USING PRECISE MATHEMATICS LANGUAGE TO ENGAGE STUDENTS IN MATHEMATICS PRACTICES

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This study examined discussions centered on precise mathematical language use in two fifth grade classrooms. Drawing on episodes from lessons in which teachers focused on encouraging mathematics reasoning, our analysis examines the relationship between precise language use and mathematical justifying. We present three classroom episodes that illustrate facets of the relationship between precise language use and mathematical argument, and explore how blurring the borders of two types of pedagogical tools (attention to precise mathematical language and engagement of students in justifying) promotes enculturation of students into mathematics.

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Using mathematical language precisely and with understanding is a complex endeavor and one with which many students struggle. In order for students to use language precisely, they must have a clear understanding of the underlying mathematical meanings and relationships associated with particular terms, as well as how terms may be used differently in common language.

Theoretical Framework

Learning Mathematics Language

Learning mathematical language is complicated by many factors. Mathematical terms are frequently used for specific and narrow ideas, whereas common language often uses terms broadly. Such terms may be confusing because the mathematical meaning may differ from the non-mathematical meaning, because multiple terms may be used to describe the same mathematical concept, or because two words sound the same (Kenney, Hancewicz, Heuer, Metsisto, & Tuttle, 2005). Thus, it is possible that a common meaning of a term may be more familiar to students than the mathematical meaning.

Furthermore, using mathematical language precisely can be a challenge for students because meanings of terms may have been obscured in prior lessons and discussions. Language used in mathematics classes has often focused more on the appearance of notation than on the mathematical ideas and relationships represented. These challenges may also lead teachers themselves to struggle with consistently using terms precisely and correctly.

Word knowledge is incremental, multi-dimensional, and interrelated, and words have multiple meanings. Students need multiple encounters with words in a variety of contexts and opportunities to build background knowledge that relates to the words in use (Spencer & Guillaume, 2006). However, vocabulary is not often taught in this manner. Common instructional practices provide students with definitions or ask students to look up definitions. As many definitions are written using words that are not meaningful to students, such approaches rarely lead students to the desired understandings. For terms associated with complex or abstract concepts, which include many mathematical terms,

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simply asking students to passively learn definitions has not been shown effective for robust vocabulary development.

One strategy for developing meaning for mathematical terms is to engage students in discussion of their understanding of particular concepts and of the precise meanings of various mathematical terms. In developing understanding of terms, Bransford, Brown, and Corking (1999) highlight the importance of involving learners actively in the generation of word meanings. It is important that learners connect words with concepts and with their prior knowledge. Integrating vocabulary development with other aspects of instruction, so that terms are encountered in context, and discussing connections students are making between terms and concepts are effective ways to support students in making needed connections. Through classroom mathematical discourse, students can develop understanding of mathematics terms and of the underlying mathematics relationships. Such discussions can be effectively embedded in other learning activities such as problem solving or mathematical investigations. “The richer and more varied students' experiences related to particular concepts, the more finely detailed and nuanced their understanding of related terms can be expected to be” (Spencer & Guillaume, 2006, p. 208). Barnett-Clark and Ramirez (2004) remind us,

> As teachers, we must learn to carefully choose the language pathways that support mathematical understanding, and simultaneously, we must be alert for language pitfalls that contribute to misunderstandings of mathematical ideas. More specifically, we must learn how to invite, support, and model thoughtful explanation, evaluation, and revision of mathematical ideas using correct mathematical terms and symbols. (p. 56)

Once mathematical terms are understood, their succinct meanings need to be reinforced throughout students’ experiences. For many teachers, engaging students in the *Common Core State Standards for Mathematics* (CCSSM) (National Governors Association Center for Best Practices, Council of Chief State School Officers [NGA & CCSSO], 2010) eight standards for mathematical practice represents a departure from their established teaching practices. CCSSM Practice 6, *Attend to precision*, involves students in attending to precision in a variety of ways, including precise use of mathematical language. “Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning…. In the elementary grades, students give carefully formulated explanations to each other” (NGA & CCSSO, 2010, p. 8).

Teachers can use development of precise language as a way to engage students in mathematical practices. While attending to precision and engaging students in mathematical practices such as justifying are both considered desirable teaching pedagogies (NGA & CCSSO, 2010; NCTM, 2014), they need not be considered as separate, unrelated pedagogies. The borders of each can be dissolved as the two pedagogies are used in connection with one another. One aspect of this connection is the use of precise mathematical language to act as a catalyst for engaging students in mathematical justifying.

Mathematical Justification

The practice of teaching and learning via mathematical argument/justification affords a practical context with ongoing opportunities for students to deepen their understanding of mathematics and use mathematical language precisely. In mathematical argumentation, the arguer must support or refute a claim with an argument that links the claim to the underlying mathematical principles and relationships. These ideas must be specified precisely and clearly in order for the argument to be understood and accepted by teacher and peers. The goal of peer acceptance of one’s argument presents a powerful motivator for students to learn precise meanings of mathematical terms and to use them appropriately.

Mathematical argumentation is itself an important practice for learning and doing mathematics. It

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is a key practice of professional mathematicians and is captured in CCSSM Mathematics Practice 3, *Construct viable arguments and critique the reasoning of others*. When engaged in this practice, students

> make conjectures and build a logical progression of statements to explore the truth of their conjectures….They justify their conclusions, communicate them to others, and respond to the arguments of others…. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments. (NGA & CCSSO, 2010, p. 7)

We view a viable argument as one with a clearly stated claim which is supported by reasoning showing how the claim follows logically from mathematical definitions and ideas that are known and accepted (Adams, Ely, & Yopp, in press). Yopp and Ely (2016) remind us that the argument's viability can only be ascertained when the definitions of the operations and objects in the claim are clear and agreed-upon.

Engaging students in mathematical argumentation benefits their learning in multiple ways. Yopp (2011) suggested that proving, one form of mathematical argument, can be used to understand a mathematical definition. Researchers have found that in mathematics classrooms where students are encouraged to explain and justify their thinking, learners demonstrated both greater achievement and more complex thinking (Boaler & Staples, 2008; Kazemi & Stipek, 2001). Sharing arguments promotes the mathematical understanding of the arguer and of the listening peers. Middle school teachers observed that when making an argument, students must interact with mathematical ideas, determine why a mathematical process works, connect ideas in new ways, refine their own thinking, and develop their mathematical communication skills (Staples, Bartlo, & Thanheiser, 2012). Additionally, student justification allows teachers to discern student thinking (Knuth, 2002) and provides information they can use in making instructional decisions. The teachers in Staples et al.'s (2012) study noticed that students who engaged in justification also developed general learning skills such as perseverance, independence, critical thinking, communication, and the expectation that ideas be supported.

Research Question

Through work with teachers to support students in mathematical reasoning, we have observed many instances of teachers engaging students in discussions of the precise meanings of mathematical terms. In this study we focus on mathematical discourse that centers on the precise meaning of mathematical terms or phrases. We examine the relationship between precise use of mathematical language and student justification. Specifically, we asked, what is the role of discussion of precise mathematical language in supporting student engagement in justification?

Methodology

The findings presented in this proposal are from an NSF-funded research and professional development project that focused on engaging students in mathematical generalization and justification. As part of the larger qualitative study, we examined two fifth grade mathematics lessons that contained class discussions of mathematical language. Both lessons were videotaped and occurred during the first month of the school year in rural school districts in the northwestern United States. While both teachers had an inclination towards focusing on precise mathematical language and justification, one teacher, Ms. K, had gone through two years of professional development as a result of the NSF-funded project. The other teacher, Ms. L, had been involved in a number of unrelated professional development experiences, but had not started work with the NSF-funded project. We selected three episodes from the two lessons in which discussion of mathematical language arose in conjunction with whole-class mathematical justifying. Each episode was analyzed

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to determine if and how the discussion of precise language was related to the mathematical justification. The three primary authors discussed the findings regarding the relationship between precise mathematical language and justifying until a consensus was reached.

Findings

We present here three episodes of classroom discussion that illustrate how conversations about precise mathematical language can be used to further student mathematical understandings and lead to student justification. All of the episodes depict a teacher asking students to clarify the mathematical meaning behind terms being used in class and all result in some type of mathematical justifying. The first two episodes are from Ms. K's lesson, and the third is from Ms. L's lesson. Episode one depicts a 15-minute discussion that capitalizes on students' lack of clarity to engage students in justifying why specific fractions are equivalent. Episode two depicts a 15-minute discussion that arose from a student's imprecise use of mathematical language, resulting in a justification of a commonly used generalization (attaching a zero to the end of a whole number when multiplying by 10). Episode three depicts an hour-long discussion that arose from multiple students' use of imprecise mathematical language, resulting in a justification of place value relationships. All names are pseudonyms.

Episode One: Equivalent fractions

In this episode from Ms. K's class, students were presented with a visual pie divided into 8 pieces, 4 of which were shaded, and asked, "What fraction of the pie is shaded?" Multiple students presented answers of 4/8, 2/4, and 1/2. In providing these multiple answers, students implicitly claimed that the three fractions represented the same quantity. Ms. K explicated this claim, writing on the board, "4/8 = 2/4 = 1/2". She then created disequilibrium by stating that the 3 fractions were all different. In making this challenge, Ms. K appealed to the appearance of the fractions as they were written, saying, "4/8 is not the same as 2/4. It's not the same as 1/2. What do I mean by *same*? What does he [student] mean by *same*? Four is not the same as two. … So how is this the same?" Through this challenge, Ms. K prompted students for a justification of why two fractions are equivalent. The opportunity for this justification arose from use of the imprecise and ambiguous term, *same*. While students did not actually use this term, Ms. K introduced it when explicating the students' implicit claim of fraction equivalence. It is apparent that Ms. K's intention during this episode was to highlight the distinction between the appearance of a fraction and the quantity it represents.

At the start of the justifying episode, a student used the phrase *equivalent fractions* as a justification of why the three fractions could be considered the same. While *equivalent* is a more precise term than *same* because it clarifies how exactly the fractions are the same (i.e. in the quantity that they represent, rather than appearance), Ms. K continued to encourage students to unpack the meaning of the word: "So *equivalent* has the root meaning "equal", right? But I've already said to you that 8 is not the same as 4." She continued to press for a more detailed justification by asking how the fractions could be the same. One student offered the explanation that one could divide 4/8 by 2/2 to get 2/4. Another said that all three fractions show the same ratio, saying, "The numerator is half of the denominator." Each additional student justification provided a more complete description of equivalent fractions for the class. Ms. K. continued the lesson as she elaborated on the difference between the appearance of a fraction and the value that it represents: "Four is just a digit. It's a digit. We don't know what its *value* is. … Is four always worth the same amount?" She emphasized her point that digits can represent different quantities depending on their place value by writing 40 and 400 on the board. At the end of the episode, Ms. K connected the mathematical ideas of place value and fractions by highlighting that each fraction could be considered as a division problem, which each yielded the same result.

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Episode Two: Add a Zero for Multiples of Ten

In this episode Ms. K serendipitously capitalized on a student's imprecise use of language to launch a discussion of what it means to "add a zero." She also used the discussion as an opportunity to teach conventions for precise use of mathematical language. In the process of unpacking the meaning of "adding a zero," a mathematical claim about relationships involved in finding the product of 6 and 20 was stated and evidence that could be used to justify the claim was documented. The result was a clear statement that they were not adding a zero; they were multiplying by ten. This discussion also helped students to understand that when discussing mathematics, they were expected to explicitly describe mathematical relationships and meanings of operations.

The episode began when Carl described a method for finding the product of 6 and 20: "I multiplied 6 times 2, which is 12, and then I added zero." Ms. K took the mathematical meaning of his words literally, while ignoring the implied meaning regarding how to write the answer. She wrote 6 x 2 = 12 + 0 = 12 and then said, "He is saying the answer to 6 x 20 is 12. Do you agree? Why not?" Amber said, "Because you don't add a zero. You put a zero behind the 12." Ms. K interpreted this statement broadly, writing 12   0, and said, "That's a zero behind it." She then asked, "Why is Ms. K being so silly here?" Kylee responded that sometimes students put things in the wrong places. The teacher agreed and said, "I don't like it when people say: 6 x 2 and add a zero. You are not adding a zero. What are you actually adding?"

The students clearly had a habit of describing changes in the way the number visually appears when written rather than the mathematical meaning or relationships involved, and therefore did not understand that Ms. K's intention was to show that they were multiplying by ten, and not adding a zero. Another student said, very precisely, "You are adding a zero to the one's place." Ms. K again took the words literally and wrote: 12 + 0. She said, "I'm adding a zero to the one's place," and then prompted, "You know what you are doing. I want you to be explicit in your thinking" and asked them to talk with a partner.

Ms. K acknowledged the strategy of first multiplying by two as "brilliant" because "it uses multiplication facts we already know." She asked, "What are we really doing here? What's the difference between 120 and 12?" Anna replied, "Zero." Ms. K asked, "What is the value of that zero?" Zoe explained, "Zeros, when added onto a number, it kind of transforms the number. If you add on two zeros – in the tens and hundreds place, it transforms it to one hundred. Zeros are pretty much kind of magic. Cause without zeros, we would be stuck with pure numbers." This response gave Ms. K the opportunity to make explicit a mathematical convention for her class: that there are reasons for the way things happen, and it is important to uncover and talk about those reasons, and to explain why. She said, "Magic makes it seem like there is some trick to it, and not maybe a reason, and that's not true at all. … I always want you to know why you are using a trick. … So I wanted you to understand that when you say *add a zero* what you are really saying is what?" Liam responded, "I noticed something. It takes ten 12's to make 120." This idea is what Ms. K had been waiting for. She emphasized its importance by writing ten 12's in a column, counting them as she wrote. She also wrote 12 x 10, and repeated, "We are not adding a zero. Chloe said, "We are multiplying by ten," which ended this episode.

Episode Three: Add a Zero for Place Value

This episode began when multiple groups of Ms. L's students worked on a task and then described a pattern they noticed among a list of numbers (specifically, 3, 30, 300, 3000, 30,000, 300,000). During the class discussion, Bob said, "I was just thinking that there was ones with added zeros." Ms. L repeated his phrase "ones with added zeros," which was used by multiple students prior to the class discussion, and then asked, "Do you think that's really true though? Ones with *added* zeros? Are we adding zeros?" Bob responded, "No, you put zeros on them." Ms. L again repeated Bob's wording and implicitly asked students to explain the meaning behind the words. "You

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put zeros at the end. Okay, but I wonder what that means that we put zeros at the end." She stated, "If we start with three, and we put a zero at the end, we aren't really adding zero, are we?" and then asked the class "What are we doing if we start with three, and we write 30?" As students did not quickly answer, Ms. L encouraged them to describe the meaning of three, to which a few students responded "three ones." She followed by asking what does thirty mean, to which Alex answered "three tens." Ms. L continued the pattern and asked, what does three hundreds mean, and Frank answered "three hundred."

Continuing the justifying activity, Ms. L asked students to discuss how to "get from 3 to 30," clarifying her desire for them to use "multiplication thinking, not addition thinking." Ms. L summarized multiple student responses as she said multiply 3 by 10. Continuing the justification, Ms. L asked how students could get from 30 to 300. Following group discussions, Beth said her group multiplied 30 by 10 to get 300. Ms. L pointed to a previously recorded pattern on the board, and explained that because they said 300 is 3 times 100, they could move the parentheses and say that 300 is 3 times 10 times 10. This move is from (3 x 10) x 10 to 3 x (10 x 10), although Ms. L points rather than saying this explicitly.

As an attempt to encourage understanding of her explanation, Ms. L asked Ted about the solution to 10 times 10, and he answered "100." Another student in Fred's group added that they multiplied 3 by 10 to get 30, and 30 by 10 to get 300, and that one could keep going with that pattern. Ms. L stated that if they multiplied by another 10, it would be 3 times 10 times 10 times 10, and asked students "what is 10 times 10 times 10? " Multiple students responded, "1000," and Ms. L summarized their argument: "Here's our first 10, right? 3 times 10 is 30. And 300, you guys said just add a zero and I said I don't think we can add a zero. So then you said we're multiplying by 10. So we have 300 equals … 3 times 10 times 10. And so you guys are saying each time we multiply it by 10, we have to put in another times 10 each time, don't we?"

Ms. L rephrased her summary: "3 equals 3 times 1. 30 equals 3 times 10. And you guys told me that to get from 3 to 30, I would have to multiply 3 by 10. Then 300, you guys said add a zero, but no, we can't add a zero, so we multiplied 30 by 10 and got 300. And so we said that we could say that that's 3 times 10 times 10. So it's either 30 times 10 or 3 by 100, either way. 3000 is 3 times 10 times 10 times 10. We see a pattern developing. We have one 10 here, two 10's here. And we're not adding tens, we're doing what with our tens?" Students said "Multiplying," and Ms. L responded, "We're multiplying each time by 10. Then, you just said that 30, 000 is 3 times 100 times 100 (a student previously and silently had recorded this on the board). Can he do that? Because 3 times 10 times 10 times 10 times 10 is that the same as saying 3 times 100 times 100? Just for the sake of time, I'm gunna say it is. We will have to talk about that in a couple of days. And then another student said 300,000 is 3 times 10 times 10 times 10 times 10 times 10." A student noted, "Five times." And Ms. L capitalized on the observation: "Five times. Here we had it 4 times; here we had it 3 times; here we had it 2 times; here we had it 1 time."

As time for the class period is running out, Ms. L connected their findings to a place value chart, showing students that each time "you move to the left" you are multiplying by ten, and asked students to think about what happens if you "move to the right." Most of the students responded with "divide" and "by 10" in unison as the lesson ended.

Discussion

Eliminating the borders between the two pedagogical practices of attending to precise language and constructing viable arguments can help improve student engagement in both types of mathematical activity. For instance, the Fraction Episode illustrates how a better understanding of the mathematical meaning behind fraction notation enables students to justify why specific fractions are equivalent. The use of precise language allows for the creation of a viable argument, which subsequently allows students to better understand the appropriate use and importance of precise

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mathematical language. As these episodes illustrate, teachers can capitalize on the imprecise language often used by students as a way to engage them in unpacking mathematical meanings and justifying mathematical properties or algorithms that are often taken for granted, such as the algorithm of appending a zero to the end of number when multiplying by ten. By increasing their own awareness of students' use of imprecise language, teachers can identify opportunities to engage students more frequently in creating viable arguments or justifications. Additionally, attending to the use of precise language develops an understanding of the culture and nature of mathematical language. While many students may be able to use mathematical terms, they need the additional understandings of *how* and *when* to use mathematical terms. For instance, it is important for students to understand the distinctions between the terms *fraction* and *ratio*, and recognize appropriate circumstances for using each.

Beyond simply knowing the correct term, another insight these three episodes provide is that common language can conflict with mathematical language, and that the pedagogical tool of attention to precision allows teachers to distinguish between the two types of language. For instance, when a person in an everyday context says, "add a zero on the end of that number," it is typically understood as the action of appending a zero. However, mathematical accuracy would require one to say "multiply the number by ten." When common language contradicts mathematical language, it creates a conflict within the class and often promotes misunderstandings about the mathematics. Focusing on the correct use of mathematically precise language helps to engage students in mathematical justifying, and the resulting focus on precision will advance students' skill with future justifications. In other words, use of precise language is a requisite skill in order to develop mathematically valid justifications. Therefore, if we want students to be able to justify, we also need them to develop the ability to use precise language when they are in a mathematical context. When mathematical language conflicts with common language, teachers need to explicitly clarify the mathematical language and consistently use it in classes. This will develop students' abilities to think mathematically and use such thinking to outline mathematical arguments at the proper times. Gaining skill in using mathematically precise language will allow students to create arguments and justifications and do so using mathematical language.

The Add a Zero for Place Value episode was an hour-long lesson that developed student understanding of the precise meaning of the use of zero within the place value number system. It is clear that students assumed everyone knows what it means to "add a zero" to the end of a number, but as Ms. L asked them to unpack the meaning behind the common phrase, it became evident that the class lacked the ability to do so. This episode illustrates the difficulty of developing a mathematically accurate interpretation of common language that is used regularly and taken as understood. As conversations about mathematical meanings and the use of precise mathematical language increase in mathematics lessons, the opportunity for students to develop a deeper understanding of mathematical language and relationships increases as well.

All three episodes point to instances where students seemed to know the correct mathematical language because they used commonly understood terms at the correct time; however, when each teacher attended to the precision of the mathematical language, there was clearly more understanding to be developed. In these episodes, this was accomplished through an unpacking of the mathematical language used within the lesson. We recommend that teachers use the pedagogical tools of attention to precision and encouragement of viable arguments and justifications in conjunction with one another to help students develop deeper mathematical understandings as well as to enculturate students into the mathematical community.

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