(IR)REVERSABILITY IN MATHEMATICS

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In this theoretical paper, I consider reversibility as a defining characteristic of mathematics. Inverse pairs of formalized operations, such as multiplication and division, provide obvious examples of this reversibility. However, there are exceptions, such as multiplying by 0. If we are to follow Piaget's lead in defining mathematics as the science of reversible mental actions, such exceptions must be examined. We consider the case of multiplying by 0 by adopting Davydov's model of multiplication as a transformation of units and by investigating the underlying mental actions. Results of this investigation have implications for breaking down the barriers between various domains of mathematics.

Keywords: Cognition, Learning Theory

As mathematics educators, our ultimate goal is to support students' development of mathematical knowledge. We can define this body of knowledge culturally, as the processes and products in which mathematical communities engage (cf., Harel, 2008), but this definition is circular (how do we recognize these communities as mathematical?). Dictionaries define mathematics as "a group of related sciences, including algebra, geometry, and calculus"; or the like ('mathematics', 2016). Such definitions defy the unity of mathematics and the emersion of new branches of mathematics, such as algebraic topology and game theory. Compare those definitions to the definition of biology: "the study of living organisms" ('biology', 2016). Like mathematics, biology has numerous branches, but those branches are unified under the umbrella of living organisms. If mathematics is a science, we should be able to define its objects of study in a unified manner—one that crosses the boundaries of its various branches.

Piaget (1970) defined mathematical objects as products of coordinated mental actions. In particular, logico-mathematical actions (operations) are characterized by their composability and reversibility. Composability empowers mathematical reasoning with the possibility of combining chains of mental actions. For example, students who *count on* can take a result from counting and combine it with further acts of counting to reach a new result. Reversibility guarantees perfect reliability in mathematics: "Because every operation is reversible, an 'erroneous result' is simply not an element of the system" (p. 15). Indeed, mathematics education researchers have studied reversibility as a critical aspect of students' development of mathematical reasoning (Greer, 2011; Hackenberg, 2010; Simon, Kara, Placa, & Sandir, 2016). By inverting mental actions, such as those involved in counting, students can return to the previous result, from which they can count on, again, with assurance that they will reach the new result again. Note that no other science has perfect reliability because the objects of study (e.g., living organisms) are derived from experimental observations (rather than mental actions)—experiments that cannot be repeated with perfect precision.

As examples of mathematical objects, consider the cube and the number 5. According to many definitions of mathematics, these objects are categorically different (shape and number), but Piagetian theory demonstrates that both arise from the coordination of composable and reversible mental actions. No one has ever seen a cube; what we see are two-dimensional projections. However, we know the three-dimensional object perfectly, through coordinated mental actions, such as rotations (Piaget & Inhelder, 1967; Roth & Thom, 2009). "Children are able to recognize and especially to represent, only those shapes which they can actually reconstruct through their own

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actions. Hence, the 'abstraction' of shape is achieved on the basis of co-ordination of the child's actions and not, or at least not entirely, from the object direct" (Piaget & Inhelder, 1967, p. 43).

Mental rotations are composable and reversible, and by coordinating them, we can imagine the whole cube at once. Likewise, 5 arises through the coordinated activity of pointing and reciting a verbal number sequence, "one, two, three, four, five" (Piaget, 1942). This activity generates a one-to-one correspondence between items that are taken as units of 1 and elements of the verbal number sequence. On that basis, students develop the mental action of iterating a unit of 1 "five" times—a mental action that can be reversed by partitioning the collection into the constituent five units of 1 (Steffe, 1992).

As mathematical objects, students can act on the cube and the number 5 in new ways—ways that might not be possible with physical objects. For example, they can reflect a cube about a plane through its center (see Figure 1). This action is an involution; it reverses itself. Acting on mathematical objects in new ways provides the basis for constructing new mathematical objects, at a higher level, and this is the sense in which mathematics builds upon itself.

Consider the number 5 again. As a unit containing five units of 1, 5 is an object that can be acted upon through mental actions associated with multiplication (Steffe, 1992). By coordinating these mental actions, products of two numbers can become objects in themselves.

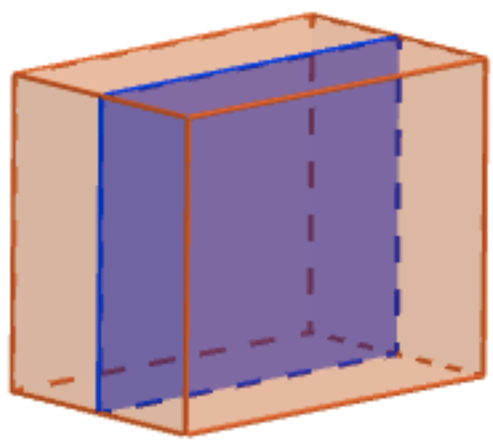

Figure 1. Reflecting the Cube about a Plane through its Center.

When we consider formalized operations, such as addition and multiplication, it may seem obvious that mathematical operations are reversible (via subtraction and division, respectively). However, reversibility does not refer to formalized operations but, rather, to the mental actions that undergird them. Moreover, some formalized operations are not reversible; consider non-invertible matrix transformations, constant functions, and multiplying by zero. If we are to define mathematical objects—including matrices and functions—as the coordination of reversible mental actions, these examples must be examined. Here, we consider the simplest example: multiplying by zero.

The purpose of this paper is to investigate the apparent irreversibility in the case of multiplying by 0. We use that example to consider other cases of apparent irreversibility. Finally, we consider the educational implications of Piaget's definition of mathematical objects. As such, we address the conference theme of questioning "borders between mathematical content areas" and how those borders "limit access to mathematical content."

Multiplying by 0

Multiplying any real number by 0 yields a product of 0. From that product, we cannot uniquely

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determine the real number with which we began. In this sense, multiplying by 0 is an irreversible formalized operation. We investigate this example further by considering the mental actions that undergird the formalized operation. We begin by defining multiplication as a transformation of units. Then, we relate that definition to the mental actions involved in units coordination. Thus, our investigation involves examining the mental actions associated with units coordination and the roles they play in transforming units.

Multiplication as a Transformation of Units

Davydov (1992) defined multiplication as a transformation from one unit of measure to another. Building from this definition, Boulet (1998) too sought to break down boundaries—boundaries between the various contexts in which multiplication arises. She demonstrated that, by defining multiplication as a transformation of units, researchers could understand the principal commonality in multiplying whole numbers, integers, rational numbers, and irrational numbers. Whereas repeated addition fits concrete models often used to introduce multiplication, the transformation of units explains what distinguishes multiplicative reasoning from additive reasoning.

Consider the product, B×A=C, as illustrated in Figure 2. We know that C is B measures of the unit A, but we want to transform the relationship to determine C as measured in units of 1 (Davydov, 1992). Thus, we are interested in a transformation of units, from units of A to units of 1. This model explains the efficacy of repeated addition when A and B are whole numbers, but the transformation also highlights what is essentially multiplicative about multiplication. Namely, to reason multiplicatively, students need to coordinate three different *levels* of units (C in units of A, and A in units of 1), not just two levels (A as A 1s) being repeated at level of 1s. Research on students' development of units coordination helps us to understand the distinction, as well as the mental actions associated with transforming units.

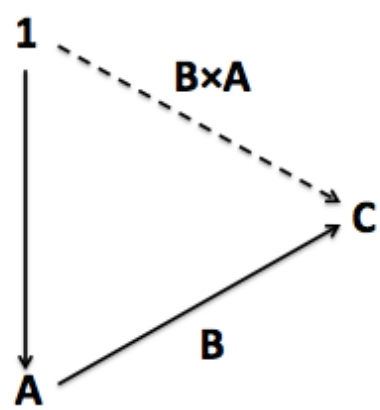

Figure 2. Davydov's Model of Multiplication.

Units Coordination

Units coordination refers to the number of levels of units that a student maintains when acting in a numerical situation (Steffe, 1992). Levels refer to the way numbers are embedded within each other. At the lowest level, there are 1s, from which all other numbers are constructed, primarily through counting. From units of 1, students can construct composite units—units containing other units. For example, students who coordinate two levels of units can consider 5 as a composite unit, made up of five 1s, any of which can be iterated (repeated) five times to produce 5. As a composite unit, 5 as five 1s is immediately available to the student in conceptualizing 5. Furthermore, students

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can act on 5 as an object. For example, students can iterate the composite unit, 5, to produce a unit of units of units.

A student who has constructed composite units can also determine the number of 5s in 35 without relying on memorized facts and without having to start from 1s. She can iterate 5 seven times, simultaneously keeping track of these iterations as iterations of 5 and iterations of five 1s—two different levels of units—while building up the third level (35 as seven 5s). As such, she knows that when she has produced seven 5s, she has also produced 35 1s (see Figure 3). Thus, by iterating the composite unit, 5, she has produced 35 as a unit of seven units of 5, each of which contains five 1s—three levels of units.

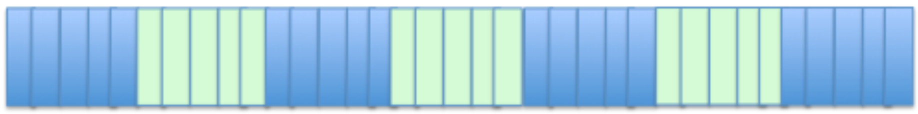

Figure 3. 35 as a Unit of Seven Units of Five 1s.

Mental actions that support units coordination include unitizing, disembedding, partitioning, iterating, and distributing (Steffe, 1992). *Unitizing* refers to the mental action of taking an item, or collection of items, as a whole unit that can be further acted upon. *Disembedding* refers to the mental action of taking a sub-collection of items without destroying the whole; the sub-collection exists simultaneously as part of the whole and as a part out of the whole. As such, unitizing and disembedding can be organized as inverse actions within a structure for assimilating and coordinating units: a collection of parts can be unitized as composite unit and, inversely, any number of those parts can be disembedded from the composite unit while maintaining a part-whole relationship.

Partitioning and iterating from another pair of reversible operations (Wilkins & Norton, 2011). *Partitioning* refers to breaking a whole into equally sized parts; *iterating* refers to making connected copies of a part. These operations are inverses because iterating a part can reproduce the whole, and partitioning an iterated part reproduces that part.

Distributing refers to inserting the units within one composite unit, into each of the units in another composite unit (Steffe, 1992). With regard to Figure 3, the five units of 1 within 5 are inserted into each of the seven 1s within 7. This can involve iterating the composite unit, 5, seven times, or recursively partitioning each of the seven parts (1s) within 7 into five new parts. Either way, distributing relates to Davydov's definition of multiplication, as illustrated in Figure 4.

Figure 4. Coordinating Units of Measure.

When A and B are positive integers, we can consider the yellow bar as a composite unit containing A units of the blue bar, and we can consider the red bar as B iterations of that composite unit. Thus, there are three levels of units to consider: the blue bar as a unit of 1; the yellow bar as a composite unit; and the red bar as the result of iterating that composite unit. As such, the yellow bar

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acts as an intermediate unit for measuring the red bar. We know the measure of the red bar in units of the yellow bar, but we want to know its measure in units of the red bar. To determine this measure, we can transform the yellow bar as the unit of measure into the blue bar as the unit of measure by partitioning each iteration of the yellow bar into A units of the blue bar. If the partitioning were done sequentially, it would amount to nothing more than repeated addition. However, if the partitioning occurs across all of the yellow bars at once, we have a multiplicative transformation of units.

Mental Actions that Comprise Multiplying by 0

In Davydov's (1992) definition of multiplication, commutativity is not taken for granted. Thus, the case of multiplying by zero occurs as two subcases: 0×A, where 0 is the multiplier; and B×0, where 0 is the multiplicand (see Figure 5). We consider these subcases separately.

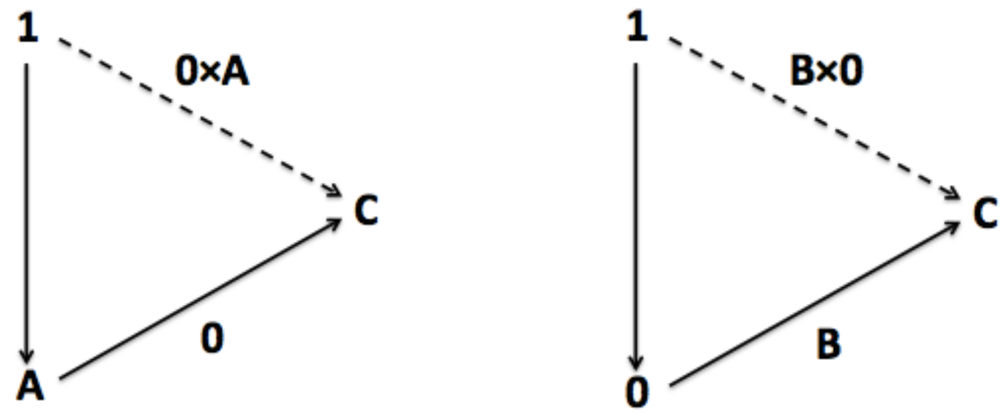

Figure 5. Zero as Multiplier (left) and Multiplicand (right).

When 0 is the multiplier, the transformation of units (between units of 1 and units of A) is reversible. This reversibility relates to partitioning and iterating as inverse mental actions: the composite unit, A, is A iterations of 1, which can be partitioned into A parts to reproduce 1. Whether we iterate A zero times or iterate 1 zero times, we produce the same result (0). This kind of many-to-one mapping is the root cause for irreversibility of formalized operations, in general, and we return to that issue in the discussion section. However, in the subcase under consideration, the transformation of units itself is reversible.

When 0 is the multiplicand, the transformation of units is irreversible because a unit is lost; the unit of 1 cannot be recovered from 0. In fact, strictly speaking, there is no transformation of units because 0 is not a unit of measure; it has no quantity, measure, or extent. We must rely upon other forms of logico-mathematical reasoning to determine the product, thus completing the formalized system of multiplying two non-negative integers. We know that C is B units of 0 and, from that, we deduce that C is 0 units of 1. Here, reversibility takes the form of reciprocity, rather than inversion.

Piaget (1970) distinguished two forms of reversibility: inversion and reciprocity. The integration of these two forms undergirds children's construction of number (Piaget, 1942). Up to this point, we have been considering inversion alone, wherein one mental action undoes another (or itself, in the case of involution). Reciprocity constitutes a form of reversibility wherein one mental action compensates for another. For example, among ordering relations, "5 succeeds 4" is reversed by way of reciprocity, in the form, "4 precedes 5." In the subcase at hand, reciprocity demands that B×0=0 because B×0 must map 1 to 0 again in order to compensate for the unit lost in the initial transformation. The mental action represented by each mapping/transformation is *projection*, which

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generally conflates units but also can annihilate them. Thus, in the second subcase, we do not have a transformation of units, but an annihilation of units, by way of projection.

Discussion

According to Piagetian theory, mathematical objects arise through coordinated mental actions, and actions on mathematical objects can become coordinated as new objects (Piaget, 1970). This is the sense in which mathematics builds upon itself. Composite units become objects for students, through coordinated mental actions of unitizing, disembedding, partitioning, and iterating (Steffe, 1992). Once students have constructed composite units, they can act on those objects through new mental actions, such as distributing. By transforming units, this mental action supports the construction of a formalized operation for multiplication, as well as its products (Davydov, 1992).

A key aspect of coordinating mental actions is the reversibility of those actions. Reversibility provides for perfect reliability in mathematics (Piaget, 1970). In some cases, this reversibility is apparent in the formalized operation. However, in cases like multiplying by 0, the formalized operation is irreversible. Nonetheless, when we consider the underlying mental actions in one of the subcases (0×A), the transformation of units is reversible, through inversion of partitioning and iterating. In the other subcase (B×0) there is no corresponding transformation of units. Rather, we must rely on a different form of reversibility—reciprocity—to generate the product.

Determining Bx0 without a transformation of units is a means of completing the formal system of multiplication for non-negative integers. The system can be further extended to all integers, and even complex numbers, by considering directed quantities (Boulet, 1998). In the subcase of B×0, the trouble in transforming units arises from the fact that the primary unit of measure, 1, is lost. Geometrically, we can think about this subcase as a projection of the entire continuum to a single point, 0. This is precisely what happens in the case of non-invertible matrix transformations and constant functions; an independent unit is lost by way of projection. In fact, the same projection can be represented by the 1×1 matrix, [0], or the constant function, $f(x)=0$.

When we consider how projection affects units and values based on those units, the result is irreversible, as it is for any many-to-one relation, because there is no way to uniquely recover any of the many values from the one value. However, the mental action of projection, itself, is reversible. In geometry, we see the inverse action in the form of sweeps: A point (*P*) can be swept to produce a line segment (*l*); that line segment can be swept in another direction (*l'*), to produce a square area; and that square area can be swept in yet another direction to produce a cubic volume (see Figure 6). Inversely, projections in those directions collapse the cube into a square; the square into a line segment; and the line segment into a point.

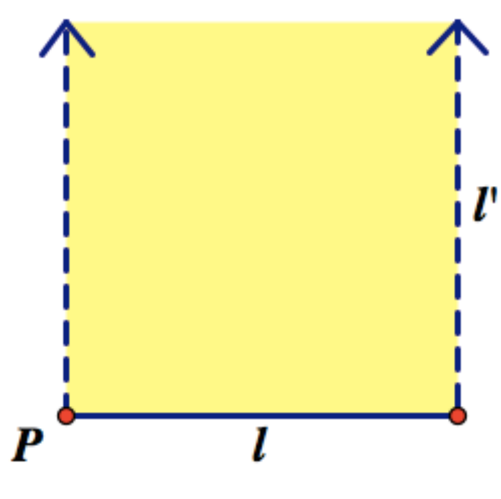

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Figure 6. Sweeping Point *P* and Segment *l*.

Implications

Borders between content domains limit access to mathematical content by focusing students' attention on superficial features within each domain, rather than supporting students' development of mental actions that cross those domains. As the study of reversible mental actions, mathematics is a unified science, and we can teach it as such. For example, units coordination involves the coordination of several mental actions that undergird students' knowledge of whole number (Steffe, 1992), fractions (Steffe, 2002), integers (Ulrich, 2012), algebra (Hackenberg & Lee, 2015), and geometry (Battista & Clements, 1996; Wheatley, 1992). Moreover, we can support students' construction of higher levels of units within various domains (Boyce & Norton, in press; Norton & Boyce, 2015). In the case of multiplying by 0, we find further connections to projections in geometry.

If reversible mental actions are the objects of study in mathematics, then students' mental actions need to be the focus of research within mathematics education. What mental actions are available to students? What activities will help students to reverse those actions and to compose them in new ways, in order to construct mathematical objects? Conveying mathematics as a unified science to students can support their creativity, as it has for professional mathematicians. Many of the intractable problems in mathematics have been solved by crossing domains. Specifically, proofs concerning the geometric construction problems of antiquity (e.g., the impossibility of trisecting angles with compass and straightedge) came millennia later, in the form of Galois theory, within abstract algebra.

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