an ongoing and selective basis.

Codes were developed out of the mathematics education literature as it related to identity, positioning, social justice, Latin@ and black adolescents, as well as out of our observations of the PSMTs as they developed over the years. We coded particular transcripts using an open-ended coding scheme, beginning with mentoring sessions and later moving to seminar sessions. I focus here on common themes that arose between participants.

Findings

Through readings and interactions with professionals, many of the PSMTs in this study developed an understanding early on that creative insubordination was necessary if they were to become the kinds of mathematics teachers they intended—providing their students with a deeper, conceptual understanding of mathematics and also supporting robust mathematical identities, not just higher test scores. They identified ways in which schools prevented students from expressing themselves (e.g., by assuming all students will learn in the same way and assessing them in standardized ways); by not acknowledging that some students know algorithms from other countries and not allowing them to use such algorithms in the US. Participants suggested such solutions as helping students learn to "play the game of school" but also "changing the game" by allowing students to speak in their home languages and changing up what counts as "showing your work" (e.g., allowing algorithms not commonly used in the US).

Professing the Benefits of Creative Insubordination to Others

Participants not only discussed their inclination to use creative insubordination with the research team in meetings, they professed these beliefs to peers in the regular teacher education program who had not received opportunities to develop political knowledge. After their first year, some participants moved beyond merely understanding what creative insubordination is or why it would be beneficial to taking the message to others and challenging their resistance to speak up. We observed participants in other courses publicly counseling peers on why "going along with the system" (e.g., covering the intended curriculum at a particular pace because you are merely a student teacher and are not really in charge) would be considered unprofessional.

Interrogating School Policies

Not only were participants in this study able to articulate their beliefs about the importance of creative insubordination, they were able to state how they would resist school initiatives that went against student identity or learning. One incident involved a PSMT during student teaching that was faced with a colleague who had given out over 150 referrals to black males for sagging pants, a school policy that enforced the dress code. Frustrated that this policy was disproportionately affecting black males and students in his mathematics class, he suggests the best course of action is not to confront the colleague (who wouldn't likely listen to his argument), but rather to bring the policy up at a faculty meeting and engage others in a discussion about the potentially unintended consequences of a seemingly reasonable policy. Other forms of creative insubordination involved PSMTs standing up to their cooperating teachers and fighting for mathematical rigor in the curriculum (e.g., emphasizing concepts over procedures) when their colleagues did not believe students could handle such rigor.

Challenging Deficit Discourses in Faculty Meetings

PSMTs in this study also showed evidence of standing up for one's students, even if it involved risking one's standing with colleagues or superiors. For example, one PSMT faced a

director who claimed in a school meeting that the achievement gap was due to black students' culture of disengagement. Outraged at the idea that he was blaming the students, but also fearful of losing her job, the participant in this study chose her words carefully, "I know this isn't what you mean to say… (that) it's their fault" and stood up to the comment, "So, I just want to make sure we are all noting the long history of oppression with respect to that group." In other situations, PSMTs noted the importance of word choice (e.g., raising issues in the form of questions rather than assertions), the timing of actions (e.g., not always acting immediately when people may be defensive), and also the need to choose carefully which battles were worth fighting (e.g., keeping the long game plan in mind).

Discussion

This study examined pre-service secondary mathematics teachers' abilities to understand and develop repertoires of creative insubordination for the purposes of supporting Latin@, black, and low-income students to develop conceptual understanding and robust mathematical identities. The participants were able to not only identify deficit-based discourses operating in schools and see the need for creative insubordination, but they were able to consider contextual factors (e.g., nature of relationships, word choice) that allowed them to employ their political knowledge. This study provides evidence that attention to political knowledge can help programs better prepare mathematics teachers to successfully negotiate their working climates and advocate for marginalized youth. Professional development needs to move beyond pedagogical content knowledge to consider the politics of teaching.

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[i] I use the @ sign to indicate both an "a" and "o" ending (Latina and Latino). The presence of both an "a" and "o" ending decenters the patriarchal nature of the Spanish language where is it customary for groups of males (Latinos) and females (Latinas) to be written in the form that denotes only males (Latinos). The term is written Latin@ with the "a" and "o" intertwined, as opposed to Latina/Latino, as a sign of solidarity with individuals who identify as lesbian, gay, bisexual, transgender, questioning, and queer (LGBTQ).

MEASURING MOTIVATIONAL CONDITIONS IN MATHEMATICS CLASSROOMS

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Wlodkowski and Ginsberg (1995) developed a framework of four motivational conditions that when present in classrooms support students' intrinsic motivation, which is strongly related to student learning. We conducted research to develop and validate a survey to measure student perceptions of these conditions in mathematics classrooms. The study relied on the theoretical framework to determine construct validity and included data from high school and under-graduate students to determine the internal consistency of the four measures in the survey, which confirmed the reliability of the four measures for these populations. Using the data we collected, we outline some potential uses of the survey data for mathematics teachers and researchers.

Keywords: Affect, Emotions, Beliefs and Attitudes, Equity and Diversity, Research Methods

In our experience, many mathematics teachers would like their students to be more motivated and are eager to learn how to increase the motivation of their students. Wlodkowski and Ginsberg's (1995) Motivational Framework provides a theoretical tool for helping teachers accomplish this goal. In order to make the Motivational Framework more functional for mathematics teachers, we operationalize the framework into a Likert-type survey that teachers can give students. For this study, we examine completed surveys from students in high school mathematics classes and undergraduate mathematics service courses in order to test the validity and reliability of the survey in these contexts. In this brief research report, we provide an overview of our theoretical perspective, describe our item construction in the survey, report evidence on the validity of the survey, and outline potential uses of the survey to understand the motivational conditions in mathematics classrooms.

Theoretical Perspective

Wlodkowski and Ginsberg's (1995) identify four conditions in their Motivational Framework: establishing inclusion, developing attitude, enhancing meaning, and engendering competence. *Inclusion* is evident when students feel valued and respected by their teacher and peers. A *positive attitude* is indicated when students have a favorable disposition towards learning. *Meaning* exists when students perceive themselves to be engaged and challenged by the classroom learning. *Competence* is apparent when students feel confident in their ability to learn and value their knowledge as a means to both an authentically possible and hopeful future. These conditions result from students' perceptions and feelings as opposed to being implemented by specific teacher behaviors. For example, inclusion is supported when teachers treat all students respectfully and fairly. However, the degree to which respect and fairness exist is a function of students' perceptions of whether they are treated respectfully and fairly rather than of the teacher's intentions or actions.Therefore, in order for teachers to understand these conditions in their classrooms, they must understand how students perceive them.

The four conditions of the Motivational Framework support students' intrinsic motivation, when developed and maintained in a context, such as a mathematics classroom (Ginsberg & Wlodkowski, 2009). Middelton and Spanias (1999) state succinctly that "motivations are reasons individuals have for behaving in a given manner in a given situation" (p. 66). They

characterize motivation as being extrinsic or intrinsic. Extrinsic motivation occurs when people engage in an activity either to avoid punishment or to pursue a reward. For example, someone might complete an assignment because they want a good grade, not because their ultimate goal is to learn from doing the activity. Someone who engages in the assignment for the sake of learning is said to be intrinsically motivated. Intrinsically motivated people engage in activities because they have decided that the activity itself is rewarding. Middleton and Spanias report that intrinsically motivated students generally exhibit more interest in what they are learning, are more concerned with the quality of their learning, and are more persistent in their efforts to learn.

In developing the essential environmental conditions for intrinsic motivation, Wlodkowski and Ginsberg draw on philosophy, sociology, the study of spiritual ideology, economics, linguistics, anthropology, political science, education theory, and other disciplines (Ginsberg, 2005). Turner, Warzen, and Christensen (2011) review theories of motivation and motivation in mathematics education research and select for their research four motivational constructs that related to mathematics instruction and learning: “fostering students’ competence, autonomy, and belongingness in the mathematics classroom and emphasizing the meaningfulness of mathematics” (p. 721). The similarity of these constructs with the motivational conditions in the Motivational Framework supports its applicability in mathematics classrooms.

Methods

The primary goal of the project was to develop a survey to measure the four motivational conditions of the Motivational Framework (Ginsberg & Wlodkowski, 2009; Wlodkowski & Ginsberg, 1995) and to validate the survey in two populations: secondary students and undergraduates. In this section we describe the sampling used to select participants from the two populations to validate the Motivational Conditions in Mathematics Classrooms $(MC)^2$ Survey. A description of the measures of the four motivational conditions follows.

Sampling and Participants

First, we obtained a convenience sampling of secondary students. Specifically, secondary mathematics teachers involved in a jointly delivered master’s program in the Rocky Mountain region administered the survey to their students as part of their work in the Mathematics Teacher Leadership Center (Math TLC), NSF-funded grant project. We requested that 30 of the teachers involved in the project survey students in at least three courses they taught in spring 2012. Eighteen teachers (60%) responded with 1344 completed surveys of secondary students.

Second, we obtained a convenience sampling of undergraduates from a doctoral granting university in the Rocky Mountain region. We contacted the 18 instructors of three courses typically taken by first-year undergraduates: college algebra, liberal arts mathematics, and introduction to statistics. Twelve instructors (67%) allowed researchers to administer the anonymous survey in their classes, which resulted in 351 completed surveys.

Survey

The $(MC)^2$ Survey initially contained 43 items intended to measure the four motivational conditions (Wlodkowski and Ginsberg, 1995). Each item was a statement about some aspect of one of the four conditions and a four-point Likert-type scale to which students selected strongly disagree, disagree, agree, or strongly agree. We measured each motivational condition by calculating the mean score of all of the items associated with the condition. Thus, the survey produced four scores ranging from 0 (strongly disagree) to 3 (strongly agree) measuring the extent to which students thought that the teacher (and students) established inclusion, developed attitude, enhanced meaning, and engendered competence in the classroom.

Results

Validating the $(MC)^2$ Survey through factor analysis proved unsuccessful since the four motivational conditions are interrelated as supported by the theoretical framework. Specifically, the factor analysis produced two orthogonal factors that included items from each theoretical motivational condition. So, we retained the four motivational conditions based on the theory of Wlodkowski and Ginsberg's (1995) Motivational Framework, considering the results of the overall factor analysis as consistent with the strong theoretical correlation among the conditions. We then conducted factor analyses on each individual condition. This rationale formed the basis of the construct validity of the survey.

Inclusion

The inclusion condition included all 14 constructed items involving feelings of respect and connectedness in the classroom. It had a Cronbach's alpha measure of .90 for secondary students and .87 for undergraduates, indicating a very strong internal reliability among the items. Two items included in the survey are as follows:

- I feel my mathematical ideas are respected by my teacher.
- I feel safe voicing my thinking in this class.

Attitude

The attitude condition included all nine constructed items involving feelings of choice and the relevance of the content in the classroom. It had a Cronbach's alpha measures of .85 and .84 for secondary students and undergraduates, respectively. Example survey items include the following:

- My teacher encourages me to express my mathematical ideas.
- The math we learn in this class relates to my own interests, experiences, or plans.

Meaning

The meaning construct included items that indicated whether students thought the material was challenging and engaging to them. A factor analysis indicated that two items were dissimilar from the others. As a result we discarded the two items: "I think I will have to know this math in the future" and "I think the math we learn in this class will be useful for future coursework." The resulting meaning condition contained 10 items that had very good internal reliability ($\alpha = .86$) for both secondary students and undergraduates. Two items retained in the meaning condition included the following:

- The instructor of this course expects me to make sense of what I am learning.
- I feel the content of this course engages me.

Competence

The competence construct included all eight constructed items reflecting students' feelings of being capable of doing the mathematics in the classroom. It had a Cronbach's alpha measure of .86 for secondary students and .89 for undergraduates, indicating a strong internal reliability among the items. The competence survey statements included the following:

- It is okay to make mistakes in this class.
- I feel I can be successful in this class.

Discussion

Even though the sampling used in this study involved two distinct convenience samples, the strong internal reliability measures for both samples indicated that the survey reasonably measures the condition indicated by the theoretical framework. Although validating the survey using more rigorous sampling techniques is warranted, preliminary data analysis suggests that

the survey is acceptable and provides meaningful data.

Based on the data we collected, researchers and teachers could use and analyze $(MC)^2$ data and the types of information the data in at least three ways. First, our data included students in grades 9-16, which allowed us to analyze trends across grade levels. Second, an outstanding question is how the motivational conditions vary for a given teacher. Data we collected would allow us to investigate patterns across courses for an individual teacher. Third, $(MC)^2$ data may also be used in statistical analysis. For example, our data may be used to compare the four motivational conditions among the three undergraduate courses: Introduction to Statistics, Liberal Arts Math, and College Algebra.

Conclusion

The Motivational Framework (Wlodkowski & Ginsberg, 1995) offers instrumental guidance to educators for developing environments that support the students' intrinsic motivation to learn. We created the $(MC)^2$ Survey to 1) refine the Motivational Framework for mathematics classrooms and 2) provide a means of collecting valid and reliable data for the four motivational conditions. Survey results can be used at the classroom level for an individual teacher or more broadly across courses or grade levels. While the survey data does not explain why students have particular perceptions of the motivational conditions, it can serve to indicate the context and motivational conditions that might warrant attention.

Wlodkowski and Ginsberg's (1995) work can help in interpreting $(MC)^2$ Survey results as well as inform explorations of the causes of particular trends. However, Turner and Meyer (1999) argue that motivation is not independent of content. This may be particularly true with mathematics because of the strong personal and societal beliefs associated with mathematics and the specific characteristics of mathematics instructional practices and mathematics tasks. Thus, when investigating the influences on students' perceptions of the four motivational conditions, researchers should consider both general and mathematics-specific factors.

Because of the important relationship between intrinsic motivation and learning, it behooves teachers and administrators to understand students' intrinsic motivations. The $(MC)^2$ Survey provides a practical means of starting this process.

Acknowledgement

This material is based upon work supported by the National Science Foundation under Grant No. DUE0832026. Any opinions, findings and conclusions or recommendations expressed in this manuscript are those of the authors and do not necessarily reflect the views of the National Science Foundation.

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DIVERSITY IN METHODOLOGY: DIFFERENT POSSIBILITIES FOR DATA COLLECTION, ANALYSIS, AND REPRESENTATION

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Mathematics education research over the past half century can be understood as operating in four distinct yet overlapping and simultaneously operating historical moments: the process–product moment (1970s–), the interpretivist–constructivist moment (1980s–), the social-turn moment (mid 1980s–), and the sociopolitical-turn moment (2000s–). Each moment embraces unique theoretical perspectives as it critiques or rejects others. Moreover, because methodology is inextricably linked to theory, each moment calls forth unique methodological perspectives. Using exemplars of research articles from each moment, the authors illustrate how each moment provides different possibilities for data collection, analysis, and representation.

Keywords: Research Methods

Introduction

Elsewhere, in an attempt to make sense of the complexities of divergent theoretical perspectives in mathematics education research, we, Stinson and Bullock (2012a, 2012b), identified four distinct yet overlapping and simultaneously operating (therefore no end dates) historical shifts or moments in mathematics education research over the past four decades: the process–product moment (1970s–), the interpretivist–constructivist moment (1980s–), the social-turn moment (mid 1980s–), and the sociopolitical-turn moment (2000s–). We showed that each moment (more or less) embraces unique theoretical traditions as it rejects others. We also made an argument for a hybrid critical postmodern theoretical approach to conducting mathematics education research where the researcher continually and simultaneously negotiates the praxis of the critical and the uncertainty of the postmodern (see also Stinson, 2009). Here, given that methodology is inextricably linked to theoretical perspective (LeCompte, Preissle, & Tesch, 1993), we extend our previous discussion to explore possibilities for data collection, analysis, and representation—that is, methodological possibilities—through the four shifts or moments. We claim that each of the four moments of mathematics education identified can be mapped more or less to one or two paradigms of inquiry—predict, understand, emancipate, and/or deconstruct (see Lather, 2006, p. 37)—which, in turn, provide different possibilities for data collection, analysis, and representation. We use “effective” or “good” mathematics teaching as just one example of a research strand in which the differences and commonalities among methodological approaches might be highlighted.

Methodologies across the Moments: Research on Effective Mathematics Teaching

Process–Product Moment

The process–product moment (1970–) is characterized by linking processes of classroom practice to student achievement outcomes or “products.” Clearly positioned in the *predict paradigm of inquiry* (Lather 2006, p. 37), theoretically and methodologically, researchers in this moment rely primarily on quantitative statistical inference as a means “to ‘predict’ social phenomena by ‘objectively’ observing and measuring a ‘reasonable’ universe” (Stinson & Bullock, 2012a, p. 43). An exemplar of process–product research is Good and Grouws’s (1979)

article "The Missouri Mathematics Effectiveness Project: An Experimental Study in Fourth-Grade Classrooms." It reports a research project that sought to create a single picture for all contexts of what the effective mathematics teacher does in the classroom. Initial data collection for the project included pre- and post-test data on student achievement to select teachers across a school district who were "consistent and relatively effective or ineffective in obtaining student achievement results" (p. 355). Once "labeled," these teachers were observed in their classrooms for approximately three months, and based on analyses of tallied behaviors observed a behavioral profile was created for each teacher. Good and Grouws then separated the teachers who they had labeled as "effective" and "ineffective" from the achievement test data and created a composite profile of both groups. They used the differences between those profiles to develop a set of characteristics of teacher effectiveness. Data representation consisted of a table indicating "Key Instructional Behaviors": observed behaviors from the effective teachers along with the time spent on each behavior. The table was presented as a rubric of sorts that administrators and mathematics teacher educators might use to "train" teachers to "perform" in ways that student achievement outcomes could be predicted.

Interpretivist–Constructivist Moment

In the interpretivist–constructivist moment (1980s–) the aim of the researcher is no longer to predict social phenomena but rather to understand it. Here, and elsewhere (see Stinson & Bullock, 2012a), due to their near-simultaneous occurrence in mathematics education research in the 1980s, interpretivist research and constructivist research is combined into a single moment. Nevertheless, it is important to note that although both of these two research strands are securely positioned in the *understand paradigm of inquiry* (Lather 2006, p. 37), they seek understanding in different ways. Therefore, they take up different theoretical and methodological possibilities.

At one end, the interpretivist researcher seeks to understand social phenomena by attempting to access the meaning(s) that people assign to social phenomena. An example is Wilson, Cooney, and Stinson's (2005) article "What Constitutes Good Mathematics Teaching and How it Develops? Nine High School Teachers' Perspectives." It reports results of a project that examined the "views of nine experienced and professionally active teachers about what they consider good teaching to be and how it develops" (p. 83). In the project, Wilson and colleagues inferred notions of good mathematics teaching from case study data related to the participating teachers' beliefs and attitudes about effective teaching. Methods of data collection comprised of conducting and transcribing three, semi-structured interviews with seasoned teachers who were mentoring student teachers. To analyze the data, Wilson and colleagues used a qualitative coding approach: developing a preliminary coding scheme in an initial analysis and modifying that scheme as they repeatedly moved through the data. Data representation consisted of several direct quotations from the interview transcripts and a modified frequency table, describing the characteristics of effective teaching that the teachers identified and how they believed those characteristics were best learned.

At the other end, the constructivist researcher understands meaning(s) as something that is constructed through experience. Or, said in another way, the focus of research is on understanding and identifying the processes of how people acquire or construct different meaning(s) over time. For instance, in "Reflective Reform in Mathematics: The Recursive Nature of Teacher Change," Senger (1998–1999) investigated how elementary teachers' changed (or constructed) their beliefs about good mathematics teaching in the context of curriculum reform. Videotaped lessons, field notes, and audiotaped interviews from a purposeful sample of elementary teachers comprised data collection. Analytical tools incorporated qualitative data

analysis software and discourse analysis as a means to ground a theory of how teachers might change their beliefs about good mathematics teaching through Deweyian reflection. This analysis "revealed that the integration of a new belief did not occur suddenly or as a single event—that is, from new information directly to new belief—but rather as a complex and thoughtful process over time" (p. 214). Data representation consisted of teacher narratives and a table comparing snapshot data from three of the teachers. Additionally, a schematic model of "Teachers' Ways of Perceiving Mathematics Reform" was presented—a flowchart of sorts of teacher change. Although Senger presented a schematic model, she did not position teachers as reaching a goal of being "good teachers" but rather used systematic teacher reflection to show progression along a continuum of teacher effectiveness.

Social-turn Moment

Researchers whose work is positioned in the social-turn moment contend that understanding social phenomena is intimately attached to the sociocultural contexts in which phenomena occurs. In that, meaning, thinking, and reasoning are understood as products of social activity in contexts (Lerman, 2000). Research in this moment can be located in the *understand* or *emancipate paradigm of inquiry* (Lather 2006, p. 37) or osculate between the two. For example, in "Culturally Relevant Mathematics Teaching in a Mexican American Context," Gutstein, Lipman, Hernandez, and de los Reyes (1997) make the social turn by placing culture and context at the center of their Freirean participatory project. The purpose of the project was "to contribute to a theory of *culturally relevant teaching*…of mathematics in a Mexican immigrant community" (p. 709). It is important to note, however, that Gutstein and colleagues saw their work as a contribution to the existing body of knowledge; they did not profess to be creating *a* theory that would predict mathematics success for all Mexican immigrant children. Several data sources were used. Demographic and contextual data (nearly two pages) about the school and participants were included as well as observations, interviews, reflections, and classroom documents. In contrast to studies in other moments, Gutstein and colleagues positioned themselves within the classroom as participant observers—including their own reflections as data—and framed the study as a form of action research—including the teachers as co-researchers. Grounded theory methods guided by literature on culturally relevant pedagogy were employed as a means of data analysis. Data representation presented extended participant quotes and descriptive vignettes, maintaining the integrity of the data by revealing the complexities of mathematics teaching and learning embedded in a Mexican American context.

Sociopolitical-turn Moment

Researchers who explore the wider social and political picture of mathematics education characterize the sociopolitical-turn moment (2000s–). This moment signals a shift toward "theoretical [and methodological] perspectives that see knowledge, power, and identity as interwoven and arising from (and constituted within) social discourses" (Gutiérrez, 2013, p. 40). Similar to the social-turn moment, research in the sociopolitical-turn moment can be located in one of two paradigms—*critique* or *deconstruct*—or osculate between the two (Lather 2006, p. 37). For instance, in "Plotting Intersections Along the Political Axis: The Interior Voice of Dissenting Mathematics Teachers," de Freitas (2004) used "fiction-as-research" to access inner dissenting voices to illustrate how the discursive practices of mathematics instruction are determined by the regulative and normative discourses that frame society. de Freitas was compelled to use fiction (as data) in her postmodern project as only through fiction can dissenting voices of mathematics teachers be explicitly heard. In that, "fiction, as a methodology, has the potential to defamiliarize, to cross boundaries, to transgress cultural norms" (p. 272).

Data analysis was storytelling, as "data representation" consisted of Agnes's, the fictional teacher of de Freitas's inquiry, reflections upon her experiences as both a student and teacher of mathematics. Agnes recalled times when, as an exemplary mathematics student, she questioned the purpose of the mathematics tasks that she encountered, surmising that the only one who stood to benefit was the teacher. As the student, Agnes believed her spoken voice was mere disruptive interference. Agnes lamented that now as the mathematics teacher she was "part of the fraudulence that torments youth" (p. 268) and expressed remorse for the students for whom she continued to surrender to normative expectations due to their exhaustion produced by resistance. Nevertheless, Agnes emerged resolutely from her guilt and confusion determined to expose the scandalous foundation of mathematics to right a terrible wrong.

Closing Thoughts

Each of the four moments of mathematics education identified—process–product, interpretivist–constructivist, social-turn, and sociopolitical-turn—can be mapped more or less to one or two paradigms of inquiry—predict, understand, emancipate, and/or deconstruct. Consequentially, each moment depends primarily on different epistemological and methodological perspectives and thus on different methods of data collection, analysis, and representation. We believe that embracing methodological diversity assists in expanding the landscape of mathematics education research so to address persistent inequities in new ways (Bullock, 2012).

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BROADENING WHAT WE PERCEIVE: A METHOD FOR ANALYZING GESTURE AND LANGUAGE

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Gesture and language are deeply intertwined, and attending to both simultaneously when examining mathematical processes is a complex yet rewarding task. We share our budding research methodology for analyzing gesture and language and discuss the methodology from a generic perspective that can be easily adapted to different contexts, participants, and mathematical domains. We further share our problem-specific gesture coding scheme as an example of the grain size and foci of such schemes. Finally, we close by discussing the importance of gesture and language to understanding mathematical justifications and proofs.

Keywords: Research Methods, Geometrical and Spatial Thinking, Reasoning and Proof.

Gesture and language are deeply intertwined: both provide channels for communicating thoughts and ideas, facilitating intersubjective understanding, and supporting various modes of cognition. Language is integral to mathematics (e.g., Hersh, 1999), as it can trace one's thinking and reveal the structure of logical and empirical thought. We often privilege linguistic and propositional accounts over other forms of mathematical and scientific reasoning (Baird, 2004; Nathan, 2012). Yet, physical manifestations of thought in the form of gesture are deeply connected to verbal language (Goldin-Meadow, 2003; McNeill, 1992; Radford, 2009). In earlier work (Williams et al., 2012), we reported on how gestures can exhibit "invisible proof" schemes that reflect analytic thought in nonverbal ways. Both gesture and spoken language contribute to multimodal channels (Arzarello, Paola, Robutti & Sabena, 2009) for communicating mathematical justifications and proofs, but their individual contributions can be difficult to synthesize and understand. In this work, we describe a methodology for developing problem-specific coding schemes for analyzing language and gesture separately and together during mathematical activities.

In the spirit of the PME-NA 2013 theme, *Broadening Perspectives on Mathematics Thinking and Learning,* this work focuses on how to achieve more nuanced insights into reasoning processes by considering gesture alongside verbal acts. We aim to provide a methodology that others can repurpose to their own ends. In the following sections, we discuss embodied cognition and its link to mathematical reasoning, and briefly share details of the research project that motivated the development of this methodology. We then detail the *generic elements* of our methodology for analyzing video and audio data generally, and then present the *problem-specific* gesture coding scheme we have developed. We conclude by connecting our research and methodology to advances in mathematical reasoning and proof practices.

Theoretical Framework and Motivation

Theories of embodied cognition posit a relationship between action and cognition (Shapiro, 2011), refuting the traditional view of cognition as composed of amodal symbol systems and instead regarding the action and perception systems as inextricably bound to thought processes (Barsalou, 1999; Barsalou, 2008; Glenberg & Robertson, 2000;). Speakers' gestures are also viewed as necessarily tied to action (Hostetter & Alibali, 2008), and as such, gestures provide evidence for the embodiment of thought. Alibali and Nathan (2012) connect theories of embodied cognition and gesture with mathematics learning, arguing that, "gestures thus provide a unique and informative source of evidence regarding the nature of mathematical thinking" (p. 274). So how can mathematics education rigorously and consistently unpack this "unique and informative source?" We developed this methodology to: identify the types of gestures that co-occur with various types of mathematical reasoning, determine how gestures support desired reasoning, and document new insights from attending to gesture and language.

Design and Methodology

We conducted an experiment with 120 undergraduate students at a large Midwestern university, who were asked to justify and prove mathematical conjectures. In this paper, we focus on a triangle conjecture:

Mary came up with the following conjecture: For any triangle, the sum of the lengths of any two sides must be greater than the length of the remaining side. Provide a justification as to why Mary's conjecture is true or false.

Prior to reading the conjectures, participants were asked to perform physical actions that were designed either to support solving the conjecture or to be irrelevant to the solution. We also varied the scale of the actions they were asked to perform, which has demonstrated importance in mathematics-related gestures (e.g., Gerofsky, 2009). For the triangle conjecture, participants formed a triangle either with their hands (*observer* scale, in which gestures are produced from a third-person perspective) or with their arms outstretched (*character* scale, in which the speakers' body becomes the character or object being described) (McNeill, 1992). Participants were directed to stand and to share their reasoning out loud. In this work, we are focusing on a subset of the data corpus: the 40 participants who solved the triangle conjecture in the irrelevant action condition, with half from each scale condition (i.e. observer vs. character). Next, we present a step-by-step summary of our generic methodology for analyzing language and gesture simultaneously. It is *generic* because it can be readily tailored to answer other research questions and apply to different mathematical tasks. Throughout this section, we provide an example from a single participant solving the triangle conjecture to exemplify the methodology (Table 1).

Generic Methodology

Our iterative coding process utilized several features of the Transana software platform. The first step involves using only the transcript and audio channels to segment the verbal stream into *speech bursts*, or continuous speech with no small pauses. Second, we code the *speech fluidity* of each segment using the audio, transcript, and waveform data, the last of which allows us to visually detect breaks in the audio stream. Speech fluidity is the degree to which a participant speaks quickly and smoothly, and our codes range along the fluidity spectrum to include: Fluid, Choppy, Slow, etc. Third, we note the number of *words per speech burst* as an additional, quantitative measure of speech fluidity. Fourth, we code the *prompt response* at the speech burst level, using the transcript and audio. This is dependent on the specific question or prompt that the participant is responding to during the task. In the example, the participant is asked to explain whether the conjecture is true or false; thus, our code for prompt response is "True" or "False."

This category allows us to note whether the participant is attempting to prove or disprove the conjecture, and to identify any shifts in this direction over the course of the task.

Fifth, we note the *gesture description, gesture code* and the gesture *length*, using the video feed, audio feed, and transcript in conjunction. Gesture description is an open-ended description of the participant's action, and gesture code assigns a problem-specific code, as explained in the following section. To code gesture length, we use both the dichotomous qualitative categories of Fleeting or Extended (*Length* column) and a quantitative measure of duration of the gesture in seconds (*# sec* column). Then, relying only on video, we next code for the *gaze* of the participant. Finally, the gesture *scale* is coded as Observer or Character using only the video.

Table 1: Analysis Excerpt (Participant G_104_Triangle)

Speech Burst	Speech Fluidity	Words/ Burst	Prompt Resp.	Gesture Description	Gesture Code	Length	# sec	Gaze	Scale
If it was like this, um	Fluid	5	True	L palm on top of R, R fingers & base of palm touch L	Forms complete triangle, 2 Hands	Ext.	1	At hands	Obs.
these two sides couldn't ever be, uh	Choppy	6	True	L index finger points to 2 sides of R hand tri.	Traces 2 sides of a triangle, single finger	Fleet.	1	At hands	Obs.
less than this. Because if they were even to it, it would be a straight line.	Fluid	22	True	Palms flatten against each other	Forms incomplete triangle, 2 Hands	Ext.	6	At hands	Obs.

Here, participants were not provided with any additional supports or tools for justifying the conjectures. However, when we use this methodology in contexts in which participants have access to such materials and representations, we add additional categories for: (1) objects/tools in use during a speech burst, and (2) actions performed with objects/tools during a speech burst.

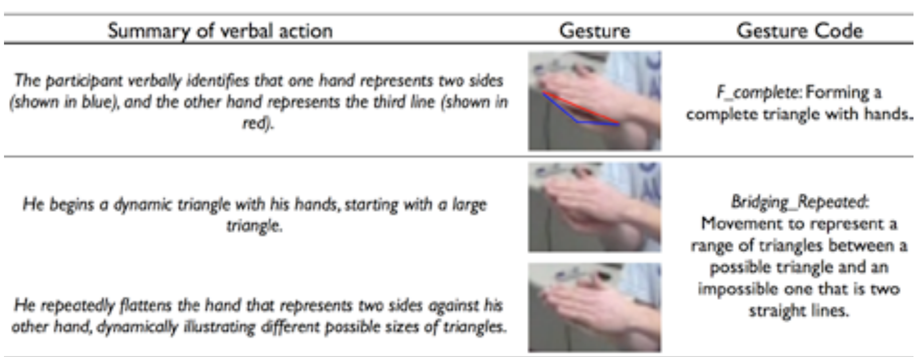

Figure 1: Sample Triangle Gestures and Codes (Participant G_104_Triangle)

Problem-Specific Gesture Coding Scheme

Our *problem-specific* gesture coding scheme was developed through repeated viewing and analysis of the data, and is based upon the Triangle Inequality conjecture. We use three broad coding categories: *tracing*, *forming,* and *bridging*. Tracing refers to gestures that are coherent only when viewed over the full course of the gesture (e.g., tracing a triangle in the air with a finger). Forming gestures, however, represent the entire object simultaneously (e.g., first row in Fig. 1). Each of these two categories includes multiple subcategories, including depicting a single line, a complete triangle, or an intentionally incomplete triangle. For each gesture, we also code which body part(s) were involved in the gesture, noting the number of fingers, hands,

and/or arms. Our third code, *bridging* (e.g., second row in Fig. 1), refers to dynamic representations of multiple triangles within the same gesture. Participants' use of bridging action is particularly intriguing because it involves a single gestural act to reason inductively.

Connecting to Reasoning and Proof

Practices of mathematical justification could be viewed as having two interwoven phases: one in which students figure out for themselves by reasoning through the relationships (*ascertaining*), and one in which they must communicate a convincing argument to a third party (*persuading*) (Harel & Sowder 2005). Our methodology is useful for characterizing phases of ascertaining and persuading by looking at indicators such as gaze, speech fluidity, and shifts in prompt response. These phases can be cross-referenced with gestures to identify the critical ways in which the body can support learners in reasoning about and communicating mathematical ideas, as well as how gestures correspond to important mathematical insights. Although it is generally accepted that gesture is integral to communication, we focus on the more novel idea that gesture triggers changes in cognitive states during reasoning.

Conclusion

Our work is progressing into identifying "invisible proof" practices (Williams et al., 2012), and thus, we have shared our methodology so that others can also begin to examine the relationships between language and gesture in mathematical communication. Research on gesture includes many examples in which gesture provides information that differs from speech, including cases of gesture-speech mismatches (e.g., Church & Goldin-Meadow, 1986). Previous research has identified some of the connections between mathematics learning and gesture (Alibali & Nathan, 2012; Abrahamson, 2004). We aim to advance the field by identifying a methodology focused on how reasoning processes can be better understood through gesture.

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A FRAMEWORK FOR IDENTIFYING MATHEMATICALLY SIGNIFICANT PEDAGOGICAL OPENINGS TO BUILD ON STUDENT THINKING

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Research has shown the benefits of building on student thinking (e.g., Fennema, Carpenter, Franke, Levi, Jacobs, & Empsona, 1996; Stein & Lane, 1996). However, recognizing which student thinking is worth pursuing and the right moment to pursue it is difficult. Based on the literature and data from our research, we conceptualize an important group of instances in classroom discourse that occur at the intersection of student thinking, significant mathematics, and pedagogical openings—what we call **M**athematically Significant Pedagogical **O**penings to Build on **S**tudent **T**hinking (MOSTs). The MOST construct provides a lens through which to identify instances of student mathematical thinking that can be used to develop student understanding. This poster focuses on how we are using a framework based on the MOST construct to identify MOSTs in classroom discourse—a process that requires identifying *that* students are thinking mathematically, articulating *what* students are thinking mathematically, and interpreting *how* students are thinking mathematically. Coders begin by identifying instances of student mathematical thinking. They then articulate the student mathematics of the instance in a complete sentence. This sentence is analyzed to determine if it is related to an important mathematical idea that the students have the background to engage with, but have not yet mastered. If it is, we examine the positioning of the student(s) articulating the thinking and the timing for engaging the rest of the class with the mathematical idea.

We will discuss the MOST framework, the process for coding videotaped classroom instruction to determine which student actions constitute MOSTs, and some of the issues that have arisen during our refinement of the framework. Additional detail about the project of which this framework is a part can be found in Van Zoest, Leatham, Peterson, and Stockero (2013).

This poster is based upon work supported by the U.S. National Science Foundation under Grant Nos. 1220141, 1220357 and 1220148. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

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COMPLEXITY IN TEACHER BELIEFS ABOUT MATHEMATICS, PEDAGOGY, AND TECHNOLOGY

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Keywords: Teacher Beliefs, Modeling

The purpose of this poster is to discuss preservice and/or inservice mathematics teachers' beliefs about mathematics, pedagogy, and technology integration in mathematics education in relation to traditional, constructivist, and other beliefs. Traditional beliefs are aligned toward absolutist view of mathematics, pedagogy, and technology (Handal, 2003). Constructivist beliefs are aligned toward fallibilist view in mathematics education (Philipp, 2007). Other views align either in between these two views or even beyond them, such as critical and postmodern views about mathematics, pedagogy, and technology. A theoretical model can be constructed to elucidate the possible belief structures of teachers' beliefs about mathematics, pedagogy, and technology integration in mathematics education. For this, the characterization of beliefs in terms of absolutist, fallibilist, or other beliefs of mathematics, pedagogy, and technology can be integrated forming all possible combinations with a model. These categories are: traditional beliefs about mathematics (TBM), constructivist beliefs about mathematics (CBM), other beliefs about mathematics (OBM), traditional beliefs about mathematics teaching (TBMT), constructivist beliefs about mathematics teaching (CBMT), other beliefs about mathematics teaching (OBMT), traditional beliefs about mathematics learning (TBML), constructivist beliefs about mathematics learning (CBML), other beliefs about mathematics learning (OBML), traditional beliefs about technology integration in mathematics education (TBTIME), constructivist beliefs about technology integration in mathematics education (CBTIME), and other beliefs about technology integration in mathematics education (OBTIME). These constructs may influence one's knowledge and practice together with perception and values (Belbase, 2012).

We can construct a matrix model of different possible combinations of these beliefs to portray how they are interrelated to each other. These combinations of teacher beliefs result into eighty-one different belief structures some of which seem practically viable and others theoretical in nature. Belief combinations that do not contradict with each other seem more viable than those contradict each other. However, a teacher may have any sort of belief combination. There may be some belief constructs that seem contradicting within themselves, for example: traditional beliefs about mathematics and technology integration in mathematics education, but constructivist belief about mathematics teaching and learning in the construct TBM-CBMT-CBML-TBTIME contradicts within it. This means the beliefs about mathematics and technology integration being traditional may not support for beliefs about constructivist teaching and learning of mathematics. However, it can be argued that such contradictory belief constructs can exist within a teacher as isolated beliefs and they may appear in practice based on circumstances. An understanding of complexity of beliefs about mathematics, pedagogy, and technology integration in mathematics education can help in modeling mathematics teacher education program in a comprehensive way to include diverse courses and activities to influence and develop beliefs in a positive way to improve teaching and learning of mathematics. This also helps in modeling an *epistemic teacher* as an *idealized model* of mathematics teacher.

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COMPARING QUALITATIVE APPROACHES: TWO RESEARCHERS, ONE DATA SET, COUNTLESS INTERPRETATIONS

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Keywords: Classroom Discourse, Data Analysis, Research Methods

Theoretical Perspectives

To enrich the understandings of a single data set, two distinct qualitative approaches were used. This poster will present the results of the two approaches as well as the researchers' experiences working collaboratively. Phenomenological analysis is designed to generate a rich description of an experience that highlights the essential nature of such experiences; it is a method for creating meaning-oriented descriptive knowledge (Wertz et al, 2011). This approach includes determining key experiential moments to parse the data into fine-grained meaning units. The culmination of the approach is to capture the experience's essences. In contrast, narrative analysis focuses on the way people make sense of the world through narrative accounts (Riessman, 2008). Narrative analysis works to emphasize the interpretive nature of narratives to engender meaning (Wertz et al., 2011). We chose a structural analysis, focusing on the story itself, and how it was told. This involves a series of 'listenings' each with a different purpose (Riessman, 2008). Our goals were to discover how a collaborative qualitative research study affects our experiences as researchers, as well as to examine how analyzing the same data through different qualitative approaches creates a more cohesive understanding of the data.

Methods

The study focused on a student – Mike –in a content course for prospective elementary teachers. Mike and a group of his peers were audio recorded participating in an activity centered around volume. For the phenomenological approach, data was individually transcribed and descriptions were developed of the focal activity. Key experiential moments were then pulled which parsed the phenomena into fine-grained meaning units. This culminated in a lived experience description that re-vivified the experience and captured its essences. The first 'listening' of the narrative approach involved the researchers placing themselves within the context of the overall story. The second 'listening' focused on Mike's positioning within the group, while the third involved examining how the content provoked specific reactions. The final 'listening' focused on Mike's use of language, word choice, and tone of voice.

Results

Data analysis is ongoing, with the intent to apply additional approaches. However in regard to the first research question, our experience as researchers was strengthened by both the collaboration and the variety of approaches. In respect to the second research question, different aspects of the data were highlighted within each approach, which is improving our overall cohesive understanding of Mike's experience.

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USING METAPHORS TO EXPLORE PRESERVICE TEACHERS' BELIEFS ABOUT TEACHING MATHEMATICS

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Metaphors permit the individual to temporarily break the limits of language by allowing them access to new ways of thinking about a particular phenomenon. Atwell-Vasey (1998) writes a metaphor "relies on the imagination of its users to see that we can only include some elusive phenomena in our talk by letting other things, more sensible to use, stand in the position of the more elusive phenomenon" (p. 11). The idea of metaphor is not new to mathematics education; some scholars claim that metaphor is a critical and fundamental facet of mathematical cognition (Lakoff & Nunez, 2000). Consider, for example, the idea of a "face" in three-dimensional geometry, "carrying" in multi-digit computation, or describing modular arithmetic as "clock arithmetic" (Pimm, 1987). In this poster presentation, we attempt to broaden perspectives on the use of metaphors to aid preservice teachers (PST) with a language to communicate their beliefs on teaching. Our guiding question was how do metaphors provide PSTs a language to discuss their beliefs about teaching and teaching mathematics?

This is part of a larger study investigating the vision (Hammerness, 2006) and beliefs of four PSTs (Alice, Eyre, Karenina, and Lenore). During the first interview, the participants were given an agree/disagree continuum for nine metaphors derived from Cooney, Shealy and Arvold (1998). We recognize that our collection of metaphors is not exhaustive. The metaphor read "an elementary school teacher is like a(n)," followed by an occupation (e.g., coach, gardener). The participant marked her position on the continuum, and then proceeded to explain her decision. This allowed the individual, through these occupation metaphors, to discuss particular aspects of teaching. Once all nine were completed, the interviewer asked the subject to look back over the occupation metaphors, this time modifying the statement to "a math teacher is like a(n)." The participant was then asked if she would change her previously marked position.

These metaphors allowed the participants to open up about their beliefs on teaching and teaching mathematics. For example, all four participants positioned themselves as agreeing with the scientist metaphor; choosing to focus on the experimental aspects of pedagogy and modification of lessons. However, there was no change of position when considering a mathematics teacher. On the other hand, with the judge metaphor, three participants chose to change their position towards *agree* when considering a mathematics teacher. These three participants talked about how teaching mathematics demands more "judging" from the teacher. How we plan to expand this study and uses for teacher educators will be discussed on the poster.

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CONCEPTUAL KNOWLEDGE IN MATHEMATICS: A REVIEW OF THE LITERATURE

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Cognitive accounts of mathematical knowledge typically divide such knowledge into two types: procedural and conceptual (e.g., Hiebert, 1986). Traditionally, research and educational practice have focused on children's procedural knowledge, or their ability to solve problems (e.g., Rittle-Johnson & Siegler, 1998). Recently, however, there has been a growing emphasis on conceptual knowledge, both in psychological research and in curricular standards (see Star, 2005). There appears to be consensus, both in psychology and education, that conceptual knowledge benefits students in ways that simple memorization of procedures does not. Research suggests that conceptual understanding aids children in evaluating procedures and using procedures more flexibly (e.g., Baroody & Dowker, 2003; Schneider & Stern, 2012).

Despite this shift toward teaching conceptual knowledge, there are several obstacles to doing so. Teachers may have inadequate levels of conceptual knowledge themselves (e.g., Ma, 1999) and are sometimes unsure of how to incorporate conceptual knowledge into lessons, in terms of content and time (e.g., Eisenhart, Borko, Underhill & Brown, 1993). An even bigger issue is that there is a lack of agreement in the literature about what exactly conceptual knowledge is.

This project reviewed how conceptual knowledge is defined and measured in three domains: cardinality, inversion, and equivalence. Both within and across domains, there are major inconsistencies in the way that researchers define and measure conceptual knowledge. Many studies lack an explicit definition of conceptual knowledge, and many of the provided definitions are poorly operationalized. Further, the tasks used to measure conceptual knowledge rarely align with a well-articulated theoretical framework.

The lack of a widely accepted definition or framework for conceptual knowledge presents a real challenge, both for teachers and researchers. To address this issue, we propose a framework that consolidates current notions of conceptual knowledge into two facets: knowledge of general principles and knowledge of the principles underlying specific procedures. Adopting these categories would facilitate thoughtful measurement of conceptual knowledge and allow us to more readily compare findings across mathematical domains.

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JUST(ICE) IN TIME FOR THE COMMON CORE: ADDRESSING ARGUMENTATION IN MATH CLASS

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In recent years there has been a push towards developing mathematically proficient students who are adept at communicating through the language and discourse of mathematics. The Common Core State Standards for Mathematical Practice (CCSSMP) (2010) argue for students who can "read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the argument." Teachers have been offered few strategies to incorporate multiple discourses in the mathematics classroom. Hence, creating a climate in a mathematics classroom that valorizes dialogue and argumentation over rote learning and procedural understandings continues to be a challenge. We argue that queer theory and queer pedagogy can be used to address this challenge, because they strive to trouble and deconstruct binaries and societal structures of gender, sexuality, race and class to create a more open and pluralistic society (Turner, 2000).

Underlying the ways that Common Core is implemented in schools is an assumption that content is the key element necessary for creating a nation of globally competitive students. But as Apple asserts, "policies involving market 'solutions' may actually serve to reproduce--not subvert--traditional hierarchies of class and race" (Apple, p. 247). We agree that "this should give us reason to pause" (Apple, p. 247). Given the conspicuous absence of a stated social justice goal in the Common Core, we argue that mathematics teachers need to actively include difference (race, gender, sexuality, etc.) in their classrooms. We further argue for their utilization of pedagogical techniques that promote the deconstruction of heteronormative traditional ideas. According to Kumashiro (2001), "the value of an inclusive curriculum comes not from full(er) knowledge, but from questions that ask us to look beyond what we already know" (p. 19). In a mathematics classroom, this can be implemented through introducing problems that critique not just the reasoning of others, but the very structures upon which those reasonings are founded.

We propose using math as an entry point into the social discourse surrounding queer identities to encourage a socially just mindset in students. In this poster presentation, we will suggest ideas for implementation of queer identities and queer pedagogy in a math classroom while still allowing teachers to follow Common Core standards.

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BELIEFS AS PART OF A GREATER GALAXY: A METAPHOR

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According to Sokolowski (2000), within a phenomenon there are manifolds of pieces and movements. Pieces can be presented outside of the whole, while movements cannot. This inspired the question: What is the whole that a belief system subsists in? With the rise of interest in professional identity and its relevance to mathematics teacher preparation, it is important for researchers to think about the interactions and overlaps between beliefs, identities and other parts of self. The construction of a metaphor for a belief system and the greater whole is presented.

Metaphor is essential for the discussion of complex phenomenon. As Sfard (1998) states, “conveyed through language from one domain to another, [metaphor] enable conceptual osmosis between everyday and scientific discourses” (p. 4). Metaphors have been used to explore many aspects of mathematics education (e. g. Parks, 2010; Primm, 1989). For this investigation, there are two components to the assembly of the metaphor: Green (1971) and Korthagen (2004). Green (1971) states that beliefs occur in sets; their relationship being dynamic and in constant flux. In addition, beliefs have varying “psychological strengths,” and exist in clusters. Korthagen (2004) adapted Bateson’s model representing the “levels of change” of an individual as an onion. The external layers of the onion are easily influenced to change (environment, behavior, and competencies), while the internal layers (beliefs, identity, and mission) are the most difficult to influence change. None of the layers are independent of one another. Consequently, a change in any level may effect change within or in other levels. For example, a change in environment could potentially influence change in all levels just as well as a change in beliefs could. To preserve the ideas of Green (1971) and Korthagen (2004), I constructed the metaphor of a belief system as a piece of a greater galaxy.

Similar to the galactic center, the mission(s) or core of the individual emits the most influential force throughout the system. Additionally, clashes between the celestial bodies’ exerted forces correlate to the ways that beliefs and other components of self interact with one another. The application of the galaxy metaphor to teacher education is discussed further during presentation.

Acknowledgment

Parts of this paper were presented at the 2012 Bergamo Conference in Dayton, OH.

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WOMEN IN MATHEMATICS GRADUATE SCHOOL: RETENTION AND NARRATIVES OF SUCCESS AND FAILURE

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Keywords: Affect, Emotions, Beliefs and Attitudes, Equity and Diversity

The underrepresentation of women in mathematics has received a lot of attention in the past decade (Blickenstaff, 2005; Kerr et al., 2004; Wiest, 2009). Researchers and educators have identified some of the obstacles that women face in mathematics and STEM fields in general, and recognize the dangers in ignoring the gap in participation between men and women at upper undergraduate, graduate, and professional levels, such as lack of access to high paying job, to higher status positions within the field, as well as to self-esteem.

According to the latest report of the AAUW "[w]orkforce projections for 2018 by the U.S. Department of Labor show that 9 of the 10 fastest-growing occupations that require at least a bachelor's degree will require significant scientific or mathematical training" (Hill et al., 2010). Therefore, if nothing is done to remedy the problems that women face in mathematics, society will remain in a situation in which the most prestigious and higher paying positions are offered to men. Women will be kept away from these positions at a higher rate than men, and therefore will not have equal economic opportunities. Not only can this be detrimental to the United States, which aspires to be more and more competitive in a growing technological world, but it is also a problem of equity that we need to address if we want to create a more just society.

This study – which will be part of a dissertation – revolves around the following research questions: *How do gendered experiences get produced in graduate mathematics? How does success (and failure) get produced and gendered through non-begin communities of practice in graduate-level mathematics? How do women's trajectories of participation through graduate math programs shift as a result?*

To theorize women in relation to graduate programs in mathematics, I will draw on the theory of non-benign communities of practice. I claim that the field mathematics can be viewed as a community of practice that has developed rules of participation that often lead to the marginalization of certain groups (e.g., non-dominant students, females, etc…). Although I do not think that the field of mathematics is acting against certain groups on purpose, I claim that the subtle and often hidden ways in which it is organized have led to different pathways to participation and success for members of different groups.

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MAPPING KNOWLEDGE COHERENCE: A CASE IN THE CLINIC AND IN THE CLASSROOM

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In this mixed methods study, we explored the potential for Epistemic Network Analysis to examine the relationship between two teachers' knowledge of proportional reasoning for themselves and the knowledge they invoked in their teaching of the same content. Findings from this study indicate that this analysis model offers insights into the organization of teacher knowledge that may be promising for further analysis.

Research Goal & Theoretical Perspective

The goal of this pilot effort was to understand how *epistemic network analysis* (ENA; Shaffer et al., 2009) might support our developing understanding of particular aspects of teachers' knowledge, particularly connections among different components of knowledge. In this way, we considered whether ENA would be an appropriate approach to understand knowledge organization.

Methods

For this pilot study, we relied on clinical interview and classroom teaching data from Walt and Carrie, both experienced sixth grade teachers. We used emergent coding informed by research on proportional reasoning (e.g., Lamon, 2007) to identify the knowledge resources being used. This yielded 6 frame elements, which were used in the ENA analysis.

Results

One notable finding is that the equiloads for each teacher and their classroom were similar in shape, suggesting that the same resources were pulling on the utterances regardless of setting. We interpret this as being evidence that the resources a teacher invokes when personally engaged in solving mathematics have a clear relationship to the knowledge that a teacher is likely to invoke in a classroom setting.

Acknowledgements

This work was supported by the National Science Foundation. The views expressed here are those of the authors and do not necessarily reflect those of the NSF.

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ITERATIVE RESEARCH: DEVELOPING QUANTITATIVE REASONING ASSESSMENTS FOR 6TH TO 12TH GRADES

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Theoretical Framework

Research has suggested that learning progressions can help inform curriculum design and professional development, as well as advance effective adaptive instruction teaching techniques (Duschl, et al., 2007; Corcoran, Mosher, & Rogat, 2009). The central component of this study is to develop assessments for 6th to 12th grades which target the content of a learning progression (LP) on students' quantitative reasoning abilities in environmental science. Our proposed LP consists of three overarching components: quantitative act (QA), quantitative interpretation (QI), and quantitative modeling (QM).

Method

Creating assessments and a LP is an iterative research process which – for us – began about two and a half years ago. Initially, a hypothesized LP was constructed based on extensive literature reviews. This hypothesized LP was tested empirically via semi-structured interviews (N=39). The qualitative data were analyzed using grounded theory, which led to a revision of the LP as well as the interview assessments. Subsequently, our revised assessments were tested (N=14) and that data informed our LP again. During this phase our interview data was used to strengthen our LP through student exemplars giving meaning and examples to our components in each level of our LP. The next step in this iterative cycle was to develop mostly closed-form assessments based on the qualitative data and revised LP to help inform our LP on a larger scale (N≈500). This data collection will take place spring and summer 2013.

Results

The result of this iterative research will be a revised LP for QR in environmental science as well as revised assessments which can provide a potential formative assessment tool for teachers. We will present how our qualitative data informed our LP and assessments as well as our qualitative and quantitative data in synthesis to emphasize the iterative research cycle. Our previous results have not indicated a consistent progress in students' QR abilities as they move from 6th to 12th grade. The quantitative data will inform whether or not this trend seems to be true based on a larger sample of students.

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AN INTERDISCIPLINARY PERSPECTIVE ON MATHEMATICS LEARNING: THE CASE OF THE CONCEPT OF VARIABLE

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One way to broaden perspectives on mathematics thinking and learning is to take an interdisciplinary perspective. In this poster, I ask: How can we utilize cognitive science methods to further understand how students understand the concept of variable?

Recently, researchers have advocated for an interdisciplinary approach to investigating mathematics thinking and learning. For example, scholars have argued for the union of mathematics education with cognitive neuroscience, to the benefit of both disciplines (e.g., De Smedt & Verschaffel, 2010). On one hand, mathematics education theories can guide the design of cognitive neuroscience studies. On the other, cognitive neuroscience provides a different level of measurement that can build on and probe findings from student assessments and observational data (De Smedt et al., 2010).

Students' understanding of the concept of variable is an ideal candidate for this type of union. Studies suggest pervasive student misconceptions about the concept of variable between beginning algebra and more advanced mathematics (e.g., Trigueros & Ursini, 2003). This difficulty may stem in part from the internal representations that students associate with variables (Rosnick, 1982). Additional research is needed on the cognitive processes related to when and how internal representations, particularly of quantity, are linked to variables. This line of research is ripe for utilizing cognitive science and cognitive neuroscience methods, which have been used to investigate similar questions for Arabic numerals (e.g., Cohen Kadosh & Walsh, 2009). In this poster, I present an adapted cognitive science paradigm that can be useful in uncovering when and how students form internal representations for variables.

The intersection of mathematics education research and cognitive science can help illuminate differences in cognitive processes related to students' internal representations for variables compared to Arabic numerals. The nature of such differences is not apparent from observational data alone. By building on our current understanding of students' conceptions of variables, an interdisciplinary lens can help unveil how and why students may struggle with certain aspects of algebraic thinking.

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A FRAMEWORK FOR STUDYING RETENTION OF SECONDARY MATHEMATICS TEACHERS IN URBAN HIGH NEED SCHOOLS

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Over a two-year period, we embarked on the use of a Multi-User Virtual Environment (MUVE), as a platform for professional development of mathematics secondary teachers and research on the retention of these teachers in urban high need schools. The MUVE was conceptualized within Second Life, a *Google Groups*, and a *Wiki Space*. Second Life (SL), accessible via the Internet, is a *virtual world*, a genre of an online community. Its users called *Residents* interact with each other through *avatars*, computer users' representations of themselves or alter egos modeled in three-dimensions (Linden Lab, 2009). *Google Group* is a service from Google that supports discussion groups. We used the *Google Group* for posting discussions, general announcements, meeting schedules within SL, and resource sharing. Our *Wiki Space* was designed for sharing notes, projects and assessments. MUVE is fast becoming a pedagogical tool for higher educational institutions (Andersen, Hirstov & Karimi, 2008).

Although there is not a large body of literature on its effectiveness, it is believed that MUVE enhances distance education and learning because it allows for the interaction and engagement that some of the other methods of distance learning lacks. The major question that informed our project was: How does the use of a MUVE as a platform for professional development and research impact our goal of retaining high-quality secondary mathematics teachers in urban high need schools? The focus of this session is on the design of our conceptual framework. We will discuss assumptions about knowledge and reality as they relate to research, MUVE, professional development practice (Webster-Wright, 2009) and learning. Our conceptual framework which is built around major concepts that inform the professional learning opportunities and research of the activities involved in the process will be discussed with respect to how it was used to undergird our study and facilitate data analysis. The major tenets of our framework are symbolic interactionism, sociocultural theories of learning, adult learning, professional development, and motivation and retention within distance education. Our framework provided an optimal structure for exploration of the meanings we are making, our interpretations of the meanings, the various interactions, and the outcomes of the use of SL as a platform for professional development (Fournillier, Thomas, Junor Clarke & Vidakovic, 2011).

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Approximately 7% of children and adolescents have a mathematical learning disability (MLD) and another 10% show persistent low achievement in mathematics despite average abilities in most other areas. Research on these two groups of students with math difficulties (MD) has traditionally focused on procedural skills. This working group is rooted in a twofold premise: (1) students with MD are capable of and need to develop conceptual understanding and mathematical reasoning skills, and (2) special education instruction and assessment needs to transition toward this focus. Participants will (a) develop a research agenda for the group, (b) brainstorm specific research questions that will address that agenda, (c) explore research methodologies that can answer the potential research questions, (d) discuss collaborations to carry out these studies, and (e) set up a plan for publishing and securing funding.

Keywords: Equity and Diversity, Learning Trajectories, Assessment and Evaluation

Brief Overview of the Working Group

The purpose of our working group is to explore issues of research around the intersection of mathematics education and special education. Substantial work exists that focuses on mathematical cognition, development, and reasoning of students in general education. However, much less is known about the mathematical development of students with disabilities or how to support the learning of these students. The absence of research addressing this subset of students may be due in part to the incompatibility of the theoretical perspectives driving research and practice of mathematics education versus special education. Our working group is designed to create sustainable opportunities for researchers and practitioners interested in disability and mathematics to move this important dimension of the psychology of mathematics education forward.

Understanding how students with disabilities develop mathematics concepts and skills has several implications for both research and practice. First, practitioners in both general and special education can gain essential knowledge of how to approach instruction for diverse learners who may rely upon alternative pathways of understanding mathematics concepts. Second, researchers stand to gain a richer understanding of how cognitive processes involved in learning essential mathematical concepts emerge by studying atypical development. Finally, active study of the development of mathematics concepts and skills for students with disabilities

provides both researchers and practitioners with mechanisms for moving toward a methodological focus on pedagogy rooted in assessment of what students with disabilities are capable of learning.

For the purposes of continuing the conversation around mathematics in special education, this group is concerned with students that have significant issues with math, including:

- learning disabilities specific to mathematics
- students with cognitive differences in how they understand and process number
- students who are placed in special education and have difficulties with mathematics

We refer to these students as having "math difficulties" (MD) in the remainder of this paper.

Brief History of the Working Group

In 2012, 15 researchers (faculty and graduate students) and 2 practitioners met during PME-NA in Kalamazoo, MI. This first meeting was specifically focused on better understanding mathematical learning disabilities (MLD). The working group began with a discussion of the issues around identification and definition of MLD. In particular, the group discussed the unique characteristics of students with MLD (e.g., slow speed of processing despite average reasoning; fundamental issues with number sense; over learning of procedural knowledge at the expense of mathematical reasoning) and implications for instruction and assessment. We took up a theoretical stance that positioned disability as an issue of diversity and considered the origin of the disability as the inaccessibility of instruction rather than a defect within the individual. Members shared videotapes of various students with MLD solving problems in assessment and teaching situations and discussed the need for teachers to target and teach toward the specific mathematical strengths and weaknesses demonstrated by the student. We further discussed at what point(s) the learning paths of students with MLD may differ from what is documented among students in general education, how existing developmental trajectories may or may not fit the population of students with MLD, and the need to expand or further document current trajectories to include students with MLD. Moreover, discussions focused on issues surrounding motivation related to the design and use of instruction, mathematical tools, and mathematical tasks. A rich discussion was held concerning the nature and sequencing of mathematical tasks, the use of concrete and pictorial representations and the extent to which they are and are not supportive of the abstraction of mathematical concepts for this population, and the need for increased research to inform the creation of practitioner tools and resources.

In the first year of our working group our focus was specifically on mathematical learning disabilities - those students with a biological and cognitively-based difference in how their brain processes numerical information. Based on our discussions during the first year of our working group we decided to expand from a narrow focus on mathematical learning disabilities to a more inclusive focus on students in special education who struggle with mathematics. This not only avoids the definitional issues at the forefront of the field (i.e., the lack of assessments to accurately identify students with MLDs and the resulting conflation of low achievement and MLDs), but also more accurately reflects the diversity of interests of the members of this group.

As a result of the contacts that were made during last year's working group two members of the working group have been working on a collaborative project. During this collaboration the two faculty members worked together on a teaching experiment about fraction knowledge. Their collaboration resulted in each bring unique expertise; the mathematics education faculty member brought insight into the mathematical thinking of the student, while the special education faculty

member brought insight into learning differences. It is collaborations like these that this working group is designed to foster.

In this coming year we plan to continue and expand collaborations between members of this working group, by focusing discussions around two central themes: (a) math concept development and corresponding methodologies for studying its emergence in students with special needs, and (b) designing research questions and writing a research plan around this topic. We invite interested researchers and educators to participate.

Issues Relating to Psychology of Mathematics Education

Mathematics Intervention in the field of Special Education: A Case for Conceptually Rich Teaching and Learning for Students with MD

Historically, special education researchers and teachers focused almost exclusively on students' mastery of procedural skills, such as basic number combinations and ability to execute mathematical algorithms (Jackson & Neel, 2006; Fuchs et al., 2005; Geary, 2010; Swanson, 2007; Kameenui & Carnine, 1998). A recent literature review comparing instructional domains for students with disabilities found that the majority of research conducted in the field of special education addressed basic computation and problem solving, with the primary focus placed on mnemonics, cognitive strategy instruction (e.g., general heuristic four-step strategy: read, plan, solve, and check), or curriculum-based measurement (Van Garderen, Scheuermann, Jackson, & Hampton, 2009). Instructional practices either focused on task analysis (breaking up skills into decontextualized steps that need to be memorized and followed), flash cards, or general heuristics that do not help with domain knowledge learning and concept development (Cole & Washburn-Moses, 2010). In particular, the focus on primarily procedures-driven instruction and rote memorization of skills seems to result in students' incomplete and inaccurate understanding of fundamental mathematics concepts as well as a lack of retention and/or transfer (Baroody, 2011).

Importance of both conceptual and procedural knowledge. Crucial for rich mathematical understandings that enable retention and transfer of fundamental concepts is the iterative development of conceptual understanding along with procedural proficiency (Rittle-Johnson, Siegler, & Alibali, 2001; Rittle-Johnson & Koedinger, 2005). Rittle-Johnson and Alibali (1999) noted that conceptual knowledge supports procedural generalization. In particular, conceptual knowledge could aid children in mindfully avoiding the use of procedures that fail to work in novel situations. Additionally, an ability to understand and manipulate different mathematical representations to conceptually navigate a mathematical context contributes to conceptual understanding and procedural skill (Ball, 1993; Kaput, 1987; Rittle-Johnson et al., 2001). It seems that any investigation into mathematical cognition, whether related to disability or not, must fundamentally engage with issues of conceptual understanding.

A focus on procedural skills limits students with disabilities' access to the general education curriculum, which is a requirement of the Individuals with Disabilities Educational Improvement Act (Maccini & Gagnon, 2002). In mathematics, access to the general education curriculum means addressing problem-solving, mathematical modeling, higher order thinking and reasoning, and algebra readiness as required by the new Common Core Standards (CCSSI, 2012). To accomplish these Standards, mathematics educators need to actively engage students in making conjectures, justifying and questioning each other's ideas, and operating in ways that lead to deep levels of mathematical understanding (Kazemi & Stipek, 2001; Lampert, 1990; Martino & Maher, 1999; Yackel, 2002).

Exemplar interventions with students with disabilities. While much work needs to be done to understand the conceptual development of students with MD: two prior instructional approaches are worthy of note. During the past decade or so, schema-based instruction has been adapted to teach elementary mathematical word problem solving to students with MLDs (Jitendra & Hoff, 1996). Schema-based instruction emphasizes semantic analysis of word problems and mapping of the problems into schematic diagrams (Marshall, 1995) that are specific to different problem types (e.g., change, group, and compare). The semantic analysis of word problems and categorization of problem types originated from the framework of Cognitively Guided Instruction (CGI) (Carpenter, Fennema, Franke, Levi, & Empson, 1999). With schema-based instruction, students need to make distinctions among various problem types (e.g., "change," "group" or "combine" or "total," and "compare" or "difference" in the case of additive word problems) and then represent the problem in its corresponding schematic diagrams. Afterwards, students rely on solution rules such as "Total is not known, so add" or 'Total is known, so subtract" (Jitendra, 2002, p. 36) to create a math sentence for the solution. While recent studies in schema-based instruction have shown promise in helping students with MLD solve word problems (e.g., Fuchs, et al., 2008), the emphasis was on decoding problems into subtypes (e.g., change, total, difference) based on semantic analyses of contextualized word problems. Connections among problem subtypes or mathematical ideas were not made explicitly, which may have contributed to non-significant effect on skill transfer (measured by standardized tests).

The second approach, which emerged from schema-based instruction and built on her empirical work in math intervention with students with MLD (Author) and cross cultural curriculum analyses (Author), Author (Author; Author) has transformed semantic representation of contextualized word problems in schema-based instruction to Conceptual Model-based Problem Solving (COMPS) to facilitate mathematical model-based problem solving. With COMPS, the emphasis is on representing story problem in mathematical model-based diagram equations (e.g., Part+ Part = Whole; Unit Rate x # of Units = Product; Author) on the basis of students' understanding of the mathematical relationship in the problem. Author also developed a set of word problem story grammar heuristic questions (Author) that guide students' attention to mathematical problem structure to facilitate model-based problem representation and solving. Preliminary empirical studies including both single-subject design and group comparison design (Author; Author; Author; Author) indicate that COMPS has shown promise in improving students' problem-solving skills as well as pre-algebra concept and skills. While COMPS is promoting an intervention approach that moves away from traditional problem-solving paradigm that focuses on the choice of operation for solution to a model-based problem-solving approach (required by the new Common Core), how students with MLD conceive of and make sense of the abstract mathematical model warrants further study. Nevertheless, it is important for students with MLD to transition from operating with concrete or semi-concrete models to flexible use of cohesive and symbolic mathematical models for acquiring generalized problem solving skills.

There is a need for a collaborative effort (from the fields of special education, math education, and other related fields) in research that focuses on further understanding of mathematical concept development and higher-order thinking/reasoning of students with MLD in order to design intervention programs that will enhance students' response to the intervention. While schema based instruction and the COMPS model begin to take up some issues in the design of instruction – much more research is needed focusing on how students make sense of the underlying mathematical concepts and procedures.

Conceptual Diagnosis Based Pedagogy - Assessing Student's Mathematics and Evaluating Effectiveness of Interventions

A pedagogical approach to be explored and advanced during this Working Group's meetings is one that focuses on promoting conceptual learning in students with MD. This approach is rooted in a constructivist stance (Piaget, 1985; von Glasersfeld, 1995), particularly the notion of assimilation, which stresses the need to build instruction on what students already know and are able to think/do. That is, teaching needs to be sensitive, relevant, and adaptive to students' available ways of operating mathematically (Steffe, 1990). To this end, teachers must learn how to: (a) diagnose students' available conceptions, and (b) design and use learning situations that both reactivate these conceptions and lead to intended transformations in these conceptions.

Building on Simon's core idea of hypothetical learning trajectories, Author has articulated such an adaptive pedagogy, which revolves around the Teaching Triad notion: (a) students' current conceptions, (b) goals for students' learning (intended math), and (c) tasks/activities to promote progression from the former to the latter. Key here is that in designing every lesson one proceeds from conceptual diagnosis of the mathematics students are capable of thinking/doing. That is, assessment methods need to focus on dynamic (formative) inquiry into student understandings, as opposed to on testing correct and incorrect answers per se. This day-to-day diagnosis, obtained via engaging students in solving tasks and probing for their reasoning processes, gives way to selecting goals for students' intended learning. Building on this diagnosis, a mathematics lesson begins with problems that students can successfully solve on their own, which Vygotsky (1978) referred as the Zone of Actual Development (see also Author). Recent studies of mathematics teaching in China (Jin, 2012; Author) revealed a strategic, targeted method, Bridging, which is geared specifically toward both: (a) reactivating mathematical conceptions the teacher supposes all students know, and (b) directing their thinking to the new, intended ideas. While working with students with MLD, Author (Author; Author; Author; Author; Author) have been piloting and studying this adaptive approach with high levels of success in promoting substantial conceptual advances (e.g., concept of number, multiplicative reasoning). In the working group we will explore adaptive pedagogy (conceptual diagnosis based) approach, as we believe it can become a core methodological approach for teaching and studying the conceptual understandings of students with MD.

Plan for Working Group

The aim of this working group is to facilitate collaboration amongst researchers and educators concerned with mathematics education for students with disabilities. The main goal is to promote basic research into how students with special needs think about mathematics and develop mathematical concepts. This working group intends to accomplish the following: (a) develop a research agenda for the group, (b) brainstorm specific research questions that will address that agenda, (c) explore research methodologies that can answer the potential research questions, (d) discuss the logistics of collaborations to carrying out these studies, and (e) embark upon collaborations leading to publication and funding opportunities.

These goals are further outlined across sessions as follows:

Session 1: Concept Formation

GOAL: Generation of research questions that are important to the group and/or sub-groups

- Introduce focus for the working group
    - What types of problems would members like to explore?

- View short video clips of students working through various mathematical concepts to better understand the students' thinking
- After viewing video data, lead a group discussion of potential research questions attending to:
  - the overall purpose/goals of this working group
  - a focus on grade levels Kindergarten through eighth grade
  - various mathematical domains of study
- Form research questions that can cross domains and use questions to form collaborations based on each members' area of interest and expertise.

Session 2: Methodologies
GOAL: Explore appropriate research methodologies

- Examine variety of methodologies
  - View videos of work already conducted to highlight possible methodologies for future studies.
  - Discuss other potential methodologies not highlighted during the video viewing
  - Discuss how to design robust study
- Small groups
  - Discuss work already done
  - Develop research agenda for the group
- Large group discussion
  - Share out research agendas from small group discussions

Session 3: Planning and Writing
GOAL: Embark on collaborations

- Small groups
  - Work on written product of research agenda
    - Develop shared conceptual framework and the relationship of our framework to what is currently being done.
    - Identify target journals and outlets or grants and funding sources
- Large group discussion
  - Share progress and commitments from small group discussion
  - Finalize a plan for individual groups to continue updating progress to the larger group (website)

Anticipated follow-up activities

Throughout the year, the members of this working group will continue working on research problems of common interest. They will contribute to a common website in which they will update other members of the working group about the progress of the various research collaborations. In the future this working group will propose a special issue to a leading journal in the field.

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ADDRESSING EQUITY AND DIVERSITY ISSUES IN MATHEMATICS EDUCATION

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As the title suggests, this Working Group has a dual focus on issues of mathematics teaching and learning and issues of equity and diversity. Following on the topics discussed at the Working Group in 2009, 2010, 2011, and 2012, this year we are going to focus on the next steps for mathematics teacher education that relate to equity. Session 1 will consist of a panel discussion with senior scholars, who will share their perspectives on the state of the field and elaborate next steps in the research toward equity in mathematics education. During sessions 2 and 3 there will be round table discussions to provide attendees with the opportunity to network, plan potential collaborative work, and/or discuss issues raised by the panel.

Keywords: Equity and Diversity, Teacher Education-Inservice/Professional Development, Teacher Education-Preservice, Teacher Knowledge

Brief History

This Working Group builds on and extends the work of the Diversity in Mathematics Education (DiME) Group, one of the Centers for Learning and Teaching (CLT) funded by the National Science Foundation (NSF). DiME is a group of emerging scholars who graduated from three major universities (University of Wisconsin-Madison, University of California-Berkeley, and UCLA). The Center was dedicated to creating a community of scholars poised to address some critical problems facing mathematics education, specifically with respect to issues of equity (or, more accurately, issues of inequity).

The DiME Group (as well as subsets of that group) has already engaged in important scholarly activities. After two years of a cross-campus collaboration dedicated to studying issues framed by the question of why particular groups of students (i.e. poor students, students of color, English learners) fail in school mathematics in comparison to their white (and sometimes Asian) peers, we presented a symposium at AERA 2005 (DiME Group, 2005). This was followed by the writing of a chapter in the *Handbook of Research on Mathematics Teaching and Learning* which examined issues of culture, race, and power in mathematics education (DiME Group, 2007). Further, in an effort to bring together and expand the community of scholars interested in this work, DiME, at AERA in 2008, sponsored a one-day Professional Development session examining equity and diversity issues in Mathematics Education. In addition, DiME members have joined with other scholars in joint presentations and conferences. A book on research of professional development that attends to both equity and mathematics issues has recently been

published (Foote, 2010). Many DiME members as well as other scholars contributed to this volume. A book on teaching mathematics for social justice (Wager & Stinson, 2012) also included contributions from several DiME members. In addition, several DiME members have published manuscripts in special issues on equity in mathematics education.

Moreover, the Center historically held DiME conferences each summer. These conferences provided a place for fellows and faculty to discuss their current work as well as to hear from leaders in the emerging field of equity and diversity issues in mathematics education. Beginning in the summer of 2008, the DiME Conference opened to non-DiME graduate students with similar research interests from other CLTs such as the Center for the Mathematics Education of Latinos/as (CEMELA), as well as graduate students not affiliated with an NSF CLT. This was initially an attempt to bring together a larger group of emerging scholars whose research focuses on issues of equity and diversity in mathematics education. In addition, DiME graduates, as they have moved to other universities, have begun to work with scholars and graduate students including those with connections to other NSF CLTs such as MetroMath and the Urban Case Studies Project in MAC-MTL whose projects also incorporate issues of equity and diversity in mathematics education. Funding for the DiME project has ceased and the PME Working Group has become a major way in which to keep the conversation going.

It is important to acknowledge some of the people whose work in the field of diversity and equity in mathematics education has been important to our work. Theoretically we have been building on the work of such scholars as Marta Civil (Civil & Bernier, 2006; González, Andrade, Civil, & Moll, 2001), Megan Franke (Franke, Kazemi, & Battey, 2007), Eric Gutstein (Gutstein, 2006), Danny Martin (Martin, 2000), Judit Moschkovitch (Moschkovich, 2002), and Na'ilah Nasir (Nasir, 2002). We have as well been building on the work of our advisors, Tom Carpenter (Carpenter, Fennema, & Franke, 1996), Geoff Saxe (Saxe, 2002), Alan Schoenfeld (Schoenfeld, 2002), and again Megan Franke (Kazemi & Franke, 2004), as well as many others outside of the field of mathematics education.

A significant strand of the work of the DiME CLT included implementing professional development programs grounded in teachers' practice and focusing on equity at each site. The research and professional development efforts of DiME scholars are deeply intertwined, and much of the research thus far produced by members of the DiME Group addresses issues of equity within Professional Development. Additionally, since the majority of the DiME graduates, as new professors are engaged in teaching Mathematics Methods courses, the integration of issues of equity with issues of mathematics teaching and learning in Math Methods has become a site of interest for research. We have learned through experience that collaboration is a critical component to our work.

We were pleased for the opportunity offered by the first four years of being a Working Group at PMENA 2009 – 2012 to continue working together as well as to expand the group to include other interested scholars with similar research interests. We were encouraged that our efforts were well received; more than 40 scholars from a wide variety of universities and other educational organizations took part in the Working Group each of the past four years.

Focal Issues

Under the umbrella of attending to equity and diversity issues in mathematics education, researchers are currently focusing on such issues as teaching and classroom interactions, professional development, pre-service teacher education (primarily in mathematics methods classes), student learning (including the learning of particular sub-groups of students such as

African American students or English learners), and parent perspectives. Much of the work attempts to contextualize the teaching and learning of mathematics within the local contexts in which it happens, as well as to examine the structures within which this teaching and learning occurs (e.g. large urban, suburban, or rural districts; under-resourced or well-resourced schools; and high-stakes testing environments). How the greater contexts and policies at the national, state, and district level impact the teaching and learning of mathematics at specific local sites is an important issue, as is how issues of culture, race, and power intersect with issues of student achievement and learning in mathematics.

There is much existing research that either focuses on professional development in mathematics (e.g., Barnett, 1998; Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Kazemi & Franke, 2004; Lewis, 2000; Saxe, Gearhart, & Nasir, 2001; Schifter, 1998; Schifter & Fosnot, 1993; Sherin & vanEs, 2003), or professional development for equity (e.g., Sleeter, 1992, 1997; Lawrence & Tatum, 1997a). Less research exists, however, which examines professional development or mathematics methods courses that integrate both. The effects of these separate bodies of work, one based on mathematics and one based on equity, limits the impact that teachers can have in actual classrooms. The former can help us uncover the complexities of children's mathematical thinking as well as the ways in which curriculum can support mathematical understanding in a number of domains. The latter has produced a body of literature that has helped to reveal educational inequities as well as demonstrated ways in which inequities in the educational enterprise could be overcome.

To bridge these separate bodies of work, the Working Group has begun and will continue to focus on analyzing what counts as mathematics learning, in whose eyes, and how these culturally bound distinctions afford and constrain opportunities for students of color to have access to mathematical trajectories in school and beyond. Further, asking questions about systematic inequities leads to methodologies that allow the researcher to look at multiple levels simultaneously. This research begins to take a multifaceted approach, aimed at multiple levels from the classroom to broader social structures, within a variety of contexts both in and out of school, and at a broad span of relationships including researcher to study participants, teachers to schools, schools to districts, and districts to national policy.

Some of the research questions that the Working Group will continue to consider are:

- What are the characteristics, dispositions, etc. of successful mathematics teachers for *all* students across a variety of local contexts and schools? How do they convey a sense of purpose for learning mathematical content through their instruction?
- How do beginning mathematics teachers perceive and negotiate the multiple challenges of the school context? How do they talk about the challenges and supports for their work in achieving equitable mathematics pedagogy?
- What impediments do teachers face in teaching mathematics for understanding?
- How can mathematics teachers learn to teach mathematics with a culturally relevant approach?
- What does teaching mathematics for social justice look like in a variety of local contexts?
- What are the complexities inherent in teacher learning about equity pedagogy when students come from a variety of cultural and/or linguistic backgrounds all of which may differ from the teacher's background?
- What are dominant discourses of mathematics teachers?
- What ways do we have (or can we develop) of measuring equitable mathematics instruction?

- What is the role of both teachers' and students' academic and mathematics identity in achievement?
- How do students' out-of-school experiences influence their learning of school mathematics?
- What is the role of perceived/historical opportunity on student participation in mathematics?

Plan for Working Group

The overarching goal of the group continues to be to facilitate collaboration within the growing community of scholars and practitioners concerned with understanding and addressing the challenges of attending to issues of equity and diversity in mathematics education. The PME Working Group provides an important forum for these scholars to come together with other interested researchers who identify their work as attending to equity and diversity issues within mathematics education in order to develop plans for future research. Some areas we have identified and intend to continue to examine include: incorporating out of school practices, explicitly examining race, analyzing broader social structures (teaching math for social justice), and examining educator identity around race and teaching about race. Our main goal for this year, then, is to continue a sustained discussion around these key issues (theoretical and methodological) related to research design and analysis in studies attending to issues of equity and diversity in mathematics education. We will do this by bringing together senior scholars to share their perspectives on the field, and then provide space and time for smaller groups to discuss, reflect on, and amplify ideas from the presentation.

Our plans for PME 2013 we will proceed as follows.

SESSION 1:

- Review and discussion of goals of Working Group.
- Introduction of participants
- Panel discussion on the state of and new directions for mathematics education with an equity focus

SESSION 2:

- Round table discussions, networking, and collaboration

SESSION 3:

- Continued round table discussions, networking, and collaboration

Previous Work of the Group

The Working Group met for four productive sessions at PMENA 2009, PMENA 2010, PMENA 2011, and PMENA 2012. In 2009, we identified areas of interest to the participants within the broad area of equity and diversity issues in mathematics education. Much fruitful discussion was had as areas were identified and examined. Over the past four years subgroups met to consider potential collaborative efforts and provide support. Within these sub-groups, rich conversations ensued regarding theoretical and practical considerations of the topics. In addition, graduate students had the opportunity to share research plans and get feedback. The following were topics covered in the subgroups:

- Teacher Education that Frames Mathematics Education as a Social and Political Activity
- Culturally Relevant and Responsive Mathematics Education (CRRME)
- Creating Observation Protocols around Instructional Practices

- Language and Discourse Group: Issues around Supporting Mathematical Discourse in Linguistically Diverse Classrooms
- A Critical Examination of Student Experiences

As part of the work of these subgroups, scholars have been able to develop networks of colleagues with whom they have been to collaborate on research and manuscripts. As a result of our growing understanding of the interests of participants (with regard both to the time spent in the working group and to intersections with their research), last year we offered our first panel in which scholars shared issues they had encountered in publishing manuscripts that considered equity and mathematics. The success of the panel discussion and feedback from attendees led us to plan to continue this structure again this year, using the working group as a sight for discussion and planning of collaborative work as well as reflection on ideas of senior scholars.

Anticipated Follow-up Activities

As has happened following previous years of this working group, we anticipate that scholars who make connections at the working group sessions will maintain contact and at least in some cases, this will lead to collaboration on manuscripts or research projects.

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DEVELOPING ELEMENTARY TEACHERS' MATHEMATICAL KNOWLEDGE FOR TEACHING: IDENTIFYING IMPORTANT ISSUES

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Lack of appropriate and adequate mathematical knowledge in elementary teachers is a major concern in mathematics education. At their first meeting in Kalamazoo the working group on ***Developing Elementary Teachers' Mathematical Knowledge for*** **Teaching** *identified five significant issues to explore from multiple, diverse perspectives: (1) selecting/creating/modifying and implementing mathematical tasks, (2) noticing/understanding children's ways of thinking, (3) describing mathematical knowledge for teaching, (4) beliefs and affect, and (5) developing mathematical habits of mind. Focus groups were formed around each of the topics, bringing together mathematicians and mathematics educators to discuss each. At the second meeting of the working group in Chicago these focus groups will share their research/writing from*

throughout the year related to their designated topics. New members are encouraged to participate in the discussions; however, a writing group may already be organized. The long-term goal of the working group is to develop an edited book that examines issues around the development of mathematics knowledge for elementary teaching within the context of university courses from varying perspectives.

Keywords: Mathematical Knowledge for Teaching, Elementary School Education, Teacher Education-Preservice, Affect, Emotion, Beliefs, Attitudes

Background

The PME-NA XXXIV Working Group on *Developing Elementary Teachers' Mathematical Knowledge for Teaching* was organized by Lynn Hart and Susan Swars (Georgia State University, USA), Susan Oesterle (Douglas College, Canada) and Ann Kajander (Lakehead University, Canada). At PME-NA XXXV in Chicago the group will meet for the second time.

The purpose of this working group is to examine how teacher preparation experiences, particularly those focused on developing specialized mathematical content knowledge, can support the development of mathematical knowledge for teaching in elementary teachers. The initial session in Kalamazoo was well attended with over 35 participants. At that meeting we discussed the purpose of the group and identified the long-term goal of developing an edited book on the focus topic. At the first session, organizers as well as participants from the 2010 working group on *Preservice Elementary School Teachers' Content Knowledge in Mathematics* shared results of their efforts/research to date. The group then broke into smaller groups to allow for in-depth discussion of significant issues around the broad theme. At the end of the small-group discussion, individuals were asked to write down what they considered to be the most important factor affecting the development of elementary teachers' mathematical knowledge. After the meeting, the session organizers sorted these factors into themes to share with the group the following day.

Although fewer people attended the second 90-minute session, those attending were highly committed. They were divided into focus groups based on the five themes identified the previous day. These focus groups were: (1) selecting/creating/modifying and implementing mathematical tasks, (2) noticing/understanding children's ways of thinking, (3) mathematical knowledge for teaching, (4) beliefs and affect, and (5) developing mathematical habits of mind. Each team met for the remainder of the session and organized into research/writing teams to begin work on their topic. Each focus group reported at the end of the session and deadlines were generated for work over the following year. It was determined that the audience for the book would be individuals teaching mathematics content courses for elementary teachers. With that in mind, we identified organizations such as the Mathematical Association of America (MAA) or the American Mathematics Association (AMA) as preferred potential publishers. A summary of each group's work follows.

Focus Group #1. Selecting, Creating, Modifying, and Implementing Tasks in Mathematics Content Courses for Preservice Teachers: An Illustrative Example

Background. Mathematical tasks are essential elements of mathematics content courses as they lay the foundation for mathematical discussions in the classroom. Our

group decided to ground mathematical tasks in tasks designed for children (i.e. tasks from current curriculum such as *Investigations*) and then modify those tasks to be appropriate for university content courses for preservice elementary teachers. We chose to modify a task designed for children as this approach allows for an exploration of the mathematics while simultaneously being an authentic activity (Newman, Wehlage, & Lamborn, 1992) for preservice elementary teachers. Our re-design of the task focused on: (a) ensuring the task has high cognitive demand (Smith & Stein, 1998), (b) helping preservice elementary teachers develop their conceptions, while addressing their incoming conceptions (Bransford, Brown, & Cocking, 1999); and (c) allowing preservice elementary teachers to deepen their own understanding while at the same time making sense of and learning about children's mathematical thinking.

In a recent triple special issue of the *Journal for Mathematics Teacher Education*, Watson and Mason (2007) summarized a taken-as-shared perspective on *task development* that was based on all 111 submissions they received. According to their perspective, a *task* is more than a set of questions or prompts and also includes ways in which a task can be presented and ways in which follow-up activities are organized. (By contrast, an *activity* arises as a result of engaging with the task and includes what the learner actually does.) In the same issue, several papers introduce frameworks for the task-development cycle – e.g. one framework consists of *design, enact, reflect,* and *modify/re-design* phases (Liljedahl, Chernoff, & Zazkis, 2007). When using this framework, the initial task design builds on preservice elementary teachers' incoming conceptions and a hypothesis of how those conceptions develop. After the enactment of the task, the reflection cycle allows for consideration of various aspects of the task. For example, the instructors identify ways in which the task implementation followed or strayed away from the original plan and identify factors that may have contributed to the implementation straying. The instructors also suggest modifications as needed, which are then considered when the task is being redesigned.

Report. The focus group on *the role of mathematical tasks in mathematics content courses for preservice teachers* has met regularly over the past year. Two outcomes from their work include: (a) an ongoing review and synthesis of current literature on task design (which will be shared during the working group meeting); and (b) a collaborative research project that produced an illustrative task appropriate for use with preservice elementary teachers in mathematics content courses. The task we developed was based upon a set of rational number problems created for children. The original task was modified in ways to address the development of preservice elementary teachers' mathematical knowledge for teaching. The process of developing this task was in itself useful and will be of benefit to instructors in mathematics content courses for preservice elementary teachers. So far, two members of our group have enacted the task within their own content courses for preservice elementary teachers. These two members collected data and shared their results and reflections with the other members of our group. As a group, we then reflected on the outcomes of these enactments and modified the task accordingly. The revised task was then implemented by a different set of group members, and the group again reflected on the implementation and refined the task accordingly. Outcomes from this work will be shared during the working group.

Focus Group #2. Noticing/Understanding Children's Mathematical Thinking in Mathematics Content Courses for Preservice Teachers

The collective interest of the focus group on *understanding children's mathematical thinking in mathematics content courses for preservice teachers* is on the use of children's mathematical thinking as a tool to promote elementary preservice teachers' mathematical content knowledge (emphasizing content knowledge rather than pedagogical knowledge). Children's mathematical thinking can be represented through different types of media, e.g., video of children discussing their problem-solving strategies, interviews of individual children, or artifacts taken from the classroom such as copies of student work. These representations of children's thinking provide real contexts in which teachers can apply their mathematical understanding. To help set the stage for the work of the focus group, two seminal studies were identified from a review of the research literature. They are Carpenter, et al. (1989) and Peterson, Fennema and Carpenter (1991).

Further review of more current research related to the use of children's mathematical thinking as a tool to develop preservice elementary teachers' mathematical content knowledge resulted in 6 papers the group believed would be helpful in providing supporting relevant research (Bartell, Webel, Bowen, & Dyson 2012; Goggins, 2007; Jacobs, Lamb, & Philipp, 2010; Kazemi & Franke, 2004; Philipp, 2008; Philipp Thanheiser, & Clement, 2002). One or more group members summarized each study. The outcome of the findings from these papers will be supported with personal stories from members of the sub-group who teach content courses for elementary teachers. Example tasks that make use of children's mathematical thinking to further develop preservice teachers' mathematical content knowledge will be provided by group members. It is intended that the examples will provide sufficient detail to be considered for implementation or adaptation by other faculty teaching mathematics courses for preservice teachers.

Focus Group #3. Mathematical Knowledge for Teaching for Prospective Elementary Teachers: International Case Studies

Teachers' specialised content knowledge is often described in general terms, such as by saying that it is *more* than general mathematics knowledge. Such descriptions are hardly useful to course designers or instructors. In order to provide substantive and detailed examples of what *mathematical knowledge for teaching* (MKT) might mean to post-secondary mathematics course design, this chapter will provide three case studies illustrating how pre-service teachers in different programs/countries are supported in their MKT development. It is intended that the level of content-detail of the examples will be illustrative to other faculty charged with developing such courses for prospective teachers.

Case 1. MKT development in a Canadian teacher education program. The Canadian province of Ontario, as elsewhere, has seen substantive changes in the elementary mathematics curriculum over the last 15 years. In our region, this change was not supported by widespread professional development. On-going research over the last eight years (e.g., Holm & Kajander, 2011) suggests that incoming preservice elementary teachers at our institution rely almost exclusively on a rule-based approach to standard algorithms, and their mathematical thinking is supported by virtually no conceptual understanding of the models and ideas that underpin these computational methods. Hence we have embarked on an ambitious agenda of new course development, and have introduced several courses in *mathematics for teaching* within our Faculty of Education.

While these new courses are most certainly *mathematics* courses, the topics and content, as well as pedagogy, are specifically aligned with our understanding of the needs of elementary teachers.

Related to this course development, there is on-going discussion at our institution as to whether specific course topics are knowledge of *content* (meaning standard mathematics content that other students would also study, or that prospective teachers might have been expected to learn in previous schooling), or *specialised* knowledge (that is, knowledge needed specifically by teachers to teach mathematics effectively). We find this a moot point. Via our previous research, (for example, Kajander, 2010a), we have identified a substantive body of knowledge (see also Kajander & Boland, forthcoming) needed by elementary teachers to support deep conceptual classroom mathematics learning. Our research further suggests that prospective teachers at our institution are not fluent with this knowledge (Kajander, 2010b). Hence, regardless of whether such mathematics is "standard" (implying they should have learned it previously), or "specialised", we find ourselves in the position of needing to support this knowledge development to prospective teachers regardless of how it is defined. For example, the area model of multiplication might be one such topic. A rectangular area model can be used to represent a multiplication fact such as 3×4, by constructing a rectangular region 3 units by 4 units, the area of which is necessarily the product. This model later generalises to two-digit products (and hence can be useful in developing multiplication procedures), and later is also useful for both fraction products (and procedure development) and even binomial product representations and procedure development. While this model is certainly helpful in supporting conceptual classroom teaching (and thus arguably is specialised knowledge), the area model is in fact mentioned in our provincial curriculum document as a *content* expectation for students (Ontario Ministry of Education, 2005). Hence, whether content knowledge or specialised knowledge, the area model and how it connects to multiplication of different numbers and terms, as well as how this model can ultimately be used to generate procedures, is just one example of the mathematical knowledge for teaching needed by preservice elementary teachers. The Canadian case study will provide several detailed examples of such content, as it is included in our courses.

Case 2. MKT development in a US teacher education program. Concern about teachers' knowledge of mathematics has been evident in much of mathematics education research in the US (Ball, Lubienski, & Mewborn, 2001). More recently, efforts have been made to articulate research agendas that specifically focus on teachers' mathematical knowledge for teaching (MKT) (Ball, Thames, & Phelps, 2008: Silverman & Thompson, 2008). Needless to say, there are a multitude of perspectives on the development of teachers' mathematical knowledge for teaching. Our case focuses on how a first mathematics course for prospective elementary school teachers (PSTs) can support their development of MKT. Our course is unique in that we designed our materials using Realistic Mathematics Education (RME) instructional design heuristics. In an RME approach, instruction is designed to provide opportunities for students to reinvent mathematics. In an RME reinvention approach, mathematical concepts are not presented to students in a top-down manner, as in traditional instruction. Rather, instructional sequences are designed so that the concepts emerge as students engage in the instructional sequence (Gravemeijer, 2004). The instructional sequences

used in our course for PSTs parallel the RME sequences that have been developed by our colleagues Erna Yackel, Koeno Gravemeijer and Paul Cobb to support the mathematical reasoning of children in grades K-8. Our case description begins with the first course as we immerse PSTs in the *Patterns and Partitioning Sequence* and *Structuring Numbers Sequence* that were initially developed by Cobb, Yackel and their colleagues (see Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997; Whitenack, 1995; and Yackel, Underwood, & Elias, 2007). The difference between these sequences and our course is that our PSTs are immersed in these sequences in a Base 8 environment. This environment forces them to think about the concepts in a less-familiar numerical context, allowing for more opportunity for realistic reinvention of the ideas.

Case 3. MKT in Norwegian teacher education. Norwegian teacher education is regulated by the National Curriculum Regulations for Differentiated Primary and Lower Secondary Teacher Education Programmes for Years 1 – 7 and Years 5 – 10. These curriculum regulations are manifested in sets of national guidelines. In the national guidelines, it is a clearly stated goal that pre-service teachers should develop mathematical knowledge for teaching. However, the concept is used without any direct reference to any mathematical knowledge for teaching (MKT) framework such as that developed by faculty at the University of Michigan (e.g. Ball, Thames, & Phelps, 2008). However, the descriptions in the national guidelines can be interpreted to lead in such a direction. In this case description, we use the framework of Ball and colleagues (ibid.) to discuss an interpretation of MKT in Norwegian teacher education.

Focus Group # 4. Beliefs and Affect

Over the years, research has established a robust relationship between aspects of teachers' affect (including feelings, emotions and beliefs) and their instructional practices, especially teacher thinking and behaviors such as instructional decision-making and use of curriculum materials (Buehl & Fives, 2009; Clark & Peterson, 1986; Philipp, 2007; Romberg & Carpenter, 1986; Thompson, 1992; Wilson & Cooney, 2002). Affect has been conceptualized as on a continuum, with feelings and emotions at one end, characterized as short-lived and highly charged, and beliefs at the other end, typified as more cognitive and stable in nature (Philippou & Christou, 2003). Over time, feelings and emotions develop into beliefs, and these beliefs are difficult to change. There is "broad acceptance that mathematics teachers' beliefs about mathematics influence the ways in which they teach the subject" (Beswick, 2012, p. 127). Results from Wilkins' (2008) support this claim, as he found that teachers' beliefs have the strongest effect on teachers' practices when compared to other factors.

Teachers' beliefs and affect develop over time (Richardson, 1996), beginning with their own experiences as students in K-12 classrooms, during what Lortie (1975) terms the *apprenticeship of observation* and are well-established by the time they enter college (Pajares, 1996). Philipp (2007) underscores the importance of beliefs about mathematics when he asserts, "For many students studying mathematics, the feelings and beliefs that they carry away *about* the subject are at least as important as the knowledge they learn *of* the subject" (p. 257). The beliefs and affect of prospective teachers have an influence on how and what they learn and should be targets of change during the teacher preparation process (Feiman-Nemser, 2001; Richardson, 1996), though programs are constrained by the limited time available to effect changes.

There are significant concerns about prospective elementary teachers' beliefs and affect toward mathematics. A recent study confirmed that mathematics anxiety remains a pervasive problem in large numbers of elementary teachers (Bekdemir, 2010). Notably, the considerable amount of research on teachers' affect testifies to prospective elementary teachers' tendencies to enter their preparation programs with mathematical affect that is less than optimal for both their own learning and their future as teachers (Philipp, 2007). As mathematics for elementary teachers content courses are among the last mathematics courses that most prospective elementary teachers will take, these courses provide an important context for influencing mathematical beliefs and affect during elementary teacher preparation.

This focus group will conduct a review of the state of the research within a specific timeframe related to prospective elementary teachers' mathematical beliefs and affect. The review will be limited to studies focusing on beliefs and affect toward mathematics as a subject itself and not toward the teaching and learning of mathematics. The group has already conducted an initial search of the literature for studies involving elementary prospective teachers in university mathematics content courses that focused on mathematical beliefs and affect, and very few were located. Therefore, the group will expand its search, possibly focusing on all studies involving elementary prospective teachers across the past five years in targeted journals which examined mathematics anxiety, beliefs about mathematics (e.g., nature, self-efficacy, etc.), and/or attitudes toward mathematics. Once the group has an initial pool of studies, the group will analyze these studies, examining the soundness and rigor of the research, possibly using the criteria in *Standards for Reporting on Empirical Social Science Research in AERA Publications* (AERA, 2006). The group will read and analyze each article and eliminate those that do not meet the criteria. Then, for the next pool of articles, the following questions are possibilities for analyses: What interventions and outcomes are investigated? What are the major findings? What claims of impact are made? It is anticipated that at the conclusion of the analyses, the group will be able to discuss contributions and limitations of the entire body of research, as well as implications and suggestions for elementary prospective teachers' learning of mathematics in light of beliefs and affect.

Focus Group #5. Developing Mathematical Habits of Mind

Mathematical habits of mind can be broadly thought of as productive approaches and ways of looking at problems (and the world around us) that are typical of practicing mathematicians. Over the last several years, there has been an increasing awareness of the importance of developing such habits of mind in mathematics learners. The latest curriculum efforts in both Canada (the Western & Northern Canadian Protocol, 2011) and the US (Common Core Standards Initiative, 2010) specifically address "mathematical processes" and "standards of practice". Walshaw and Anthony (2008) in their review of research into mathematics classrooms, observe that, "effective teachers develop their planning to allow students to develop *habits of mind* whereby they can engage with mathematics productively" (p. 540). In order to support teachers' abilities to effectively initiate students into the discipline of mathematics, the report of the Conference Board of the Mathematical Sciences (2012) recommends that: "All courses and professional development experiences for mathematics teachers should develop the *habits of mind* of a mathematical thinker and problem-solver, such as reasoning and

explaining, modeling, seeing structure, and generalizing" (p. 19). In keeping with this recommendation, this focus group seeks to draw out what instructors of preservice teachers should know about this topic.

Although the CBMS (2012) citation offers a few specific examples, the theme of *mathematical habits of mind* is a potentially broad one. Along with literature referencing this particular designation, there are other closely related areas of investigation, including mathematical dispositions, mathematical thinking, and even socio-mathematical norms. Since November, the members of this focus group have been reviewing this literature, sharing their findings electronically, and narrowing the scope of what can and should be addressed within this project.

Furthermore, in keeping with the intent of the original Working Group to explore issues from a variety of viewpoints, the focus group has contacted and invited several mathematicians to contribute their thoughts on mathematical habits of mind/mathematical dispositions/mathematical thinking. These submissions will be incorporated into the eventual book chapter.

At the 2013 meeting in Chicago, the group will report out on its findings and share the received comments of the contributing mathematicians.

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COLLABORATING TO INVESTIGATE LIVED AND LIVING MATHEMATICAL EXPERIENCES: THE DIME WORKING GROUP

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The "Developing Investigations of Mathematical Experience" (DIME) working group, initiated 2011, has focused on building a research-based understanding of the interiorized experiential world of idiosyncratic mathematics. The major goal of this research is to characterize experiences in ways that acknowledge a person's active and reflective thinking efforts within mathematical contexts linked to emotive dimensions of their lived and living mathematical experiences, in order to inform mathematics teaching and curricular practices. To probe these largely unexplored complex domains, we are using both phenomenological and radical constructivist theories and methods. We seek to identify and relate expressed indicators of intended and actual mathematical experiences, as they appear to occur in the complex relationships of participants involved in mathematics teaching and learning. The primary goals of the PME-NA Working Group are to involve interested scholars in the ongoing work of the DIME Research Team through further conceptual analyses related to the phenomena of lived and living mathematical experiences, to extend our epistemological and psychological perspectives for characterizing critical elements, and to identify and address methodological issues inherent in investigating interiorized mathematical experiential phenomena.

Brief History of the Working Group

A core membership for this PME-NA Working Group was established through the initial invitational "Planning Conference for WISDOM^e" (Wyoming Institute for the Study and Development of Mathematical Education) conducted September 8-10, 2010 at the University of Wyoming (Chamberlin, Hatfield, Belbase, 2011; Hatfield, 2011). The background context for that conference was the establishment of three collaborative, interdisciplinary Research Teams connected to four new Ph.D. program research identities: QRaMM (quantitative reasoning and mathematical modeling), TTAME (technology tools and applications in mathematics education), and DIME (lived and living mathematical experience). Each of the teams consists of UW mathematics education, mathematics, and educational cognate faculty, mathematics education doctoral students, and now over forty active Mathematics and Science Education researchers from a variety of national and international universities.

The work of, and involvement in, the DIME Research Team has continued and grown in three subsequent conference venues. Through the conduct of the initial Working Group sessions at the October 2011 PME-NA (Reno) conference, we established an important venue to continue and expand discussions of the background perspectives and issues related to investigations of lived and living mathematical experience, to share information, issues, and problems related to ongoing research, to promote interest and potential participation in furthering these and other disciplined inquiries into these phenomena, and to provide continuing support to the team members to collaborate within and across ongoing and future research. From those opportunities, nine new scholars joined the initial twelve DIME participants.

Research on mathematical experience was one of the four themes on which an International STEM Research Symposium on Quantitative Reasoning in Mathematics and Science Education was conducted (May 31-June 2, 2012 in Savannah GA; co-sponsored by WISDOM^e). Attended

by 85 leaders and advanced research students, a major plenary paper for DIME was presented by Bal Chandra Luitel (Kathmandu U., Nepal), and extensive discussions in three DIME working group sessions led to nine new participants joining the Research Team. [Symposium papers are being edited as Volume 3 of the WISDOM[e] monograph series (Mayes, Hatfield, Mackritis, in press); DIME-related essays from Volumes 1-3 will be distributed to all intended WG participants prior to the Chicago conference.]

The third continuation and expansion of our work and participations occurred in the 2012 PME-NA (Kalamazoo) DIME Working Group sessions. That program included five presentations of work underway or completed; these were actively and extensively discussed in the context of the group addressing needed research and potential collaborations. In that venue, we saw thirteen returning and seven new participants; several of these have indicated they will be attending next November, with nine of these being included in presentations planned in this proposal.

Focal Issue: "How Might We Investigate Mathematical Experience?"

We consider research on lived and living mathematical experience to be a relatively new, and yet to be defined, domain of explicit attention and scholarship in Mathematics Education. As such, we have found a need to address basic questions related to how it would be studied---

- Why and with what aims and purposes should "mathematical experience" be investigated?
- Where and with whom could we conduct such research?
- How might we access the elusive phenomena related to "lived or living mathematical experience?"
- In what ways can a researcher connect to understand the interiorized "experiential world of the other?"
- How can we portray or characterize phenomenological experiences "of the other?"
- What analytical approaches or tools might be appropriate to investigate "experience?"
- How can we interpret such "data" in ways that seek to balance our inherent, idiosyncratic subjectivities with goals for building an evidential basis for penetrating to understand the nature of the "experiences," per se?

For this conference, we will address collaboratively the "how" for our embryonic research ideas. Our goal is to share a variety of perspectives on approaches through a set of essays to be written and distributed to intending participants (to be identified from registration information) at least two weeks prior to the conference; printed copies will also be available in the session. Thus, in the conduct of the sessions we will assume prior reading, so little "presentation" time will be used. Rather, the focus will be upon questioning, commenting, discussing, elaborating, and refining what is presented in the essay. These discussions will be framed by the following overview to present a detailed set of "how" questions and issues (also to be distributed ahead and briefly highlighted in the opening organizational discussion by the organizers; again, printed copies will be available in the session).

Overview for DIME Research: Some Framing Questions, Issues, and Approaches

The DIME team has grappled deeply with methodological issues, set against our parallel struggles to have greater construct clarity for our overall questions---trying to be clear about what we think we are setting out to study, but questioning how we can do so in ways that will build a solid evidential basis for any findings or claims about "lived mathematical experience."

Fortunately, we have found helpful ideas in the research literature, and we have identified and addressed the following questions in order to formulate some initial investigations and strategic approaches. [The following framework includes important ideas discussed in the 2012 DIME Working Group, drawing therein on an earlier version submitted in the Working Group proposal and as originally posed by Hatfield (2012a).]

1. In what ways is it possible to study the "lived mathematical experiences" of anyone: oneself? The "other?"

From our review and discussion of the literature (e.g., van Manen (1990); Moustakis (1994); Hurlburt & Akhter, 2006); Petitmengin (2006)), we have identified these guiding principles:

A. A research methodology for studying "lived experience" includes the theoretical precepts behind the methods---the values and assumptions of phenomenology and radical constructivism.

B. Our overall framework is adapted from van Manen's structure "…seen as a dynamic interplay among six research activities:

(1) Turning to a phenomenon which seriously interests us and commits us to the world;
(2) Investigating experience as we live it rather than as we conceptualize it;
(3) Reflecting on the essential themes that characterize the phenomenon;
(4) Describing the phenomenon through the art of writing and rewriting;
(5) Maintaining a strong and oriented pedagogical relation to the phenomenon;
(6) Balancing research context by considering parts and whole. (van Manen, 1990, 30-1)

C. Some views of phenomenology aim at being "presuppositionless," warding off a tendency to construct or enact a predetermined set of fixed procedures or techniques that would rule-govern the research. We plan to engage our observation and analysis with general acceptance of this view, while also making efforts to stipulate and articulate, *a priori*, as many of our individual values, assumptions, beliefs, and attitudes about the phenomenon of "mathematical experience" as we can and seems relevant.

D. In research contexts where we aim to stimulate new experiences, we will adopt the dynamics of radical constructivist orientations.

E. We will seek to address the phenomenon of "mathematical experience" in a variety of ways. In doing so, we will seek attentively to *orient to the phenomenon* as we strive to deepen our *formulation of the phenomenological questions*.

F. We are building a team approach with a purposeful aim of including a variety of perspectives and voices, and this brings opportunities beyond research conducted by one, or even two collaborating scholars. As such, we have adopted views and tactics to mirror what we perceive as formative, developmental research, and to be alert to elements of our inquiry that includes aspects of team building, per se.

2. Whose experiences should be studied, and why? Who are to be the subjects of the research? How are they chosen?

Because mathematics as a human endeavor in society and globally is so pervasive and seemingly universal, we foresee a full range of research participants who experience mathematics in a wide array of situations and for a diverse set of reasons. Of course, one *orientation* we bring to this is the "enterprise" of mathematical education, and this will greatly influence our choices for whose experiences we will try to investigate, and also determine how we frame the contexts and the templates of analysis and interpretation. We want to include at least these in our sampling of "lived mathematical experiences"---ourselves, other mathematicians, other mathematics teacher educators, other mathematics education researchers,

pre-service and in-service mathematics teachers across levels of school mathematics, mathematics students across levels of school mathematics, and parents of the students.

3. What are sources or forms of "data of lived experience?" How can these be generated in ways that yield penetration into phenomena? Be seen to be accurate (true to the phenomena)? Valid? Reliable?

We accept the following views about the nature of "data of lived experience." The world of "lived mathematical experience" is for us both the source and the object of our research. We bring strong *orientations* to it. But, we share a fundamental assumption: experiential accounts are never identical to lived experience itself. Van Manen (1990) states this so well:

> All recollections of experiences, reflections on experiences, descriptions of experiences, taped interviews about experiences, or transcribed conversations about experiences are already *transformations* of those experiences…So, the upshot is that we need to find access to life's living dimensions while realizing that the meanings we bring to the surface from the depths of life's oceans have already lost the natural quiver of their undisturbed existence. (p. 54)

4. In what ways can research participants be directly engaged in experiential mathematical situations while informing the researcher about what they are experiencing?

It is one of the basic assumptions of phenomenology that experience will be changed within an attempt to introspect---to "rise above" and give attention to the experience while it is occurring, and that the distinctions between what is introspection and retrospection are blurred. While we accept this assumption in theory, we want to explore this phenomenon, also.
As researcher-observers interactively engaged in mathematical situations we've posed for the purposes of engendering active involvement by, say, a student, we intend to become a part of the student's experiences, per se---to get into "the flow" (Csikszentmihalyi, 1990; Hatfield, 2012b) of what the student is experiencing. "Being there" can mean (to the student) that, as a part of their unfolding experience, we ask questions. While these questions will primarily focus upon their activity and their "thinking aloud" verbalizations related to it, at times we will ask a question pointing more directly to their conscious reflection upon their experiences, per se.

5. When or how might a research intervention or analysis or interpretation influence or alter or "contaminate" an experiential context under study? How can this be avoided?

This concern admits that there may be ways in which our research methods could negatively impact upon what is being experienced, both in immediate or longer-term ways. Indeed, one characterization of instruction are the intentional impacts that alter otherwise natural experiences of students. As such, these would not be avoided.

6. How do we, as researchers, conduct analysis and interpretation of data to build accurate portrayals of "lived mathematical experience?"

Key to phenomenological and radical constructivist research is analysis and interpretation as an observer, or teacher-researcher. Critical to either are the struggles to maintain, as much as possible, open thinking in which one consciously acknowledges potential biases and avoids "projecting" one's own experiences onto the situation.

We believe that through a team approach in which multiple descriptions can be generated independently and then discussed and debated, we will likely achieve more sensitively accurate interpretations of the phenomena---"negotiated meanings." Across experiential episodes we will look for consistencies as well as variation, thus being attuned to elements of cross-validation of the qualities to be found in a person's "lived mathematical experiences." Also, in these we will look for how the nature of experiences for each research participant may change; again, as

educators we seek and expect change---growth and development as a consequence of something we call "mathematical experience." In our studies, we will focus not only on indicators of change in relation to mathematical knowledge, but also on the emotional elements within experience and how these occur within the dynamics of various qualities of lived experience.

7. In what ways can we "make sense of" our study of the observed "lived mathematical experience" in relation to its implications and potential applications?

This question speaks to the important intent that our research results lead beyond information about the interior "mathematical life," although as I pointed out, such research-based information is generally lacking today. Our goals include the possible implications for such new insights and understandings about "lived mathematical experience" to impact upon future mathematical experiences. We foresee possible benefits to the ways that mathematics teachers seek to stimulate and engage their students to engender certain qualities of experiences they might have.

Developing a DIME Research Program: Possible Sources of Data

In our embryonic investigations, we are formulating a variety of approaches for collecting data of "lived/living experiences," including written personal protocols, samples of professional literature, and clinical interview interactions and observations. We are conceptualizing and exploring appropriate research methodologies for analyzing, interpreting, and communicating these elusive phenomena. One longer-term goal is to construct epistemic models of "mathematical experience."

Our current research efforts have begun with plans involving the following sources or forms of data of "lived mathematical experience." Our Working Group papers will address examples of these contexts, within which we will focus on our approaches to the research methods (the "how" of investigations) for gathering, framing, describing, analyzing, and interpreting the kinds of experiential data we are each studying.

1. Using recalled personal lived mathematical experiences as a starting point.

We can reflect, individually and independently, on our overall "mathematical life" to write a lived-experience description of this "bigger picture" of our own past. Our purposes in doing this include the following:

- What we might learn about the nature of reflective activity related to experience;
- What we might learn from what we choose to identify and value through our reflective activity;
- What we might learn about how a person can develop a written description of experience within a focus on simply describing while setting aside inclinations to analyze, explain, defend, or interpret;
- What we might subsequently learn about the "essences" of our own "lived mathematical experience" identified from the description we wrote;
- Later, and in the context of our efforts to study and "make sense" of observed experiences of "the other," how we might recognize our possible biases or distortions affecting our analyses and interpretations as connected to our own prior experiences.

2. Using experiential descriptions of "intended experiences" found in literature.

From our reading, using, and sometimes contributing to the literature of Mathematics Education, we acknowledge that there is a widespread rhetoric of implied "mathematical experience." The ancient writings related to Platonic views of the growth of mathematical knowledge with respect to the slave boy, Meno, are only an early example. Indeed, while most published Mathematics Education literature items, including journal articles, textbooks, reports

of commissions, or curriculum standards and resources, present ideas that often fall short of explicitly discussing "experience," they are rich sources of ideas for the activities of students and teachers in behalf of promoting certain content or goals or outcomes. As such, we believe our understandings of what might be considered a quality mathematical experience can be enhanced by an analysis of selected literature samples.

We are identifying two categories of possible "data of experience" from Mathematics Education literature:

(a) Items containing descriptions of mathematical lesson events and discourse (possibly including fragments of verbatim exchanges, either imaginary or captured protocols); and

(b) Items in which "activities" are described in ways that seem to imply particular kinds of experiences for the students and their teacher.

3. Interviewing to inquire about "lived mathematical experiences."

A different approach will involve direct interviews with another person about their "lived mathematical experiences." One purpose will be to facilitate through dynamic interactions the capacities of another person to engage in recall of what they see as significant events and identify and elaborate the impacts and consequences within their personal past "mathematical lives." As a team, we have begun to consider the nature of these interviews as a way of accessing the "lived experience of the other." While we may pose particular questions or situations to provoke memories and re-collective accounts, these will be "open, unstructured" interviews aimed at stimulating and guiding the person in their spontaneously generated descriptions. These matters may become a topic of interaction across interviews, wherein clarifications, amplifications, and connections among described experiences might be sought. These interviews will be video recorded, to serve efforts toward analysis and possibly to support stimulated recall within subsequent interviews with the same person.

Here again, we need to explore the procedure described above---reviewing and analyzing the taped interview session first independently to produce separate descriptions ("second order") which will be shared and analyzed as a team to produce a "third order" description. We anticipate that our capacities for analysis and interpretation in these particular kinds of interviews will be grounded in what we experience and construct.

4. Using "reflective journals" (diaries or logs) as subject-generated descriptions of recalled lived experiences.

We are also exploring how we might use reflective journals as another source of experiential data. As teacher educators, many of us include the use of such journals as a course requirement, based upon research evidence that reflective practitioners tend to be among the more effective teachers; these could be used especially in our research with teachers (both pre- and in-service), and we will explore such reflective journals with others, such as students engaged in mathematics courses.

In these approaches with reflective journals, we are interested in seeing how the raw data of "lived experiences" may differ from completely open, unstructured but regular journaling compared to writing journal entries in response to specific prompts that focus on particular events or situations. Further, after analyses of data from journals we foresee conducting interviews (as per item #4 above) with journal authors to explore further, and possibly to clarify or validate the essences of our experiential descriptions through their testimonies.

5. Witnessing "living mathematical experiences of others" to write experiential narratives from observations.

Through direct observations and subsequent analysis of videotaped episodes of a variety of contexts in which one or more participants (including regular "math" lessons in classrooms) are engaged in posed mathematical tasks, problems, and situations, we will seek to characterize the nature of the "lived mathematical experiences of others." Here, our primary aim is to see what quality of description we can generate from this level of source---our "second order" experience. In doing this, we will seek to hone our capacities to produce descriptions that illuminate the essences of the observed "living/lived experiences of the other."

To build confidence in such perceptive analytical capacities, we can engage in multiple stages of comparison to cross-validate, and we may eventually interview participants as they watch the videotaped episodes, and later ask them to confirm or refute our descriptions of their "lived mathematical experiences." But, the focus here is upon observation, and our construction and reconstruction of our analyses of these observations..

6. Using constructivist/phenomenological interviews or "teaching experiments" to inquire about real-time living mathematical experiences as they are being lived.

In this context, we would shift from a focus on observation (unobtrusive witnessing) to engaging in a dynamic of interaction with the "other who is experiencing." In this, we seek to develop both our researcher capacities to engage others within/during "their living mathematical experiences" and to develop the capacities of the "other" to connect with the essences of their own ongoing "living mathematical experiences" as they are lived, in ways that will allow them to plumb those introspectively, and to share those with us. In this, we are seeking to challenge, or clarify, the classical phenomenological assumptions.

7. Witnessing and interviewing of classroom "living mathematical experiences" to inquire about "actual" school mathematical experiences of students and their teacher.

Because the nature of mathematical education is so very heavily and fundamentally invested in "life in classrooms," we feel drawn to studying the (mostly overwhelmingly) complex phenomena of "living/lived mathematical experiences" of those engaged in mathematics classrooms! While we will approach that as carefully as we can, we feel completely ill prepared at this time to address every participant's experiences during a mathematics lesson.

Instead, we will begin by seeking to gather certain "real time" data from a few selected students in the group, these responses collected in relation to particular prompts during events as they unfold. We will use these data to structure post-lesson interviews with each student (or, in small group interactions). Further, we will seek data from the teacher, within an interview prompted by her/his viewing of our video records of the lesson. Then, after we construct indicators of students' "lived experiences" in the lesson, we will again interview the teacher about those, to gain further indicators of the teacher's recalled, perceived experiences set against how she/he hears about and reacts to the student accounts of their experiences.

Later, and across time, we will work with teachers using this detailed feedback about students' "lived mathematical experiences" as a basis for them to adjust their teaching practices to become more responsive to reported student experience. In this, the aim will be to study the influences upon teaching resulting from teacher knowledge of student experiences.

Specific Plans for DIME Sessions at PME-NA 2013

Overall, the three sessions will be conducted in "workshop/working session" format, structured to inform and orient new, interested participants, while allowing returning WG participants to report and share, to interact actively toward clarifying basic questions, issues and challenges in our methods of conducting research on lived/living mathematical experiences, and

to extend individual and collaborative plans for activities that would develop and occur prior to PME-NA 2014. Specifically, the following plans have been developed in collaboration with prior DIME WG participants.

The following nine papers will be written for distribution PRIOR to the conference (by November 1st) to all past DIME participants and all PME-NA'13 registrants selecting the DIME WG.

Investigating lived mathematical experiences of pre-service secondary mathematics teachers
Shashidhar Belbase, U. of Wyoming

I have explored pre-service secondary mathematics teachers' mathematical experiences and beliefs in a task situation of teaching geometric transformations using Geometer's Sketchpad. Mathematical tasks on teaching reflections, rotations, translations, and dilations were used as contexts to interact with them in one-on-one interviews. I posed to them situations of teaching these mathematical concepts using Geometer's Sketchpad to interact with them and discuss what they experienced when they worked through the task situations. I posed questions in relation to those teaching situations to reveal their experiences and beliefs. For example, one of such situations was: "Let's assume that you are going to teach concept of reflection to high school students. Will you please think of an activity to introduce the concept of reflection? How will you demonstrate the concept of reflection in the dynamic geometry environment of Geometer's Sketchpad? How will you engage students?" I asked them to reflect on their experiences through these situations and interactions.

Story narratives of lived mathematics lessons: Composing & decomposing intended mathematical experiences of students & teacher Larry L. Hatfield, U. of Wyoming

What kinds of experiences might school mathematics students have that could lead to knowledge of mathematical modeling? How might they experience a contextual situation and a problematic approach, purposed to guide them to build-up mathematical models as abstracted re-presentations, and then use those to verify and extend their understandings of the problem context? How might a teacher stimulate and guide precollege students toward such educational goals and experiences? How might a researcher describe, analyze, and interpret such pedagogical and psychological elements as lived/living mathematical experiences? In this paper (Hatfield, 2012b), I sought my answers to these questions, first by telling a "teacher's mathematics classroom story" written as a composite retelling from my memories of dozens of my actual lived lessons with the same "view tube" exploration. Then, I offered a metaphorical paradigm for discussing "states of being" within experiential "flow." Finally, I discussed four "layers of meanings" I can identify and infer from an analysis and interpretation of my "view tube" experiences. In this DIME WG session we will discuss the rationales, potentials, techniques, and issues in this approach.

A consideration of methodological approaches for capturing students' experiences with mathematics curriculum: Implications from a review of the literature Amanda Jansen, U. of Delaware

The purpose of my paper will be to present methodological approaches to capture students' experiences of mathematics curriculum, responding to a call by Erickson and Shultz (1992) from over two decades ago that has not been achieved by mathematics educators. Inspired by a previous review of research literature that I conducted (over 125 articles from three major mathematics education journals from over 20 years), prior research on mathematics curriculum does not construct students in active, dynamic, or nuanced ways that capture their experiences with the process of learning. Rather, students are typically described in this work as static outcomes, or as one who learns from teachers' curriculum implementation. In this paper, I extend what I learned from conducting this literature review to elaborate upon possibilities for future research on mathematics curriculum that moves closer to capturing students' lived experiences with curriculum, including research on how students use textbooks and participatory action research that involves student voice in curriculum development and evaluation.

Accessing the other's lived mathematical experience: Experiences from the "field"
Simon Karuku, Aga Khan University, Dar es Salaam, Tanzania

Discussions on how to access the other's lived experience normally take it for granted that one has necessarily to know the other's vocabulary and syntax. This paper is based on a study that explored students' experiences of seeking help in mathematics. The paper describes the researcher's experience of gathering phenomenological descriptions in a language in which the researcher lacked fluency relative to the co-researchers. The experience of helping the co-researcher stay within the limits of a singular experience, as well as the difficulty of re-presenting prosodic features in the participants' descriptions are discussed. The researcher had to bracket all established investigative techniques and constantly invent anew an approach that seemed to fit most appropriately the situational context of the study (van Manen, 2011).

An investigation of lived mathematical experience: A teacher's intended experiences versus students'

idiosyncratic lived experiences” Matthew V. Mackritis, U. of Wyoming

Based upon my experiences as a mathematics educator I make the assumption that, prior to providing a mathematics lesson, a teacher has in “mind” intended experiences for their students; noting that such experiences may be explicit or implicit. However, it seems what is not well understood is the manner in which the teacher’s intended experiences are manifested in the experiential world of the student. To contribute in explicating this area of research, I investigate two fundamental constructs: (1) the intended experiences of a middle-grades teacher prior to giving a mathematics lesson on ratio and proportion, and (2) the lived mathematical experiences of three students that participated in the lesson.

What Distinguishes Mathematical Experience from Other Experiences? Anderson Norton, Virginia Tech

Investigating the lived mathematical experiences of students presents dual challenges for the researcher. On the one hand, we must respect that students’ experiences are not directly accessible to us and are likely very different from our own experiences. On the other hand, we might not want to rely upon the students’ own characterizations of what constitutes mathematics because these characterizations could be limited to the formal products students learn in school. I suggest a characterization of mathematics as objectified action, which would lead the researcher to focus on students’ operations—mental actions organized as objects within structures so that they can be acted upon. These operations and structures can be used to characterize both mathematics and students’ experiences of mathematics. Teaching experiments and clinical interviews provide means for the researcher to infer students’ available operations and structures on the basis of their physical activity (including verbalizations).

Researching Literatures of Lived Mathematical Experiences: Extending Our "How To's" Beyond Textbooks

Travis A. Olson, U. of Nevada-Las Vegas & Lisa A. Kasmer, Grand Valley State U.

We will discuss our current work in utilizing and extending a framework for textbook analysis regarding post-Common Core textbooks. Through this work, we will present a framework for "doing research" as content analyses in the literature domain of mathematics textbooks (K-College). We will then posit theoretical frameworks for extending content analyses into other literature domains (journals, reports, supplemental materials, digital resources, etc.) with respect to comparative and descriptive analyses that have the potential to establish baselines for further research on examining the use of such materials lived experience(s).

Refining and supplementing research approaches to study mathematical experiences

Lisa Rice, U. of Wyoming

Investigating mathematical experiences requires a reflective and evolving approach for me. In studying how Craig and T experience mathematics, I also look at how I am investigating mathematical experience. With Craig, my methods did not capture the range and depth of his experiences I feel are possible. Thus, I propose supplementing my analysis approach with strategies from grounded theory, discourse analysis, and narrative analysis. For example, Craig’s use of short stories from his past and present is one way he appears to make meaning of his mathematical experiences and so narrative analysis may be useful. The goal is augmenting previous methods will aid deeper investigation of this phenomenon.

Researching mathematical experience from the perspective of a second person empathic observer

Elaine Simmt & Martina Metz, U. of Alberta

Tom Kieren often paraphrased Warren McCulloch: What are learners that they can know mathematics what is mathematics that learners can know it. This question has interested philosophers, psychologists and anthropologists for hundreds of years. But I am an educational researcher. I am a teacher. This has two immediate implications, the first is that I have first-person experience with the “subjects and phenomena” I do mathematics, and I teach youth mathematics. Hence I, as a researcher am positioned differently than the philosopher, psychologist or anthropologist who is interested in children’s cognition and sees mathematics as a particularly fruitful site for exploring their question. The distinction is that I have been part of the interactions in which a learner has brought forth mathematics. Secondly, as a person who has taught / who teaches and who is concerned with teaching I ask, “Can it be taught?” (c.f Metz, 2012). As a researcher, I need a pragmatic theory, one that has implications for teaching and learning. Enactivism (c.f. Varela, Thompson and Rosch, 1991; Maturana and Varela, 1991) provides me with a means to investigate mathematics knowing from my first hand experience and as second person empathic inquirer (Varela and Scharmer, 2000; Metz, 2012). In this work I explore how this orientation might make a difference (that makes a difference) when studying mathematical experience.

Thus, all identified participants will have a copy of every paper to be read in advance; for new participants arriving in the session, copies of all papers will be distributed at the start. During the sessions, to promote active participation and interaction the papers will NOT be presented, per se. Rather, each paper author(s) will lead an open group discussion with primary attention to the methods used in the paper for generating, describing, analyzing, and interpreting

data---all framed by the seven key questions posed as the core issues for our WG. Recorders will produce brief summaries of these discussions to be distributed to all participants immediately after each session ends. These summaries will promote continued thinking and further discussions of key ideas between and beyond these sessions.

The concluding session will engage all participants to discuss feedback toward revising essays to be published in a WISDOM^e Monograph (Volume 6), tentatively titled "*Developing Investigations of Mathematical Experience: Rationales, Methods, & Issues*," to be distributed prior to PME-NA'14. Also, new participants will be invited to propose and develop additional essays for inclusion in the monograph. Being active in this publication will foster continued engagement and interactions, plus within the dynamic of the ongoing DIME Research Team, there exists sustaining collaborative work into which new participants are invited to join.

Across the past three years, this group has continued to grow and remain interactively connected and involved. In this proposal, we have planned explicit paper development and discussion leading to the publication of our own monograph---a strong trajectory of development and sustainability for the DIME Working Group.

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REPRESENTATIONS OF STUDENTS' MATHEMATICAL CONCEPTIONS AND THEIR USE IN TEACHER EDUCATION

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We propose to convene for the fourth time the Representations of Mathematics Teaching working group, with a special focus on representing students' mathematical conceptions. This document describes how the cK¢ framework presented in Nicolas Balacheff's plenary lecture (at this PMENA conference) can be used along with representations of practice using animated cartoons or image sequences created with cartoon characters to render student conceptions in ways that can be used in teacher education.

Keywords: Cognition, Teacher education-Preservice, Technology

Brief History of the Working Group

For the past three years we have convened a working group on the use of representations of mathematics teaching in teacher education. For this fourth year and complementing the particular focus of Nicolas Balacheff's plenary lecture, we are proposing to convene the working group with a slight change of focus: How can representations of students' mathematical conceptions be of use in mathematics teacher education?

The idea of this working group emerged during a series of three-day conferences on representations of mathematics teaching held in Ann Arbor, Michigan in 2009, 2010, 2011, and 2012 (and earlier workshops in 2007 and 2008) organized by ThEMaT (Thought Experiments in Mathematics Teaching), an NSF-funded research and development project directed by Herbst and Chazan. ThEMaT originally created animated representations of teaching using cartoon characters to be used for research, specifically to prompt experienced teachers to relay the rationality they draw upon to justify or rebuff actions in teaching. The original workshops were conceived to begin disseminating those animations to be used in teacher development. The four Representations of Mathematics Teaching (RMT) conferences from 2009 to 2012 gathered

developers and users of all kinds of representations of practice to present their work and discuss issues that might be common to them. These conferences included users of video, written cases, dialogues, photographs, comic strips, and animations. An outcome of the 2009 RMT conference was a special issue (Volume 43, issue 1, 2011) of the journal ZDM—The International Journal of Mathematics Education, guest edited by Herbst and Chazan. Outcomes of the 2010 Conference included two sessions at the 2011 NCTM Research Presession. We convened the RMT conference in 2011 and 2012 and continued with a PME-NA working group focused on the design and investigation of a pedagogical framework for teacher development that makes use of representations of practice and work toward an edited book on the subject. In previous meetings of the working group we have emphasized the use of representations of practice to depict teaching moves and the use of technology to support teacher candidates' study of teaching moves.

One area of inquiry presented at the RMT conferences has been the representation of K-16 students' work through videos, sequences of photographs, and annotated records of written work. This includes Johnson's (2012) use of annotated video clips of students that allow teachers to examine students' work on a covariation problem while it is in progress rather than in a final state as well as Lesseig's (2011) use of video case studies of a professional development session that involved teachers' work on proof problems in order to identify the elements of Mathematical Knowledge for Teaching involved in mathematical proof. Using animated representations of classrooms, Aaron (2011) has presented work looking at how teachers perceive the work that students do in classroom. The ThEMaT project itself has used this venue to present preliminary work on a database of student conceptions, including a poster by Chieu, Herbst, and Dimmel (2012). The database they proposed would allow researchers (and eventually teachers) to upload records of student work to a shared archive, where the records would be tagged with searchable metadata and could be browsed, annotated, and coded by other researchers.

Connections to PME-NA Issues

A central issue for the psychology of mathematics education is how to represent observational data and research results on students' conceptions in a manner conducive to their use in teacher education. Transcripts, video, case narrations, tables, scans, and photography are some of the tools that have been used; they all have affordances and limitations for communicating knowledge of students' conceptions to teacher candidates and novices. It seems to us that the field can still explore other media and conventions. The importance of this issue is clear given the interest in the mathematics education community writ large on having teachers notice student thinking, as exhibited in the recent volume by Sherin, Jacobs, and Philipp (2011). This is of particular importance for courses on teaching methods and learning theories. There is a substantial body of research on students' thinking, documented in a variety of forms (e.g., case studies, responses to items, or descriptions of schemes, etc.). Steffe (1988; 1994) conducted multi-year teaching experiments in elementary school that investigated how children develop a conception of rational number arithmetic from schemes for counting. Other research shed light on the regularity of some errors students make when they compare decimal numbers (e.g., Resnick, Nesher, Leonard, Magone, Omanson & Peled, 1989). At times the data for such work has been made available in a video format (e,g., Carpenter, Fennema, Franke, Levi, & Empson, 1999) or in online collections (e.g., the Video Mosaic Collaborative, http://videomosaic.org/).

The cK¢ framework introduced by Balacheff and Gaudin (2002, 2010) has provided an important perspective for organizing the existing research on students' mathematical thinking. In that framework a key idea is to represent conceptions not in terms of mental, invisible structures but by a careful organization of the visible evidence from occasions in which students' approaches to a problem appear to be stable or consistent even if errorful. The cK¢ framework places significant value on problems as the epistemological source of conceptions: They are not solely means of diagnosis and assessment of conceptions, but they can also be the triggers of evolution of conceptions. Within cK¢, a conception is modeled as a quadruplet whose elements are a set of problems (P), a set of operators (R), a system of representation (L), and a control structure (S). *Problems* are the occasions in which conceptions manifest themselves, providing opportunities for students to use specific operators, representations, and controls to complete them. *Operators* are the means through which students do work on a problem--when operators are fed with particular givens of a problem they can instantiate operations (algebraic, graphic, geometric, etc.). The *system of representation* is a set of literal, symbolic, and visual signs through which problems, operators, and controls are mediated; they could also include gestures, movements, postures, or prosodies with specific mathematical meaning. The *control structure* includes the various means through which students monitor and warrant the work they do--they include descriptive statements, in particular definitions and theorems, that students might use to check their results or to decide on the application of an operator; controls include also metacognitive strategies like those described by Schoenfeld (1985). The cK¢ model thus offers a way to model students' thinking based on the evidence available in students' work. In addition, the (P, R, L, S) categories offer a framework for creating records of student work that could represent particular conceptions.

One form such records of student work could take is that of image sequences made with cartoon characters. These image sequences can be used to represent a student's work on a problem as a story, where operators (in the form of actual operations) and controls (possibly represented as students' reflective comments) are events in the story, and the system of representation is realized in the form of the objects, inscriptions as well as embodied signs used in the problem. A sequence of images can in that way boil down to a few key moves the way a particular student may have gone about solving a problem and thus be used to represent the conception itself as something that unfolds in time (yet without subsuming into the particular way one student enacted it, which might include more complexities incidental to individual students' psychological makeups). We note, incidentally, that in Balacheff's (1988) classic paper on conceptions of proof one can also find a set of comic strips written by the author (and drawn by Eric Coulomb) to represent some of those conceptions.

For the past decade the ThEMaT project, directed by Herbst and Chazan, has pioneered the use of cartoons to represent instructional practices, initially through animations of classroom scenarios represented with cartoon characters. Later the project also produced a tool (Depict; Herbst & Chieu, 2011, included in the LessonSketch platform) that gives researchers and teacher educators the opportunity to represent scenarios as image sequences, each of which is realized with cartoon characters. These can be exported as images and composed in a traditional comic book format or can be displayed as slideshows in LessonSketch experiences. In either form image sequences can be created to represent student work as it evolves over time along with the accompanying actions of the teacher or other students. In such image sequences, the operators that could be used to solve a problem would be represented as the mathematical operations students complete as they work on the problem, while the controls would be the propositions that

warrant those operations. For example, while working on a given geometry problem, a student might draw an auxiliary parallel line in a diagram, in order to create alternate interior angles to use in a congruence argument. The operator could be described as $PL_t\,(x, l)$, where *PL* is "draw parallel line", the subindex *t* refers to a particular tool (say a ruler that slides parallel to itself) that further characterizes this operator, and *x* and *l* are the operator's arguments. In the depiction, one could see a student having a particular *x* as the point through which the line is drawn, and a particular *l* as the line drawn parallel to. The student would be shown drawing a parallel by sliding a ruler parallel to itself from *l* to the point where the ruler passes through *x*, then drawing the new line (say, *m*) with the pencil. Various control structures might be used, including the postulate that says that for any given line and any point not on that line there exists a unique line through the point parallel to the given line and the notion that two parallel lines are equidistant. We surmise that narrating a conception in the form of a story of a student solving a problem would make these details quite more explicit and meaningful. The system of representation would include not only drawings of the line *l* and the point *x* and their labels, but also the ruler, the pencil, etc. In an algebra setting, operators might include things like the steps that a student takes to solve a problem—such as multiplying one equation in a system of linear equations by a scalar—while the controls could be the properties of equality and the number system that gives those steps their validity.

Teacher educators could use such representations of student work to help teacher candidates learn about student conceptions and in developing teachers' capacity to notice student thinking (Sherin, Jacobs, & Philipp, 2011). Various designs have been explored that look into the effects of experiences with student thinking (Jacobs, Lamb, & Philipp, 2010). The literature on the subject attests to the complexity of getting novices to attend to student thinking (see van Lehn, Ohlsson, & Nason, 1994). It seems that the community still needs more work that focuses on how to create representations of student thinking that can be educative for novice teachers and teacher candidates and on how to use those representations in courses for novice teachers or teacher candidates.

Teacher candidates often have an inadequate understanding of the types of difficulties that students can face when learning a new concept (Stacey et al., 2001) and tend to be more familiar with arithmetic bugs than deeper conceptual errors (Tirosh, 2000). Many teacher candidates can only see correct ways of solving problems and tend to think about student errors in terms of forgetting to do what they were supposed to do (Crespo, 2000). With this mindset it can be quite hard for the teacher candidates to interpret students' written work or videotapes of students doing a problem--their expectations of what students should be doing tend to cloud how they read the writing or view the video. The image sequences can make the particular mathematical work more apparent than in its mere written record and more streamlined than in a videotaped record. With such image sequence, teacher candidates may be better able to appreciate the way the student actually attacked the problem in its own coherence and consistency even if it is mathematically incorrect or inefficient.

To help teacher candidates learn to notice students' mathematical thinking and to intervene productively in that thinking is one of the purposes of mathematics teacher education. Representations of student thinking using video records of children at work or written records of what they have done are useful but can also be too complex—the former for being too rich in nuance, the latter for collapsing a temporal process (the creation of the marks on the page) into a static one (the completed page itself). In the case of video, because of the speed with which students can work and the sometimes partial, subtle traces they leave of how they complete tasks,

teacher candidates may miss key parts of what students are thinking (e.g., a student muttering about “unique parallel line” may get lost amidst other details of the scene). In the case of written records of student work, because of the idiosyncratic ways students organize their writing, it can be difficult (at times even impossible) to reconstruct how a student solved a problem when that work is presented all at once, rather than in sequence. These challenges notwithstanding, video and written records of student work also clearly have advantages: videos can show mathematical work being developed over time and in classroom situations—where the teacher, other students, and resources such as tools are used by students to make the work; while written records provide concise representations of the work students deem sufficiently important to actually write down. The cartoon slideshows described above are one resource that can combine the situated, sequential nature of video with the conciseness of written records to produce representations of student work that show how the key features of that work developed over time. Teacher educators could use such cartoon depictions to connect discussions of student thinking to specific representations of student work.

If teacher educators could either create or have access to a library of depictions of how students produce solutions to problems they could use them to (1) educate teacher candidates on how conceptions manifest in student work and (2) engage teacher candidates in thinking about how to respond to students or how to change problems slightly to induce different kinds of student work. The cK¢ framework provides elements to inspect actual samples of student work (e.g., written records) and turn them into depictions of how students may have solved the problems. The teacher educator, operating as something like a materials developer, could first identify in a piece of student work indication of its problems, operators, representations or system of representation, and controls. The teacher educator could then create a cartoon depiction that models how a student may have created a solution. In doing that the cK¢ framework would suggest making explicit operators, elements of the systems of representation, and controls. These could eventually be used in activities with teacher candidates. For example, teacher candidates could be asked to critique a student's use of an operator. In the case of the “parallel line” operator described above, a teacher educator could make a depiction that shows a student drawing an over-determined parallel line, such as a parallel line that also intersects a particular point (like the midpoint of some other segment or a specific vertex of an angle). One goal could be for the teacher candidate to identify what the student did wrong and to consider what the mistake indicates about the student’s conception of parallelism.

Plan for Engagement

The working group will meet during PMENA in Chicago to explore how the creation of representations that depict students’ work on problems might proceed. This could begin as a collaborative project where members of the working group share a stack of student work that they want to represent and use these to develop a set of depictions (done with LessonSketch’s Depict tool or with other tools) that explore various students' conceptions of the same mathematical object. We have a collection of student work on linear functions that we will make available to the group for that purpose. The working group meetings could allow members to collaborate on the depiction of a subset of this student work. Then the working group could determine the types of metadata that would be useful to associate with those depictions.

Anticipated Follow-up Activities

One goal of the working group is to establish plans for participants to use these depictions in settings where they engage in teacher education, collect data regarding how well the depictions open space for reflection upon student thinking, and then reconvene at future PMENA conferences to share their use of the depictions and continue to refine these tools for further use. Over time we expect these kinds of development might converge to having documented libraries of student conceptions that can be used in teacher preparation.

Building on Prior Work

The proposed work builds on prior work of this group in that it continues to address issues on teacher preparation for which representations of practice are usable. It also proposes an entirely new direction, possibly more useful for PMENA researchers interested in disseminating results from research on student cognition.

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MATHEMATICS SPECIALISTS: THEIR ROLE IN SCHOOLS

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This working group addresses the emerging role of the mathematics specialist in schools as a leader for mathematics teaching and learning both formally and informally. In addition, elementary math specialists may have advanced mathematics preparation for teaching. Questions for mathematics education remain about the knowledge and preparation of the specialist and their role in schools as a bridge to research-based practices for teaching.

Keywords: Mathematical Knowledge for Teaching. Teacher Education-Inservice/Professional Development, Teacher Knowledge

Mathematics specialists have been rapidly emerging as a new endorsement licensure area for teachers in at least eleven states (e.g., Virginia, Maryland, West Virginia, California) as of 2011 (http://www2.mcdaniel.edu/emstl/certification.html, 03/12/2013). There have been multiple calls for the placement of mathematics specialists in schools as a means for improving teaching and learning with school-based professional development (e.g., Cobb & Jackson, 2011; McGatha, 2008). However, research remains limited about the role, knowledge, and leadership development for mathematics specialists as states and localities are implementing different models for these professionals (e.g., Cobb & Jackson, 2011; Cornett & Knight, 2008; McGatha, 2008; Nickerson & Moriarty, 2005). The primary focus of this working group will be to raise questions about the mathematics specialist in K-12 schools and begin to aggregate research and research questions related to the role, identity, knowledge, and leadership skills inherent in the position. We draw on the expertise of the group leaders' work with mathematics specialists and development of mathematics specialist education programs over the last 10 years to guide this discussion.

Issues for Mathematics Education

Role of the Math Specialist

Mathematics specialists for the purposes of this working group encompass a wide range of emerging and ongoing roles in schools. The roles may include working with students part-time (e.g., small groups of remedial students, 1-2 class periods per day of teaching whole classes), working with teachers to improve instruction, or teaching only math in the elementary school setting. As stated by the Association for Mathematics Teacher Educators, "Whatever the setting or responsibilities, [elementary mathematics specialist] professionals need a deep and broad knowledge of mathematics content, expertise in using and helping others use effective practices, and the ability to support efforts that help all students learn important mathematics." (p. 1, 2010). There are math specialists across elementary, middle and high schools. Their work may be

funded at the school or district level. They may work at multiple schools within a district. The Elementary Mathematics Specialist Leader Project has begun aggregating descriptions of district models (http://www2.mcdaniel.edu/emstl/district_models.html). In general, the role is defined as improving mathematics teaching and learning from a school-based position. The titles may include math coach, resource teacher, lead teacher, or specialist. Specialists for literacy have long had a role in schools, most often one in which they interact directly with students often in the context of remediation. The emerging role of the math specialist is trending toward work with teachers rather than direct work with students, however many math specialists also teach students directly either part-time or full-time. For the purposes of this paper, we will use "mathematics specialists" generally to refer to their role as mathematics leader within the school whether they are teaching in classrooms or primarily working with teachers. "Math coach" will be used to refer specifically to their work with teachers' professional development.

The mathematics specialists' role is often viewed as a bridge between multiple stakeholders in the school setting. For example, Fairfax County (a suburb of Washington DC) schools recently adopted an online textbook series for mathematics. The job of the math specialist in many schools is to support teachers' use of the new textbook series. In other schools, the math specialist may be called upon to conduct beginning of the year assessments and placement tests, lead professional development workshops, support professional learning communities or grade level teams, etc. In essence, they are the resource for mathematics knowledge related to teaching, learning, curriculum and assessment in the school. Another way the specialist serves as a bridge is between the principal (or leadership team) and teachers. In this role, the specialist both helps the principal understand mathematics teaching and learning within the school as well as helping to set or implement a vision for mathematics instruction that may be set by the leadership (Felux & Snowdy, 2006).

A significant component of the role is that it is school-based (even if the specialist works at multiple schools). The goal is to move beyond professional development as situated only in summer institutes or periodic workshops but rather to have an ongoing, locally-available resource for teachers to support math instruction (Neufeld & Roper, 2003). This means that, by design, the specialist’s role may evolve as needs in the school or district change. The specialist is also then a conduit for research and reform (Cobb & Jackson, 2011). For example, the math specialist should be familiar with and knowledgeable about best practices, resources and latest information about math teaching. Hence, the specialist can serve to interpret and disseminate that research for and to teachers in the building, thus supporting teachers' development of new practices (e.g., supporting teachers' use of discourse related to problem solving in the classroom).

Knowledge of the Mathematics Specialist

Current recommendations from AMTE (2010(Campbell & Malkus, 2011)) and NCTM (2012) for mathematics specialists knowledge and skills include mathematics content, leadership, pedagogy, curriculum and assessment. The math content includes number, rational number, geometry, algebra, and statistics (including probability). In addition, math specialists are intended to be experts in math pedagogy including problem solving, communication, reasoning, and proof. A typical graduate program for math specialists includes both coursework in the math department and in education (e.g., curriculum and instruction). While a non-evaluative role in schools, math specialists do need to understand leadership in schools and means for teacher observation and feedback. For math specialists whose primary role is teaching, the endorsement program may include only the mathematics courses with some additional coursework in mathematics pedagogy taught from a foundation in mathematics for teaching. This raises a

variety of questions about the possible differences in education or preparation for math coaches and elementary math specialist teachers.

A significant part of the work of the math coach is working with teachers individually or in groups (Whitenack & Ellington, 2007). This type of work has indicated impact on students' mathematics achievement (Campbell & Malkus, 2011). Recommendations for individual coaching (e.g., Felux & Snowdy, 2006; West & Staub, 2003) focus on the using mathematics content and students' thinking to support teachers' improving their work with students. Individual coaching may also include co-teaching or modeling for the teacher preceeded by a planning meeting and followed by a debriefing meeting. Anecdotally, scheduling such meetings is sometimes challenging for math specialists due to the constraints of the school day (McGatha, 2008). As a second component of their work, math coaches are also called to participate in professional learning communities, department teams or grade-level teams of teachers who work together (Nickerson & Moriarty, 2005). The goals of this type of teacher collaboration is to facilitate planning, analyze assessment data and create consistency across similar courses. The math coach may lead such meetings or may only participate in the meeting as support for the teachers work. Lesson study is one model for professional development in which the math specialist may play a role in supporting teachers' ongoing understanding of teaching and learning. Research is emerging related to the specific tasks a math specialist may take on during their work (Salkind, 2010; Whitenack & Ellington, 2007). A common recommendation is that the math specialists should support teachers' analysis (individually or in groups) of students' work (e.g., formative assessment, summative assessment, standardized testing data).

Challenges of the Math Specialist

There are a variety of challenges that emerge as the math specialist assumes a formal or informal leadership role in the school. The first is that the math coach is intended to work primarily with teachers rather than directly with students. One transition new math coaches need to make, then, is working with adults and learning to facilitate professional development with teachers. If the math specialist is working at a school where previously they were a teacher, this can create both opportunities (e.g., a prior relationship exists which can facilitate leadership) and challenges (e.g., shifting in roles from a "peer" teacher to a member of the leadership team). Related to this is a shift in personal identity as a teacher to a teacher/leader of other teachers. Anecdotally, some math coaches report missing daily interaction with K-8 students. However, the role of math coach also represents an opportunity for a teaching professional seeking new challenges who may have been teaching for a long period of time (e.g., 10-15 years) to continue working in schools with teachers and students while exploring new career opportunities.

As an informal leadership role, the math teacher at the elementary level may be called upon to be the lead teacher or point-of-contact for a grade level team who's focus is on mathematics. This can include being the "expert" on the team for mathematics learning and pedagogy. This role may also be as an elementary teacher who only teaches mathematics (or mathematics and science). This can be an opportunity for an elementary teacher to explore teaching only mathematics in a focused way or develop expertise in mathematics that can create informal leadership opportunities in the school.

A second challenge also relates to the leadership role of the math coach. The coach is not intended to be evaluative even though they often do conduct teacher observations and may be called on to work with struggling teachers. The specialist may also be part of the leadership team and work closely with the principal. This creates a need for balancing the role of trusted colleague to teachers in the school with the role of an instructional leader in the school. If

teachers feel like the coach is reporting on them or only works with struggling teachers, this can increase resistance to the coach's presence in the classroom or the team meeting. However, if the coach can be recognized as trustworthy and a positive resource for teachers; they can bridge their role as school leader and teacher colleague.

Specialists in other disciplines, especially reading and technology, have been in schools for a long time. The math specialist is a new role and may carry with it additional challenges. Much research has demonstrated that elementary teachers' may lack math knowledge or have significant math anxiety (unlike reading). This negative disposition to mathematics creates challenges for the math coach that other specialists may not face. First, the math coach may need to construct professional development activities to enhance teachers' knowledge of mathematics and pedagogy. In addition, coaches may meet significant resistance to trying new reform practices (e.g., problem based learning or problem solving) because teachers' anxiety makes them resistant to attempting new mathematics strategies.

Activities for the Working Group

For the working group, we have identified two areas to begin discussion drawing from current and ongoing research related to math specialists. The first major area is the role and work of the math specialist. For this discussion, Author’s findings from a study of high school professional development including a math specialist will start a discussion about the role of the math specialist at different grade levels. Similarly, Author’s work with math coaches as facilitators and supporters of lesson study groups will start a conversation about school-based professional development opportunities. These discussions will help articulate what is a mathematics specialist by identifying the various roles and responsibilities of this group of educators. The second major area relates to the identity and development of mathematics specialist professionals. Author will lead a discussion about the identity of the math specialist across two models for specialists (math teacher and math coach). Author will discuss the creation of online courses for mathematics specialists and the opportunities and challenges therein for their development as leaders of mathematics. Author will focus on preparation for specialists as elementary math teachers. These discussions will focus attention on what is needed to prepare educators for mathematics specialist roles and the various contexts and factors that may influence their development.

What We Know About the Day-to-Day Work (Author)

How might we describe the mathematics specialists’ day-to-day responsibilities in the elementary school building? The list of responsibilities is long as we consider proposals by Campbell and Inge (2006), Fennel (2007), Reys and Fennel (1997), Yow (2007) or the Virginia Mathematics and Science Coalition, to name a few. Reys and Fennel, for instance, describe two models to describe this daily work: lead-teacher model or the specialist-teaching-assignment model. When the mathematics specialist serves in a leader-teacher role, he or she is “released from classroom instruction to assume mentoring and leadership responsibilities at the building or district level” (Reys & Fennel, p. 280) (c.f., Rowan & Campbell, 1995). One might expect a mathematics specialist to plan, co-teach, make observations, model lessons, and so on (Reys & Fennel). By way of contrast, mathematics specialists that serve in the specialist-teaching-assignment role assume the primary responsibility for teaching mathematics at a particular grade level, for instance (Reys & Fennel). Reys and Fennel suggest that in the latter case, the classroom teacher develops a more narrow set of competencies and responsibilities.

The Virginia Mathematics and Science Coalition [VSMC], too, offers insight into the mathematics specialist’s role:

[K-8] Mathematics Specialists are teacher leaders with strong preparation and background in mathematics content, instructional strategies, and school leadership. Based in elementary and middle schools, mathematics specialists are former classroom teachers who are responsible for supporting the professional growth of their colleagues and promoting enhanced mathematics instruction and student learning throughout their schools. They are responsible for strengthening classroom teachers' understanding of mathematics content, and helping teachers develop more effective mathematics teaching practices that allow all students to reach high standards as well as sharing research addressing how students learn mathematics. [VMSC, 2007]

As the VSMC suggested, the mathematics specialist assumes responsibility for promoting and supporting professional growth for their colleagues that lead to supporting or enhancing student learning. The mathematics specialist has a unique set of challenges, a range of responsibilities that can affect the daily work of administrators, teachers and their students. In our discussion, we borrow from VMSC's description of the role of the mathematics specialist to define more broadly the mathematics specialist's work.

Mathematics specialists engage in various types of work during the school day. One of the important ways that she works with teachers is to serve as a coach, working side-by-side with them in their own classrooms (Hall & Simeral, 2008). As she coaches individual teachers, she helps them (1) more deeply understand the mathematical ideas that underpin the curriculum they teach, (2) teach content more effectively to support their students' learning, (3) refine skills so that they can better interpret and assess their students thinking, and (4) develop lesson plans that support their students' learning (West & Staub, 2003).

So how can the specialist accomplish these goals? As one approach, she plans the lesson with the teacher, observes or co-teaches the lesson and then debriefs with the teacher following the lesson. This process of planning, supporting the teacher during the lesson, and debriefing about the lesson is often referred to as a coaching model (West & Staub, 2003). As the mathematics specialist works with teachers, she can implement each of these three components of the model, perhaps in varying degrees, when she coaches. Her efforts to provide ongoing professional development serve teachers and their students. As such, the specialist has the opportunity to effectively support reform recommendations that have been made by the mathematics and mathematics education communities.

Math Specialists at the High School (Author)

Licensure of mathematics specialists in the state of Virginia covers kindergarten through eighth grade. Question arises about the role of the mathematics specialist on the high school level. Are they needed? Jones (1997) states "secondary mathematics teachers are, in general, mathematics specialists and so subject knowledge ought to be secure (p.22)." NCTM's content recommendations for math specialists reveals what a mathematics specialist on the secondary level should know and be prepared with, depth and breadth, to support teachers (2012). Topics include Number, Algebra, Geometry, Trigonometry, Statistics, Probability, Calculus, and Discrete Mathematics. Do teachers entering the classroom have the depth and breadth of knowledge, content and pedagogical, expected? The study conducted by Jones (1997) suggested that when students graduate and enter the classroom they often think of teaching mathematics as a list of facts with little connections to the relationships between concepts. Jones goes on to state that "This has the effect of making the transformation to effective pedagogical content knowledge all the more difficult" (p. 30).

The NCTM NCATE standards (2012) list the topics so what coursework is needed so that teachers will be comfortable with implementing the NCTM Process Standards (2000) with

confidence. During a case study involving professional development of in-service teachers with regard to rich tasks, a specialist supporting high school mathematics teachers stated that he did not feel confident or competent to help the algebra teachers as they dove into facilitating instruction based on big ideas and rich tasks (Author, 2013). The specialist was in the process of completing course work for the K-8 mathematics specialist license and was a previous high school mathematics teacher in the same building. Responsibilities of the specialist in this case study centered on data analysis of benchmarks and state assessments. Curriculum and instruction was also part of the duties but ones that the specialist did not interfere, allowing the teachers to continue providing instruction as they have always done. Even though this is only one instance involving a high school specialist questions, arise as stated above. This case and other questions bring forward multiple questions about high school math specialists. When the decision has been made that specialists on the high school level are needed then what would be the expectations? What course work would be beneficial to enable the specialist to provide the needed assistance to teachers? Would the coursework required for K-8 specialist be beneficial for those involved on the high school level?

Mathematics Specialists Serving as Lesson Study Facilitators (Author)

The Japanese Lesson Study model (Lewis, 2002) was introduced to the mathematics education community more than a decade ago and captured the attention of mathematics educators in the United States for many of its appealing affordances. Lesson Study offers a sustainable teacher-led professional development model that is job-embedded and focuses on student learning. It encompasses many of the important professional activities for teachers such as researching and studying important mathematics content more deeply, co-designing rich problem-based lessons, doing mathematics, anticipating common misconceptions, planning ways to differentiate for diverse learners, evaluating multiple solution paths and assessing student thinking. However, implementing the Japanese Lesson Study in its true form has been challenging due to cultural differences, different infrastructure in schools and limited professional learning time allocated for such activities like co-planning, observing and debriefing during the academic year. However, with some creativity, passion and efforts by teachers, administrators, and university educators, Lesson Study has become a staple professional development model in many schools. In fact, the recent push to establish professional learning communities (PLCs) has been a window of opportunity that we have leveraged to establish school based Lesson Study.

With the establishment of many professional learning communities, many teachers have been mandated by the district to have PLCs and have been provided the time and space for teachers to collaborate. Some schools have used this designated PLC time to implement Lesson Study as a form of professional development. However, many of the teams need a facilitator especially when they are first being introduced to this model. In our professional development project (Author, 2012), we have designated mathematics specialists to serve as Lesson Study Facilitators for each school-based team. We define the Lesson Study Facilitator as an important role, who not only serves as the ‘knowledgeable other’ (often present in Japanese Lesson Study) but also a teacher-leader who can guide a meaningful mathematics professional development and who can maximize the professional learning. Unlike the ‘knowledgeable other’ who usually is an outside expert who comes to the research lesson and comments at the end of a debrief, our Lesson Study Facilitators take on the role of helping teachers from the start of the process by helping the team develop a goal based on their student data, formulate the research theme and select a rich task that will elicit important mathematical thinking.

Typically, mathematics specialists have taken courses to earn their endorsement through the state that requires coursework in three main areas: 1) mathematics content courses which focuses on deepening the content knowledge while identifying big ideas and common pitfalls; 2) pedagogy courses, which focuses on teaching and learning with an emphasis on assessing student learning; 3) leadership courses, which focuses on being a change agent in schools and district by leading professional development and school change. The specialized training offers the mathematics specialists the pedagogical content knowledge necessary to lead teachers and give access to the research-based practices. As the team moves along the Lesson Study cycle of studying, planning, teaching and observing, and debriefing, the Lesson Study Facilitator can offer resources that promote reform teaching practices and support teacher development throughout the phases. For example, during the studying and planning phases, the Lesson Study Facilitator can help teachers select a rich problem, do the mathematics themselves as learners and evaluate multiple approaches as they prepare and anticipate students' responses.

One result from our research lessons is how the mathematics specialist contributes to the lesson analysis. In many cases, the mathematics specialist is able to select important pedagogical math content knowledge to bring to the forefront of their discussion. While the teacher observers were readily able to identify misconceptions or confusion occurring during a lesson, they were not able to easily make a connection to the source of the problem or determine how the lesson could be enhanced to bring better understanding. The mathematics specialists' analysis of the lesson usually had more depth and with their background and training in analyzing student thinking and unpacking mathematical ideas, were better able to offer a rationale or connect the teachers' noticings, discoveries or pitfalls with important pedagogical math content knowledge. An example of this occurred during one of the research lesson called the Mango problem when teachers were noticing students having difficulty working with a finding the number of mangoes the character started with in the fruit bowl. Many students worked backwards, guessed and checked but many had major confusion understanding what made a whole. The teachers discussed how the students had difficulty understanding what 1/2 of what was left meant. Some students read the problem as ½ of what was left and interpreting it as 1/2 of one mango, instead of interpreting that it was 1/2 of the remaining fruit in the bowl. During the debrief, the mathematics specialist took on the role as the "knowledgeable other" and shared that the source of the confusion stemmed from students' partial understanding and confusion about unitizing. She shared with the teachers the importance of unitizing as an understanding that numbers can be arranged in groups and that these groups can then be counted as individual units. Some students were stuck on fraction as a part of a whole unit (mango) instead of thinking of it as a part of a whole set. These students needed to have experience of unitizing which would allow them to see a unit could be made up of discrete sets, like a dozen eggs can be considered one carton, thus 1/2 of the carton would equal 6 eggs. Often times, teachers did not have the terms to identify the mathematical ideas that represented an important pedagogical content knowledge.

Through these research lessons, we are learning more about the specialized knowledge needed for mathematics specialists as they lead professional development. We are also seeing in practice that the preparation for mathematics specialists affords them the knowledge and skills needed to facilitate professional learning structures like Lesson Study and allowing for this rich professional development model to be a viable structure for teachers to engage in as a job-embedded professional experience. Some research questions that could be explored include: (a) What other courses or experiences within the Mathematics Specialist course sequence can better prepare our Mathematics Specialists to do the work of leading professional development? (b)

How can Mathematics Specialists use the PLC time set aside in many schools to maximize the professional learning of teachers that impacts student learning? (c) How are other Mathematics Specialists using Lesson Study as a vehicle for school change?

Professional Identity of Mathematics Specialists (Author)

The West Virginia State Board of Education approved standards for Elementary Mathematics Specialists in January 2012 allowing for two options: Elementary Mathematics Endorsement (K-6), an area of specialization attached to an Elementary certification, and Elementary Mathematics Specialist, requiring a master's degree. Both options include 13 process, pedagogy, and content standards; the Specialist degree includes two additional standards related to curriculum and assessment and leadership. The intent of these different options is to prepare EMS professionals for different positions. The Elementary Mathematics Specialist path would prepare an elementary teacher to be an expert in mathematics teaching and learning at a classroom level. This could be an added endorsement to an existing license or be part of an initial certification program (in apparent contradiction to national recommendations for EMS professionals describing their "expertise and successful experience at the elementary level" as being "critical" to success (AMTE, ASSM, NCSM, and NCTM, n.d.). The second path, the Elementary Mathematics Specialist, would prepare an experienced teacher to be a school- or district-based leader in elementary mathematics.

Students populating programs intended to prepare them for these various paths may be undergraduate students seeking initial certification, licensed teachers wishing to add to an existing certification, or experienced teachers seeking leadership positions. Thus, individuals in these programs will differ in terms of a variety of factors: identity (how they perceive themselves as learners of mathematics, how they perceive themselves as teachers of mathematics), levels of experience (as teachers, leaders, and learners), available resources, and motivation for pursuing an elementary mathematics specialist path. These differences, while providing challenges for program implementation and design, also provide a rich background for research on issues of identity and knowledge development and transference in an elementary mathematics specialist context. The following questions could be explored: (a) How do different participants experience and make sense of courses and learning experiences designed to prepare them to be experts in elementary mathematics? (b) How do they position themselves as learners? As knowledgeable participants? (c) How do they then enact the knowledge and skills developed in these programs in their work (teaching and/or coaching), i.e., how do they transfer this knowledge to different contexts? (d) What are the influences of context (social, political, autobiographical, etc.) on how different participants experience and enact their knowledge and experiences?

Building Professional Learning Communities in Online Courses (Author)

Mathematics specialists are typically practicing teachers or already working as coaches. One aspect of their new identity as coaches is as leaders. A goal of our program is to foster collaboration and learning from teachers in other schools and districts (e.g., urban, suburban, rural, public, private). Our math specialists report that engaging with teachers in other contexts broadens their understanding of mathematics education and helps them consider other models. They also have the experience of learning to become coaches in collaboration with other teachers in the same situation. Given there are usually only 1-2 coaches per school building (or a few coaches per district), there is a significant chance of isolation as leaders. At the same time, they have asked for flexibility and are looking for online courses to complete their licensure. To address both the needs of the working professional teacher and foster professional learning

communities across multiple teaching contexts, we have begun to offer online courses for math specialists. This discussion will focus on the types of challenges created in the online learning environment as well as the opportunities e-learning can generate for collaboration and conversation across contexts that are geographically dispersed. A challenge for education reform has been to reach urban and rural communities. Online learning platforms are one means of bringing university programs to these settings. In addition, given the relatively small number of mathematics specialists (and hence students), online education is a means to reach a wider audience.

Possible research directions include the following questions: (a) How can we develop professional learning communities in an online learning context for math specialists? (b) What kinds of leadership and mathematics knowledge can be developed in different learning formats (e.g., online, synchronous, asynchronous, face-to-face)? (c) How do mathematics teacher educators develop their practice for online teacher development contexts?

Specialists as Elementary Mathematics Teachers (Author)

Another aspect of the math specialist role is an enhanced mathematics knowledge for teaching either as an elementary math teacher or in more informal leadership roles (Association for Mathematics Teacher Educators, 2010). Utah has an Elementary Math Endorsement designed to help a classroom teacher to be a better classroom teacher of mathematics because they develop their mathematical knowledge for teaching. The endorsement is made up of 6 courses: 5 Mathematics courses (Number, Rational Numbers, Algebra, Geometry, Data Analysis and Statistics) and one course in Assessment and Intervention. While there is significant research about pre-service teachers knowledge of mathematics and the development of that knowledge, this is an innovative model for extending their mathematics knowledge beyond the standard expectations for elementary licensure. The design of an additional endorsement for math teaching raises a number of questions about developing teachers with advanced knowledge in mathematics. For example, what are the impacts on their mathematics knowledge for teaching after having participated in extensive mathematics preparation? How does the additional coursework influence their work as elementary teachers of mathematics? Are their informal teacher leader roles that would be available for teachers with a elementary math endorsement (e.g., math lead teacher for a grade level team)?

Future Work/Products of the Group

Given a number of reports and articles point to the lack of research about the preparation and professional work of mathematics specialists, our focus will be, first, on determining what existing work may be occurring; and second, to identify possible areas of collaboration for that work. To that end, the working group will start to define salient issues for further research and investigation. The second goal will be seeking possible collaborations among participants for joint papers or chapters regarding any current research efforts regarding math specialists or teacher leaders.

As a role that in some cases has been in schools for some time but is also relatively new as states begin to define standards and licensure requirements for mathematics specialists, the overarching goal of this group is to continue conversation about the knowledge and activities that make up the various roles of math specialists in schools from teaching to professional development to leadership. Their role in school-based professional development follows suggested best practices for teacher development as school-based, job-embedded and responsive to teacher needs. Their role as mathematics teacher brings more mathematics knowledge for teaching into classrooms to support student learning. As a professional opportunity for teachers

at multiple grade levels, the mathematics specialist represents an opportunity to develop the abilities of those teachers with existing interests in mathematics teaching and learning.

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MATHEMATICS TEACHER NOTICING: A HIDDEN SKILL OF TEACHING

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Expertise in noticing is a critical component of effective mathematics teaching, yet this skill has often remained hidden, making it challenging to study and develop. Increased attention to mathematics teacher noticing in recent years prompted the creation of this new Working Group. Our goal is to gather individuals interested in teacher noticing, as either a focus of research or a tool for professional development. We hope to deepen our collective understanding of noticing by discussing key issues, providing opportunities for participants to share related research and professional development efforts, and collectively engaging in the practice of noticing with classroom artifacts.

Keywords: Instructional activities and practices; Teacher Education-Inservice/Professional Development; Teacher Education-Preservice

Imagine observing or teaching a mathematics lesson, what do you notice?
What do others notice? What noticing is most productive?

Noticing is used in everyday language to indicate the act of observing events, objects or interactions. The term can also be used more narrowly to capture strategic ways of observing that reflect the expertise of a particular profession. It is this more narrow view of noticing that we refer to here, specifically the noticing of mathematics teachers. Teachers work in complex environments and are constantly confronted with a "blooming, buzzing confusion of sensory data" (Sherin & Star, 2011, p. 69). A key component of teaching expertise, therefore is being able to decide, in the midst of classroom interactions, where to focus one's attention, and to be able to distinguish between more productive and less productive ways of noticing. Recently, the construct of noticing has increasingly captivated mathematics education researchers as evidenced by the rising number of conference presentations and publications on the topic (see Author for a compilation) and an entry in the *Encyclopedia of Mathematics Education* (Author). This new Working Group is being created to capitalize on this widespread interest in noticing, promoting collaboration and sharing of research and professional development efforts.

Background on Mathematics Teacher Noticing

The field of mathematics education currently embraces a range of conceptualizations of teacher noticing. However, this construct has generally been associated with two components: (a) *attending to particular instructional events*, and (b) *making sense of those instructional events*. The first component emphasizes the complexity of the classroom environment, and the need for teachers to attend to this complexity in a discriminating manner. The second component highlights that, in addition to focusing on particular events, teacher noticing involves making sense of what stands out. Researchers differ, however, on what constitutes *making sense*, with some focusing exclusively on teachers' interpretations of events whereas others include both how teachers interpret events as well as how teachers reason about their possible responses to these events. Another major difference in the conceptualizations of noticing is whether

researchers are interested in documenting everything teachers find noteworthy or whether they focus on teachers' noticing of particular aspects of instruction identified as important, such as students' mathematical thinking or specific mathematical ideas.

Despite the range of conceptualizations of noticing, many mathematics education researchers were initially inspired by similar work, namely Goodwin's (1994) ideas about *professional vision*, Mason's (2002) *discipline of noticing,* and the research on expertise in which noticing seems to play a large role (National Research Council, 2000). The basic idea underlying all these bodies of work is that individuals looking at the same thing may see it in different ways, and these differences are important for the individual's understandings and future actions.

Work on noticing also connects with several current efforts in mathematics education. First, researchers are trying to decompose teaching so that it can be productively discussed and practiced (Ball & Cohen, 1999; Ball, Sleep, Boerst, & Bass, 2009; Grossman & McDonald, 2008; Lampert, 2001). Noticing can be considered one of the core activities resulting from this decomposition. Noticing research and professional development has not only made this often hidden part of teaching visible but also given teachers language that enhances their ability to collaboratively reflect on the act of teaching. Second, researchers have been working to conceptualize teaching as adaptive and responsive to the ongoing nature of a lesson (National Council of Teachers of Mathematics, 2000; National Research Council, 2001). In particular, teachers are now expected to attend to their students' mathematical thinking and adjust the lesson so that it builds on the students' ideas. This type of teaching must be, almost by definition, informed by teachers' noticing in the moment of instruction. Thus, the work on noticing supports the vision of mathematics teaching endorsed by the reform movement and is consistent with the goals of the *Common Core State Standards in Mathematics* (CCSSO/NGA, 2010). Third, researchers have been emphasizing the promise of helping teachers learn from their own teaching (Franke, Carpenter, Fennema, Ansell & Behrend, 1998; Hiebert, Morris, Berk & Jansen, 2007). Teacher noticing supports this goal because the principles a teacher extracts from an experience of teaching depends strongly on noticing—to what instructional events does a teacher attend and how does he or she make sense of them.

Showcasing the Range of Work on Mathematics Teacher Noticing

To provide a sense of the different approaches researchers currently take in the study of mathematics teacher noticing, we offer an example based on a 40-minute lesson taught in a combination class with Grades 1 and 2. At the beginning of the lesson, a child posed the problem "We have 19 children, and 7 are hot lunch. How many are cold lunch?" (*Hot lunch* was the classroom terminology used to refer to students who were buying their lunches and *cold lunch* referred to students who had brought their lunches that day.) After the problem was posed, the teacher allowed the children to work on the problem individually or in pairs, using any tool they wanted, and solving the problem any way that made sense to them. During this time, the classroom was noisy and children exhibited on and off-task behavior. Three correct strategies were then shared with the group, and each showcased different strategies, different tools, and different levels of understanding. Specifically, one strategy involved counting back 7 from 19 on fingers, and another involved the drawing of 19 individual (not grouped) tallies, erasing 7 and counting the remaining tallies. The third strategy was shared by a child who used a counting frame (10 rods with 10 beads on each rod). He built 19 (as 9 sets of 2, and 1), removed 7 beads, and counted the remaining beads by subitizing 4 and then counting by 1s to 12. During the sharing, the teacher extensively questioned the children about their reasoning and corrected some off-task behavior (example drawn from Author).

The complex environment represented in this example is something teachers face every day. There are many—too many—important things to notice and what teachers do notice will determine not only how they respond in the moment but also what they take from the lesson to inform the planning of future lessons. How might various noticing researchers consider using this lesson (or an excerpt of this lesson) to either assess teachers' noticing or support the development of teachers' noticing expertise?

Different goals and different conceptualizations of noticing would likely drive different uses and perspectives. For example, some might show the lesson video to prospective teachers to learn the range of lesson components that are most salient to them (Star, Lynch, & Perova, 2011); Do they attend to the arrangement of desks in the room, the teacher's classroom management style, the mathematical task posed by the child, the communication among students, and so on? Others working with prospective teachers might, with the help of a video analysis tool, use this lesson to support prospective teachers in developing an ability to notice specific aspects of the classroom, namely students' mathematical thinking, teachers' roles, and discourse (Author). The video tool would be designed to help prospective teachers identify noteworthy events and use evidence from the video to support that these events are noteworthy as well as interpret what these events mean in terms of the lesson and student learning.

In noticing work with practicing teachers, researchers often focus on a particular type of noticing such as noticing specific mathematical ideas or noticing students' mathematical thinking. For example, some might emphasize the noticing of mathematical ideas that the teacher in the lesson did while she was teaching (Schifter, 2011). Specifically, did she notice key mathematical concepts such as ideas related to problem solving, part-whole relationships, and so on? Other researchers might emphasize the noticing of children's mathematical thinking. For instance, in the work on *professional noticing of children's mathematical thinking*, the focus would be on understanding practicing teachers' expertise in the three interrelated skills of attending, interpreting, and deciding how to respond (Author). First, did teachers *attend* to the details of the three strategies presented? Second, can teachers *interpret* children's understandings reflected in those strategies? Third, did teachers *decide how to respond* on the basis of children's understandings (or on the basis of other things). Note that this third skill is intended responding—the reasoning prior to the actual execution of the response—and is included because it is integrally connected to the first two skills.

Other researchers who emphasize noticing students' thinking might use the lesson in a video club in which teachers meet in small groups to improve their ability to attend to and make sense of student thinking. On the basis of a three-part learning-to-notice framework, teachers would be encouraged to (a) identify noteworthy aspects of the classroom situation, (b) use knowledge about the context to reason about the classroom interactions, and (c) make connections between the specific classroom events and broader principles of teaching and learning (Author). Still others might use the lesson in professional development in conjunction with a lesson analysis framework in which teachers would be asked to focus on the lesson's learning goals, conduct an analysis of student learning, and generate alternative strategies (Santagata, 2011).

Finally, some noticing researchers might be interested in the characteristics of the common tools used to development teachers' noticing expertise, namely the features of the artifacts discussed. In this case, they might consider whether the lesson should be shared on video or only through written work and when it would be desirable for the teachers in professional development to be familiar with the teacher and students in the lesson being discussed (Goldsmith & Seago, 2011).

In this Working Group, we would explore the rich variety of both the conceptualizations of noticing and its uses. We are not promoting a single approach to the construct of noticing but instead believe it beneficial to jointly consider the affordances and constraints of various approaches for various target audiences (e.g., prospective and practicing teachers) and various research and professional development goals.

Plans for the Sessions and Active Engagement of Participants

Our overall goal is to deepen our collective understanding of noticing and its role in mathematics education, become better acquainted with each others' related research and professional development efforts, and collectively consider directions into which noticing research might productively progress. Throughout our interactions, we will elicit and highlight what underlies the growing fascination with noticing and its potential to positively affect mathematics education.

Session 1: Issues in Teacher Noticing

In this first session, we will provide an introduction to the construct of noticing and engage participants in discussion of three key issues related to the study and use of noticing. One additional goal of this session is to identify individuals who would like to share their work in Session 2.

Conceptualizations of noticing. There are affordances and constraints with any conceptualization of noticing, and thus some conceptualizations will be better for particular purposes than others. The group will discuss several conceptualizations and consider when their application may be most beneficial (or problematic).

Methods of capturing noticing. Noticing happens "in the moment" and is generally invisible to observers, so capturing it is challenging. Investigating noticing while it is occurring may change the noticing itself, but reflecting on noticing carries with it the danger that subsequent events will change what individuals think they noticed. Recent advances in technology have opened new avenues for capturing noticing in the moment, from the teachers' perspective, such as when teachers wear portable video cameras that allow them to capture noteworthy moments of video immediately after they occur. However, the use of this technology and interpretation of the results are far from trivial (Author). The group will explore a variety of methods for capturing noticing, debating the affordances and constraints of each.

Development of noticing skills. Given the increasing consensus that expertise in noticing is essential for effective mathematics instruction, the question becomes how do individuals develop noticing expertise and how can we support that development? Research has shown that noticing skills can improve with support, and the group will explore a variety of frameworks detailing the development of noticing expertise (e.g., Author; van Es, 2011) as well as consider professional development efforts that have (or could) support this development. Video clubs (Author), lesson analysis (Santagata, Zannoni & Stigler, 2007), and strategic use of classroom artifacts (Goldsmith & Seago, 2011) are some of the types of professional development efforts that may be considered for supporting the development of noticing.

Session 2: Research and Professional Development Efforts in Teacher Noticing

This second session will provide an opportunity for group members to share their research or professional development efforts related to mathematics teacher noticing. They may choose to share past work, ongoing work, or future plans. The goal will be twofold. First, we want to create an opportunity for everyone to be aware of active noticing researchers and research efforts, and to have the opportunity to identify any emerging trends. Second, we want to continue our discussion of the three key issues highlighted in Session 1. Toward this end, we

will invite each participant to provide a short summary of their work and pose a discussion question related to one of the three key issues. (Note that the facilitators have several noticing projects, with different conceptualizations and methodologies, that can jumpstart the conversation if needed.)

The organization of this session will depend on who volunteers to share their work (and how many people). Some categories we would like to highlight include (a) work on prospective teacher noticing, (b) work on practicing teacher noticing, (c) the use of records of practice (video or written work) to support the development of teacher noticing, and (d) methodological contributions to the study of teacher noticing. Depending on the number of participants and presenters, we may remain as a whole group or split into smaller interest groups to explore each other’s work. We recognize that the field of teacher noticing draws upon the work of several senior scholars, but we also note that junior researchers, including graduate students, have adopted and adapted noticing frameworks. Explicit efforts will be made to include a few graduate students in sharing their work.

Session 3: Engaging in Noticing

We will provide a few carefully selected classroom artifacts (i.e., video or student written work) to allow the group to “notice” together. Based on the two earlier conversations, we may choose to notice with a particular lens or using a variety of lenses. We will consider noticing for both research and professional development purposes, and how our purpose may change our noticing and related concerns. Our hope is that by actually engaging in noticing together, we can further clarify and investigate some of the issues discussed earlier. This activity may also lead some participants to consider future collaborations around similar issues.

We will conclude by revisiting what we have learned about our three key issues—conceptualizations of noticing, methods of capturing noticing, and development of noticing skills—and identify any new issues that still need attention in noticing research or professional development.

Acknowledgments

This research was supported in part by a grant from the National Science Foundation (ESI0455785). The opinions expressed in this article do not necessarily reflect the position, policy, or endorsement of the supporting agency.

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WISDOM[e]: QUANTITATIVE REASONING AND MATHEMATICAL MODELING (QRaMM) WORKING GROUP

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This is the third meeting at PME-NA of the Quantitative Reasoning and Mathematical Modeling (QRaMM) working group. QRaMM is one of three research strands initiated by the Wyoming Institute for the Study and Development of Mathematical Education (WISDOM[e]). The idea for this working group developed from an invitational planning conference for WISDOM[e] held at the University of Wyoming in September 2010. In 2011 and 2012 we addressed the following focusing issues: (1) secondary students' development of quantitative reasoning and mathematical modeling, (2) creation of QR learning progressions to explain such development, and (3) the impact and interplay of QR and modeling on students' development in mathematics and science. A focal point emerging in 2012 related to how an individual's perspective on context might impact the individual's quantitative reasoning. For the 2013 Working Group we plan to expand the discussion related to context. Participants can expect to engage in dialogue focusing on research issues related to design, implementation, and analysis of tasks that could support and/or investigate secondary students', university students', and teachers' quantitative reasoning.

Keywords: Cognition, Modeling, Reasoning and Proof, Design Experiments

Brief History of QRaMM Working Group

A core membership for this continuing PME-NA Working Group was established through the initial Invitational Planning Conference for WISDOM[e] held in September 2010 at the University of Wyoming. During that conference three collaborative, interdisciplinary research teams were established: Quantitative Reasoning and Mathematical Modeling (QRaMM), Developing Investigations of Mathematical Experience (DIME), and Technology Tools and Applications in Mathematics Education (TTAME). The QRaMM and DIME research teams went on to host working groups at PME-NA in 2011 and 2012. In 2012, WISDOM[e], the NSF Pathways Project housed at Colorado State University, and Georgia Southern University hosted the International STEM Research Symposium: Quantitative Reasoning in Mathematics & Science Education. The conference, which gathered members of both mathematics and science education, was held in Savannah, GA. During that conference a Quantitative Reasoning Working Group convened. Issues discussed during that working group included distinctions between quantitative reasoning and quantitative literacy, interplay between quantities and quantification, and roles of context in quantitative reasoning. The 2012 PME-NA Working group continued discussion related to roles of context in quantitative reasoning. A focal point emerging from 2012 related to how an individual's perspective on context involved in mathematical and scientific tasks might influence the individual's quantitative reasoning.

The opportunity to conduct a third Working Group at PME-NA provides an important venue to continue and expand discussions of quantitative reasoning in relation to individual's (students, teachers, and researchers) perspectives on mathematical and scientific tasks. Namely, the third QRaMM Working Group at PME-NA provides a setting to discuss issues related to design, implementation, and analysis of tasks that could support and/or investigate how secondary students', university students', and prospective teachers' quantitative reasoning is influenced by the contextual perspective they take. During the Working Group participants will share information, issues, and problems related to ongoing research in relation to context and task design. Our goal is to promote interest and potential participation in disciplined inquiries related to quantitative reasoning and mathematical modeling, and to provide continuing support for collaboration within and across ongoing and future research efforts.

Focal Issues

Researchers (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Oehrtman, Carlson, & Thompson, 2008; Smith III & Thompson, 2008; Thompson, 2011, in press) have called attention to the importance of quantitative reasoning in the teaching and learning of mathematics at the secondary and university levels. Research investigating secondary students', university students', and prospective teachers' quantitative reasoning suggests that quantitative reasoning could support students' consideration of rate of change (Carlson et al., 2002; Johnson, 2012), linear function (Ellis, 2007, 2011), quadratic function (Ellis, 2011; Lobato, Hohensee, Rhodehamel, & Diamond, 2012), exponential function (Castillo-Garsow, 2012; Ellis, Ozgur, Kulow, Williams, & Amidon, 2012), combinatorics (Tillema, 2012, in press), and angle measure (Moore, 2013). In each of these studies, the context in which tasks were presented played a central role in the researchers' investigation of quantitative reasoning. The present working group explores the interplay of quantitative reasoning and task design by investigating how task contexts influence quantitative reasoning.

Quantitative Reasoning

When we refer to quantitative reasoning, we mean the mental actions involved in conceiving a situation such that the situation entails measurable attributes – called quantities – and relationships between these quantities (e.g., Smith III & Thompson, 2008; Thompson, 1993, 1994; Thompson, 2011). The following assumptions guide our design of tasks to investigate students' quantitative reasoning.

- Conceiving and coordinating measurable attributes of objects are a basis for quantitative reasoning.
- Conceptions of quantities vary from individual to individual.
- Conceiving of measurable attributes is distinct from evaluating amounts of quantity.
- Quantification involves a process of assigning a measure to some quantity and giving meaning to this measure—the measure may or may not be a specific numerical amount.
- The process of quantification is inseparable from quantitative reasoning.
- Quantitative reasoning, including quantification, is an emergent process.

By foregrounding reasoning processes and conceptions of quantities, we intend to communicate that the basis of quantitative reasoning is not determining numerical amounts and performing calculations. Although determining amounts of quantity can be involved in quantitative reasoning, it is neither necessary nor sufficient to constitute quantitative reasoning.

Task Design

By task, we mean a problem (or sequence of problems) for use in research or teaching settings involving a range of participants (e.g., secondary students, pre-service teachers, and in-service teachers) (e.g., Sierpinska, 2004). By design, we mean an iterative process of task development and implementation, including a researcher's decisions regarding a task, a student's (or students') interpretation(s) of a task, and the setting in which a task is implemented. Included in a researcher's decisions regarding a task is his or her current research purposes (e.g., investigating quantitative reasoning or engendering quantitative reasoning to support learning). For instance, a researcher might intend that a task *support* students' envisioning of quantities and relationships between quantities. On the other hand, a researcher might intend that a task draw out meanings that do not involve quantitative reasoning in order to contrast these meanings with those that do entail quantitative reasoning. Also in relation to an individual's interpretation of a task, we use the phrase *context of a task* to be how an individual interprets the situation in terms of activity to be done (van Oers, 1998). We recognize that individuals might interpret a situation in terms of measurability, yet we acknowledge that individuals may not necessarily consider quantity when working on a task designed to support and/or investigate students' quantitative reasoning. The idea of context also emphasizes our stance that quantitative reasoning is not an inherent feature of a task. No matter how strategic a researcher or teacher is in designing a task, individuals' interpretations of the task will vary from individual to individual and quantitative reasoning is never a given.

Consider the following example: a researcher investigating how students conceive relationships between two co-varying quantities might design a series of tasks that prompt students to reason about co-varying quantities in different contexts and representations. After implementing the tasks, that researcher might form hypotheses about students' reasoning that inform subsequent task design for research and/or instruction. For instance, the researcher might conjecture that the students did not reason about co-varying quantities when approaching the tasks. Based on this finding, the researcher forms hypotheses of how the students were reasoning and designs subsequent tasks to perturb such meanings with the intention of promoting quantitative reasoning. The purpose of this example is to illustrate how we consider task design to be an emergent and evolving activity, such that task implementation informs subsequent task development. From our perspective, a task does not exist independently of the individual(s) designing or working on the task. The task is whatever an individual or group of individuals makes of the task, whether those individuals are teachers, researchers, and/or students. For this reason, research is a critical aspect of task design, as research enables educators to develop tasks that are informed by students' or teachers' activities and interpretations of the tasks.

Research Questions

A central question for the QRaMM research team is: How do students reason quantitatively and what impact does that reasoning have on their learning in mathematics and science? For the 2013 Working Group, we address the design and implementation of tasks to support students' quantitative reasoning. The research questions below will provide focus for discussions taking place during the 2013 Working Group. When we use 'students' in the research questions below, we intend for 'students' to include secondary students, university students, and/or prospective teachers. When we use 'teachers' in the research questions below, we intend for 'teachers' to include both prospective and practicing teachers.

- How do students' interpretations of the context of a task influence their quantitative reasoning?

- How do students' and teachers' perspective of the nature of mathematics and science (e.g., mathematics is about rules and procedures) influence their quantitative reasoning and their interpretation of tasks?
- How do models of students' quantitative reasoning inform the teaching and learning of mathematics and science, particularly in relation to task design?
- How does teaching focused on quantitative reasoning support students' development of a robust understanding of mathematical and scientific concepts?
- How do teachers develop and engage in practices that support their use of tasks to engage students in quantitative reasoning?

Plan for Working Group

The three QRaMM working group sessions will actively engage the participants in discussion of quantitative reasoning and task design focused by the above research questions. Workgroup sessions will address both theoretical and practical considerations related to the design and implementation of tasks to investigate students' quantitative reasoning and support students' use of quantitative reasoning to develop understanding of mathematical and scientific concepts. Below is our plan for each session:

Session 1

- Overview of QRAMM Working Group (Heather Lynn Johnson and Kevin Moore) – 15 minutes
- Presentation 1: Designing tasks to perturb pre-service teachers' meanings (Kevin Moore) – 10 minutes
- Presentation 2: Establishing a power meaning of multiplication using combinatorics problems: A quantitative approach (Erik Tillema) – 10 minutes
- Whole group discussion – 10 minutes
- Breakout sessions with subgroups addressing issues and questions – 30 minutes

Session 2

- Presentation 1: Designing tasks to assess quantitative reasoning in the context of environmental science (Robert Mayes and Franziska Peterson) – 10 minutes
- Presentation 2: Designing task sequences to support students' quantitative and covariational reasoning (Heather Lynn Johnson) – 10 minutes
- Whole group discussion – 10 minutes
- Breakout sessions with subgroups addressing specific research problems – 30 minutes

Session 3

- Presentation 1: Designing modeling activities to develop students' quantitative reasoning (Eric Gaze) – 10 minutes
- Whole group discussion – 10 minutes
- Breakout sessions with subgroups addressing collaborative efforts – 30 minutes
- Whole group discussion – synthesizing breakout group discussions, looking ahead to the future – 25 minutes

QRaMM Follow-up Activities

We will sustain the efforts of the working group through the WISDOM^e research initiative, in collaboration with the NSF Pathways Project. We envision potential outcomes of this working group to be conference papers and/or publications involving quantitative reasoning related research. Each of the previous working groups has supported and fostered continuing

collaborations among individuals doing research in the area of quantitative reasoning, including the International STEM Research Symposium held in Savannah. We expect a similar outcome from the proposed working group, particularly in relation to the pressing need of better understanding the interplay of the contexts used in designing tasks and quantitative reasoning.

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CLASSROOM MATHEMATICS DISCOURSE: BROADENING PERSPECTIVES BY INTEGRATING TOOLS FOR ANALYSIS

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This working group explores tools for analyzing mathematics classroom discourse across two projects with different, but complementary perspectives. The goals of the working group include generating interaction about the theoretical lenses that we use to analyze and discuss classroom mathematics discourse and the relationships between these different theoretical frameworks. Participants will engage with the individual frameworks in the first two sessions and discuss interactions of the two frameworks in the third session.

Keywords: Classroom Discourse, Instructional Activities and Practices, Teacher Knowledge

History of the Working Group

This working group focuses on the overarching question, *What are the relationships between various tools and frameworks for analyzing classroom mathematics discourse?* The question emerges from two perspectives: the general proliferation of theoretical frameworks for analyzing classroom mathematics discourse, and specifically the interactions between two research groups interested in determining interesting intersection points between their two analytical frames. The group does not have an existing history at PME-NA, but stems from three prior working groups: the *Mathematics Classroom Discourse* (Herbel-Eisenmann, Staples, Choppin, & Wagner, 2005-2007), *Investigating Mathematically Important Pedagogical Opportunities* (Leatham, Stockero, Van Zoest, & Peterson, 2010), and *Measuring Instruction in Relation to Curriculum Use* (Kim, Remillard, Steele, Blunk, Piecham, & Lewis, 2012). We model the organization of this proposed working group on that of the 2012 group, which brought together researchers from three projects to engage participants in conversations around their frameworks for measuring curriculum use, analyzing data using each framework, and having a cross-cutting conversation about the affordances and constraints of each approach.

Our working group brings together researchers from two projects interested in capturing important interactions in mathematics classrooms and describing the ways in which these interactions might represent generalizable, visible patterns that move a teacher's mathematical and social goals forward in interesting ways. Such interactional moments would have great potential as common sites of inquiry and as tools for planning and reflection by teachers, teacher educators, and educational researchers. The Leveraging MOSTs: Developing a Theory of Productive Use of Student Mathematical Thinking Project (MOST) is an NSF-funded collaboration between researchers at Western Michigan University, Brigham Young University,

and Michigan Technological University that aims to support teachers in identifying and leveraging important instances of students' mathematical thinking. The project focuses on the identification of MOSTs: **M**athematically Significant Pedagogical **O**penings to Build on **S**tudent **T**hinking and progressively investigates MOSTs across four contexts: student thinking, teacher interaction with student thinking, teachers' learning about student thinking, and shareable products for teacher learning.

The Mathematics Discourse in Secondary Classrooms (MDISC) Project is an NSF-funded collaboration between Michigan State University and the University of Delaware to develop a set of case-based professional development materials focused on secondary mathematics classroom discourse. Specifically, the materials to support teachers in becoming more *purposeful* about fostering *productive* and *powerful* discourse in the mathematics classroom. *Productive* discourse focuses on the ways in which teacher-student and student-student discourse moves the mathematics forward in the classroom. *Powerful* attends to the ways in which discourse can help further and attend to social goals and positioning. At the heart of the materials are six Teacher Discourse Moves (TDMs), which are identifiable discourse tools that can serve to structure productive and powerful discourse in secondary classrooms. The theoretical concepts of mathematics register and positioning are used as lenses through which to interpret what happens when one uses the TDMs.

Both the MOST and MDISC projects have developed conceptual frameworks for identifying specific classroom discourse moments that appear to be powerful and have the potential to influence student learning. In this working group, each project will briefly introduce their theoretical framework to participants and invite engagement with the framework through the collective analysis of classroom episodes. Following this deep engagement with each project's framework, the working group will facilitate a conversation comparing the affordances and constraints of the frameworks, identifying interesting intersection points between them, and more broadly discussing the importance and challenges of identifying and analyzing classroom mathematics discourse, with a particular focus on secondary classrooms.

MOST Project Overview

The MOST project focuses on the work of facilitating and researching teachers' mathematically-productive use of student thinking. We developed the MOST construct based on characteristics of "teachable moments" that emerged from the literature (Davies & Walker, 2005; Davis, 1997; Jaworski, 1994; Schoenfeld, 2008)—student thinking, significant mathematics and pedagogical openings. We define these characteristics in the following sections.

Student Mathematical Thinking

Because the MOST construct is designed to help articulate productive use of student mathematical thinking, we begin by defining what we mean by *student mathematical thinking*. Foremost, the thinking underlying a MOST must come from a student. In addition, the thinking must be mathematical. Although we recognize our inability to access directly the thoughts of students, we make inferences based on our observations of what they say and do. Thus, when we use the phrase *student mathematical thinking* we refer to observable evidence of student mathematical thinking, which we define as any instance where a student's actions provide sufficient evidence to make reasonable inferences about what they are thinking mathematically. In the classroom setting, this evidence most commonly is visible in actions such as verbal utterances, gestures, or written work (including on the board).

Mathematically Significant

In order to be a MOST, the mathematics in an instance must warrant use of limited instructional time; that is, it must be what we call *mathematically significant*. We use the term *mathematically significant* in the context of teachers engaging a particular group of students in the learning of mathematics. Specifically, mathematically significant instances contain mathematical ideas that, when they become the object of discussion, can be used to further the students' understanding of mathematics. To begin the mathematically significant analysis, we formalize the student mathematical thinking to articulate the mathematics of the instance (MI)—the important mathematical idea to which the student thinking is related. We then consider the instance in relationship to two key criteria: (1) the appropriateness of the MI for the mathematical development level of the students, and (2) the centrality of the MI to the mathematical goals for the students.

Meeting the *appropriate mathematics* criterion requires two things. First, the MI must be accessible to the students given their prior mathematical experiences; they must have adequate background knowledge to engage with the mathematical idea. Second, students at that mathematical level would not be expected to have mastered the MI. If they had, pursuing that idea would not likely further their understanding of mathematics. The *central mathematics* criterion requires that the MI be related to a central mathematical goal for student learning in that class. The mathematical goals for the classroom encompass both mathematical content and mathematical practices. A goal can meet the centrality criterion by being a lesson goal or by being a broader mathematical goal that is central to the discipline of mathematics. The further a goal is from the lesson, the more central it has to be to the discipline of mathematics to meet this criterion.

Pedagogical Opening

Conscientious teachers continuously seek evidence of their students' engagement with a wide variety of instructional goals. They take cues from actions big and small, making adjustments and pushing students to elaborate, explain and justify their thinking. Not all student actions, however, are "critical moments" (Walshaw & Anthony, 2008, p. 527) that create "potentially powerful learning opportunities" (Davis, 1997, p. 360). In the interest of differentiating student actions that meet this higher threshold, we define *pedagogical openings* as observable student actions that provide compelling opportunities to work toward an instructional goal. Determining whether an opening has been presented requires considering both the *positioning* and the *timing* of an observable student action. Building on the notion from the discourse analysis literature (e.g., Davies & Harré, 1990), we define *positioning* as the way in which an observable student action positions that student with respect to the content of an instructional goal. Students are positioned well with respect to an instructional goal if their action has a "deep" connection with the content of that goal as opposed to remaining "at a surface level." Whereas good positioning is determined by a particular student's engagement with the content of an instructional goal, good *timing* is determined with respect to the preparation of other students in the class to engage with the idea being raised in ways that support, rather than supplant, overall instructional goals.

Putting the Theory into Action

When determining whether a MOST has occurred, the focus of our analysis is an "instance"—an observable student action or small collection of connected actions (such as a verbal expression combined with a gesture). Typically an instance is one conversational turn or physical expression (such as writing a solution on the board), but it can involve multiple turns.

Determining whether an instance qualifies as a MOST involves a systematic analysis of whether the instance embodies the three MOST characteristics (see Figure 1); if any criterion is not met, the analysis ends and the instance is not a MOST. The analysis begins with questioning whether the students' mathematics can be articulated. Focusing first on this characteristic stems from the perspective that what students say or do during a lesson is critical and should inform the teacher's actions. If the students' mathematics can be articulated, it is formalized into a statement of the MI. The MI is then analyzed to determine whether the instance is mathematically significant; that is, whether it satisfies the appropriate and central mathematics criteria.

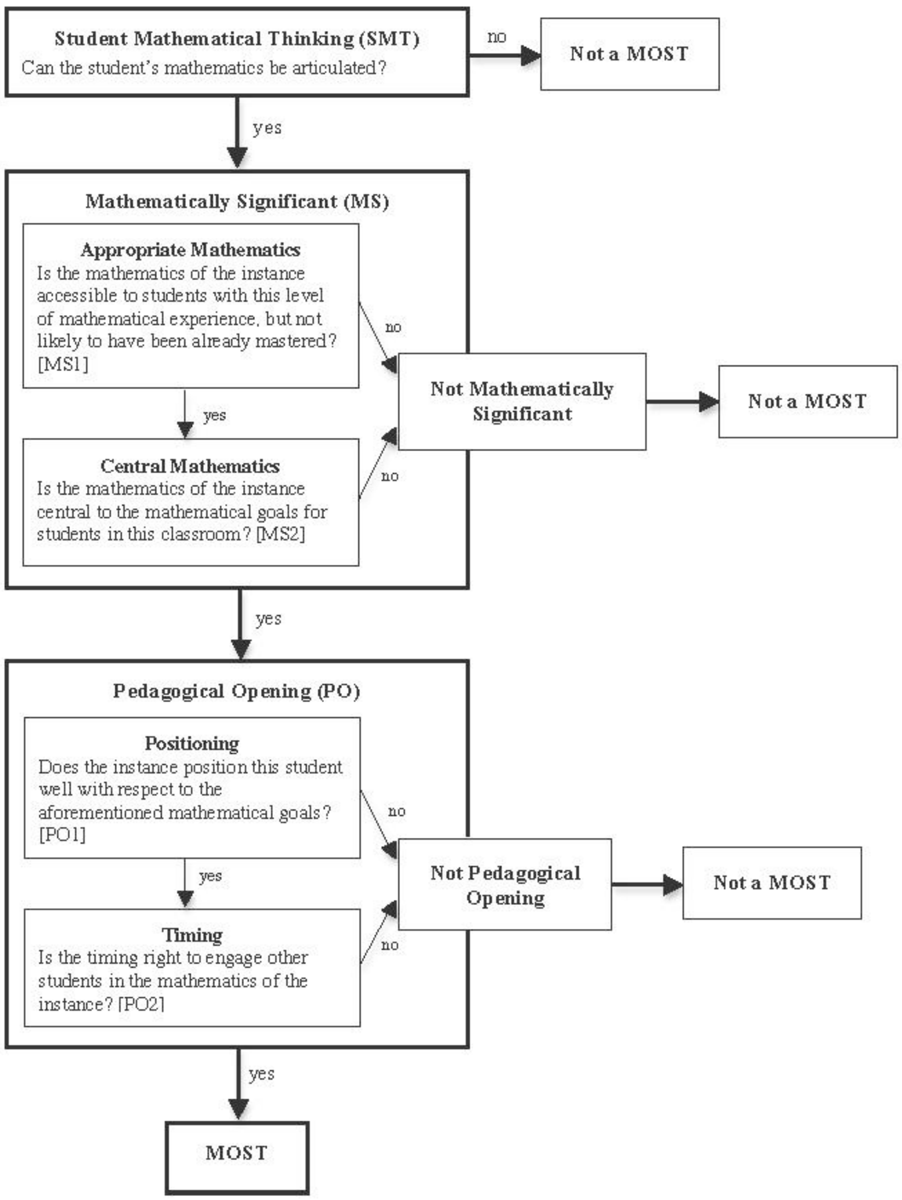

Figure 1: Analysis Process for Determining Whether a Classroom Instance is a MOST.

This mathematical analysis of the instance distinguishes our work from more general work on classroom discourse or even "teachable moments" in that we focus on instances that are likely to advance students' development of mathematical ideas. If an instance is mathematically significant, it is analyzed in terms of whether the positioning and timing are right to create a pedagogical opening. If so, the instance has met the criteria for all three characteristics and is deemed to be a MOST. We have found that taking this flowchart approach to the analysis of an instance brings structure and simplicity to an often chaotic and complex task.

Conclusion

By clearly defining three critical characteristics that distinguish instances that provide high-leverage opportunities to advance students' mathematical understanding from those that do not, the MOST construct has the potential to become a tool to make sense of classroom interactions. In particular, the construct provides both a means for systematically analyzing instances of classroom discourse and a vocabulary for discussing the mathematical and pedagogical importance of student thinking that arises within such discourse. Engaging in this analysis provides a mechanism for researchers and teacher educators to frame teachers' practice in terms of their use of high-leverage instances of student mathematical thinking. This framing shifts the focus of the work from *whether* a teacher is using student thinking, to *what* student thinking a teacher is incorporating into a lesson and *why* that incorporation is valuable.

MDISC Project Overview

The Mathematics Discourse in Secondary Classrooms (MDISC) project seeks to provide tools to secondary mathematics teachers to enhance the quality and power of their classroom mathematics discourse through the design of professional development materials. The set of tools at the core of the materials are the *Teacher Discourse Moves* (TDMs), revised from Chapin, O'Connor, and Anderson's (2009) construct of talk moves, extensively researched and disseminated at the elementary level (see Herbel-Eisenmann, Steele & Cirillo, 2013, for additional detail on the TDMs). A suite of concepts from sociolinguistics and discourse studies situate the TDMs in ways that help teachers consider how their principled use can influence students' conceptions of the purpose of classroom mathematics discourse, their communication development, and their identities as mathematical learners and doers. These ideas illuminate aspects of discourse related to both productive and powerful discourse, where *productive* relates to mathematical goals and *powerful* relates to social goals.

Teacher Discourse Moves (TDMs)

A suite of six TDMs (see Table 1) are introduced as a part of the professional development materials, each serving a variety of roles in shaping classroom discourse related to mathematical and social goals.

These moves, used individually or in combination, can advance a teacher's mathematical goals (productive discourse), social goals (powerful discourse), or both. The frames of productive and powerful discourse help teachers analyze the ways in which their use of the TDMs influences students' opportunities to learn mathematics and their developing mathematical identities.

Table 1: The Teacher Discourse Moves

TDMs and Intended Purposes	Examples
Waiting can... ● Provide students with time to process a question or response ● Hold students accountable for thinking and doing mathematics ● Encourage broader participation	● [pause without saying anything] ● I want you to think individually, without saying anything yet. ● Think about this for a few seconds and write down any questions you have.
Inviting Student Participation can... ● Initiate a discussion ● Elicit multiple student perspectives ● Position a student as someone whose ideas are worth considering	● Who is ready to share their thinking? ● What do other people think? ● Does anyone have a question for <student>?
Revoicing can... ● Amplify or draw attention to an idea ● Provide more mathematically precise or correct language ● Mark the value of individual students' thoughts for exploration	● The way I'm interpreting what you are saying is.... . Is that what you meant? ● So, I heard you say two things: _____ and _____. ● For those of you who may not have heard, <student> was just saying... .
Asking Students to Revoice can... ● Amplify student ideas ● Encourage students to use more or less mathematical language or precision ● Position a student as someone whose ideas are worth considering	● Can someone else say that in his or her own words? ● What did you hear <student> just say? ● Can everyone look at what <student> just wrote on the board? I want someone else to explain her strategy.
Probing a Student's Thinking can... ● Allow the learner to transform, modify, or correct their contribution/thinking ● Assist students in further articulating ideas by elaborating or justifying ● Make a student's thinking available to other students for follow-up	● Why does that work? ● Can you say a little bit more about your thinking? I am not sure that we are all clear on what you are trying to say. ● Can you come up to the board and show us what you mean?
Creating Opportunities to Engage with Another's Reasoning can... ● Allow students to take someone else's approach and use it for yourself ● Allow students to disagree/agree with someone else's idea, approach, or explanation	● Do you agree or disagree? Why? ● Can someone continue with <student's> train of thought? ● In what situations do you think that <student's> method would be more efficient than the one we discussed yesterday?

Productive Discourse

Productive discourse supports students' opportunities to engage with mathematical content and mathematical ways of understanding. The Language Spectrum, which contains four Communication Contexts, helps teachers analyze their use of the TDMs in promoting productive

discourse. Communication Contexts describe the settings in which classroom mathematics discourse is produced (e.g., small group, whole class, written solutions, textbook excerpts). The Language Spectrum, developed from Gibbons' (2009) mode continuum, considers the relationship between communication context and expected characteristics of a text that would be produced in that context. For example, in small groups, a typical text features context-dependent language (e.g., here and here, this, it) and gesture. Students might use less formal language, such as, "on the top," or "under here." Their common small-group experiences facilitates shared understanding in this context. In contrast, textbook excerpts draw heavily on the mathematics register (Halliday, 1978; Pimm, 1987) the style of communication valued by the mathematics community. Human actors are rare; instead, mathematical objects or processes serve as the subject of sentences which are constructed in passive voice using timeless present tense. The textbook necessitates communication to an audience who is not present, which leads to language that is not reliant on immediate context. The examples given here are two of the set of four Communication Contexts described in the Language Spectrum. Increasing awareness of the relationship between communication context and the expected kind of text that is produced is important as teachers are more likely to evaluate student work as correct if it includes characteristics of the mathematics register (Morgan, 1998). It is important to note that, in this range of texts, one way of communicating is *not* better than another. Instead, the Language Spectrum illustrates how communication context affects the kind of language that students use, and, by extension, illuminates how important it is to provide communication contexts in which students can use mathematically complex language.

The Language Spectrum and Communication Contexts are important aspects of communication to make explicit because teachers and other students also have been shown to treat students differently depending on whether or not they consistently use features of the Mathematics Register correctly (e.g., Esmonde, 2009). So, it is important for participants to be aware of the particularities of the Mathematics Register and to openly support students' use of it.

Powerful Discourse

Powerful discourse attends to how students are positioned both socially and as knowers and doers of mathematics. Messages about how students are perceived by others, themselves, and what they come to understand about what it means to know and do mathematics are embedded implicitly and explicitly in the discourse. The idea teachers use to analyze how use of the TDMs influences powerful discourse is *positioning*, "the ways in which people use action and speech to arrange social structures" (Harré & van Langenhove, 1999). That is, students, teachers, and the content of mathematics are being positioned through the interactions in the classroom all of the time.

We introduce the concept of positioning in three different ways. Teachers are asked to attend to student-to-student interactions to consider positioning. For example, when other students say, "let's try that," or "that will never work," they are positioning one another or indicating one another's status (Featherstone, Crespo, Jilk, Oslund, Parks, & Wood, 2011). Teachers are asked to consider teacher-student interactions because these also position people in classrooms in various ways. Teachers are both *in* authority in their classrooms and *an* authority in their classrooms. Teachers are also asked to consider how the practice of mathematics is positioned through a consideration of the kinds of tasks, activities, process, and practices students engage in during mathematics class. For example, students might come to see mathematics as an individual endeavor or a collaborative one. Questions, such as, "Who is considered 'smart' in my classroom?," "Who is talking (the teacher, which students specifically)?," and "What kind of

mathematical practices (e.g., argumentation, explanation, just answers) do we engage in?" provide teachers with an opportunity to surface ideas related to positioning about their own classroom practices.

Similar to the roles of the Language Spectrum and the Communications Contexts for productive discourse, teachers can consider positioning as they analyze the ways in which their use of the TDMs might support students' identity development (both as people and as mathematical knowers and doers). These three primary concepts, along with the TDMs, comprise a set of analytical tools teachers can use to better understand, plan for, and implement rich classroom mathematics discourse practices.

Plan for Working Group Sessions

At the beginning of the first session Dan Heck of Horizon Research will briefly introduce the purpose of the overall working group. As a lead external evaluator on each of the NSF projects from which these tools are taken, Dan is positioned well to facilitate discussion about the potential advantages to analyzing the tools in tandem.

The first two sessions of the working group will then be organized around the tools the two projects have developed--the MOST project in the first session (Keith, Blake, Shari, Laura, Isaí and Lindsay) and the MDISC project in the second (Kate, Mike, and Beth). Each of these sessions will consist of a) a description of the framework for analyzing classroom mathematics discussion, b) a coding activity in which all participants use the framework to analyze an excerpt of classroom mathematics discourse, and c) a group discussion of participants' answers to the following questions:

1. What aspects of classroom mathematics discourse does this tool foreground and background?
2. What are the affordances and constraints of foregrounding and backgrounding these aspects of classroom mathematics discourse?
3. How and under what conditions could this tool be used by mathematics educators in their work, both as researchers and as mathematics teacher educators?

During the third session Dan will present summaries of the previous sessions' discussions. Participants will then engage in re-analyzing the excerpts from the previous two sessions with each tool. The aforementioned summaries and the coding experience will then be fodder for discussion around the following questions:

1. In what substantive ways are these two tools similar and different?
2. What are potential advantages or disadvantages to attempting to use these research tools in tandem?

Anticipated Follow-up Activities

We anticipate two types of follow-up activities as a result of the working group - additional activities on the part of each of the two projects, and further engagement of other researchers and research groups interested in analyzing classroom mathematics discourse. Through engagement in the working group discussions, each of the two project groups will refine their suite of tools based on the discussions across the three days. In addition, the two projects will identify intersection points in the two sets of tools and frameworks and continue dialogue about ways in which the projects can work together to better understand classroom mathematics discourse.

In addition, the discussions are likely to spark interest from other researchers in making use of and connections to the projects' tools and frameworks. The work of analyzing classroom mathematics discourse has been an increasing focus of research and practice in the last two

decades. The discussions in the working group will contribute to this research agenda by engaging participants in the consideration of two sets of tools that can be mobilized in a wide range of research and teacher education settings.

Acknowledgments

The MOST and the MDISC projects are supported by the National Science Foundation (grants DRL-1220141 and 0918117, respectively). Any opinions, findings, and conclusions or recommendations expressed in this article are those of the authors and do not necessarily reflect the views of NSF.

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EXPLORING MATHEMATICS METHODS COURSES AND IMPACTS FOR PROSPECTIVE TEACHERS

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This working group continues to develop a research program and scholarly inquiry focused on the study of mathematics teacher educators' (MTEs') practices and their impacts on preservice teachers (PSTs). The research agenda contains two strands of inquiry exploring (1) empirical links between PSTs' development and MTEs' practices and (2) variation in MTEs' practices and the evolution of methods course activities over time. Participants will discuss and dissect (a) existing literature illuminating the impact of methods activities on PSTs' development, (b) methods of documenting and exploring MTEs' practices and (c) the next steps to be taken in the development of the research agenda and the design of scholarly inquiry. Dialogues and collaboration among working group members will be encouraged. We will provide opportunities for individuals or teams to engage in the development of facets of the emerging research agenda.

Keywords: Teacher Education-Preservice, Instructional activities and practices

Background: Exploring the Content of Methods

The 2012 "*Content of Methods"* working group was established and began exploring one central question: "What does the research literature reveal about mathematics methods courses in terms of frameworks, activities, and residues?" (Kastberg, Sanchez, Edenfield, Tyminski, & Stump, 2012) . Central to our working group was a commitment to answering repeated calls to the field (Arbaugh & Taylor, 2008; Cooney, 1994; Lee & Mewborn, 2009; Mewborn, 2005) for research studies and a research agenda focused on informing and supporting the work of mathematics teacher educators (MTEs). In 2013, the working group will continue its explorations of methods courses framed by a research agenda developed through studies of existing activities conducted in 2012 and 2013 as well as reviews of scholarship investigating methods courses. Such work is focused on the empirical investigation of MTEs' practices and goals as they relate to preservice teachers' (PSTs') learning to teach mathematics and is grounded in the "psychological aspects of teaching and learning mathematics and the implications thereof" (www.pmena.org).

In the remainder of this proposal we identify how the working group has contributed and will continue to develop a research agenda for the content of methods. We first outline the history of the group including a summary of contributions to date. Second, we articulate a research agenda for the group, which draws from our group's prior work. Finally, we outline the working sessions and discuss the goals of methods courses as well as the potential impact of selected methods course activity-types on PSTs' learning and practice.

History of the Working Group

Prior empirical explorations of the content of methods (Harder & Talbot, 1997; Taylor & Ronau, 2006; Watanabe & Yarnevich, 1999) have uncovered some commonalities in activities used in mathematics methods courses, but have done little to move the field forward. Our research group was formed with the intent of moving toward a research agenda for studies of methods. Informed by the studies above, the working group leaders collected data from a voluntary survey of 79 MTEs, in which we asked respondents to identify 2-3 “important” activities from their methods courses. This survey differed from previous studies in that respondents identified important activities, rather than researchers inferring the importance of activities from course syllabi or providing respondents with *a priori* categories of important activities which they rated with respect to importance. Consistent with earlier findings, planning was identified as the most used activity in methods; 32 percent of the 205 “important” activities identified by MTEs were focused on planning. The survey results also suggested there is substantial variation across and within the activities MTEs use in methods courses. This result was consistent with earlier studies. However, the effect and meaning of this variation identified in all studies of methods is not clear. Taylor and Ronau (2006) suggested that further study of such variation is needed as MTEs do not understand the benefits of including or not including various activities within a methods course. The survey also identified activities used in methods not included in prior studies. One such activity focused on providing PSTs with opportunities to explore students’ mathematical thinking through the exploration of student work or by interacting with students. Such opportunities made up 10% of the activities MTEs reported in the survey. The identification of activities in the survey data not included in prior studies suggests that methods course activities may have changed as MTEs’ understandings of the complexity of learning to teach have evolved. Yet scholarship in mathematics education has not tracked or explored such changes. These investigations suggest that further investigation into MTEs’ practices and the effect of such practices on PSTs’ development is needed to inform and support MTEs as they design, enact and investigate practices for a methods course, as well as to document the variation in and evolution of practices used by MTEs.

Our initial discussions of existing literature and the examination of the survey findings resulted in a presentation at the Association of Mathematics Teacher Educators (AMTE) (Kastberg, Edenfield, Sanchez, & Tyminski, 2012). Participants engaged in active discussions of the frameworks and activities they used in methods courses and encouraged our development of a working group. Thus, session presenters organized a PME-NA 2012 working group to develop a research agenda for the exploration of mathematics methods courses.

Watson and Mason (2007) suggest a difference between tasks as conceived by teachers and the experiences of the students, as the intended purpose may be enriched or diminished in student experience. We differentiate between activities as they are experienced by PSTs during a course and more long-term effects we call *residue*. In a mathematics course, residue refers to the mathematics retained by students as a result of solving problems (Davis, 1992). In methods courses we define residue as evidence of the impact of an experience beyond methods courses. Experience and residue form the overall *impact* of an activity on PSTs. With impact on PSTs’ practice as our ultimate goal, the working group sessions were organized around the following “two threads of inquiry” (Kastberg, Sanchez, Edenfield, Tyminski, & Stump, 2012, p. 1264) for the exploration of methods courses: “Framework-Activity-Residue” and “Activity-Framework-Residue.” We suggested MTEs could take one of two approaches in their design of a methods course. One approach is to begin with an overall theoretical framework, select and enact

activities that support the development of that framework, and examine the impact of these enactments on PSTs. The second approach is to select and enact an activity, employ a particular framework as a lens for the activity, and examine the impact on PSTs. Each approach could be an avenue for empirical investigation. The exploration of the Framework-Activity-Residue strand was motivated by Stump's (2012) presentation of her study of the frameworks MTEs use in developing and teaching methods courses. The exploration of the Activity-Framework-Residue strand was motivated by Tyminski's (2012) presentation of his study of one activity used in his methods course and the impact of this activity on PSTs. Discussion unearthed additional foci for future work including identifying learning goals, exploring how MTEs build instructional activities to achieve goals, and the possibility of building a repository including thick descriptions of activities MTEs use. At the conclusion of the final session, a Dropbox, including resources generated during the conference, and a contact list were created and attendees were invited to join.

A Research Program

MTEs have developed practices by drawing on their own experiences teaching teacher learners, through practical knowledge (Arbaugh & Taylor, 2008), and scholarly practices (Lee & Mewborn, 2009) adapted from literature on mathematics learning and teaching. Building from this research base, our 2012 discussions and work completed to explore current literature on the impact of MTEs' activities, Kastberg, Tymiski, and Sanchez (2013) have proposed two strands of a research program intended to further the knowledge base of mathematics teacher education. The first strand focuses on building the empirical basis for what Lee and Mewborn (2009) referred to as scholarly inquiry: explorations of "issues and practices through systematic data collection and analysis that yields theoretically- grounded and empirically-based findings" (p. 3). In this way, scholarly inquiry would support the creation of scholarly practices. MTEs' development, selection and enactment of activities should be informed by research that reveals links between MTEs' instructional practices and PSTs' development. We identify reports of scholarly inquiry exploring such links as contributing to the *learning strand* of the proposed research program. Inquiry focused on the development of PSTs' pedagogical concepts (Simon, 2008) or on PSTs' development of ambitious practices (Kazemi, Franke, & Lampert, 2009) are examples of research that would contribute to the learning goal.

The second strand involves inquiry into the variation in MTEs' practices. Reports describing methods activities must be created and disseminated in significant detail in order to allow for replication by other MTEs wishing to foster similar outcomes, and more importantly, enabling researchers to look across reports and gain insights about the evolution of MTEs' practices and to explore the variation in activities used in methods. The purpose and limitations of current research venues prohibit this level of detail. We identify this as the *landscape strand* of the research program. Reports of the development and descriptions of MTEs' activities would involve theoretical frameworks and perspectives held by the instructor, discussions of iterations of the activity enacted, and decision points in the construction of the activity. Enabling access to this information is paramount to the development of scholarly practice in mathematics education.

Progress Since 2012 Working Group

Sanchez, on behalf of the working group, presented a session at the 2013 annual AMTE conference "*Building a Theoretically-Grounded Practice of Methods Instruction.*" All active members of the 2012 PME-NA working group were invited to attend. Attendees discussed outcomes of the PME-NA working group and findings from work completed by the working group leaders. This work included the identification of common activities used by MTEs and

findings from a review of empirical studies focused on those activities and published in the *Journal for Mathematics Teacher Education* (JMTE). To continue to build understandings of the empirical support for existing MTEs' practices, we completed a systematic review of select journals (*Journal for Research in Mathematics Education, Cognition and Instruction,* and *Educational Studies in Mathematics*) for articles focused on methods activities after the AMTE session. This effort was extended to the development of papers summarizing existing research focused on three activities used by MTEs: dynamic interactions, video cases, and task analyses. Findings were presented and discussed at the 2013 National Council of Teachers of Mathematics' Research Presession during a symposium titled "*Building Scholarly Inquiry and Practices for Mathematics Methods Courses.*" Dr. Denise Spangler served as the session's discussant, providing critical feedback on the ideas from the papers and the significance of the work. The research summaries, revised based on discussant feedback, will be used during the 2013 working group to stimulate questions and discussion about the learning strand of the research agenda. In addition these summaries will serve as a springboard for discussions of further studies that are needed. Brief overviews of the three summaries are provided below as context for the description of the proposed 2013 working group sessions.

Dynamic Interactions with Students

The reports exploring experiences of PSTs resulting from dynamic interactions with K-12 students and the mathematical thinking they generate were varied in their approach and illustrate that PSTs' development is progressive (Crespo, 2000) and linked to existing mental structures such as beliefs (Ambrose, 2004). These interactions may be synchronous, as in student interviews, or asynchronous, as in letter writing. Student interviews often employ various forms and structure and have been shown to impact PSTs' beliefs (e.g., Ambrose, 2004), interpretive listening skills (e.g., Jenkins, 2010) and questioning ability (e.g., Moyer & Milewicz, 2002). Letter writing, as a form of interactions with student thinking, provides PSTs with more time to examine student work and decide how to respond. Crespo's (2000) use of letter writing between PSTs and fourth grade students also addressed PSTs' developing interpretive listening skills as well as PSTs' emerging abilities to pose problems. Studies of letter writing at the secondary level include how such interactions support PSTs' abilities to develop cognitively demanding tasks (Kastberg, Tyminski, & Sanchez, 2013). Questions raised include: How should MTEs design and sequence interactions with students' mathematical thinking to impact pedagogical concepts? How might experiences for elementary PSTs be different from those designed for secondary PSTs?

Task Analysis

Although frameworks have been developed for tasks designed to be used with PSTs (Baturo, Cooper, Doyle, & Grant, 2007), there are no clear guidelines for MTEs regarding how to design activities that support PSTs' development and analysis of tasks. Teachers' use of tasks and the evolution of task demand during classroom discussions were initially highlighted by Silver and Stein (1996) as a result of findings from the Quasar project. As a result, attention to the cognitive demand of tasks has resulted in a focus on analysis of different dimensions of tasks in mathematics methods courses. PSTs' analyses and design of tasks have been linked to their content knowledge (Osana, Lacroix, Tucker, & Desrosiers, 2006) and mathematical play and exploration (Crespo & Sinclair, 2008) respectively. What is less clear is how analyzing and developing tasks might be framed to impact PSTs' future practice. In particular, how might MTEs support PSTs in ways that produce experiences with task design and analysis that encourage PSTs to attend to issues such as cognitive demand?

Video Case Analysis

Video case analysis as an activity in methods courses took many forms. Evidence from reports suggested that video cases can be used by MTEs to impact many dimensions of PSTs' development including their beliefs and views of teaching (Friel & Carboni, 2000), attention to student thinking (Masingila & Doerr, 2002), focus on links between teacher actions and student learning (Santagata, Zannoni, & Stigler, 2007), reflective stance (Stockero, 2008), and professional and mathematical norms (Van Zoest, Stockero, & Taylor, 2012). The use of video cases provides opportunities for PSTs to revisit teacher actions and student utterances and to build meanings for what they see in interactions with colleagues. The impact of this activity depends largely on the goals of the MTE in selecting and structuring the activity as well as how the activity is implemented. Two explorations of the impact of video case analysis identified residue from early use of the activity in methods courses (Stockero, 2008; Van Zoest, et al., 2012). These reports are of significant importance to MTEs planning methods instruction with the intent to impact PSTs' future practice.

Plan for PME 2013 Sessions

Although the review of existing literature provided insights that can be used by MTEs in selecting and implementing activities in methods, there is a need for more inquiry in this area. For example, MTEs' use of planning activities suggests that scholarly inquiry exploring these activities and their impacts (experience and residue) is needed. Questions about how best to structure lesson plans have emerged from existing literature (Zazkis, Liljedahl, & Sinclair, 2009). Further exploration of variations within activities used by MTEs is needed. To continue the effort begun in 2012, the 2013 working group sessions will introduce new attendees to the essential question of the group: "What is the content of methods?" and will provide a brief history of the progress of the group (Figure 1). In addition, members of the group will present findings from summaries of research reports contributing to the learning strand and will share draft versions of reports MTEs could potentially create to contribute to the landscape strand. These presentations will be used to introduce the strands of the research program and to motivate attendee discussion.

The 2013 working group will focus on two guiding questions:

1. How do we envision and implement scholarly inquiry designed to explore the nature of the residue for a particular framework and/or activity?

This question focuses on understanding the experiences PSTs bring to and develop in methods and how they interact with experiences in other facets of teacher education programs and teaching experience. MTEs' roles in teacher education programs uniquely position them to ask questions about the influence of empirical research in programs of study and to motivate the use of empirical study in program design.

2. How can we encourage MTEs to leverage their current practice in order to contribute to the development of scholarly inquiry for methods?

This question focuses on the exploration and consideration of MTEs' existing practices developed through experiences in local contexts, "practical knowledge" (Arbaugh & Taylor, 2008), or designed by drawing on research findings. MTEs should share, structure, and review reports of these practices. Developing a repository of MTEs' practices can facilitate a systematic exploration of the evolution and variation in such practices.

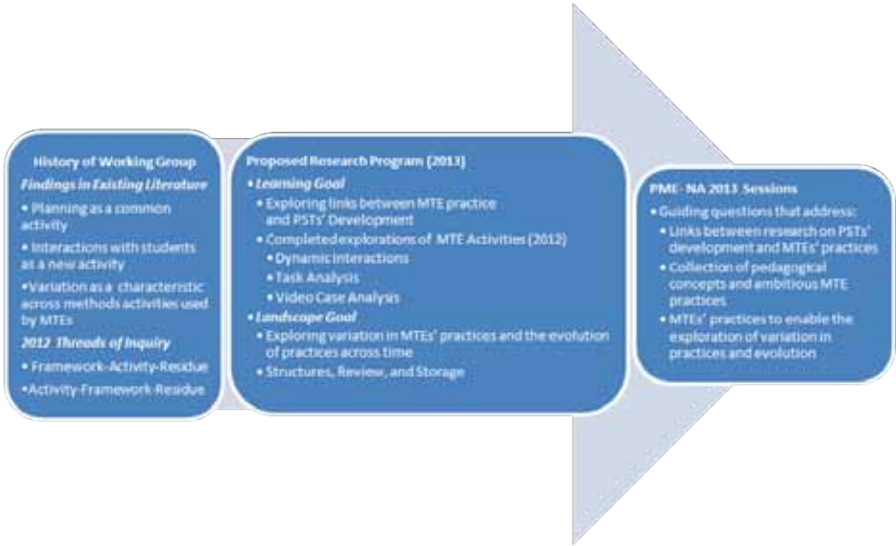

Figure 1: Evolution of Working Group Themes and Aims

Outlines of Working Group Sessions

Day 1. Our first session will include an introduction and overview of the working group (Figure 1). We will begin with a presentation describing the background and goals of the group. We plan to present the disparate knowledge base about methods courses and prior efforts to explore the content of methods courses. To orient the attendees to the discussion of the strands for the research program, we will share examples of reports that could contribute to the landscape strand and to the learning strand of the research program.

We address the landscape goal by presenting a report of one MTE's implementation of letter writing derived from scholarly inquiry (Crespo, 2003) describing the context and impacts of this interaction on PSTs. This report will be offered as an example of MTEs' reports of their scholarly practices. The characteristics of such reports and criteria for their review will be discussed (referred to as structures and review in Figure 1). In addition, plans for dissemination of such reports will be explored (referred to as storage in Figure 1). Attendees will then move to the learning strand. Pedagogical concepts (Simon, 2008) and processes such as routines to support ambitious practices (Kazemi, Franke, & Lampert, 2009) will be introduced to frame the discussion. The three research summaries described above of reports addressing the learning strand will be shared. Attendees will use these reports as a springboard to suggest additional lines of inquiry focused on linking MTEs' practices and PSTs' development. Attendees will be invited to contribute short presentations of their research or instructional activities on Day 3.

Day 2. The goal of the second session is to explore a conceptual plan for moving forward in addressing the learning and landscape strands. The session will begin with two presenters describing studies conducted with PSTs. The first study explored an assignment in which PSTs analyzed a video of their own teaching as reflection on the type of discourse the PST generated with students. The second study examined PSTs' capacity for lesson planning. In the study, PSTs wrote lesson plans using three different modes of production, namely synthesizing, creating, and modifying, and the study compared the quality of the lessons PSTs delivered. Noticing curriculum design as a primary topic in mathematics methods courses, the study questions whether or not curriculum design needs to be an essential part of the methods course. These two studies will serve as seeds for the ensuing discussion about the learning and landscape strands for the proposed research program. Discussion of the learning strand will be focused

with questions such as: What pedagogical concepts or ambitious practices are these studies exploring and are those pedagogical concepts or practices central to a methods course aligned with reform-oriented teaching? What pedagogical concepts or ambitious practices should be included in a methods course aligned with reform-oriented teaching? How do we prioritize the pedagogical concepts or ambitious practices to address the needs of beginning teachers? Discussion of the landscape strand will be focused on the question: What information do MTEs need from these studies in order to make use of these results in their own practice?

Day 3. Time will be allotted in our third session for working group participants to share their own methods activities and discuss scholarly inquiry related to these activities. In addition, the group will discuss what information would need to be provided about presented activities and their enactment for MTEs to make decisions about implementing the activity in their own methods practice.

The working group leaders intend to seek funding to underwrite a conference aimed synthesizing, supporting the development of, and disseminating scholarly inquiry focused on MTEs' practices in methods. On Day 3, participants will discuss the conference aims and make suggestions regarding participants and structures.

We will close out the working group sessions by discussing the development of appropriate venues, such as a repository, for MTEs' descriptions or reports of activities they use and their evolution. This topic was of primary interest to attendees in 2012. Time will be devoted to the development of a subgroup interested in pursuing the creation of a repository. The subgroup will be provided with sample reports drawn from research literature (e.g., Duffie, Akerson, & Morrison, 2003). In addition, we will discuss the logistics of follow up activities for the working group. The group will decide the best way to communicate and share files over the upcoming year (e.g., Google Group, Dropbox, etc.).

Anticipated Follow-up Activities

As we move forward in explorations of MTEs' practices our goal remains that of encouraging the development and dissemination of scholarly inquiry and practice within mathematics methods activities. Although there have been discussions about standardizing the preparation of teachers (for one discussion see, Richardson & Roosevelt, 2004) we view such movements associated with mathematics methods practices as premature. Instead, the working group seeks to understand MTEs' practices, impacts of practices, variations in practices, and evolution of practices. We claim that scholarly inquiry focused on experiences of PSTs in methods and the evolution of PSTs' understandings of those experiences across their careers has the potential to inform MTEs' practice. We further anticipate that the creation of publication venues that encourage MTEs to share the development and enactments of activities, with attention to frameworks, contexts (such as program size, student population, characteristics of space), critical decision points, as well as impacts, will support MTEs' development of scholarly practices and research explorations of variation within and between MTEs' practices.

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issues and next steps related to research on teaching and learning integers. This working group provides that opportunity.

The Working Group

This working group will come together to discuss the state of the field in relation to what we already know about the teaching, learning, epistemology, and history of integers. We will discuss a range of topics—including participants' own research related to integers—and seek to come to consensus on major research strands involving integers. We imagine an initial set of categories of integer-related research that includes

- Integers and the role of contexts (Stephan & Akyuz, 2012; Whitacre, Bishop, Lamb, Philipp, Schappelle, & Lewis, 2012)
- Integers and historical development (Bishop, Lamb, Philipp, Whitacre, & Schappelle, 2013; Gallardo, 2002; Hefendehl-Hebeker, 1991)
- Integers and algebra (Gallardo, 2002; Thomaidis & Tzanakis, 2007; Vlassis, 2002)
- Integers and preservice and inservice teacher content knowledge (Charalambous, Hill, & Mitchell, 2012)
- Integers and k-12 students' understanding (Bishop, Lamb, Philipp, Schappelle, & Whitacre, 2011; Bofferding, 2010)
- Integers and models used (Janvier, 1985; Liebeck, 1990)

The goal of the working group will be to identify, make connections among, and categorize the research studies in this area, to identify what "next" questions the field might address in relation to this topic, and to make a contribution toward what constitutes integer understanding. By coming together to share ideas, participants can help to situate the research strands in such a way as to establish a more unified and directed research agenda, rather than pursuing a more individualized approach whereby different groups work independently toward separate goals.

Plans for Three Working Group Sessions

Across the three sessions, participants will engage with facilitators about both the current state of the field and the state of the research literature through brief presentations, and will have many opportunities for debate, discussion, critique, and analysis. In order to build on the emerging knowledge base, we will begin the first two sessions with brief presentations, followed by extended discussion. We will reserve substantial time for discussion during the third session. We describe each session below.

Session 1: Research About Children's Understandings and Classroom Interactions

Session 1 will begin with a brief overview of the literature (by Lamb and Bagley) in order to situate the efforts and goals of the working group across the three sessions. However, the overriding focus of Session 1 will be on research about children's understandings of integers and classroom interactions. To initiate and sustain worthwhile discussion, facilitators will share their research in three brief presentations and will identify what they believe are the next steps in their particular line of research. The presentations will serve as a springboard for identifying existing research in the field and making connections across studies, to support the group in collectively categorizing current research strands. Facilitators will briefly share their research as indicated in the next section.

Constructive and limiting roles that contexts play in students' integers development, Michelle Stephan. It is generally accepted that using real world contexts to ground students' mathematical activity can be a powerful instructional technique. In fact, one of the major tenets of the instructional design theory of Realistic Mathematics Education (RME) is that students' experiences should move from reasoning with concrete, real-world contexts toward more abstract mathematical activity, including modeling and symbolizing in personally meaningful ways. While RME argues that the initial contexts should be experientially-real to the students, there is much less guidance about what contexts are both experientially-real and also serve as didactically-rich scenarios that support the mathematics that students should learn. In this session, we will explore and discuss three issues related to real-world contexts. First, what are the characteristics of real-world scenarios that make them didactically rich? Second, what makes certain contexts limiting for students' mathematizations? Third, is it the context that is rich/limiting or is it the tasks posed within the context that are rich/limiting? Given that many textbook publishers are cashing in on "real world math", it is time that we explore instruction that does more than simply make word problems into "world" problems. The results of our session can have significant impact on what "real world" means in mathematics instruction, how to choose pedagogically rich contexts, and how to pose tasks within those contexts to maximize students' learning. We explore these issues by looking at various contexts that have been studied for integer instruction (e.g., assets/debts, disco halls, positively/negatively charged particles). Additionally, we share the results from a classroom teaching experiment that used finance as the context. Both the supports and constraints of the context will be described with an eye towards building a set of characteristics that can be used to choose supportive contexts.

Children's use of number lines in operating with integers, Laura Bofferding. Two important supports that elementary students use to make sense of which numbers are larger or smaller and to add and subtract positive integers are the forward and backward counting sequences (often represented by number lines or number paths) (Case et al., 1996). In this part of the session, Bofferding will explore the difficulties and successes first grade students experience as they transition to using counting sequences and number lines that extend into the negative integers. To highlight their reasoning about negative numbers and operations involving them, she draws on data (including videotapes) of the first graders' pre- and post-interviews and partner activities during an instructional intervention. Fifth graders' solutions to the same problems will be used to contrast different types of number line reasoning and lead to a discussion about effective uses of number lines for supporting integer reasoning.

Children's ways of reasoning, Jessica Bishop. Consider the problem, $3 - 5 = \square$. How might a child solve this problem? Would you expect different strategies from a 7th grader and a 4th grader? Drawing from a cross-sectional study of 160 K-12 students in US schools, we will share five broad categories of integer reasoning we have identified that children in grades 2, 4, 7 and 11 use when solving decontextualized number sentences. These categories are: order-based reasoning, analogical-based reasoning, computational reasoning, formal reasoning, and alternate/limited reasoning. In this presentation we will define these categories and show video clips as exemplars of particular ways of reasoning. For example, some children might approach $3 - 5 = \square$ using an order-based strategy, counting down five numbers starting at 3 and ending at negative 2. Others might answer 0 or not possible, arguing that 5 is larger than 3 so how can one remove more than one has or remove anything less than 0 (an example of what we call an alternate/limited way of reasoning). And still others might use a more computational approach by

computing the difference between 5 and 3, assigning the "negative" to the 2 and explaining the answer of -2 by saying that because 5 is larger than 3 so it has to be a negative solution. We will consider limitations and affordances of different ways of reasoning and discuss potential implications for instruction.

Guiding questions to spur discussion in Session 1 may include those below.

1. What are the common findings about children's understanding of integers?
2. What does the research say about integers' instruction?
3. What issues are absent from the research on children and classrooms as they relate to integers teaching and learning?

Session 2: Teachers' Understanding of Negative Numbers

During Session 2, the group will build on the discussion from Session 1, integrating the research and discussion from Session 1 with the research shared during Session 2. The focus of Session 2 will be on the research related to how teachers' understanding of negative numbers influences instruction. Two facilitators will share their work to open up a discussion. Similar to Session 1, at the end of each presentation, facilitators will identify what they believe are the next steps in their particular line of research. Below we describe the research each facilitator will share.

The relationship between teachers' understanding and teaching of negative numbers, Jennifer Lewis. This session examines the mathematical knowledge for teaching (MKT) that teachers have regarding integers. MKT includes common mathematical knowledge that all adults may have, but it also encompasses specialized knowledge unique to the demands of mathematics instruction (Ball, Thames, & Phelps, 2008). In this session, we analyze teachers' MKT specific to integers and we examine their pedagogical choices for teaching integers. Using interview and questionnaire data, as well as videotapes of classroom lessons, we consider the knowledge base needed for teaching integers and what kinds of metaphors, representations, and tasks teachers use and might use in effective instruction.

The role of teachers' content knowledge: One teacher's journey in coming to identify and sort out understandings related to negative numbers, James Brickwedde. The work presented here grows out of a classroom-based action research project that originated with first graders debating if -2 + -2 = -2 + 2 was true or false. These students then engaged with the same question as a group of fourth graders. The video transcripts of these sessions allow an analysis of a set of mathematical questions raised by the students among themselves as they grappled with articulating their thinking. An analysis of the questions and prompts used in the exchange reveal the teacher's own conceptions and misconceptions of integers. This interactive session will focus on one teacher's journey in coming to identify and sort out understandings related to negative numbers as evidenced by transcript sections from these student interviews.

Four overarching questions will guide the discussion during Session 2.

1. What does it mean to have MKT in relation to integers?
2. How is MKT for integers acquired by teachers?
3. What does the research say about representations, models, or contexts that are essential for teachers to have at their command to be able to facilitate discussions around integers with students?
4. What connections bridge research strands from Session 1 and Session 2?

Session 3: *Integer Sense* and Next Steps

During Session 3, Lamb, Bishop, Philipp, and Whitacre will share their thinking about what constitutes robust integers understanding (what they call *Integer* Sense) for discussion, critique,

and revision. They use this term to encompass the understandings, intuitions, metaphors, models, representations, and contexts one draws upon to reason about integers and integer addition and subtraction. This framework will provide another lens from which participants can organize and synthesize existing research and identify critical areas for future research. Initial ideas related to *Integer Sense* are shared below.

Integer Sense—A Conceptualization of Integer Understanding

We view the broad construct of integer sense not as something one either does or does not have but as something people hold with varying degrees. We thus conceptualize integer sense as existing along a continuum, with one end representing those holding *little* integer sense and the other end representing those holding *extensive, integrated, and rich* integer sense. The following list comprises our current notion of the enriched end of the continuum in relation to integers and integer addition and subtraction.

1. **Students demonstrate symbolic and representational fluency with integers.**
   a. Students interpret the sign – in three ways: as subtraction (a binary operator), the sign of a number (a non-operator), and negation (or the opposite of, a unary operator). Further, they can move flexibly among the three meanings. Similarly, students recognize the sign + as both addition and the (implicit and explicit) sign of the number and move flexibly between these meanings.
   b. Students appropriately use symbols to represent situations (we define the term “situation” broadly) that could be appropriately represented with signed numbers (e.g., write a number sentence to represent a series of movements on a number line; write a number sentence to represent a scenario or context; or use signed numbers to represent numbers of, for example, Happy and Sad Thoughts).
   c. Students can map a scenario or situation to a particular symbolic equation or expression. (for example, beginning with -6 + 5, create a scenario that corresponds to the expression).
2. **Students can compare and order signed numbers and can view their relative sizes in flexible ways**. For example, students conceive of numbers less than zero and can compare the relative sizes of numbers by, say, using a number line or a context. This use might provide evidence that students are using the ordinal and sequential nature of numbers. In addition, they can also imagine how -7 might be thought of as larger than -5 if one considers magnitudes (or absolute values) only.
3. **Students use appropriate domain specificity in their reasoning**. For example, students invoke negative numbers when solving problems such as $6 - \square = 8$, $5 + \square = 2$, or when comparing x and $-x$. We highlight this characteristic because some students who invoke negative numbers when solving some types of problems (such as $5 - 7 = \square$) do not invoke negative numbers when solving the first two types of problems listed above. Furthermore, when variables are introduced, many students incorrectly compare x and $–x$ by claiming that x is always larger than $–x$ because $–x$ represents a negative number.
4. **Students exhibit multiple conceptions of integers.** That is, they conceive of integers as inverses, locations and directed movements on a number line, directed magnitudes, embodiments of real-world phenomena, and members of an equivalence class. They may use a variety of tools for making sense of integer-related problems.
5. **Students choose appropriate and varied strategies for adding and subtracting on the bases of number choice and operation. That is, they flexibly select strategies to promote**

BUILDING ON THE EMERGING KNOWLEDGE BASE FOR TEACHING AND LEARNING IN RELATION TO INTEGERS

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Students' understanding of integers has major implications for their success in algebra. However, relative to other content areas, research about issues related to the teaching and learning of integers is limited. The members of this new working group seek to explore what is known in the field in relation to integers' teaching and learning and to identify next steps for continuing to build the research base on this topic. During the first two sessions, participants will draw from facilitators' sharing of their work in order to identify existing research in the field and collectively categorize research strands. During the third session, the group will discuss directions for future research, seek to build consensus about what constitutes a robust understanding of integers, and plan next steps for continued engagement.

Keywords: Number Concepts and Operations, Learning Progressions, Instructional Activities and Practices, Middle School Education

Integers and integer operations are difficult mathematics content for students to learn but are building blocks for success with algebra. For example, to navigate algebraic equations, students must perform algebraic procedures using additive inverses, which first come into play with the introduction of integers. An integral part of the middle grades curriculum (Common Core State Standards for Mathematics, 2010; NCTM, 2000), integers mark a transition from arithmetic to algebra in part because of their abstract nature (Hefendehl-Hebeker, 1991; Linchevski & Williams, 1999). Unsurprisingly, given their abstract nature, students have great difficulty operating on them (Gallardo, 1995, 2002; Lamb, Bishop, Philipp, Schappelle, Whitacre, & Lewis, 2012; Latuszek & Wasser, 2008; Reck & Mora, 2004; Vlassis, 2002). Furthermore, difficulties in algebra have been linked to a lack of integer understanding (Gallardo, 2002, Kidd, 2007; Moses, Kamii, Swap, & Howard, 1989). Students' struggles with integers appear to be robust; even those who have completed algebra courses are challenged by problems with negative numbers (Reck & Mora, 2004; Vlassis, 2002). Given the importance of understanding integers and the difficulties that students experience, the field needs to better understand the

efficient computations. For example, one may solve 5 + ☐ = 3 by first noting that this problem involves addition wherein the result is smaller than the first addend and thus the answer must be negative (what we call *inferring the sign*); further, because the difference between 5 and 3 is 2, the answer is -2. This same student may solve -18 + ☐ = 6 by noting that adding 18 to -18 is 0 and then adding 6 more would be 24. Next, this student might solve or explain -8 – -6 = ☐ by recognizing that -8 represents 8 negative ones, and that the student must subtract 6 negative ones to find a difference of 2 negative ones. And finally, this same student may use yet another strategy to solve 11 – 17 = ☐. The student may share that 17 – 11 = 6 and because 11 – 17 = -1(17 – 11), the answer to 11 – 17 is -6. (Of course, the student would also have to implicitly (or explicitly) recognize that 11– 17 = 11 + - 17 = -17 + 11 = -1(17 – 11) = -1(6) = -6.)

6. **Students can explain underlying conceptions related to adding and subtracting negative numbers, including a particularly important and difficult subset of problems**: addition problems in which the second addend is a negative number (e.g., 6 + -4 = ☐) and subtraction problems with a negative subtrahend (e.g., 7 – -2 = ☐).
7. **Students can accurately perform operations on integers**. For example, students can solve typical textbook problems such as 5 – -2 = ☐ as well as Open Number Sentences such as 6 – ☐ = 8.

We recognize that the statements above might be read either as *conceptions people hold* that constitute integer sense or as *behaviors that we might observe* in people who hold integer sense. If we focused only on the conceptions people hold, we would refer to that which people understand (whether or not we might be able to measure that understanding), whereas if we focused only on behaviors we might observe, we would refer to behaviors people demonstrate. We are interested both in presenting a view of integer sense and measuring people's conceptions; thus at times we use language suggestive of behaviors and at other times language suggestive of conceptions. This type of framework can serve to direct future research in ways that serve to move the field forward.

We will close by seeking consensus on major themes across the research, identifying gaps in the literature, developing future sites for research, and planning for continued engagement.

Acknowledgments

This material is based on work supported by the National Science Foundation (NSF) under grant number DRL-0918780. Any opinions, findings, conclusions, and recommendations expressed in this material are those of the authors and do not necessarily reflect the views of NSF.

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QUEERING, TRANS-FORMING, AND EN-GENDERING MATHEMATICS AND MATHEMATICS EDUCATION

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The purpose of this working group session is to launch a new working group focused on using lenses from queer, trans, and gender studies in mathematics education. As such, this working group actively and directly connects to the conference theme of "broadening perspectives on mathematics teaching and learning."

Keywords: Curriculum, Equity and Diversity, Gender, Standards

Rationale

Western mathematics was traditionally thought of as a subdiscipline of philosophy or the humanities with the goal of uncovering philosophico-mathematical truths. Over the centuries, however, it has come to be viewed more as a natural science, akin to something like physics. Both the philosophical perspective and the natural science perspective view mathematics as being outside the social realm; mathematics, so this view goes, is neutral, universal, uninfluenced by society or culture, and ahistorical. As D'Ambrosio (1999) notes, "During the first half of [the 20th century] . . . mathematics and mathematics teaching were considered to be independent of the sociocultural context" (p. 48). In fact, Skovsmose (2009) points out that "[i]t is precisely mathematics that is assumed to strip away all those elements that can be associated with subjectivity" (p. 68). These days, mathematicians rarely study philosophy of mathematics anymore. This view of mathematics as separate from the social realm has shaped mathematics education. Culture and the societal context of mathematics have not been a part of the standard mathematics curriculum - outside the scope of the discipline, so to speak..

This asocial view of mathematics is so prevalent that queer studies as a discipline has virtually ignored mathematics. The fields of mathematics and mathematics education have similarly overlooked queer perspectives. As mathematics educators have begun to recognize that mathematics and mathematics education are sociocultural practices, new critical and social constructivist perspectives have shifted the way educators think about teaching math. Insights from queer and trans* studies have the potential to further these trends and provide new insights for the fields of mathematics and mathematics education.

We are a group of education researchers, graduate students, pre-service teachers, and practicing classroom educators who believe that the time has come to queer mathematics and mathematics education. To "queer" is the verb form of the noun "queer" which in its original meaning "unusual, strange, or odd." When homosexuality went from being a practice to being an identity (due to the increasing administrative and medical apparatus that was put into place along with the changing imperatives of capitalism) a new set of slurs emerged to denigrate that identity and keep homosexuals in their place. So queer became one of the key epithets used against homosexuals. In the 1980s, however, two related movements, queer activism and queer studies began a long process of reclaiming that word as a term of empowerment and as a

descriptor for a methodological approach, respectively. Concurrent with that was the shift from the noun ("a queer") and adjective (e.g., "a queer idea") to the verb, "to queer" implying the act or process of making something queer. What does it mean to queer something? Charlie Glickman year, in his piece, "Queer is a Verb," elaborated:

> To queer something, whether it's a text, a story, or an identity, is to take a look at its foundations and question them. We can explore its limits, its biases, and its boundaries. We can look for places where there's elasticity or discover ways we can transform it into something new. To queer is to examine our assumptions and decide which of them we want to keep, change, discard, or play with. This becomes a practice in transcending the habit of settling for pre-defined categories and creating new ones. And even when we leave something unchanged, we have changed our relationship to it.

Queer is a way of questioning foundations, unearthing alternate or double-meanings, and reshaping things in a new way. It doesn't have a fixed meaning; queer studies prefers to keep "queer" as fluid and ever-changing, and adaptable to new situations and new scenarios. It is what linguistists and semioticians refer to as a floating signifier; a term and an idea that isn't fixed in content, but changes as it is used to queer different things in different contexts. "Queer" as a verb and queering isn't something that is restricted to the humanities; more and more, researchers in the social sciences and natural sciences are using queer as a way of approaching their work and their research citation as examples. We, as a working group, are taking this momentum in the field of education and in natural science to transform the way we think about mathematics and mathematics education.

Focal Issues

Since our working group is in the initial stages of formation, clarifying our purpose and beginning to delineate the focal issues of the working group are high priorities. An important launching question addresses how we view the aim of our group: What does it mean to queer/ en-gender/ trans-form mathematics and mathematics education? (En-gender being the verb of gender, trans-form being the verb of trans)

Another issue of high priority include building a sustainable support network: What kinds of connections can we find among the work that our members are doing? What kind of support can we provide and do we need from one another? What kind of support do we need from organizations and how can we bring about this support? How can we find others interested in our purpose and provide and connect them with this growing support network? More specifically, what actions can we take within the next year to begin building this network in a sustainable way?

In addition to these two broad issues, our working group has consulted our members-- consisting of practicing teachers, preservice teachers, MA and PhD students in education,, teacher credential instructors, and professors of education-- to see what kind of topics they are interested in exploring under this umbrella of queer. These specific focal issues are organized below into categories.

Gender

Each of the issues in this category consider gender beyond a normative, binary view. See, for example, Esmonde (2011), who discusses how research into gender in mathematics education either insists on boys as being stereotypically masculine and proposes having more problems about sports and cars and "boy" things or insist on girls as being stereotypically

feminine and wanting more of a soft mathematics and social interaction and group work in the study. Esmonde (2011) proposes more of a complex way of looking at gender in mathematics that doesn't rely on stereotyped categories and takes into account the fluidity of gender.

- *Queering gender in mathematics*-Trans inclusion in gender pipeline projects - Are trans people being considered when we say we want "women" in education? There was recently a project started in San Francisco to train and educate trans people in software engineering and computer programming - how can we study these projects and compile the best practices and find ways to include trans issues, people, theory, and perspectives in the agenda of the gender pipeline movement?
- *Gender Complex Math Education*--What does it mean to teach math in a way that takes a complex view of gender rather than a dichotomous view of gender? One working group member presented a paper at the *Teaching Math and Social Justice* conference in 2012 related to gender complex math education.
- *Trans studies perspectives in mathematics education*--Trans studies provide insights about gender that can benefit mathematics education. For example, what might it mean to consider gender beyond a dichotomous view when conducting research about gender and mathematics learning, attitudes, beliefs, identities, or affiliations?

Research

Each of the issues in the research category relate to queering or en-gendering research about mathematics education in some way.

- *Queering research methodology* - There's a whole set of issues about being out as queer in a classroom setting - and people who do not conform to standard gender and sexuality norms are not normally given access to school settings and classrooms to do research. Our perspective and presentation influences the kind of data we collect, and it's important to be self-reflexive about how that affects our work as queer researchers. One of the authors of this proposal presented at this topic at AERA 2010 in a session on innovations in qualitative methodology.
- *Queering citation formats* - Traditional APA style removes the first names of all authors - degendering and de-racing and de-culturing their identities through this erasure of authorial subjectivity in favor of a more neutral, genderless, cultureless presentation. Also, in what way does the use of "he" and "she" when citing works reinforce binary systems of gender?
- *Gender complex research* - Not simply reducing our research to the categories of "boys" and "girls", "men" and "women" but really looking at the complexity of gender - traditional gender roles, queer gender forms such as "butch" and "femme", the study of female masculinity as it plays out in mathematics education, and really seeing how people "do" gender - engendering as a verb instead of gender as a noun. One working group member presented a paper at PME in 2012 suggesting lines of research in mathematics education from a more gender complex perspective.

Curriculum, Resources, Representations

The issues in this category address queering mathematics curricula and resources. Educators or researchers might use a queer analytical lens to critique what is normative in each of these, and how inquiry/ inqueery might move beyond what is normative.

- *Queering the common core / Queering curriculum* - At AERA this year, one curriculum studies scholar was presenting a queer approach to the Common Core Standards in English and

expressed a desire to collaborate with anyone who would like to extend his approach and methodology to exploring the Standards in Mathematics. Another member of our working group has looked at the foundations of STEM education in the common core and looked at the ways that the agendas and requirements of capitalism are reflected in the rationale and justification for the mathematics standards and for reform mathematics.

- *Queering materials and resources available, images, texts* - why is it always heterosexuals that get pictured in texts and materials? how do you know this, can you tell someone is queer just by looking at them? if so, how... Why are there no queer sidebars? what does this mean? Why is homosexuality or gender nonconformity not discussed in discussions of historical mathematicians?
- *Queering representations:* for example, who makes it into textbooks - pictures, exercises, problems, etc; popular culture representations of mathematicians and math teachers/students. Do gays and lesbians make it into textbook problems? Transgender people? When problems discuss a family setting - What about two mothers or two father families? Families that have multiple generations at home? Gay men coparenting with lesbians? Who gets portrayed as mathematicians in media, and how is their gender and sexuality portrayed? An important issue here is the assumption of heterosexuality - when someone's homosexuality is unremarked upon in film, television, or books, they are assumed by readers to be heterosexual. But when they are portrayed as gay, what are the implications of that for readers' understanding of the text?) this is its own field of study

Pedagogy and Teacher Education

The topics in this category focus on pedagogical processes and perspec for example, is Alan Turing or Hardy portrayed as gay, what does it mean to portray them in that way in a particular context?

- *Queering math pedagogy/ Mathematical Inqueery*--How might a queer theoretical lens shape math pedagogy? One working group member has written articles and book chapters about what they term "mathematical inqueery".
- *Queering "Math Methods" (teacher education)*--How can teacher educators teach queer their math methods courses? What do queer and trans perspectives have to offer math teacher preparation courses? Are math teachers prepared to work with queer and trans students? One working group member has explored some of these issues when working with teaching candidates.
- *"Teaching math for social justice" and sexuality/ gender diversity*--A body of scholarship focuses on teaching math for social justice. How can educators teach math in ways that work toward social justice for LGBT folks? One working group member recently wrote an article suggesting a middle school project in which students analyze and collect survey data related to how often peers stand up for one another when they hear a negative comment about someone's gender expression.
- *Queering mathematical assessment*--One working group member recently presented a paper about queering assessment. A next step is to consider queering mathematical assessment.
- *Queering mathematics disability* - One working group member presented at the DC Queer Studies Symposium this year about what it would mean to queer mathematics disability and to rethink how we view difference when it comes to students who perceive numbers and mathematics different from the norm.

Mathematical Concepts and Content Areas

The topics in this category consist of specific mathematical concepts topics. The topics listed serve as a beginning place for considering what a queer perspective might offer mathematical content. For example, Elizabeth Freeman (2010) has explored the concept of time through using a queer lens and suggested the concept of chrononormativity. Working group members have considered in their research how this concept might be incorporated into mathematics education by getting students to rethink who makes it into historical timelines (and in what ways) and how time is used to structure their classroom experiences. In related fields, a physicist and feminist studies scholar named Karen Barad (2012) has considered what it might mean to queer physics and biology-- starting with queer animals and the hidden history of homosexuality in nature and eventually even contending that the atom itself is queer. Similar consideration might be given to the other mathematics concepts on the list, as well as other concepts not yet added to the list.

Queering:

- time
- infinity
- space
- topology
- knots
- numbers
- measurement
- statistics
- place value
- alternate number bases
- modular arithmetic
- composition and decomposition
- cardinality
- counting
- differentiation, integration
- zero
- irrational numbers
- rational numbers
- dimension
- binary
- polygons
- polyhedra
- spheres, tori, double tori
- gluing
- imaginary numbers
- complex numbers, the complex plane, complex analysis
- linear algebra

Plan for Working Group

Session 1: What does it mean to queer/en-gender/trans-form mathematics?

Our first session will provide the opportunity for group members to collaboratively explore

perspectives on the purposes of the working group. The working group has been actively working on coming up with these definitions during the preparation of this proposal; we hope to draw a lot of new participants into this conversation and re-work our shared definitions. Participants will have the opportunity to explore the session focus questions through discussion, and to review the list of focal areas and revise them based on their current understandings of their own research. From this, the participants will also brainstorm projects related to the specific focal issues. The working group will continue to develop our extensive list of participants' topical interests to further build our research network.

Session 2: How can we begin to build a sustainable network to support our work?

In the second session, the group will further develop our ideas for building and maintaining a sustainable network to support the ongoing work of group members. In addition to informal connections among participants and making accessible to one another potential research interests, the group will consider the types of support needed from various organizations and how to go about putting that support into place. For example, the group will consider questions such as the following: What kinds of changes would we like to see in organizations such as PME, PME-NA, NCTM, and AERA in order to enable our network to become sustainable? Might we want to consider forming affiliate groups or special interest groups within these organizations? What other strategies can we develop to foster sustainability?

Session 3: What does our activist agenda look like for next year?

In the third session, the group will develop more specific plans for the next year. What collaborative projects do we want to begin? What kind of product could we create collectively (for example, a queer math methods text or an anthology on queer mathematics)? What specific actions will we take to put our plan for sustainability in place?

Anticipated Follow-Up Activities

Follow up activities will entail enacting the plans developed in Session 3. These will involve engaging in a collaborative project or projects developed in Session 3. We will also take action to put our plan for sustainability in place to allow our supportive network to continue to grow. Finally, we anticipate proposing another working group session for PME-NA 2014 to allow for additional exploration of the focal issues.

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RESEARCH ON SECONDARY MATHEMATICS TEACHER PREPARATION: UNDERSTANDING HOW TEACHERS LEARN

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Research on secondary mathematics teacher preparation (SMTP) is limited in scope. Studies have focused on specific aspects of SMTP such as mathematical content knowledge (Ferrini-Mundy & Findell, 2001; Monk, 1994) or knowledge of the use of technology in a mathematics classroom (Lee & Hollebrands, 2008). However, it is time to focus our efforts on understanding the entire SMTP process in order to determine the effectiveness of SMTP programs and to allow research to move SMTP programs forward. The purpose of this working group is to promote a collaborative SMTP research agenda that can serve as catalyst for understanding and improving secondary mathematics teacher preparation. The goals of the group are as follows: 1) Initiate a SMTP working group interested in developing and researching innovative strategies for secondary teacher preparation; 2) Share SMTP structures and strategies in order to better understand SMTP at participating institutions; 3) Brainstorm innovative SMTP strategies and research possibilities; 4) Develop and share common research instruments and collect data across institutions; and 5) Develop a collective agenda for future SMTP work.

Keywords: Secondary Teacher Preparation, Teacher Knowledge, and Teacher Learning

History and Goals of the Working Group

The purpose of this working group is to promote collaborative research that can serve as a catalyst for understanding and improving secondary mathematics teacher preparation. The group will consist of those interested in how teachers learn, teacher knowledge, teacher preparation, and developing and testing innovative strategies for meeting the changing needs of our current and future teachers.

Although this group has never formally met, the organizers have informally discussed the need for a working group focused on secondary mathematics teacher preparation. The common theme we encounter is the fact that we are all implementing new and varied strategies for teacher preparation without an arena to collaboratively share ideas, develop research instruments, and to collect data on large numbers of subjects from multiple institutions. In addition, there is a lack of research involving secondary mathematics teacher preparation; the majority of the research in this area tends to be at the elementary level. The establishment of this working group would lead to the collaborative development of innovative strategies for secondary mathematics teacher

preparation and provide valuable information on the implementation of these strategies in a variety of institutional structures. The goals of the group are as follows:

- Initiate a Secondary Mathematics Teacher Preparation (SMTP) Working group interested in developing and researching innovative strategies for secondary mathematics teacher preparation.
- Share SMTP structures and strategies in order to better understand SMTP at participating institutions.
- Brainstorm innovative SMTP strategies and research possibilities.
- Develop and share common research instruments and collect data across institutions.
- Develop a collective agenda for future SMTP work.

Rationale and Relevance to PME-NA

Research on secondary mathematics teacher preparation combines several distinct areas within the psychology of mathematics education, which include, but are not limited to, understanding how students learn, how teachers learn, teacher knowledge, the impact of curriculum on students and teachers, why pre-service teachers persist or fail to persist to program completion, etc. A PME-NA working group unites individuals with expertise in these important areas of the psychology of mathematics for the common purpose of improving secondary mathematics teacher preparation. In addition, the goal of developing and testing innovative strategies for SMTP aligns well with the PME-NA theme of Broadening Perspectives on Mathematics Thinking and Learning. The following sections make an argument for the needs of a working group on SMTP identify the challenges that drive this important work.

Recent studies, such as *Rising Above the Gathering Storm: Energizing and Employing for a Brighter Economic Future* (NAP, 2007), have sounded the alarm over the shortage of qualified mathematics and science teachers and are calling for a significant effort to recruit and prepare additional teachers. Despite the recent economic downturn, the U.S. Department of Education (2011) still identifies mathematics and science as *high needs* fields, as of the academic year 2011-2012. This shortage of qualified mathematics and science teachers can be seen as eroding the scientific and technological building blocks that are critical to our economic leadership in a more competitive world. A primary recommendation of *Rising Above the Gathering Storm* (NAP, 2007) is to increase America's talent pool by vastly improving K-12 science and mathematics education. Unfortunately, recent studies such as the Teacher Education and Development Study in Mathematics (TEDS-M) suggest that future teachers in the United States receive relatively weak preparation, especially in mathematics (IEA, 2012). Thus, in order to improve secondary mathematics education, we need to have a concerted effort to develop and test innovative strategies for SMTP.

The challenges to SMTP are daunting. First, secondary mathematics education majors often have difficulty integrating their knowledge of mathematics and pedagogy in meaningful ways for effective instruction. They often see their methods and mathematics courses as discrete, unrelated topics, often taught in different colleges (Ball, 2000). It is common for future secondary mathematics teachers to take a series of mathematics courses designed for mathematics majors, with mathematics specific pedagogical courses tagged on to the end. Papick (2011) said:

> "They [teachers] should know how to represent and connect mathematical ideas so that students may comprehend them and appreciate the power, utility, and diversity of these ideas, and they [teachers] should be able to understand student thinking (questions, solution

strategies, misconceptions, etc.) and address it in a manner that supports student learning." (389-390)

Second, future secondary mathematics teachers often lack a deep understanding of the high school mathematics they will teach and the capability to meaningfully analyze that content (CBMS, 2012; author, 2003). Secondary mathematics education majors often see little, if any, connection between the mathematics of their university courses and the mathematics they will teach at the high school level (Thompson, 1992). Part of this student perception of non-connectedness may stem from the type of mathematics courses taken, more so than the number of mathematics courses in the program. Mathematics courses that pre-service high school teachers take are often designed to prepare students for graduate work in mathematics or careers in business rather than teaching (CBMS, 2012). Finally, secondary mathematics education majors have difficulty applying the lessons learned from university courses to the high school classroom (Frykohlm, 1996, 1999). The previously mentioned challenges are but a few of the many tasks facing the work of SMTP.

Although there is some agreement as to the issues and challenges of SMTP, "little consensus exists regarding the features of mathematics teacher preparation that promote teacher and student learning" (Steele & Hillen, 2012). Each of the challenges to SMTP listed above affords the members of this working group an opportunity to create and explore innovative approaches to SMTP. The development and testing of innovative strategies involving clinical experiences, integrating knowledge (Steele & Hillen, 2012; authors, 2013), capstone courses (Artzt, Sultan, Curcio, & Gurl, 2012; CBMS, 2012), the role of technology, and re-evaluation of the role of methods courses evolved out of the challenges of preparing future secondary mathematics teachers and has the potential to improve SMTP in our country.

Wilson, Floden, and Ferrini-Mundy (2001, 2002) examined the research on teacher preparation via a meta-analysis of the studies conducted on teacher education to that time. They examined research on the effects of subject matter preparation, pedagogical preparation, and clinical experiences on teacher effectiveness. In the studies examined for the meta-analysis, few focused on SMTP. Moreover, in some cases, findings from studies were found to be contradictory. With respect to pedagogical preparation, Wilson, Floden, and Ferrini-Mundy note that, "We found no research that directly assess what teachers learn in their pedagogical preparation and then evaluated the relationship of that pedagogical knowledge to student learning or teacher behavior" (Wilson, Floden, & Ferrini-Mundy, 2002, p. 193). The authors make several recommendations for research on teacher preparation. First, research on teachers' mathematical preparation should be connected to teachers' knowledge, skills, and practice in the classroom. Second, research on methods courses should aim to describe how what is learned in methods courses interacts with what pre-service teachers learn in their mathematics courses and clinical experiences. Finally, research should focus on the structure of pre-service teachers' clinical experiences and how those clinical experiences affect teaching. A PME-NA work group in SMTP can address the recommendations found in Wilson, Floden & Ferrini-Mundy (2001, 2002).

Organization and Plan for Engagement

The working group will be organized into a series of discussions that allow for both whole group interaction on general ideas that permeate secondary mathematics teacher preparation and breakout groups that allow participants to discuss particular ideas and strategies. We

acknowledge that our plan is ambitious and therefore may not be able to complete all planned activities. An outline of the discussions follows:

Discussion #1: Purpose and Goals of the Working Group on Secondary Mathematics Teacher Preparation.

The purpose of this discussion is to clearly define the vision of the SMTP working group. The initial goals stated in the introduction will be presented as a starting point, but the group will be given the opportunity to revise the initial goals, suggest additional goals and directions, and to reach a consensus.

Discussion #*2*: Sharing SMTP Structures, Strategies, and Challenges.

The purpose of this discussion is to allow participants to learn about different secondary mathematics teacher preparation structures and strategies. This dialogue will hopefully promote discussion and collaboration among institutions that are implementing similar strategies or are facing similar challenges. The initial discussion will be whole group and address the following questions: What is the current structure of your secondary mathematics teacher education program? What challenges do you face? What strategies/ideas/courses have you found effective in preparing secondary mathematics teachers? How do "we" provide the kinds of experiences necessary for preservice teachers to develop their mathematical knowledge and then transfer this knowledge into the teaching arena? This will be followed by two rounds of breakout sessions that will be determined by the content of the whole group discussion.

> **Breakout Round 1**: Divide institutions into common structures to continue a discussion of challenges.
> **Breakout Round 2:** Divide participants into groups that address a specific SMTP strategies/ideas/courses to discuss implementation.

Discussion #3: Identifying Foundational Aspects of SMTP to Examine as Teams.

Following the discussion of structures, strategies, and challenges the group will work to identify specific areas of SMTP (such as, but not limited to, capstone courses, clinical experiences, integrating knowledge) to examine. The working group will then form subsets around the specific areas of SMTP according to their research interests and/or situation at their university. Our initial plan is for this discussion to comprise a significant portion of the devoted working group time. The discussion will alternate between break out groups and whole group debriefing.

> **Breakout Round:** The subsets will discuss future directions for examining their particular area of interest in SMTP. Research strategies may be discussed and developed at this stage.

Discussion #4: Sharing/Developing Common Research Instruments

After participants have identified an area of focus within SMTP, we will then transition to the sharing and development of research instruments that may be useful for research on SMTP. Given that we want to be able to share strategies, it is our desire to make this set of instruments more uniform. The first step is to discuss instruments that the group is currently using to examine

SMTP issues. It may be the case that the group will identify the need for additional instruments to be developed. A discussion on how to proceed will follow.

Discussion #5: Developing a Collective Agenda

The final discussion will summarize the progress that was made during the working group and to establish a set of next steps. Given the large undertaking of this working group we do not anticipate completing all of our discussion prompts. However, we will conclude with this final discussion.

Anticipated Follow-Up Activities

The anticipated follow-up activities will partly be determined by the progress made during the working group. The main purpose of the group is to promote collaborative research that can serve as a catalyst for understanding and improving secondary mathematics teacher preparation. Hence, one of the major follow-up activities is to encourage and promote collaboration beyond PME-NA. This will be encouraged by the working group organizers in multiple ways through: a) email exchanges & interactions via social media, such as Facebook or Google +; b) establishing a means (such as a Dropbox or other file sharing mediums) to exchange instruments, curricula, etc.; c) informal gatherings at other mathematics education meetings; and d) hopefully the continuation of the PME-NA working group.

In addition, a future goal we have for the working group is to promote grant writing to help fund this important work on SMTP. To this end, we envision having future discussions via email, or future working groups, that provide information regarding potential funding sources and assistance in writing and submitting grants. Along these lines we are currently exploring the option of writing an NSF grant to fund a conference focused on innovative strategies for secondary mathematics teacher preparation that would extend the work, and provide further assistance, to the PME-NA working group.

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COMPLEX INSTRUCTION WORKING GROUP: INVESTIGATING COMPLEX INSTRUCTION IN MATHEMATICS TEACHER EDUCATION

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Complex instruction (CI) is a pedagogical framework that provides tools to create equitable mathematics classrooms. Based on the assumption that participation is essential to learning, CI considers how status, or rankings of people based on perceptions (and not actual assessment) of competence, prevents access to activity. CI also provides resources for addressing status, including supports for helping students value each other's diverse intellectual contributions. While researchers have examined the role of CI in students' mathematical learning, we have not yet adequately explored how teachers, including mathematics teacher educators, come to understand, implement, and teach about CI. The purpose of this working group is to take an initial step into this relatively unknown territory by using CI to first extend our own understanding of the pedagogy and then support collaborative efforts to examine how mathematics teacher educators, preservice teachers, and inservice teachers come to make sense of and implement CI strategies in their K-12 classrooms.

Keywords: Equity and Diversity, Teacher Education-Inservice/Professional Development, Teacher Education-Preservice, Instructional Activities and Practices

Mathematics educators have developed and explored many theoretical and pedagogical pathways for analyzing and achieving equity in mathematics education for marginalized students. One successful, but relatively unknown, equity tool is the pedagogical framework of *complex instruction* (CI). CI is a set of instructional strategies founded in the research of sociologist Elizabeth Cohen (1994) that explicitly attend to and address inequitable participation in the classroom, especially when students work in cooperative groups. Researchers have documented high levels of mathematical achievement for marginalized students in CI classrooms (Boaler & Staples, 2008). In spite of this success, CI has not become widely used in mathematics classrooms. However, this situation is beginning to shift as mathematics teacher educators (MTEs) use CI in their work with preservice and inservice teachers (e.g. Featherstone et al., 2011; Fede, Mojica, & Civil, 2013). As more MTEs take up CI, it's important to understand what support MTEs and their preservice teachers and inservice teachers need as they make sense of and enact CI strategies. This working group takes up this exploration by engaging working group participants first in activities and discussions about CI and then in thinking about steps to advance teaching about and research on CI. We focus specifically on three central strands of CI: status, smartness, and equity.

This working group has four goals:

1. Support MTEs who are new to CI in coming to understand the theory and pedagogical strategies of CI (particularly as related to status, smartness, and equity) as well as some ways of introducing CI to inservice and preservice teachers.
2. Expand the research base by initiating studies of learning to teach with CI (particularly as related to status, smartness, and equity).
3. Expand teaching support available to MTEs by sharing existing resources and building additional tools (particularly as related to status, smartness, and equity).
4. Develop a professional network of MTEs in order to support research on and sharing of CI with teachers.

Issues in the Psychology of Mathematics Education

Background on Complex Instruction

CI is a pedagogical approach drawn from the work of Elizabeth Cohen and Rachel Lotan (Cohen, 1994; Cohen & Lotan, 1997). CI seeks to support all students in engaging in rigorous academic work by addressing the ways in which being perceived as not smart can be a barrier to participation. CI has many essential components that are intertwined to support equitable learning outcomes. We proposed to concentrate on three of these components as part of our focus in this working group: status, smartness, and equity. We describe each of these CI strands below and suggest ways in which the working group might expand upon each.

Cohen and Lotan's initial research focused on social studies contexts in elementary and middle schools. Their work was later elaborated for use in secondary mathematics classrooms by the teachers at a school known in the research literature as Railside High School (Boaler, 2006 & 2008; Boaler & Staples, 2008). In our discussion of CI below, we consider both the original framing and the elaborations as all part of the CI umbrella.

Status

CI emerged from Cohen and Lotan's (1997) concern about students who were not participating in classroom activities and were consequently missing opportunities to learn. These students had interesting ideas and were, in many instances, eager to participate, but were denied access to classroom activities by their peers. Using a sociological lens, Cohen and Lotan studied how students construct hierarchies of competence (called status rankings) in which some students are seen as smarter and more capable than others. This hierarchy is established through unspoken agreement about external markers seen by students as tied to competence. These external markers are usually unrelated to students' actual intellectual abilities. For example, students might decide that the smartest students are those who appear to be European-American, wear the right clothes, speak first, or have a Midwestern accent. Students enact this competency ranking by denying low-ranking students access to materials or time to talk. In contrast, high-ranking students receive floorspace to share their ideas, credit for other's ideas, and the privilege of being seen as right, even when their ideas may be wrong or misleading.

Status is self-perpetuating (Featherstone et al., 2011). Students who have high status have opportunities to articulate their ideas, providing these students with feedback on their ideas and practice mastering the content discourse. Because their ideas are most likely to be taken up by others, they become the most knowledgeable about activities, supplying further evidence of their more advanced skills. Also, because their talk is most likely to be heard, when they restate the ideas of others, their peers are likely to assume that the ideas were generated by the high status student, further reinforcing the image of this student as a smart student.

In contrast, low status students have few opportunities to demonstrate their intellectual abilities (Cohen, 1994). They usually have limited access to materials and are physically prevented from participating by their peers. Without opportunities to voice their ideas or to contribute to academic tasks, low status students are left with few resources for demonstrating what they know and for using participation as a means for making further gains in understanding. As a consequence, it can be very challenging for a low status student to demonstrate their skills and abilities and improve their status ranking.

Status differences can be particularly hard to address in classrooms as they are outcomes of social norms shared by most members of the classroom (Cohen, 1994; Featherstone et al., 2011). Addressing status difference means asking a classroom of students (and perhaps even the teacher) to shift beliefs about who people are and what they are capable of. Also, because high status students have resources to share their ideas, they also have many means of reinforcing their status by undermining others' efforts to reframe what it means to be competent.

The challenges of addressing status provide many areas for exploration during the working group time. Questions the working group might pursue include: How do teachers make sense and take up the construct of status? What kinds of supports and experiences can support the decoupling of status from ability? How do teachers notice, interpret, and address status issues in their classroom? How do MTEs understand and address status issues within their own teacher preparation classrooms or across groups of teachers?

Smartness

Cohen and Lotan developed several strategies that address status issues by helping students see each other as competent and as supports for each other's learning (Cohen 1994, Cohen and Lotan, 1997, Featherstone et al., 2011). These CI strategies are predicated on the assumption that all students are capable and that student "smartnesses" are varied and potentially masked by a history of identifying as not smart. Two important ideas underlie this assumption: one about the nature of being smart and the other about ways of being smart.

Carol Dweck (1999) described two different understandings of smart. Some people imagined being smart as a fixed, innate attribute: People have a certain inherent amount that does not vary regardless of activity or effort. In contrast, other people see intelligence as something that can be developed and improved through hard work and effort. This latter characterization matches both research (see, for example, Sternberg, 1998) and the assumptions of CI.

A second important idea about smartnesses is the ways of being smart. CI recognizes that accomplishing challenging and interesting academic work requires many different intellectual resources. Mathematical smartness is more than quick computation. It is even more than the strands of mathematical proficiency described in *Adding It Up* (Kilpatrick, Swafford, & Findell, 2001). While conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition play important roles in working on mathematical tasks, also important are the skills of identifying which mathematical concept is still confusing, determining the best resource for getting help, and listening and making sense of others' ideas, to name just a few.

The work to redefine smartness can be as challenging as working on status and, as such, offers a rich area for exploration during the working group. First, there is some debate as to what word to use. Some educators have questioned whether redefining smart is possible and have suggested that ability, skill, or strength might be better terms. The working group might explore different meanings and values assigned to different synonyms (or near-synonyms) for smart. Also, supporting MTEs, teachers, and students in reconceptualizing what it means to be smart

requires renegotiating school structures (such as assessment tests and report cards) and discourses (such as high, middle, or low student) that reinforce more static and narrow understandings of being smart. Working group participants may choose further explore teaching strategies or research questions around this issue.

Equity and CI

Research on CI has demonstrated its success with marginalized students. The school that pioneered CI in mathematics classrooms, Railside High School (Boaler, 2006; Boaler & Staples, 2008) was a diverse school with 30% of students labeled as English language learners, 40% of students identified as Latin@, and 20% as African-American. After two years of CI math classes, Railside students were outperforming students at other, less linguistically and culturally diverse schools. Additionally, Railside students were more likely to take Calculus in their fourth year of high school than students at other schools, and differences in achievement between different ethnic groups at Railside were reduced. Students at Railside also reported more enjoyment and interest in mathematics (Boaler & Staples, 2008). The research on Railside High School offers just one example of the success of CI in supporting more equitable learning outcomes for all students. The early research done by Cohen and Lotan showed similar results in diverse elementary and middle school classrooms (1997).

In addition, the gains made by students are more than academic (Esmonde, 1999). Students not only learn content, they also learn to value each other's diverse contributions, something Boaler (2006) named *relational equity*. Rather than use difference to inform perceptions of smartness, students in CI classrooms see diversity as contributing to their learning. Students understand that different perspectives mean that everyone is simultaneously both a learner and a teacher and that the outcome was better learning (Boaler & Staples, 2008).

What is intriguing about these equity outcomes is that they were achieved without explicit curricular attention to students' diverse cultural and class backgrounds. Unlike equity approaches in which teachers work to specifically include culturally responsive materials that address student diversity, CI tasks may be entirely embedded within the culture of the discipline (Boaler & Staples, 2008). This is not to imply that CI classrooms are culture- or class-blind. Instead, these classrooms are purposefully inclusive through their explicit attention to each student's smartness. Tasks are designed so that multiple smartnesses are required for success and students learn to turn to each other as resources for accomplishing academic work (Cohen 1994, Featherstone). This *multidimensionality* (Boaler, 2006) means that every student finds that their skills are not only useful in completing the task, their skills are essential to the success of other students.

Students find that work in CI classrooms supports them in enacting the most important aspects of their identities, even when these identities seem nonacademic (Jilk, 2007; 2009). By allowing students to find themselves, whoever that self might be, CI enables students to excel in academics. By including multiple aspects of who students are, CI acknowledges and embraces the full complexity of students' differences including cultural, linguistic, gender, family, and class diversities.

CI's multidimensional focus suggests many questions working group participants might explore: How does CI intersect with other equity pedagogies and theories? What aspects of CI might be enhanced by specifically intertwining it with other equity pedagogies and theories? How do teachers make sense of and implement the equity agenda undergirding CI strategies? How is using CI to achieve equitable learning in K-12 schools compare to using CI to teach about (and achieve) equitable learning outcomes in teacher preparation programs?

Research on Teachers and CI: The Literature Gap

There are only a handful of studies examining how teachers learn to teach mathematics with CI. Horn (2005) studied mathematics teachers' learning in and from their day-to-day interactions with other teachers in two schools, one of which used CI for mathematics, and found that teachers in the school that used CI came to frame problems of student learning in terms of relationships between students, teachers, and the content (as opposed to the other group which framed problems more narrowly in terms of teacher-student relationships). Horn and Little (2010) studied two groups of high school teachers using CI in order to better understand teachers' learning situated in their day-to-day practice; one of the groups was a group of mathematics teachers using CI. They found that the CI provided teachers with a shared set of conceptual tools and language that the other group did not have, and that teachers used this to engage in conversational routines that opened up learning opportunities in teacher communities. Horn (2010) further described ways in which these same CI teachers used storytelling to continually make sense of their practice.

Other researchers have started to examine how teachers learn about CI from professional development sessions or teacher preparation coursework. Crespo and Featherstone (2012) wrote about how elementary PSTs learn about status in mathematics classes. They reported that adopting the language of status allowed PSTs to reframe problems of student participation as teaching problems that they could work on instead of problems of student characteristics, which were outside of their control. Fede, Mojica, and Civil (2013) studied inservice teacher learning resulting from professional development. Finally, Oslund (2009, 2010, & 2012) focused on the interplay between teachers' mathematics identities and their understandings of the principles of CI, as studied through teachers' storytelling about a CI professional development course, their practice, and their life stories. These studies are a promising beginning, but much remains to be learned about (a) how preservice and inservice teachers make sense of CI and learn to implement the strategies, (b) the best resources for working with preservice and inservice teachers, and (c) how MTEs make sense of, implement, and teach about CI. This working group is an initial step towards realizing these goals.

Plan for the Working Group Sessions

Our goals during this working group time are to increase MTEs' familiarity with CI and to begin a conversation about theoretical and pedagogical concerns related to supporting teachers (both inservice and preservice) in taking up the pedagogy and strategies of CI, specifically along the themes of status, smartness, and equity. We will use social media tools to capture and share attendees' responses to activities and questions, helping us to both quickly assess attendees' reactions and to help establish an electronic community that can support our collaboration beyond this working group.

- Session 1: CI Introduction
  - Determine levels of understanding across attendees (using social media)
  - Overview of CI - theoretical framework, research, and practice
  - Engage in CI task frequently used with preservice and inservice teachers
    - Focus on status, smartness, and equity during the task
- Session 2: From Introduction to Teacher Education (Inservice & Preservice)
  - Debrief task from Session 1
    - How does this task emphasize status, smartness, and equity?

What might an MTE consider before, during, and after using this task with teachers?

What aspects of status, smartness, and equity are particularly challenging for:

- mathematics teacher educators to make sense of and use in teaching?
- preservice teachers to make sense of and use in teaching?
- inservice teachers to make sense of and use in teaching?

Session 3: Research Agendas and Plan for the Future

Explore research on CI in mathematics classrooms

Overview of research

What are additional potential research trajectories tied to status, smartness, and equity?

What other frameworks might be useful in making sense of teaching and learning CI along the three main strands?

Organize into teaching and research groups

Make commitments for future teaching and research activities

Introduce social media tools (webpage, twitter feed, blogs, and online tasks)

Follow-up Activities

Working group participants may use the CI website and twitter feed as a means of staying connected, sharing teaching strategies and tasks, and organizing research activities. As working group members develop teaching strategies, we anticipate that these may be resources published in the online journal Mathematics Teacher Educator. We hope to continue to develop research themes and studies and extend our collaboration through not only social media tools, but also through additional face-to-face meetings at other conferences.

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Errata

The following papers were not included in the proceedings and were published online on October 24th, 2013 and November 19, 2013, respectively:

"Pre-Service Teachers' Ability To Pursue Children's Mathematical Thinking Through Formative Assessment Interviews" by Mi Yeon Lee.

"Handling Student Difficulties When Going Over Homework in Secondary Mathematics Classrooms" by Samuel Otten, Beth A. Herbel-Eisenmann and Michelle Cirillo

PRE-SERVICE TEACHERS' ABILITY TO PURSUE CHILDREN'S MATHEMATICAL THINKING THROUGH FORMATIVE ASSESSMENT INTERVIEWS

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This study investigated how pre-service teachers in an innovative field experience developed their abilities to pursue children's mathematical thinking. Participants conducted weekly interviews with pairs of children. Videos and documents from these interviews were analyzed as a multiple case study. Findings suggest that the relationship between pre-service teachers' interview skills and their Prediction Assessment scores was stronger than relationships between other measures. Also, the higher their Prediction Assessment scores, the more conceptual interview skills pre-service teachers demonstrated. Implications are discussed for the improvement of pre-service teachers' ability to pursue students' thinking through clinical interviews in order to prepare better elementary teachers.

Keywords: Pre-service Teacher Education, Students' Mathematical Thinking, Formative Assessment Interviews

Several teacher education programs have focused on students' thinking (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Jacobs, Lamb, & Philipp, 2010; Schifter, Bastable, & Russell, 1999; Wright, Martland, Stafford, &Stanger, 2006). Such programs provide some evidence that a focus on understanding students' thinking can be effective in advancing teachers' beliefs, teachers' instruction, and students' achievement (Fennema et al., 1996; Philipp et al., 2007; Wright & Ellenmor-Collins, 2008). Also, studies have shown the importance of teachers' ability to understand students' reasoning in order to plan lessons based on students' prior knowledge and thus provide authentic learning opportunities. However, several questions about teachers' development of understanding children's mathematical thinking remain open to further research. One of those questions is how teachers develop questioning skills and predictive ability in teacher education programs. *Questioning skills* refer to the techniques of questioning in order to elicit students' thinking. *Predictive ability* refers to the ability to conjecture how students would respond to a task similar to one already experienced. This study addresses these issues, particularly as they apply to pre-service teachers' (PSTs') understanding of children's mathematical thinking in an early field experience.

The setting for this study was the Iterative Model Building (IMB) project, which incorporates two innovations in the pre-service field experience: one is for PSTs to focus on children's mathematical thinking by making models of students' knowledge through clinical interviews (Hunting, 1997), a process called Formative Assessment Interviews (FAIs); and the other is for them to reflect on their teaching practices in lesson study groups. This study focused on the first intervention, and the purpose of this paper is to report how PSTs' abilities to pursue children's mathematical thinking developed in the IMB field experience depending on their Prediction Assessment scores. The Prediction Assessment (PA) is a questionnaire based on video clips, designed to measure PSTs' ability to predict a student's thinking. In the assessment, the PST engages in an analysis of a video of a student solving mathematical tasks, builds a model of the child's mathematical thinking based on the analysis, and then uses that model to predict how the

student will respond to a subsequent task (Norton, McCloskey, & Hudson, 2011). The research questions guiding this study are:

1. Is there a significant difference in pre- and post- Prediction Assessment scores between pre-service teachers in the IMB group and those in a control group?
2. How do eight selected pre-service teachers pursue students' mathematical thinking in constructing FAI questions, in conducting FAIs, and in reflecting on their FAIs?
3. What is the relationship between PSTs' performance in the FAI sessions and their Prediction Assessment scores?

Theoretical Framework

Researchers have concluded that, along with knowledge of mathematical content, an understanding of students' mathematical thinking is one of the most important factors for successful teaching (Sowder, 2007; Steffe & Thomspon, 2000; Wright et al., 2006). It can support the achievement of better student-centered teaching (Fennema et al., 1996; Philipp et al., 2007), which is consistent with a constructivist perspective on learning (von Glaserfeld, 1995; Simon, 1995). Therefore, developing an understanding of students' thinking has become both a goal and a strategy of teacher education (Jacobs et al., 2010; Schifter, 2001; Sowder, 2007). In programs promoting this focus, student thinking is considered an "interpretive lens that helps teachers think about their students, the mathematics they are learning, the tasks that are appropriate for the learning of that mathematics, and the questions that need to be asked to lead them to better understanding" (Sowder, 2007, p. 164). In order to help teachers inform their teaching goals and strategies by focusing on students' thinking, researchers have investigated the effects of teachers conducting clinical interviews with children (Jacobs et al., 2010; Franke, Carpenter, Levi & Fennema, 2001; Schifter et al., 1999; Wright et al., 2006). The clinical interview helps teachers engage in interactive communication with a child, comprehend the child's level of understanding of a concept, and interpret a given problem from the child's point of view (Hunting, 1997).

The clinical interviews in this study, called Formative Assessment Interviews or FAIs, were divided into three time periods: before FAIs, during FAIs, after FAIs. A week before conducting the FAIs, PSTs were asked to create interview questions, and during the FAIs, they interviewed a pair of children. After the FAIs, the PSTs were expected to write reflections on their interview in terms of students' thinking and how to use the students' thinking to design next lesson.

To analyze the PSTs' FAI questions, I used five criteria derived from a literature review: (a) the degree to which FAI questions aligned with the FAI goals (Hunting, 1997); (b) the degree to which "how and why-questions" were used (Hunting, 1997; Mewborn & Huberty, 1999); (c) whether manipulatives were used (Hunting, 1997); (d) whether iterative opportunities to examine students' thinking were provided in different ways (Labinowicz, 1985); and (e) whether gear-up and gear-down questions were considered (Hunting, 1997).

In addition, to analyze the PSTs' questioning skills during FAIs, I referred to two studies by Schifter (2001) and Labinowicz (1985). Schifter identified four skills as necessary for teachers to understand students' thinking: (a) attending to students' mathematical thinking; (b) assessing the mathematical validity of students' ideas; (c) listening for the sense in students' mathematical ideas; (d) identifying the conceptual issues the students are working on. Labinowicz suggested skills required to interview students effectively, such as repeating or rephrasing the questions. I combined the third and the fourth skills from Schifter's research into one category because these are consecutive skills and so it was difficult to determine a distinct boundary between them. Thus, I identified four skills – three from Schifter and one from Labinowicz – to evaluate how

PSTs pursue children's thinking in FAIs. The skills from Schifter are related to PSTs' mathematical knowledge, and so I called them *conceptual* interview skills, while the skill from Labinowicz is related to general interview techniques, so I called it a *procedural* interview skill.

Finally, to analyze the PSTs' FAI reflections, I used three points provided in the FAI handout that specified what PSTs were expected to address in their reflections: (a) giving at least two specific examples of students' thinking; (b) providing inferences based on the examples; and (c) suggesting modifications for whole class lesson. Also, based on my preliminary data analysis, I extended the inference criterion by adding the question of whether PSTs used various sources to support their interpretations of students' thinking.

Methods

Participants

To address the first research question, 71 elementary PSTs enrolled in mathematics and science methods courses concurrently with a field experience course at a large Midwestern university in the Fall of 2010 were invited to participate in this study. All participants were either juniors or seniors and enrolled in either IMB groups (34 PSTs) or comparison groups (37 PSTs) in their field experience. Then to address the second and third research questions, eight participants were selected from IMB groups using three criteria: The Prediction Assessments results, consultation with the PSTs' field experience supervisors, and the researcher's field notes.

The Prediction Assessments were administrated to all participants twice, at the beginning and at the end of semester. Then two pairs of researchers assessed the Prediction Assessments using a rubric which had been specifically developed for this project by several math educators. After this grading, I classified PSTs' Prediction Assessment scores into high and low categories. Then depending on their pre-test and post-test scores, I divided the PSTs into four categories: High-High, High-Low, Low-High, and Low-Low category. For example, the PSTs who got high score on the pre-test and low score on post-test was assigned to the High-Low category.

Setting of the Field Experience

The IMB field experience placed six PSTs in a classroom and involved weekly cycles of four activities: formative assessment interviews (FAIs); model building (MB) sessions; planning, teaching and observing lessons; and debriefing lessons in lesson study sessions. Prior to teaching their lesson, pairs of PSTs conducted FAIs in order to understand students' thinking, and then the data from the interviews were used to make models of children's reasoning and to help plan instruction for whole class lessons. Then a pair of PSTs developed their whole-class lesson plan based on the FAIs and MBs. When doing the lesson planning, they considered collaborative discussions which had taken place in the lesson study meeting. Then the pair of PSTs taught the lesson the following week, while the others in the lesson study group observed. When teaching, one PST took a leading role and the other took a supporting role. After teaching the lesson, the study group debriefed how well the lesson went and discussed how to improve it in future teaching. At the end of the lesson study meeting, one pair of PSTs presented a draft of the next lesson to the study group, and the lesson study members suggested ideas for improving the lesson. Among these four activities, this study focused on the first component, FAIs.

Data Collection and Data Analysis

Three data sources from the Fall of 2010 were used – Prediction Assessments Scores, FAI videos and FAI handouts. To analyze the data, I used *mixed methods* (Creswell, 2008). For the first research question, I quantitatively analyzed Prediction Assessment scores using the t-test. Based on the result, I selected eight PSTs to address the second and third research questions, using the qualitative method of *a multiple case study* (Yin, 2009). First I recursively watched the

eight PSTs' videos and reviewed the documents collected in the FAI sessions by conducting recursive rounds of note taking (*holistic analysis*). Then I developed coding schemes based on the notes and literature (*analysis of themes*). After that, I analyzed each PST's data with the coding schemes (*within-case analysis*). Finally, I looked across all PSTs' data in four categories to find the differences and commonalities among them (*cross-case analysis*).

Findings

Differences in Prediction Assessment Scores between IMB group and comparison group

The IMB group's mean score on the pre-Prediction Assessment was higher than that of the comparison group; this difference was statistically significant at an alpha level of 0.05 (see Table 1). There was no difference between the means of the IMB group and the comparison group at the post-test. Table 2 shows the analysis of the change in Prediction Assessment scores within each group. Numerically, the mean of the post-Prediction Assessment in the IMB group decreased by the end of the semester while the mean of the comparison group increased. However, neither group showed a statistically significant change in prediction ability. These results were counter to the expectation that the scores of the PSTs in the IMB would increase since they had participated in specific interventions to improve the ability of predicting children's thinking throughout the semester.

Table1: Independent T-test Analysis of Significance in the Pre-and Post- Prediction Assessments Between the IMB Group and the Control Group

Test	Group	Mean	Std. Deviation	df.	t_{cal}	Sig.(2-tailed)
Pre-test	Control (N=37)	6.22	2.405	69	-2.055	.044
	IMB (N=34)	7.26	1.880			
Post-test	Control (N=37)	6.89	1.745	69	.206	.838
	IMB (N=34)	6.79	2.213			

*P < .05

Table 2: Paired T-test Analysis of Significance in the Pre-and Post- Prediction Assessments within the IMB Group and the Control Group

Group	Test	Mean	Std. Deviation	df.	t_{cal}	Sig. (2-tailed)
Control (N=37)	Pre-test	6.22	2.405	36	-1.685	.101
	Post-test	6.89	1.745			
IMB (N=34)	Pre-test	7.26	1.880	33	.1.252	.219
	Post-test	6.79	2.213			

*P < .05

Concerning this unexpected result, I offer three possible explanations. First, these results might reflect the insufficient number of participants required for sound statistical analysis. In this study, there were 34 participants in the IMB group and 37 participants in the comparison group. However, the IMB group and the comparison group were each divided into two sub-groups in order to control the different difficulty levels of the pre-and post- Prediction Assessments. For this reason, I did not have enough participants in the sub-groups for a T-test, which requires at least 30 participants for viable analysis (Creswell, 2008). Second, it is possible that the pre-service teachers did not take the post-Prediction Assessments as seriously as they did the pre-test because they knew that the scores did not affect their course grade. Finally, the cause may be a

problem transferring knowledge in that the environment for taking Prediction Assessments was different from that of the real activities that the PSTs participated in during the IMB field experience. Thus, I conducted a qualitative phase of the research in order to deeply investigate the effects of the IMB field experience on PSTs' ability to understand students' thinking.

PSTs' Pursuit of Children's Mathematical Thinking in the Field Experience

To address the second research question, I examined how eight PSTs pursued students' thinking. The results of the analysis are presented for each participant in pairs of PSTs in the following order: Two PSTs in the High-High category, two PSTs in the Low-Low category, two PSTs, two PSTs in the High-Low category, and two PSTs in the Low-High category. I first evaluated the FAI questions they developed based on the five criteria that I mentioned before. Then I investigated what characteristics they showed in interviewing their students in terms of three *conceptual* interview skills and one *procedural* interview skill. Finally, I assessed how the PSTs pursued their students' thinking in their FAI reflections in terms of examples, inferences and modifications.

Before FAIs: creating FAI questions. In their FAI questions, overall the eight PSTs satisfied three criteria including alignment with the FAI goals, the use of manipulatives, and the provision of iterative opportunities, although John (L-L) only partially met the three criteria. However, the PSTs showed differences in the extent to which they met the three criteria depending on their categories. More specifically, seven PSTs provided FAI questions aligning with their main FAI goals, but some FAI questions written by the PSTs in the Low-Low category did not align with the goals. In addition, the eight PSTs planned to use manipulatives, although the frequency with which and the way they used them were different. The PSTs in the High-High category, Low-High category, and High-Low category started each interview by asking students to do an activity with manipulatives and planned to use various materials continuously throughout the interview. However, the PSTs in the Low-Low category planned to use only one type of manipulatives for one or only a few questions, although manipulatives seemed to be effective in investigating children's thinking across all FAI question contexts.

Moreover, the eight PSTs provided students with iterative opportunities to show their understanding. That is, most PSTs provided similar but slightly different FAI questions by changing the numbers given in the questions. In terms of gear-up and gear-down questions, seven PSTs provided them although the nature of these questions was different across the PSTs. That is, the PSTs in the High-High category provided gear-up and gear-down questions using various strategies, but other PSTs provided them using only a simple strategy such as scaling up or scaling down the given numbers.

Finally, in the FAI questions, all eight PSTs showed weakness in the second criterion, evaluating the degree to which "how" or "why" questions were used. Only about 25 % of the FAI questions that the eight PSTs planned to use were "how" or "why" questions. Most questions in the FAI plans consisted of "how many" or "what" questions, which are usually used to seek answers rather than to elicit students' thinking in the process of determining answers. This tendency is consistent with prior research in which it was found that many pre-service teachers were limited in planning their ideas about what questions to ask in order to encourage children's mathematical thinking (Moyer & Milewicz, 2002).

During FAIs: conducting interviews. The PSTs were evaluated in terms of the *conceptual* interview skills and *procedural* interview skills they demonstrated while conducting the FAIs. The PSTs in the High-High category and in the Low-Low category showed distinct differences in these interview skills. The PSTs in the High-High category demonstrated strengths in both

conceptual interview skills and procedural interview skills and explored students' mathematical thinking by connecting the two. That is, Joy (H-H) and Cathy (H-H) attended to students' mathematical thinking and assessed the mathematical validity of students' ideas. Also, both of them flexibly re-directed interview questions according to students' responses. However, Cathy demonstrated more strength in her conceptual interview skills, while Joy showed more strong points in her procedural interview skills.

In contrast, the PSTs in the Low-Low category demonstrated weakness in both conceptual interview skills and procedural interview skills. In particular, John (L-L) and Melissa (L-L) struggled with investigating students' mathematical thinking. They also did not deal with a pair of students well. In addition, they often used the Initiate-Respond-Evaluate (IRE) discourse pattern. That is, they initiated a question, their students responded, and they evaluated the student's response (Bush & McGatha, 2010). By using this pattern, they focused on students' answers rather than on their mathematical thinking, and John sometimes went further by attempting to impose his thinking onto students. This finding is consistent with prior research showing that the IRE discourse pattern does not give students sufficient chances to develop their conceptual understanding or mathematical reasoning (Bush & McGatha, 2010).

The PSTs in the High-Low category showed strengths in their procedural interview skills and weaknesses in their conceptual interview skills. That is, Sunni (H-L) and Sara (H-L) neither attended to students' mathematical thinking nor checked the mathematical validity of students' thinking. However, they managed a pair of students well enough for the children to keep engaging in solving problems, although Sara sometimes experienced difficulty asking questions clearly or paraphrasing the questions.

The PSTs in the Low-High category both showed weakness in their procedural interview skills. However, the two PSTs differed in conceptual interview skills. That is, Lisa (L-H) demonstrated strength in conceptual interview skills by attending to students' mathematical thinking and assessing the mathematical validity of students' thinking. However, she did not give students enough time to think about her questions. Instead, she tended to shift abruptly to the next question after hearing what she wanted to hear or to immediately answer the question herself. On the other hand, Mary, also in the Low-High category, showed weakness in both procedural interview skills and conceptual interview skills, unlike Lisa (L-H).

After FAIs: writing FAI reflections. The FAI reflection was made up of three parts: examples of students' thinking, inferences about the examples, and modifications for the following lesson. Each PST completed one FAI reflection every week for six weeks. Among six FAI reflections for each PST, I examined only two – the one written after the PST had assumed the lead role in composing FAI questions and the one written after s/he conducted the FAI. The two reflections that I examined showed PSTs' tendency to improve in their second FAI reflections, which were completed in later weeks. However, there were not distinct differences between categories except that PSTs in the High-High category showed strengths across all criteria by providing examples of students' thinking, drawing inferences and making lesson modifications, while PSTs in all other categories showed both strengths and weaknesses. Thus, it was hard to find any patterns in the FAI reflections of PSTs in the latter three categories. In particular, the FAI reflections of PSTs in the Low-Low category indicated relatively less difference from those of PSTs in the other categories than did their FAI questions and FAI skills.

Discussion

Taking the results of FAI questions, FAIs, and FAI reflections together, these findings indicate that PSTs' interview skills in the FAI sessions showed a stronger connection with their

Prediction Assessment scores than did creating FAI questions or writing FAI reflections. That is, the PSTs showed rather distinct differences in FAI interview skills depending on their categories as assigned by their Prediction Assessment scores.

In the FAIs, the PSTs in the High-High category explored students' mathematical thinking by appropriately using both conceptual interview skills and procedural interview skills. The PSTs in the High-Low category incompletely explored students' mathematical thinking because of the lack of conceptual interview skills. The PSTs in the Low-High category explored students' mathematical thinking in limited ways owing to their lack of procedural interview skills. The PSTs in the Low-Low category directed students' mathematical thinking by focusing only on answers because of their lack of both conceptual interview skills and procedural interview skills.

However, the PSTs in the three categories did not differ appreciably in their FAI questions and FAI reflections, although the PSTs in the High-High category consistently presented superior work, and John (L-L) demonstrated weaker work. The explanation of why interviewing rather than other factors tended to differentiate PSTs and to be more correlated with Prediction Assessment scores might be that prediction ability, which involves conjecturing how students might think in a similar context based on existing information about the students, is a key ability for investigating students' mathematical thinking in interviews. Also, the interviewing situation is more impromptu than creating FAI questions or writing FAI reflections. That is, interviewing requires an ability to spontaneously cope with students' unexpected answers by providing appropriate responses in order to elicit students' mathematical thinking. Thus, an internalized ability to predict students' thinking might be necessary to effectively elicit students' thinking during interviews (Moyer & Milewicz, 2002). In a similar vein, PSTs might need such an impromptu ability to predict students' thinking in Prediction Assessments because the PSTs have to instantly predict students' thinking in another task after watching a partial video clip of students' work, although the students are not familiar with them (Norton et al., 2011).

Concluding Remarks

Successful mathematics teaching requires an ability to understand children's mathematical thinking. Many people assume that PSTs' ability to understand children's thinking will improve as they gain more teaching experience in their professional lives. However, according to research on mapping a trajectory of the evolution of teachers' expertise in understanding students' thinking (Jacobs et al., 2010), teaching experience and engagement in professional development do not necessarily guarantee improvement of the ability to understand students' thinking and decide how to respond on the basis of it. That is, the ability to understand children's mathematical thinking does not develop naturally as teaching experience increases. Thus, in order to start PSTs on the right course to becoming successful teachers, it is important to provide specific interventions to help them understand children's mathematical thinking.

Toward this end, I provide some suggestions to improve PSTs' ability to pursue students' thinking through clinical interviews. First, we need to encourage PSTs to consciously use "how" or "why" questions as much as they can in the field experience, taking into account that "how many" or "what" questions are focused more on answers than on students' thinking. Second, we need to help PSTs increase their predictive ability so that they can effectively interview children by focusing on children's thinking. The stronger relationship between Prediction Assessments and FAI skills in this study shows that conjecturing how students might think in a similar context based on existing information about the students is a key ability for investigating students' mathematical thinking in interviews. Third, we need to encourage PSTs to connect comprehensive sources such as previous FAIs and frameworks of children's mathematical

thinking studied in their methods course with their reflections after interviews. In the FAI reflections, most PSTs showed weaknesses in using such comprehensive sources to interpret students' thinking. Thus, it would be helpful to share PSTs' exemplary reflections using comprehensive sources with other PSTs through workshops.

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HANDLING STUDENT DIFFICULTIES WHEN GOING OVER HOMEWORK IN SECONDARY MATHEMATICS CLASSROOMS

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Many mathematics teachers spend a substantial amount of class time going over homework with their students. This study analyzes the classroom discourse of the going over homework activity structure in eight secondary (grades 6–10) mathematics classrooms. In particular, this brief report focuses on the ways in which student difficulties or errors were taken up in the discourse. In some instances, the error was raised by the teacher based on past experience, whereas, in other instances, the students shared a present difficulty or error. Teachers also responded in various ways such as by explaining an approach that avoids the error or, less frequently, by making the error an explicit focus of the discourse.

Keywords: Classroom Discourse, High School Education, Middle School Education, Problem Solving

Reviewing homework is a common activity in many U.S. mathematics classrooms, reportedly taking an average of 15% (Grouws, Tarr, Sears, & Ross, 2010) to 18% (Otten, Herbel-Eisenmann, & Cirillo, in press) of class time at the middle and high school levels. A feature that makes this particular activity distinct from other classroom activities (e.g., lecture, seatwork) is that students are presumed to have worked earnestly and independently on the homework assignment *prior* to going over homework (GOHW) in class. Because of this preparatory experience for GOHW, one might hypothesize that students have more to contribute to the classroom discourse than in activity structures where students must think or work on the spot. Past research has shown, however, that the activity structure itself tends not to be conducive to such contributions from students (Otten, Herbel-Eisenmann, & Cirillo, in press). GOHW also seems to be a potential site for students to look back and reflect on their mathematical work and also, with a teacher's guidance, engage in the Common Core State Standards for mathematical practice (National Governors Association & Council of Chief State School Officers, 2010).

Another opportunity GOHW might afford relates to students' confusions or difficulties on the homework assignment coming forth into the classroom discourse. Past research has investigated the extent to which teachers attend to and address student difficulties in general. In grade 7 mathematics classrooms, Zahner and colleagues (2012) noted that a teacher with high student gain scores tended to use incorrect responses as a launching point for discussion, whereas a teacher with low gain scores often did not build upon or respond directly to students' incorrect responses. This past work suggests that the handling of student errors is worthy of further consideration, particularly in an activity such as GOHW where students and teachers can discuss and collectively learn from difficulties in ways that are impossible through individual work or teacher feedback. Moreover, the handling of errors during GOHW is an opportunity to model certain mathematical practices and help students develop agency in working through the difficulties that arise in the learning of mathematics.

In this paper, we analyze the practice of GOHW as it occurred in eight different secondary mathematics classrooms. Our research was guided by the following question: What was the nature of discourse around student errors or difficulties?

Method

This study is situated within a larger research project focused on mathematics classroom discourse (see Herbel-Eisenmann & Cirillo, 2009). The eight participating teachers taught grade 6–10 students (aged 11–16) in various districts from a single Midwestern state. The teachers varied in their educational backgrounds and prior teaching experiences. During the project's baseline year, 148 video-recorded classroom observations were conducted, occurring in one-week segments at four different points across

the year for each teacher.

For this study, we take learning to be inseparable from the process of coming to actively participate in a discourse community (Lave & Wenger, 1991). Discourse communities, according to Lemke (1990), comprise many different *activity structures*, that is, patterned or organized social activities that members of the discourse community recognize and which shape the interactions therein. Lemke (1990) identified activity structures in science classrooms and Herbst and Chazan (2009) have called for a similar endeavor in mathematics classrooms. Using Lemke's work as a starting point, the corpus of classroom videos was segmented according to activity structure and the GOHW segments were collected for this particular study.

Analysis proceeded by focusing on interactions in which a mathematical difficulty or error related to the homework assignment was expressed, either by a student or the teacher. A student merely identifying an exercise number to go over did not warrant analysis; rather, we looked for instances wherein the student or teacher described or referred to a specific difficulty or mistake that had been made (or could have been made). We then organized these particular instances along two dimensions: whether the difficulty or error was based on present work by the students or past work known to the teacher, and whether the teacher's response dealt with the difficulty or error directly or indirectly. Although these dimensions are fairly large grain-sized, the nuances of how they were enacted varied. In the following section, we describe these dimensions in more detail and illustrate the range of discourse practices that we found.

Results

Across our data set, it was relatively rare for specific student errors to arise during GOHW. Typically, students simply identified problem numbers for the teacher to go over or teachers shared solutions without soliciting difficulties (Otten, Herbel-Eisenmann, & Cirillo, in press). When errors did come up, however, they were handled in a variety of ways (see Table 1).

Table 1. Varieties of Ways that Student Errors or Difficulties Played out During GOHW

	Source of the Error or Difficulty
Present	The current students' work on the assignment.
Past	The teachers' knowledge of potential errors or experience with past students' errors.
	Response to the Error or Difficulty
Indirect	A correct solution or approach is taken up in the GOHW discourse that circumvents discussion of the error or difficulty.
Direct	The error or difficulty becomes an explicit object of focus in the GOHW discourse.

Indirect Handling of Errors

By far the most common situation was for errors to arise from the *present* students and be handled *indirectly*. That is, an error would be identified from the present students' work or responses during GOHW and the teachers would respond by explaining or leading the students through a correct approach to the problem. For example, when Ms. P asked questions during GOHW and students gave incorrect responses, Ms. P often asked the same question again or rephrased her question to try to prompt a different response from students. Another example of this pattern is the following excerpt from Mr. M's 8th grade classroom. Students have written on the board answers to homework problems dealing with

areas of compound shapes (shapes made up of other simple shapes), and Mr. M is talking through each one when he notices an error.

Mr. M: This [answer] seems off. Because … this has a distance of 9, and this distance is gonna be 7 going up and down there. Because the total length is 14, this length is 7 and so that means this [other length] has to be 7 because 7 plus 7 equals 14. That makes that 63. Add those together, I think you get 203 for your answer on that. If we used the same method that … Brandon showed yesterday, you people might have boxed it in and did an overall area of 14 times 19 and come up with 266. This [smaller] area is 63 and you subtract away 63 and you still get the same answer. You have a couple of different ways of doing it.

With his response, Mr. M did not ask the student who wrote the incorrect answer or anyone else about the reasoning that might lead to the incorrect answer. He instead explained a correct method for computing the compound area and also referred to a previously discussed strategy of determining the area by subtracting away regions that are missing from a larger shape.

Teachers did raise errors from *past* experience or knowledge of the content and students. Sometimes, as they were explaining a homework problem, the teachers quickly mentioned a common error that they have seen in the past. In her advanced algebra class, for example, Ms. H sometimes briefly inserted or mentioned common mistakes she was aware of, regardless of whether she had observed the present students do it or not. For instance, when talking through the simplification of an expression, Ms. H said, "[Three squared] is six, right? That's a popular mistake." In this example she did not explicitly mention or ask about the confusion between multiplication and exponentiation, but she did refer to the potential error. Another example was found when Ms. P designed an assignment in which she gave students problems and solutions, but some solutions were erroneous. She asked students to evaluate the work and fix any errors that they found. Through this assignment students engaged with errors based on Ms. P's knowledge of students and content. During GOHW the next day, however, the discourse focused on whether students judged each solution as right or wrong and on the correct solutions rather than on having students articulate the mathematical thinking that might have underlay the errors.

Direct Handling of Errors

Although indirect handling of errors was prevalent, there were a few instances in which a teacher responded *directly* to a present student's error by making it a focus of attention in the classroom discourse. For example, Ms. A, a grade 7 teacher, was discussing a problem in which students had to find a fraction and whole number whose product was between one-half and one. She had already solicited two correct answers when she asked for another response.

Sydney: Um, I don't know if this is right or not, but five-ninths of twenty-five.
Ms. A: [writes $5/9 \times 25$ on the board] That's about what fraction, five-ninths? Close to what?
Sydney: Half?
Ms. A: So you want about half of twenty-five.
Sydney: Oh.
Ms. A: What's wrong?
Sydney: It'd be twelve and a half.
Ms. A: It would be close to that, wouldn't it? And that would not be between half and one.

Here, Ms. A responds to Sydney's incorrect response by having an interaction with her through which it becomes clear to Sydney that the response does not fit the conditions of the problem. Perhaps Ms. A could have gone further by asking Sydney why she initially thought the product would work, but regardless, this is an example of a direct handling of the error. In general, Ms. A made reasoning and sense-making type discourse moves (e.g., "Why did you do that?") during GOHW, which resulted in direct handling of many of the errors that arose. Her classroom, however, ran counter to the activity structure of GOHW in most of the other classrooms.

Discussion

In our analysis of eight secondary mathematics classrooms, we found that errors arising during GOHW were predominantly handled in an indirect way through providing a correct solution rather than

investigating the reasoning or thinking behind an error. It should be noted, however, that these teachers did not ignore student thinking in general. During other activity structures and even when handling *correct* responses from students in GOHW, the teachers often made student thinking an explicit object of focus in the discourse. Also, teachers may view direct handling of errors as too time consuming or better suited for other activities, but one could argue that handling the errors indirectly is missing opportunities for formative assessment. Moreover, when mathematics learning is viewed as coming to participate in the disciplinary practices of mathematics, direct handling of errors may provide opportunities to explore reasoning and model perseverance through difficulty, which are hallmarks of mathematical practice. This brief report reveals that such practices are not necessarily embedded in the discourse structure of GOHW.

Acknowledgement

This study was supported by the National Science Foundation (grant 0347906, Herbel-Eisenmann, PI). Any opinions, findings, and conclusions or recommendations expressed in this article are those of the authors and do not necessarily reflect the views of NSF.

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