INDIGENOUS KNOWLEDGE PROVIDES AN ELEGANT WAY TO TEACH THE FOUNDATIONS OF MATHEMATICS

Jerry Lipka
University of Alaska Fairbanks
jmlipka@alaska.edu

Dora Andrew-Ihrke
University of Alaska Fairbanks
dmandrewihrke@alaska.edu

David Koester
University of Alaska Fairbanks
dckoester@alaska.edu

Victor Zinger
University of Alaska Fairbanks
vazinger@alaska.edu

Melfried Olson
University of Hawaiʻi, Mānoa
melfried@hawaii.edu

Evelyn Yanez
University of Alaska Fairbanks
eyanez@alaska.edu

Don Rubinstein
University of Guam
don.rubinstein@gmail.com

This unlikely cast of characters, by working collaboratively in a trusting learning community, was able to identify an approach to teaching rational numbers through measuring from the everyday practices of Yup'ik Eskimo and other elders. "The beginning of everything," as named by a Yup'ik elder, provided deep insights into how practical activities were conceptualized and accomplished by means of body proportional measuring and nonnumeric comparisons. These concepts and practices shed light on the importance of measuring as comparing and the importance of relative units of measure, and helped us imagine a way to establish an alternative learning trajectory and school-based curriculum that begins with the insights gained from Yup'ik and other elders. This approach may well provide teachers a way to teach aspects of elementary school mathematics in an integrative and elegant way.

Keywords: Measurement, Number Concepts and Operations, Curriculum

Introduction

More than 30 years ago, Evelyn Yanez, Dora Andrew-Ihrke, and Jerry Lipka almost blindly began a journey to map the landscape of Yup'ik elders' knowledge, practices, and conceptions of mathematics. Evelyn and Dora were students in the Cross-Cultural Education Development Program (X-CED), University of Alaska Fairbanks, in Dillingham, Alaska, and Jerry was a field faculty member. Little did we know that insights gained from our long-term collaboration with Yup'ik elders and our more recent work with Indigenous Knowledge (IK) holders from the Yap Outer Islands, Greenland, Kamchatka, and Norway would result in a way to teach aspects of the foundation of elementary school mathematics. Nor did we realize that this would become our life's work.

From 1995 to approximately 2010, we worked with a dedicated group of Yup'ik elders and teachers. The most significant outcome of this phase of our collaborative work was the publication of the *Math in a Cultural Context* series—ten elementary school culturally based mathematics modules and ten accompanying storybooks. During this time, we also developed a culturally based and reform-oriented pedagogical approach (Lipka, with Mohatt & the Ciulistet, 1998), which incorporated aspects of Sternberg's theory of intelligence (Sternberg, Lipka, Newman, Wildfeuer, & Grigorenko, 2006). We shared our curricular and pedagogical approach in many teacher training institutes and workshops. This was our response to identifying conceptions and contexts of mathematical behavior and thinking based on Indigenous Knowledge and applying them to a classroom context. Quantitative and qualitative journal articles documented both teachers' pedagogical practices while implementing MCC curriculum and outcome studies on the efficacy of

Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.). (2015). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. East Lansing, MI: Michigan State University.

the curriculum (Kisker, Lipka, Adams, Andrew-Ihrke, Yanez, & Millard, 2012; Lipka, Hogan, Webster, Yanez, Adams, Clark, & Lacy, 2005).

Yet, we were not satisfied, as we were beginning to perceive a more cohesive approach to teaching elementary school mathematics based on Indigenous Knowledge (Lipka, Andrew-Ihrke, & Yanez 2011; Lipka, Wong, & Andrew-Ihrke, 2013). Could we identify and understand the embedded and encoded mathematics in everyday Yup'ik activities in a systematic way? We wondered what it is that enables Yup'ik and other elders to accomplish an array of everyday tasks without overburdening memory. Yup'ik people are generalists; individuals have needed to make clothing, build houses, fish racks, smokehouses, and kayaks, and orient themselves in various weather conditions when traveling on the tundra, in forests, and on rivers and open seas. Were there concepts that cohesively supported the performance of these tasks?

Our latest grants have enabled us to continue this exploration into mathematical concepts embedded in the everyday activities of Indigenous Knowledge holders among the Yup'ik and other cultural groups across the Arctic as well as an "outlier" group from Yap State in the Federated States of Micronesia. This work has allowed us to imagine what an elementary school mathematics curriculum would look like if developed from key aspects of Indigenous Knowledge. We began this phase of our research aware of the importance of symmetry and measuring in the everyday activities of Yup'ik elders (Lipka, Andrew-Ihrke, & Yanez, 2011). Measuring is the major conception of "mathematics" that elders have identified and use in their everyday activities (Lipka, with Mohatt & the Ciulistet, 1998; Lipka, Wong, & Andrew-Ihrke, 2013).This paper reports on what we have learned to date.

From the outset of our work, because numbers and operations represent such an overwhelming part of the curriculum, we were initially unable to see how a cultural group such as Yup'ik, which does not prize numbers in its daily activities, could provide us with key insights on how to teach the foundations of school mathematics. Yet, the information that follows is what we found, and it is what was right in front of us for years. Even though there are fundamental epistemological differences between Yup'ik practical activity and school mathematics, the embedded mathematical principles in everyday activity can generalize to the teaching of school mathematics. Because their measuring approach occurs in a nonnumeric environment, the embedded mathematical principles provide a generalized model for teaching aspects of rational number reasoning and other aspects of mathematics. In fact, rational number learning (fractions, ratios, and scaling) has been identified as a difficult topic for most U.S. students (Confrey, Maloney, Wilson, & Nguyen, 2010; Wu, 2011; Lamon, 1999), and rational number reasoning is considered a key concept in students' mathematical education (National Mathematics Advisory Panel, 2008).

This article describes how measuring can be perceived as a central and integrative concept across a wide range of everyday activities conducted by Yup'ik and other Indigenous Peoples. We will identify and describe a few cultural activities that highlight underlying generative cultural and mathematical principles. We briefly describe these principles and will argue that the principles embedded in activity provide an alternative pathway to teaching the foundations of mathematics. The key curricular and teaching examples demonstrated in this paper from Indigenous Knowledge connect measuring with the elders' halving algorithm and demonstrate how this can be an exemplar to teach place value in base 2.

Brief Methodological Considerations

Math in a Cultural Context (MCC) is a long-term project at the University of Alaska Fairbanks; federally funded grants have supported this work. A cohort of Yup'ik students enrolled in the X-CED Program and coauthor Jerry Lipka began working with elders in the late 1980s as a first step in understanding and connecting their everyday knowledge to elementary school teaching. This unlikely

Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.). (2015). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. East Lansing, MI: Michigan State University.

long-term collaboration (well documented in *Transforming the Culture of Schools: Yup'ik Eskimo Examples)* occurred among Yup'ik elders, Yup'ik teachers and now-retired Yup'ik teachers, and academics (mathematicians, mathematics educators, linguists, cultural anthropologists, and educators) to make teaching elementary school math more cohesive, accessible, and relevant. Critical to our long-term work has been the establishment of trust, respect, and continuity, as we have been working with some elders for over twenty years and, in the process, have become "elders" ourselves. Evelyn Yanez, a coauthor and retired Yup'ik teacher and current MCC faculty, stated that "elders trusted us enough to give us their stories and knowledge so that it may go into the future like an arrow. They knew we would prepare books for future generations; they were excited about sharing their work" (personal communication, June, 2015). The meetings were important to the elders as the following anecdote describes. Lily Gamechuk, an elder from Manokotak, Alaska, and now deceased, came to a meeting in Fairbanks some years ago. She needed to go to the hospital but refused to go until she told her story to the gathered group. Elders would often state to us how MCC was one of the few programs in which their knowledge counted.

Over these many years, insiders and outsiders have met three or four times a year for three to five days per meeting, when possible through grant funding. These meetings included storytelling, describing and performing everyday tasks such as making clothing and patterns, and discussing and simulating star navigation. Less often, we would practice these skills and knowledge *in situ*. However, more remarkable, our elder meetings at times have become Socratic in character as the discursive activities of the entire group demonstrate and refine the growing understanding of connections between Yup'ik cosmology, epistemology, and practice as a system, and as we relate this Indigenous Knowledge System to school mathematics. Koester (2014) describes the process as "getting to 'mathematical foundations' from oral accounts of activities, a discursive process that invokes metaphor, symbols, and various forms of discursive displacement."

Jerry Lipka, Principal Investigator of MCC and first author of this paper, cannot stress enough the absolute importance of the long-term collaborative inquiry, which has allowed exploration of topics in depth and in new ways and has allowed opportunity for serendipitous events to occur, ones from which completely new lines of inquiry developed. For example, a single geometrical construction performed by Dora Andrew-Ihrke, which she was not going to share because she "thought everyone knew this," led to a generalized way to construct many different planar shapes as well as three-dimensional shapes. (It is beyond the scope of this paper to explore in depth the methodological considerations or the geometrical constructions.)

The Beginning of Everything: Connecting Everyday Activity and Mathematical Reasoning

In one of our first meetings with Yup'ik elders many years ago, we asked Lily Gamechuk, a respected elder from Manokotak, Alaska, to share with the group how she made clothing. Lily asked one of the Yup'ik teachers-in-training in the room to stand. Without any instruments, and without touching her, Lily "measured" her. The measuring took place in her mind's eye. Minutes later, Lily had made a complete outfit out of butcher paper including a dress, belt, hat, and boots. The only instrument she used was a pair of scissors. How did she do this? How did she measure? She never told us directly, as it was more important from her point of view that we learn this skill in our way. What mental operations did she employ to transfer her visual perception to the practice of cutting and sewing proportionally? Little did we know that understanding how Yup'ik elders performed such everyday math would transform our own thinking.

In a recent elders' meeting with the Yup'ik cohort, Raphael Jimmy, an elder of approximately 90 years from Mountain Village, Alaska, slowly raised his hands above the table lifting them at eye level so that we could all see that he had crossed his left and right index fingers, forming right-angled

axes referred to here as a plus sign: "+" (Figure 1). Simultaneously, he stated, "What was once hidden is now revealed" (personal communication, November, 2013).

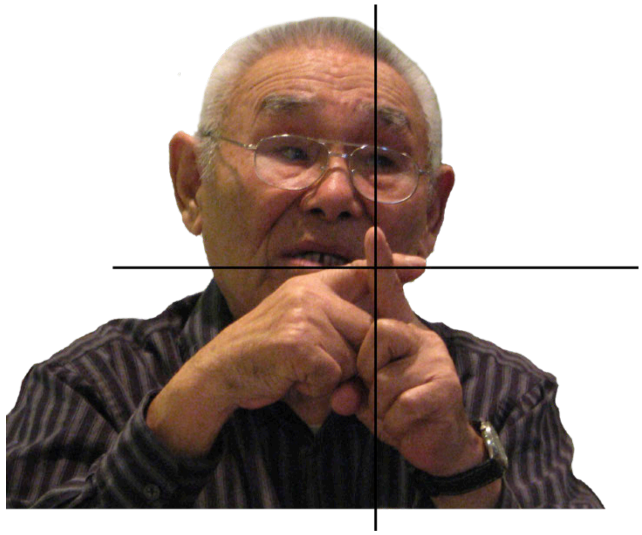


Figure 1: The beginning of everything

Mr. Jimmy explained that his crossed fingers represent "the beginning of everything." In a practical sense, this meant that the embodied abstraction was the starting point for many, if not most, practical activities. Even a length or a line segment has an implied center; folding material in half establishes a line of symmetry and two equal parts. Once Mr. Jimmy shared this concept, other members in our group realized that they too use this concept, heretofore unnamed, in their everyday constructions. We slowly realized that the concept reflected a culturally preferred way of perceiving, thinking, and performing across a wide variety of activities. Other members in our research group had also described this process, but until that moment, the process had gone unnamed. We have begun to observe the importance of "the beginning of everything" in other cultural groups with whom we are working, most notably the Yap Outer Islanders from the Federated States of Micronesia.

The cultural and mathematical activity-generating concepts signaled by Mr. Jimmy's action of crossing his fingers include the Yup'ik concepts of *qukaq* [center] and *ayagneq* [a place to begin], *avek* [halving], *tapluku* [to fold it, partitioning material], and *ayuquk* [testing and verifying congruence and symmetry or identifying equality]. This cluster of concepts and actions relate space, locating, and measuring space by invoking a line of symmetry that emerges through the action of folding material in half, or as a mental image in which the whole contains its parts, and a process of verifying equalities (is this side equal to the other side?). These words and actions establish a center and a place to begin many different projects performed by Yup'ik elders. The following few examples from Yup'ik cultural activity will illustrate how ubiquitous this concept is.

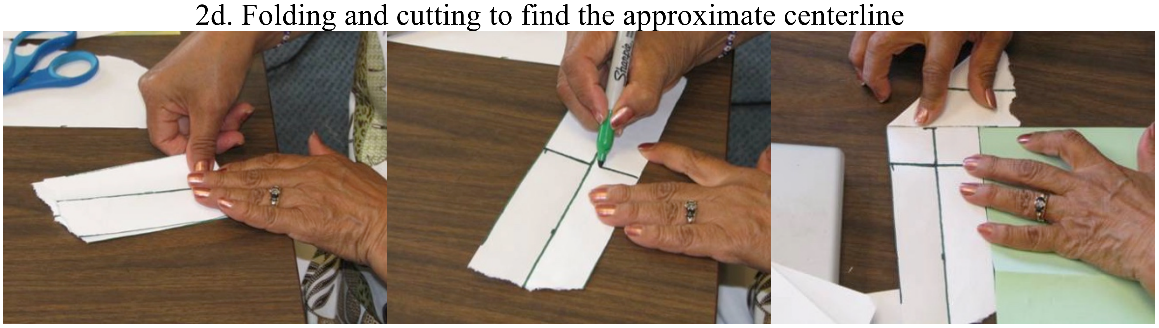
Although Dora Andrew-Ihrke, a long-term Yup'ik colleague and coauthor of this paper, did not have a name for "+," she uses the concept in her geometric constructions, such as for making a square out of irregular uneven material and in numerous other projects. Through body proportional measuring and folding, she establishes the center of the material. The "+" becomes the inner structure for a square, as shown in Figure 2. (The drawn lines are for demonstration purposes.)

The square and its center are co-constructed, as Dora refers to the initial center point as "fake," meaning approximate. The square and its center become "real" when she verifies the congruence of each fold (vertical, horizontal, and diagonals).

Dora uses these same processes and the concept of the orthogonal center to create patterns. She folds a square in half from top to bottom and side to side twice. This produces smaller squares, all the while verifying that she has maintained the orthogonal center. Figure 3 shows an example of this

2a. Irregular material 2b. One fold creates an edge; 2c. Measuring from the edge

2d. Folding and cutting to find the approximate centerline



2e. Perpendicular fold 2f. The approximate center of the square is established and Dora continues to construct the square

Figure 2: Geometric constructions using the concept of "+"

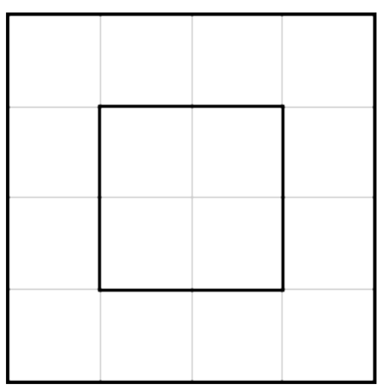


Figure 3: An example of geometrical similarity and scaling

similarity and scaling. Scaling is invoked through many everyday activities within the Yup'ik culture and is part of our school-based program.

The square is Dora's central geometrical shape, which she transforms through symmetrical folding into a circle or other planar shapes (Lipka, Wong, & Andrew-Ihrke, 2013). Aspects of the Yup'ik orientation system guide the transformation of a square into a circle by folding along lines of symmetry oriented by the winds, then the in-between winds, and once more between-those-winds until there are 16 points on the square (Figure 4).

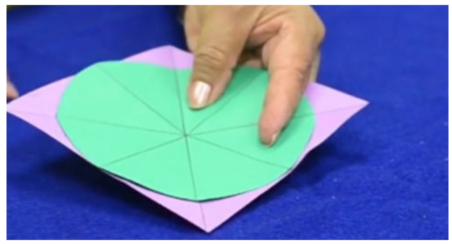


Figure 4:

When the square has been transformed into a many-sided polygon, Dora cuts a circle out of the square; the circle is inscribed in the square as shown in Figure 4 (Lipka, Andrew-Ihrke, & Yanez, 2011). All of these actions occur through the axial center.

Mrs. Nanalook, a respected elder from Manokotak, Alaska, has been a working member of the Yup'ik elders' cohort for a long time. Recently, she showed us step-by-step how she begins to weave a grass basket (Figure 5). A detailed ethnographic description is beyond the scope of this paper, but by referring to Mrs. Nanalook's key movements around the center point (see photographs in Figure 6), you can imagine how she begins weaving a grass basket.

Mrs. Nanalook takes a blade of grass, and if it is wide, she cuts it in half along its line of symmetry. As shown in Figure 6, she takes the two parts of the blade of grass and forms a cross, positioned in reference to the basket maker. Her actions of going up [*quletmun*] and going down [*acitmun*] along the vertical axes are coordinated by her actions of going left-to-right or right-to-left on the horizontal axes [*canitmun*]. She further orients the grass's motion in reference to her body as the grass is going inside toward her body [*ilutmu*n] and is going away from her body [*elatmun*]. The upward and downward and sideway folds form the loop at the center of the basket [*qukaq*]. The process is repeated along the horizontal plane, as these motions going inside/inward, and, going outward form another loop. The process is repeated six times until the beginning of the grass basket has been constructed.

Motions around the center, coupled with specialized demonstratives that aid orientation and location in reference to the speaker/weaver, reveal aspects of this generalized system. The Yup'ik language is exceedingly rich in demonstrative words that describe locating and orienting differentiating space (see Jacobson, 1984, pp. 653–662). Underlying this system is the central, bodily-situated orthogonal axis that functions as a cultural code of practical action and supports a wide range of activities from locating one's own self on the tundra or in the bays and surrounding waters, to orienting patterns and weaving the structure of a grass basket, to assisting a seamstress in measuring, cutting, and sewing clothing. In fact, even the Yup'ik counting system—base 20 and sub-base 5—uses movement across the spatially oriented four sectors of the body in relation to the axial center.

Figure 5: Yup'ik coiled grass basket

6a. Establishing the center 6b. Going downward 6c. Going horizontally

6d. Establishing the center of the basket

Figure 6: Mrs. Nanalook demonstrates basket weaving

The following examples are from both the Yapese and the Yup'ik context. These examples were chosen because they reveal how actions around the center (line of symmetry) contain both practical knowledge, used in making everyday products, and mathematical knowledge, applicable to teaching the foundation of mathematics. The illustration in Figure 7 shows Larry Raigetal, a Lamotrek (Yap Outer Islands) knowledge holder who lives on the main island of Yap, the state capital of fourteen widely dispersed atolls. Larry demonstrates to members of our research/study group how he transforms a coconut leaf into a "ruler." (Typically, a master canoe builder would use a pandanus leaf [personal communication, Cal Hachibmai, August 20, 2015].)

Larry first measures the desired length of the leaf by holding it between his thumb and index finger, stretching it over the top of the rest of his fingers and down the side of his hand until he reaches his wrist (proportional to his body) (Figure 8). He then constructs the tool by folding the measured leaf in half, and then he folds from one edge to the center and folds in half again. He

repeats this process on the other side of the leaf and then folds each of the four segments in half again until there are eight segments in total. Master carvers use this measuring tool to build boats, and such tools are used to build traditional houses. We have observed others in the Yap Outer Islands as well as in Chuuk (also part of the Federated States of Micronesia, 1500 kilometers away from Yap, with a

Figure 7:

distinct although related language and culture) use repeated halving to create tools and build canoes and houses, and as a way to fashion loom-woven cloth. For the Yap Outer Islands, this process is well documented (Alkire, 1970).

Figure 8: Measured proportional to the hand

Dora Andrew-Ihrke similarly learned how to fold material as a young girl from her mother, Annie Andrew, a well-known seamstress. Mrs. Andrew demonstrated and explained to Dora that there are two kinds of folds: easy and difficult. Easy folds are the same as those demonstrated by Larry, using either serial or recursive half-folds, producing 2, 4, 8, and so on, number of parts. Difficult folds are odd folds that include 6 equal parts, as 6 equal parts are constructed by multiplying an odd and even number.

When Dora folds a strip of paper into three equal parts, she follows her mother's folding algorithm and principle—always use the simplest fold, the half-fold. The main difficulty we had in understanding this halving algorithm was in moving away from our habituated way of seeing three parts of the whole as thirds, when in fact Dora was consistently expressing the relationship between two parts: "Is this half equal to that half?" (See Figure 9).

Stating Mrs. Andrew's folding algorithm more formally, it is n-1, when n is an odd number. For elementary school classroom purposes, we demonstrate this process using small numbers as a model

Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.). (2015). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. East Lansing, MI: Michigan State University.

for the algorithm and managing the number of folds. Thus, with the single-digit prime numbers 3, 5, and 7 using the n-1 algorithm, the number of folds reverts to the simple half-fold. Subtracting 1 from those primes results in 2, 4, and 6 parts each, and these specific examples are achievable through the half-fold. The process is illustrated and described in Figure 9.

The halving algorithm represents movements around the center, as shown in previous examples. This algorithm provides a crucial part of Indigenous Knowledge that transfers to the teaching of

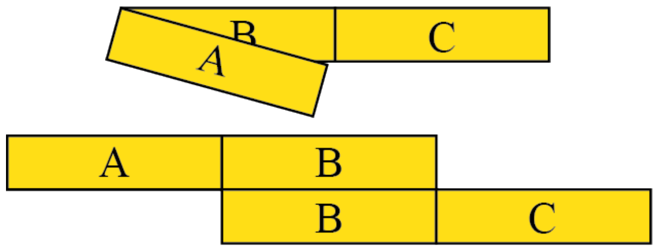


Figure 9:

school mathematics. Examples described below further develop the connection between Indigenous Knowledge and its potential in teaching rational numbers in a school context. Applying the *n*-1 folding algorithm to create three equal parts, we can either estimate Length A or Length C as the estimated third part. In the diagram shown in Figure 9, we estimated Length C as one-third of the whole. This leaves segment *n*-1, which is folded in half to create Length A and Length B. As illustrated, the halving algorithm is used twice.

When Lengths A and B were folded on top of each other, Dora treated them as a single entity and again referred to the notion: Are Lengths A and B with A folded on top of B equal to C? When we understood these actions and descriptions from her Yup'ik perspective, the centrality and the power of this halving algorithm (binary operation) became clear from within a Yup'ik cultural context. This realization opened a new and deeper understanding of how Yup'ik practical knowledge relates to fundamental aspects of numbers and relations.

The last cultural example is a variation of the halving algorithm, which both Mrs. Andrew and Dora use in their constructions. We observed other Yup'ik elders using the following algorithm, but clearly, Mrs. Andrew and Dora have refined this process at an expert level. In fact, other Yup'ik people learn from them. To find the difference between two quantities, for example Length A and Length B, Dora aligns them and then folds back the difference, Length C, so that Length A minus Length C equals Length B. (We have demonstrated this process in multiple workshops and institutes by comparing the length of Jerry's foot with that of Dora's. By finding the difference using the method shown in Figure 10, Dora eventually establishes a common unit that can measure both sets of feet.)

The difference between Length A and Length B is C. Dora uses C as the divisor. She now divides (or measures) Length A and Length B by the divisor, C. She folds until there is no remainder. In this simplified example, there is no remainder after two folds. C is now established as the common unit that can be used to measure Lengths A and B. Algebraically, B = 2C and A = 3C. This algorithm was developed in an Indigenous culture, but it is essentially Euclid's famous algorithm (https://en.wikipedia.org/wiki/Euclidean_algorithm), used to find the greatest common divisor of two numbers as well as the greatest common factor.

Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.). (2015). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. East Lansing, MI: Michigan State University.

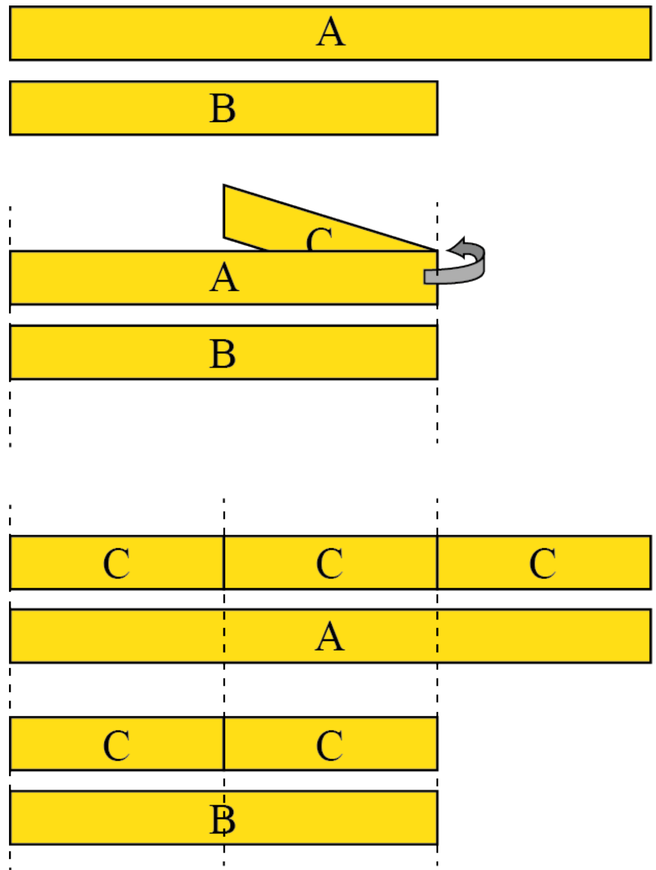


Figure 10:

Discussion: Indigenous Knowledge as a Basis for Teaching the Foundations of Mathematical Thinking

The above-described examples represent only a fraction of our data and provide evidence that in an Indigenous cultural context, measuring—as a means of comparing objects in practical activity—models the concept of ratios, expressed as rational numbers. When Dora creates a square from irregular material, or when Larry creates a measuring tool, both the square and the tool are constructed based on body measures and bodily techniques that establish ratios and proportions. The resulting products are made in proportion to the body of the person creating them. Everything is balanced between the user and the crafted product or clothing. The products and processes reflect relational thinking and relative units.

Mr. Jimmy identified the beginning of everything as a cultural code, which Mrs. Nanalook enacted while weaving her grass basket. This culturally grounded code is transferrable to the teaching of rational numbers in the elementary school. Dora's mantras are part of that cultural code: "What you do to one side you must do to the other side," and "Is this side equal to the other side?" Measuring as comparing (including body proportional measuring) reflects Lockhart's (2012) concept of measuring:

> How are we going to measure the length of two sticks? Let's suppose (for the sake of argument) that the first stick is exactly twice as long as the second stick. Does it matter how many inches or centimeters they come out to be? I certainly don't want to subject my beautiful mathematical universe to something mundane and arbitrary like that. For me, it is the proportion (that 2:1 ratio)

Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.). (2015). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. East Lansing, MI: Michigan State University.

> that's the important thing. In other words, I'm going to measure these sticks relative to each other. (Lockhart, 2012, p.32)

We believe that it is through *measuring as comparing* that we can establish an integrative approach to teaching aspects of rational numbers. Measuring proportionally is about quantitative comparisons, comparing two or more lengths (or measurable attributes of quantities such as length, area, mass, and volume). A set of seemingly simple principles reflects the generative cultural practices and ways of thinking, which we then use to establish a mathematical starting point for developing school-based mathematics. The generative principles and processes can be expressed as:

- measuring as comparing
- relative units, ratios and rational numbers
- symmetry, halving, and verifying
- scaling

Applying Indigenous Knowledge to the Development of Rational Number Reasoning

These generative principles are the active ingredients that we have identified, and based on the knowledge of these principles and processes, we have begun the development of supplemental curriculum materials and a learning trajectory. Our work is informed by the *Measure Up* program, University of Hawaiʻi at Mānoa's Curriculum Research Development Group. The *Measure Up* program follows the experimental curriculum developed in Russia by Elkonin-Davydov (Dougherty & Simon, 2014), in which first grade students explore quantitative relationships through nonnumeric comparisons as a way to establish early algebraic reasoning including generalizing. Elkonin-Davydov's curriculum was influenced by Vygotsky's cultural-historical-activity theory (Engeström, Miettinen, & Punamäki, 1999). According to Davydov (2008), "the basic task of the school mathematics curriculum is to bring the students to the closest possible understanding of the conception of real number." Davydov goes onto to state, "the properties of quantities are discovered when a person works with real lengths, volumes, weights, time intervals, and so on (even before these are expressed in numbers)" (Davydov, 2008, p. 148). Moxhay (2008), who adapted the Elkonin-Davydov curriculum, and implemented and assessed it in the Portland School District in Maine, noted:

> All the children are exposed to the same, very high level of mathematical content, which at first view looks like high-school mathematics (use of algebraic notation, counting in bases other than 10). In particular, Davydov's curriculum has the goal of developing, in all students, a scientific, or theoretical, *concept of number,* from the very beginning of Grade 1. (Moxhay, 2008, p.2)

The developmental psychologist Sophian, at the University of Hawaiʻi at Mānoa, who collaborated with the *Measure Up* program, makes a persuasive argument for why the comparison-of-quantities approach should be considered a legitimate alternative to the "counting first approach." Sophian contrasts the concept of number and quantity as follows:

> In order to clarify the contrast between these two perspectives, the concept of *number* needs to be differentiated from that of *quantity.* In the senses most pertinent to the present discussion, *Webster's New World Dictionary* (Neufeldt & Guralnik, 1994) defines number as "a symbol or word, or a group of either of these, showing how many or which one in a series"; and quantity as … "that property of anything which can be determined by measurement." (Sophian, 2008, p. 3)

She notes an ontological difference between number and quantity by explaining that quantity is associated with physical things which can be compared in a variety of ways, while numbers are not physical things but are a mental operation (Sophian, 2008). She cites the work of Gal'perin and

Georgiev (1969), that numbers arise from the comparison of a quantity and a unit, and that different numbers occur if the unit size changes (Sophian, 2008). This approach leads to the recognition that "numerical values are essentially representations of the relation between the quantity they represent and a chosen unit" (Sophian, 2008, p. 7). The Davydov approach in some ways mirrors the cultural practices of some Indigenous People. Davydov takes measuring as comparing, as an alternative approach to establishing the foundations of mathematical thinking in a school context.

Measuring as Comparing Connects to Properties of Equality

Introduction to early algebraic thinking via the comparison of nonnumeric quantities. The measuring approach has allowed us to reverse and refine the sequence of introducing early algebraic thinking prior to the concept of number. In our approach, students compare and explore with quantities, understand, and slowly formalize algebraic operations (addition, subtraction, division, and multiplication) through the comparison of nonnumeric quantities, experimenting with length segments represented by strips. Elders' comparison of quantities lends itself directly to classroom application. Such experimentation organically leads to generating, representing, and verifying the basic algebraic properties of inequality and equality such as those noted below:

Properties of Inequality

- $quantity\ a < quantity\ b$
- $quantity\ b > quantity\ a$

Properties of Equality

- **Reflexive**: $a = a$
- **Symmetric**: $if\ a = b, then\ b = a$
- **Transitive**: $if\ a = b\ and\ b = c, then\ a = c$
- **Substitution**: $if\ a = b, then\ b\ can\ be\ substituted\ for\ a\ in\ any\ expression$
- **Commutative Property of Addition**: $a + b = b + a$
- **Associative Property of Addition and Multiplication**: $(a + b) + c = a + (b + c)$
- **Division Property**: $if\ a = b\ then\ \frac{a}{c} = \frac{b}{c}$

A Curricular Example: Halving, Comparing, and Place Value

The *Measuring Proportionally* curriculum construction process follows the lessons that we learned from working with Indigenous cultures. Measuring as comparing, the binary nature of the halving algorithm, and the concept and process of measuring by *dividing a quantity by a unit of measure* is ideally suited for modeling place value understanding. We believe that Indigenous practices that highlight halving and comparing lend themselves to modeling place value through base 2. The process of "halving," which is intuitively accessible to young students, becomes a powerful algorithm for furthering students' understanding of numbers. Research has shown that many students do not have a generalized understanding of place value systems (Venenciano & Dougherty, 2014). Recent work suggests that for students to understand a positional place value system, they need to compare base 10 to other systems (Schmittau & Morris, 2004; Slovin & Dougherty, 2004). We use measuring, halving, and properties of equality, particularly equivalence substitutions, as a way to conceptually develop this understanding. We describe this process below.

Place value example. Before engaging in these activities, students would have had ample opportunity to measure lengths by units of measures; to generate numbers from comparing a quantity and a unit of measure; and to explore through project materials many of the properties of inequality and equality enumerated above. Students are provided with an unnamed length, but for purposes of

Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.). (2015). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. East Lansing, MI: Michigan State University.

this discussion, we skip the processes involved in naming and representing quantities. We will name this example Length D. Students will be given a few different strips equal in length to Length D. They will be asked to keep Length D whole. With the other strips, through a series of recursive folds (1, 2, and 3 folds), they will create strips with 2, 4, and 8 parts. The strips will be labeled as shown in Figure 11.

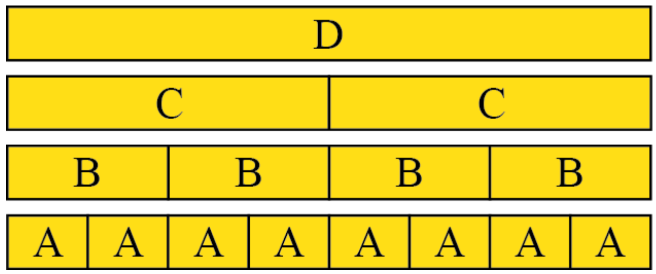
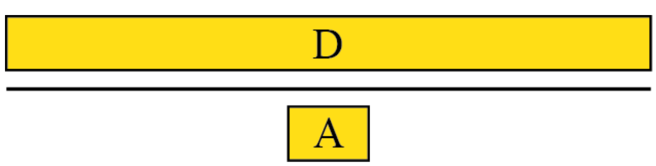


Figure 11:

Students will generate numeric values as they measure a quantity such as Length D by a unit of measure such as Length A. After they have had a chance to explore and communicate mathematically, students will be told that they are developing a Measuring Kit (Figure 12) based, in part, on the ways that Larry and Dora create their own measuring tools. Each Measuring Kit will contain only one of each length. Students will be challenged to measure various objects simply by using these lengths.

Figure 12:

Students will create a table of values by using Length A to measure the other quantities. This method results in $\frac{D}{A} = 8$, $\frac{C}{A} = 4$, $\frac{B}{A} = 2$, and $\frac{A}{A} = 1$.



$= \frac{D}{A} = 8$ and so on

Students can either create or account for the units using a recording table as they measure the specific objects. There are only two numbers in this system: a unit of measure is used or a unit of measure is not used. Classes that have piloted these lessons have used 0 to indicate that they did not use a particular length, and they have used 1 to indicate that they did use a particular length. Some classes used “Y” for *yes* (they used a particular length) and “N” for *no* (they did not use a particular

Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.). (2015). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. East Lansing, MI: Michigan State University.

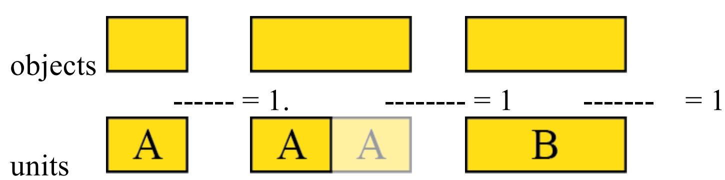
length). Each of these aspects of the activities on place value makes for interesting classroom discussions.

We challenge students to measure a set of objects. These objects are equivalent in base 10 lengths ranging from 1A to 16A. As students use their measuring kits to measure objects, they are faced with a few challenges, such as how to measure when your measuring kit has only one of each unit, and how to create a new unit of measure for your measuring kit when all of the units in the kit have been used. To facilitate solving these challenges, students receive a recording table (Table 1) and objects to be measured. For purposes of this presentation, we limited the number of objects to be measured.

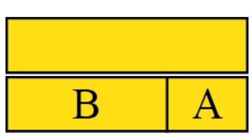
Table 1. Recording Table

Object to be measured	D	C	B	A
				1
			1	0
			1	1
		1	0	0

Length A (the length of A equals the length of the first object) is used to measure the first object one time. Students record 1 in the A column to show that one Length A was used. However, Length A is too small to measure the second object. Two Length A's would be needed to measure the second object. When students try to place 2 in the A column, they realize that there is space for one Length A and there is only one Length A in the kit. They can immediately observe that two Length A's = the second object, and that two Length A's = Length B. They can substitute two A's for Length B. Students record that they used 1 Length B and 0 Length A's. The specific measuring code for the second object is 10 (read as "one zero").



The next object is measured by (B + A) units, and thus it is recorded in the table as 11 (read as "one one").



The process continues as described above. Students measure multiple objects designed so that they continually experience the process of adding a unit, substituting the next unit from the tool kit, and moving from one place value column in a right-to-left direction.

Students continue this process until they reach a quantity that is exactly one unit longer than their entire set of measuring tools concatenated. They are then challenged to create a new unit of measure, Length E, that follows the system's exponential relationships: Length B is two times longer than

Length A; Length C is two times longer than Length B; Length D is two times longer than Length C. Now the students need to realize that Length E is twice as long as Length D. With these concrete objects and the structure of the recording table, the students are learning through measuring, building on prior knowledge related to properties of equality and equivalence substitutions. Only two numbers are allowed in each column (0 and 1), and each column to the left increases by a power of 2. In fact, this example models any positional number place value system, and may enhance understanding of base 10.

Discussion and Conclusion

This paper provides a glimpse into a long-term and continuing collaboration between Indigenous Knowledge holders—retired Yup'ik teachers who bridge Indigenous Knowledge and Western schooling—and academicians. We believe that the insights gained from everyday Indigenous activity and the ongoing intensive discussions among the Yup'ik cohort have enabled us to see how Indigenous Knowledge could have relevance for modern Western schooling. Measuring as comparing, and Dora's and Mrs. Andrew's folding algorithm, have implications for teaching aspects of the foundation of mathematics that go well beyond the scope of this paper. Our school-based curriculum development approach is attempting something not often done in academia or in the teaching of school mathematics: to use Indigenous Knowledge as a starting point for developing an approach to learning a core academic subject. Here we are building an elementary mathematics supplemental curriculum and learning trajectory from the perspective of Indigenous Knowledge. *Measuring Proportionally* is aided by insights gained from allied programs such as *Measure Up* and the work of Davydov and his colleagues. The halving algorithm and Euclidean-like way of generating the greatest common divisor are key aspects of our approach learned from Indigenous People, which distinguishes this approach from the work spearheaded by Davydov and his followers.

Yup'ik and Micronesian elders, in particular, use a process of *measuring as comparing and halving* to create tools and subunits across a wide spectrum of everyday activity. We are applying this process to our mathematical and pedagogical approach to teaching elementary school mathematics. Because the same processes are used in making the simplest of comparisons, such as direct comparisons of two quantities, to modeling place value, to comparing fractions and creating common units, our approach supports a cohesive way to teach aspects of elementary school mathematics. Similarly, our approach provides teachers with an integrative way of teaching foundations of mathematics. Measuring as comparing nonnumeric quantities establishes early notions of algebraic reasoning, and establishes relational units (as an early form of ratios). Proportional measuring and symmetrical folding are used in constructing geometrical shapes. The approach is both horizontally and vertically integrative. The same processes and halving algorithm used to compare two length quantities are also used to model and teach division of fractions, proportions, and scaling. Similarly, these cultural/mathematical principles are applied to Dora's and other elders' everyday construction of geometrical shapes and designs. Constructing a square and transforming a square into other planar shapes through symmetrical folding links the elders' knowledge to aspects of teaching geometry in school, including geometrical similarity (Lipka, Andrew-Ihrke, & Yanez, 2011). Scaling is invoked through many everyday activities within the Yup'ik culture and is taught in our program in the same fundamental ways as has been described for the comparison of quantities.

We believe that the mathematical pedagogical approach being developed by this project has potential to provide students and teachers with an elegant way to teach the foundations of mathematical thinking. This remains an empirical question.

The distance that this program has traveled—from its exploratory beginnings with elders to a more systematic collaborative study of everyday activity—continues to amaze and inspire those of us who are working in it.

Acknowledgments

This work has been generously supported by a grant from the National Science Foundation, Arctic Social Sciences Program, entitled *The Potential Contribution of Indigenous Knowledge to Teaching and Learning Mathematics*, 2012–2015 and from a grant from the U.S. Department of Education, Alaska Native Education Program, entitled *Measuring Proportionally: Elders' Wisdom Applied to Teaching and Learning Math,* 2012–2015. We also acknowledge the work of Gary Holton, linguist, University of Alaska Fairbanks, Judy Olson, mathematics education, University of Hawaiʻi, Mānoa, Keely Colby, graphics and editing, University of Alaska Fairbanks, Fran Pedersen, editing, University of Alaska Fairbanks, and Amy Topkok, project manager, Math in a Cultural Context, University of Alaska Fairbanks. We also acknowledge the Yup'ik elders who have worked with us for so many years.

References

Alkire, W. H. (1970). Systems of measurement on Woleai Atoll, Caroline Islands. *Anthropos*, *65*, 1–73.

Confrey, J., Maloney, A., Wilson, H., & Nguyen, K. (2010). Understanding over time: The cognitive underpinnings of learning trajectory. Paper presented at the *American Educational Research Association Annual Meeting*, Denver, CO.

Davydov, V. V. (2008). *Problems of developmental instruction*. Hauppauge, NY: Nova Science Publishers.

Dougherty, B. J., & Simon, M. (2014). Elkonin and Davydov curriculum in mathematics education. In *Encyclopedia of Mathematics Education*. 31 Jul 2014 ed., 204–207. Netherlands: Springer.

Engeström, Y. (1999). Activity theory and individual and social transformation. In Y. Engeström, R. Miettinen & R. Punamäki (Eds.), *Perspectives on Activity Theory* (pp. 19-36). Cambridge, UK: Cambridge University Press.

Gal'perin, P., & Georgiev, L. S. (1969). The formation of elementary mathematical notions. In J. Kilpatrick & I. Wirszup (Eds.), *Soviet studies in the psychology of learning and teaching mathematics: The learning of mathematical concepts*, 1, 189–216. Chicago, IL: University of Chicago Press.

Jacobson, S. A., & Alaska Native Language Center. (1984). *Yup'ik Eskimo dictionary*. Fairbanks, AK: Alaska Native Language Center, University of Alaska Fairbanks.

Kisker, E., Lipka, J., Adams, B. L., Rickard, A., Andrew-Ihrke, D., Yanez, E. E., & Millard, A. (2012). The potential of a culturally-based supplemental math curriculum to reduce the math performance gap between Alaska Native and other students. *Journal for Research in Mathematics Education*, *43*(1), 75–113.

Koester, D. C. (2014). Discursive path to Eureka. *International Congress of Arctic Social Sciences*, VIII, 246.

Lamon, S. J. (1999). *Teaching fractions and ratios for understanding essential content knowledge and instructional strategies for teachers*. Mahwah, NJ: Erlbaum.

Lipka, J., Andrew-Ihrke, D., & Yanez, E. (2011). Yup'ik cosmology to school mathematics: The power of symmetry and proportional measuring. *Interchange*, *42*(2), 157–183.

Lipka, J., Hogan, M. P., Webster, J. P., Yanez, E., Adams, B., Clark, S., & Lacy, D. (2005). Math in a cultural context: Two case studies of a successful culturally based math project. *Anthropology & Education Quarterly*, *36*(4), 367–385.

Lipka, J., with Mohatt, G. V., & the Ciulistet. (1998). *Transforming the culture of schools: Yup'ik Eskimo examples*. Mahwah, NJ: Lawrence Erlbaum Associates.

Lipka, J., Wong, M., & Andrew-Ihrke, D. (2013). Alaska Native indigenous knowledge: Opportunities for learning. *Mathematics Education Research Journal*, *25*(1), 129–150.

Lockhart, P. (2012). *Measurement*. Cambridge, MA: Belknap Press of Harvard University Press.

Moxhay, P. (2008). Assessing the scientific concept of number in primary school children. Paper presented at *International Society for Cultural and Activity Research.* San Diego, CA: ISCAR.

National Mathematics Advisory Panel (2008). *Foundations for success: The final report of the National Mathematics Advisory Panel*, 120: Washington, DC: U.S. Department of Education.

Schmittau, J., & Morris, A. (2004). The development of algebra in the elementary mathematics curriculum of V.V. Davydov. *The Mathematics Educator*, *8*(1), 60–87.

Slovin, H., & Dougherty, B. J. (2004). Children's conceptual understanding of counting. *International Group for the Psychology of Mathematics Education*, *4*, 209–216.

Sophian, C. (2008). *The origins of mathematical knowledge in childhood*. New York, NY: Routledge.

Sternberg, R., Lipka, J., Newman, T., Wildfeuer, S., & Grigorenko, E. L. (2006). Triarchically-based instruction and assessment of sixth-grade mathematics in a Yup'ik cultural setting in Alaska. *Gifted and Talented International 21*(2), 6–19.

Venenciano, L., & Dougherty, B. J. (2014). Addressing priorities for elementary school mathematics. *For the Learning of Mathematics*, *34*(1), 18–24.

Wikipedia, t. f. e. (n.d.). Euclidean algorithm. Retrieved August 2015 from *Wikimedia Foundation, Inc*. https://en.wikipedia.org/wiki/Euclidean_algorithm

Wu, H. (2011). On the teaching of fractions in elementary school. In *Understanding numbers in elementary school mathematics*. Providence, RI: American Mathematical Society.