STUDENTS' CONCEPTIONS SUPPORTING THEIR SYMBOLIZATION AND MEANING OF FUNCTION RULES

Nicole L. Fonger
University of Wisconsin-Madison
nfonger@wisc.edu

Amy Ellis
University of Georgia
amyellis@uga.edu

Muhammed F. Dogan
Dicle University
fatih.dogan@dicle.edu.tr

This paper explores the nature of students' quantitative reasoning and conceptions of functions supporting their ability to symbolize quadratic function rules, and the meanings students make of these rules. We analyzed middle school students' problem solving activity during a small group teaching experiment (n=6) emphasizing quadratic growth through covarying quantities. Results indicate four modes of reasoning supportive of students' symbolization of quadratic function rules: (a) correspondence, (b) variation and correspondence, (c) covariation, (d) flexible covariation and correspondence. We discuss implications for research on learning vis-à-vis students' representational fluency, as well as design principles to support quantitative reasoning.

Keywords: Algebra and Algebraic Thinking, Cognition, Middle School Education

Research Issues and Purpose

Students' understanding of functions is central to school algebra in which two perspectives on algebraic thinking are valued: a change and variation perspective, and a structural or symbolic perspective (Cai, Bie, & Moyer, 2010). An oft-cited goal of students' activity in algebra, and functional thinking particular, is to cultivate the ability to create, interpret, and connect numeric, graphic, symbolic, and verbal representations of functions (Kieran, 2007). Indeed, there is a large body of work addressing students' understanding of function vis-à-vis the study of students' abilities to translate between multiple representations, especially from symbolic to graphical representations of function (e.g., Leinhard, Zaslavky, & Stein, 1990). Such research often focuses on students' representational performances (e.g., abilities to connect multiple representations) to the detriment of understanding what representations *mean* to students and how this representational activity is supported or constrained by students' *conceptions* of mathematical ideas (Thompson, 1994b).

In this research we explore the interplay between students' representational abilities and their conceptions and meanings of functions. We focus on students' symbolization of quadratic functions of the form $y=ax^2$, a topic commonly taught using a "method of finite differences," with the meaning of a restricted to determining the steepness of the parabola (Ellis & Grinstead, 2008). In contrast to these common approaches, we ground our investigation of students' understandings of quadratic functions in a quantitatively rich context as a source of possible meaning for students' activity (Thompson, 1994a). We address three research questions:

- What is the nature of students' quantitative reasoning about quadratic growth situations?
- What ways of reasoning *support* students' abilities to symbolize quadratic function rules?
- How do students *make sense of* the symbolic quadratic function rules they write?

We first articulate three mutually supportive lenses that guided our interpretations of student thinking: quantitative reasoning, conceptions of function, and representational fluency.

Theoretical Framework and Background

Quantitative Reasoning and Conceptions of Functions

By quantities, we refer to measurable attributes of objects or phenomena (Smith & Thompson, 2007). A quantity is a mental concept, composed of one's conception of an object, a quality of the

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

Articles published in the Proceedings are copyrighted by the authors.

object, an appropriate unit or dimension, and a process for assigning a value to the quality (Thompson, 1994a). Examples of attributes that can be conceived as quantities are height, length, and area. Students engage in quantitative reasoning when they operate with quantities and their relationships, conceiving new quantities in relation to one or more already conceived quantities. Comparing quantities multiplicatively is a critical aspect of quantitative reasoning.

A correspondence perspective of function identifies function relationships such as $y = ax^2$ as the fixed correspondence mapping between the members of two sets (Farenga & Ness, 2005; Smith, 2003). From this perspective, pairs of quantities are linked by a multiplicative relationship. Although this static view of function is privileged in school mathematics curricula, some researchers argue that a covariation approach to functional thinking is important in supporting a well-developed understanding of function as a dynamic relationship between quantities (e.g., Confrey & Smith, 1994; Thompson & Carlson, in press).

One stance on covariation involves the examination of a function in terms of coordinated changes of *x*- and *y*-values, in which students move operationally from y_m to y_{m+1} coordinating with movement from x_m to x_{m+1} (Confrey & Smith, 1994; Smith, 2003). This perspective requires that students understand quantities as having a sequence of values and relate the values in each sequence additively or multiplicatively. Thompson and Carlson (in press) instead emphasize the importance of helping students envision change through the notion of *continuous variation*. Students who can think about smooth continuous variation can imagine a variable's magnitude increasing in bits while simultaneously anticipating that within each bit, the value varies smoothly. From this stance, students engage in *covariational reasoning* when they can envision the values of two quantities, such as the height and the area of a growing rectangle, varying together (Thompson & Carlson, in press). In this study we will characterize students' reasoning as *covariational* when they attend to coordinated change across two or more quantities. This does not mean the students necessarily thought about continuous variation; in many cases, their understanding of variation was likely chunky, or even discrete. However, we characterize reasoning as covariational when students used language and gestures that suggested images of a rectangle's height values and area values simultaneously varying together.

Representational Fluency and Meaning Making

Zbiek, Heid, Blume, and Dick (2007) define representational fluency as "the ability to translate across representations, the ability to draw meaning about a mathematical entity from different representations of that mathematical entity, and the ability to generalize across different representations" (p. 1192). In this study we focus on both students' translations (i.e., the correct creation and interpretation of quadratic function rules from tables, words, or diagrams), and what these symbolizations *mean* to students. What one is able to "see" in a representation is supported and constrained by what one knows (Piaget, 2001). Thus we adopt a constructivist stance in building second-order models of students' mathematics (Steffe & Olive, 2010), making inferences about students' meaning making as opposed to first-order models of a researcher's meanings of representations.

Methods

Teaching Experiment

We conducted a 15-day videotaped teaching experiment (Steffe & Thompson, 2000) with 6 middle school students in an after school setting. The teacher-researcher (TR, Ellis) taught all teaching sessions, each lasting 1 hour. All sessions were transcribed and pseudonyms were assigned to all participants. One purpose of the small-scale teaching experiment is to gain direct experience with students' mathematical conceptions and the change in those conceptions over time (Simon,

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

Articles published in the Proceedings are copyrighted by the authors.

1995). Our aim was to study the factors promoting students' algebraic generalizations as they explored rectangles that grew proportionally by maintaining the same height-to-length ratio. The teaching-experiment setting supported the development and testing of hypotheses about students' understanding in real time while engaging in teaching actions. Thus, the mathematical topics for the entire set of sessions were not pre-determined, but instead we created and revised new tasks on a daily basis in response to hypothesized second-order models of the students' mathematics.

Task design. All tasks were grounded in the growing rectangle context, in which the relationship between the height, h, and the area, A, can be expressed as $A = \mathrm{a}h^2$. Students worked with computer simulations of the growing rectangles, drew their own rectangles, created tables representing the heights, lengths, and areas of growing rectangles, and created and justified algebraic rules comparing the rectangles' areas with their heights (Figure 1).

Task a. Here is a table for the height versus the AREA of a rectangle that is growing in proportion:

Height	Area
3	6.75
4	12
5	18.75
6	27
7	36.75
8	48
50	?
h	?

Task b. Here is a table for the height versus the AREA of a rectangle that is growing in proportion:

Height	Area
2	16
4	64
6	144
8	256
10	400
65	?
1/2	?
h	?

Figure 1. Sample far prediction and generalization tasks.

Data Sources and Analysis

Data for this study included video, transcripts, and PDFs of students' written work. We analyzed all student discourse and written work in which a student stated (written or spoken) a correct rule of the form $A = \mathrm{a}h^2$. Data analysis focused on identifying the nature of students' quantitative reasoning (RQ1), ways of reasoning supportive of symbolizing function rules (RQ2), and students' meanings for the coefficient a in the function rule $A = \mathrm{a}h^2$ (RQ3).

Two coders independently coded half of the data corpus in a first round of coding, then met to discuss findings and clarify questions. In a second round, one coder (Fonger) independently compiled all data from the first round of analysis and named major code categories and subcategories. In a third round of analysis Fonger re-analyzed all compiled data, one (sub)category at a time in a constant comparative fashion (Strauss & Corbin, 1990), referring to the original transcripts as needed. In a fourth round of analysis, Fonger engaged in axial coding (Straus, 1987) to discern relationships among codes. Coding was aimed at identifying how types of quantitative reasoning may have supported students' symbolization of rules and meanings of a. For a given episode it was possible for a student to demonstrate multiple forms of quantitative reasoning; all types were coded where appropriate, except when a student demonstrated evidence of covariation, in which case we did not also code variation. In cases in which students demonstrated flexibility across ways of reasoning, we coded multiple meanings of a.

Results

Students' Quantitative Reasoning

We characterized the nature of students' quantitative reasoning about functional growth situations (RQ1) into four types: (a) *Static Correspondence*, (b) *Variation*, (c) *Uncoordinated Variation*, and (d) *Covariation* (Figure 2).

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

Articles published in the Proceedings are copyrighted by the authors.

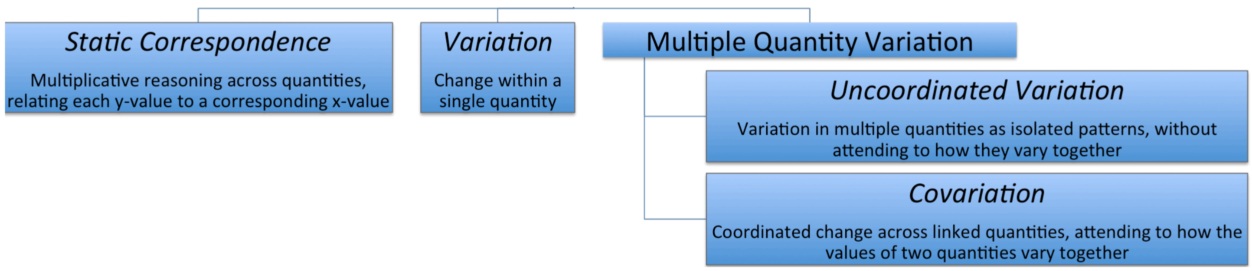

Figure 2. Quantitative Reasoning about Functional Growth Situations.

For *Static Correspondence*, students demonstrated evidence for reasoning multiplicatively across quantities, directly relating each *y* value to a corresponding *x* value. This includes both multiplicative relationships between height (x) and length (y_1) and height (x) and area (y_2); the first case was more common. For example, Daeshim said "height times 3 is length" and Bianca noted "length over height" as a static multiplicative relationship across linked quantities.

Students reasoned about *variation* when they reasoned about change within a single quantity, for instance, attending to growth in area without coordinating with growth in height. Typical ways in which students reasoned about *variation* in quantities was through attending to either differences in height or in length of the growing rectangle, but not to both. For example, for the rule $A = .75h^2$ Ally described .75 as "the rate of growth of the length." When students reasoned about variation in more than one quantity we categorized their thinking as either uncoordinated variation or covariation. *Uncoordinated variation* is typified by students' attention to variation in two or more quantities in a manner that does not coordinate simultaneous change, but rather treats the variation as isolated patterns or sequences (Figure 3).

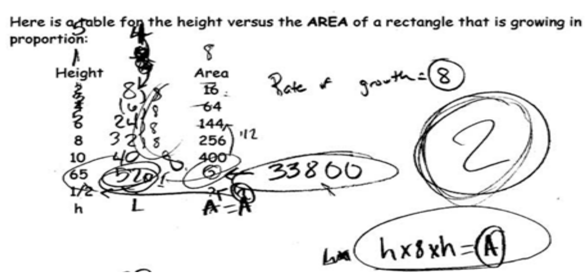

Figure 3. Ally's uncoordinated variation of length and height.

Ally's work in Figure 3 (for Task b) shows how she computed length values for each pair of height and area values. Ally attended to variation in L by computing successive differences in length (which the students called DiL). She also noted a pattern for the difference in height (which the students called DiH), writing, "you can't reduce regularly because it is going up by 2s." Thus Ally attended to DiL and DiH as isolated patterns.

Finally, we characterized students' quantitative reasoning as *covariational* when they attended to a coordinated change across more than one quantity, envisioning both quantities varying together. For example, Jim wrote the rule A=4.5h^2 and explained if you grew a rectangle "it would go over 4.5 for every time you go up the height 1."

Conceptual Supports for Students' Ability to Symbolize Function Rules

Students' symbolization of quadratic function rules of the form $A = ah^2$ was supported by four modes of reasoning (RQ2): (a) *Static Correspondence*, (b) *Variation and Static Correspondence* (when ΔH=1), (c) *Covariation*, and (d) *Flexible Covariation and Correspondence*. Each mode of reasoning is exemplified across two task types in Table 2 (save mode D, which combines modes A

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.
Articles published in the Proceedings are copyrighted by the authors.

and C). See Figure 1 for the two tasks (Tasks a and b), in which ΔH=1 for Task a and ΔH=2 for Task b. The examples in Table 2 are sample student responses to these tasks, and are paraphrased from our analyses of student thinking. These results reflect the relationships observed in students' quantitative reasoning.

Table 1: Conceptions of Functions Supportive of Symbolizing Function Rules

Mode of Reasoning	Example for ΔH=1	Example for ΔH>1
A. Static Correspondence. Student writes a rule by computing a ratio of two quantities, either (a) length to height, or (b) area to height^2.	A=.75h^2 because I take the ratio of length to height, 2.25/3, which is .75.	A=4h^2 because the ratio of the length to height, 8/2, is 4.
B. Variation <u>and</u> Static Correspondence. Student reasons about the difference in length (DiL), multiplicatively relates height and change in length (L=H*DiL), and writes the rule A=HL as A=H(DiL*H).	A=.75h^2 because .75 is the difference in length, so I can write L=.75*H, and A=LH so A=(.75H)(H)	n/a (i.e., mode of reasoning leads to incorrect rule)
C. *Covariation*. Student understands coordinated variation in two linked quantities and uses this to support writing a correspondence rule of the form $A = ah^2$. This includes change in length, change in height, and rate of growth of area.	A=.75h^2 because the length changes by .75 each time (the height changes by 1).	A=4h^2 because the length increases by 8 cm for every 2-cm increase in height.

There are two important cases to discuss regarding our results of RQ1 in relation to RQ2. First, in some cases, students' quantitative reasoning was evident in their thinking without necessarily supporting their ability to write a rule. For example, Bianca reflected on the rule A=4.5h^2 and questioned "What is the 4.5 that we came up with? What does that have to do with anything? Length over height!" Thus Bianca's reasoning about a correspondence relationship did not inform her writing of the rule (it was a retrospective connection she made). Second, some types of reasoning did not support students' ability to write correspondence rules. Recall Ally's reasoning on Task b (Figure 3). Notice that Ally wrote the rule $h\times 8\times h=A$, which was informed by her understanding that $A=HL$ and H=8L (mode B, *variation and static correspondence* for ΔH>1). This example illustrates two findings: Ally's *variation and static correspondence* reasoning for ΔH>1 led to an incorrect rule (see Table 2), and Ally's uncoordinated variation of DiL and DiH did not support writing a correct rule. Thus in what follows we focus only on the cases for which students' quantitative reasoning informed their symbolization of function rules. For brevity, we exemplify modes B and D.

In mode B, students coordinated their reasoning about variation in length with reasoning about a static relationship between height and length in order to re-write the area formula $A=HL$ in terms of these quantities. For example, on Task b, Tai found "the rate of growth of the length" to be .75 and explained "height times .75 equals the length … and then you take the length and you times it by the height." Thus Tai attended to variation in L to posit .75 as the rate of growth in the length, then stated a multiplicative correspondence relation between length and height (H*.75=L), and used the rule $A=LH$ to write .75h^2.

In mode D, students could think both covariationally and in terms of correspondence relations to symbolize function rules. Daeshim, Jim, Bianca, and Tai demonstrated flexibility in their conceptions of functions across the static correspondence view and the covariation view, which supported their abilities to symbolize correspondence rules. For instance, given a table of values in which the height increased by uniform increments of 3 cm, Jim found the rate of growth in length to be 6.75 cm for each 3-cm change in height. He wrote "2.25h^2 = Area" and explained that he divided 6.75 cm by 3 cm "because that's what you're going up by each time." Jim's attention to the coordinated change in height and length supported his symbolization of the function rule. Jim also demonstrated flexibility

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

Articles published in the Proceedings are copyrighted by the authors.

in leveraging his correspondence thinking to write rules. In one example, he created a table for the height, length and area of a growing rectangle and quickly said "$4.5h^2$ … I just did the first two numbers and that's all I need to do." Jim computed the ratio of the original length to the original height, and thus a static multiplicative relationship informed his writing of the rule.

Students' Meanings of Symbolic Quadratic Function Rules

In light of the finding that students demonstrated great variation in their conceptions supporting their ability to symbolize quadratic function rules, it is not surprising that their meanings for *a* in the symbolic function rules they wrote of the form $A = ah^2$ varied as well. We discerned five types of meanings of *a* in students' symbolic rules $A=ah^2$ as shown in the rows of Table 3. We exemplify three themes that emerged across students' meanings next.

Table 2: Students' Meanings of Symbolic Function Rules

Meaning of *a* in $A = ah^2$	Nature of Quantitative Reasoning
A Quantity	*a* = *length* when *height* is 1
Ratio of Quantities	*a* = (*original length*)/(*original height*); *a* = *length*/*height; or* *a* = *area*/($height^2$)
Variation in Quantities	*a = DiL; or a* = 1/*DiH*
Ratio of Variation in a Quantity to Value or Quantity	*a = DiRoG/2; a=(DiL)/(original height);* or *a* = *DiRoG*/(2(*original height*)^2)
Covariation of Quantities	*a =DiL*/*DiH; a = DiL* when *DiH* = 1; *a* = *DiL*/*DiH* when DiH>1; *a* = *DiRoG*/2 = *DiL*, Δ*height* = 1; or *a* = *DiRoG*/(2**DiH*)

Ratio of Quantities or a Quantity

Students made sense of the coefficient *a* both as a quantity and as a ratio of two quantities. For instance, Daeshim wrote a general expression nh^2, explaining n as "the length when the height is 1" (Task b). Bianca explained that the coefficient *a* could also represent "length over height." Students often wrote the static ratio of length to height in the general form, *a* = (*original length*)/(*original height*). The parameter *a* could also represent a static ratio of area to height squared. For example, Tai wrote a symbolic rule $4.5h^2$ by computing "the area [288] divided by *h* squared [8^2=64] equals a number [4.5]". In this case, *a* = *area*/($height^2$).

Variation and/or Ratio of Variation and Quantity

As a second theme, students reasoned about the variation in length or in the difference in the rate of growth in area (which the students called the *DiRoG*), sometimes also computing a ratio with a fixed quantity. For instance, Ally made sense of *a* as "the rate of growth of the length" (Task b), or *a* = *DiL*. Students also attended to the DiRoG in order to determine the value of *a*. Bianca explained, "I just found the DiRoG and divided it by two and then I knew what number to multiply by," so *a* = *DiRoG*/2. Other students related variation in quantities to a fixed quantity. For example, Tai explained: "First you figure out the, um, length. And then you find out the rate of growth, and then you divide the rate of growth by the height, and then you put like whatever number, height squared." In this case, *a=(DiL)/(original height).*

Covariation of Quantities

Students also conveyed meanings of *a* that were firmly grounded in a covariation perspective, both (a) as the ratio of the change in height (DiH) to the change in length (DiL), and (b) as a coordination of the difference in the rate of growth (DiRoG) with either the DiL or the DiH. As an example of the first meaning, Sarah explained, "the height squared times the DiRoG of the length equaled the area [$h^2*DiRoG_L=A$]. And then depending on how much it went up by, you would divide

Articles published in the Proceedings are copyrighted by the authors.

by that number [e.g., h^2*$DiRoG_L$/2=A].” In this case, *a* = *DiL* when *DiH=1* and *a* = *DiL*/*DiH* when *DiH* > 1. In the second case, students coordinated the difference in the rate of growth (DiRoG) with variation in height or length. For example, in one task, students had to relate the parameter *a* to the DiRoG. Both Joe and Tai made sense of *a* as the ratio of the difference in the rate of growth in area over 2 which equals the difference in the length. In this case Tai wrote DiRoG/2=DiL thus *a = DiRoG/2 = DiL, ΔH = 1*. It is notable that in these cases the symbolic equation actually represents a covariational relationship between coordinated variation in the quantities length and height.

Discussion and Conclusion

This study addresses how students' ways of thinking may support their ability to create symbolic rules from numeric tables embedded in a quantitative context, and meaningfully interpret those rules from both the correspondence and covariational perspectives. Our results suggest that if students' function activity is grounded in a continuous quantitative context, such as the growing rectangle, and guided by a purposeful sequence of tasks focused on encouraging covariational reasoning, students may come to see function rules as representations of covariation. Such a stance can encourage a powerful and flexible understanding of function rules and serve as a productive foundation for further exploration in algebra.

Our findings also suggest that students need not necessarily be flexible in moving back and forth between correspondence and covariation views. Instead, students who develop a strong covariation view of functional relationships may come to make sense of algebraic symbolic rules as statements of covariation (e.g., seeing the parameter *a* as a relationship between two quantities that vary together). For example, recall Jim's reasoning; for him, the coefficient of 2.25 was a ratio of change in length to change in height. This suggests that a correspondence relation between area and height can emerge from attention to covariation in height and length, demonstrating the power of covariational reasoning in writing function rules (cf. Thompson & Carlson, in press).

Finally, when we consider the results of RQ1 and RQ2, an important finding emerges: students' conceptions of uncoordinated variation or variation alone were generally *not* supportive of their ability to symbolize function rules. This finding contributes to the literature on students' ways of thinking that might help explain representational disfluencies (i.e., unsuccessful translations from verbal descriptions or numeric tables to symbolic rules). On the other hand, we hypothesize that students' representational fluency in symbolizing function rules—especially the meanings they developed for the rule $A = ah^2$—seemed to be supported by (a) grounding their activity in a quantitatively rich task situation, (b) designing task sequences to encourage attention to the nature of how quantities change, (c) asking students to consider how quantities are related, and (d) prompting the generalization of those relationships by extending to far cases. Our approach to focus on the interplay between students' conceptions and meaning making from multiple representations of functions grounded in a quantitatively rich setting is a productive stance that warrants further exploration and research.

Acknowledgements

Support for this research was provided in part by the U.S. Department of Education IES Research Training Programs in the Education Sciences under grant no. R305B130007, and the National Science Foundation under Award REC 0952415. Any opinions, findings, and conclusions expressed in this material are those of the authors, not the funding agencies.

References

Cai, J., Bie, B., & Moyer, J. S. (2010). The teaching of equation solving: Approaches in *Standards*-based and traditional curricula in the United States. *Pedagogies: An International Journal*, *5*(3), 170-186.

Confrey, J., & Smith, E. (1994). Exponential functions, rates of change, and the multiplicative

Articles published in the Proceedings are copyrighted by the authors.

unit. *Educational Studies in Mathematics*, *26*(2), 135–164.
Ellis, A. B. & Grinstead, P. (2008). Hidden lessons: How a focus on slope-like properties of quadratic functions encouraged unexpected generalizations. *Journal of Mathematical Behavior*, *27*(4), 277-296.
Farenga, S. J., & Ness, D. (2005). Algebraic thinking part II: The use of functions in scientific inquiry. *Science Scope*, *29*(1), 62-64.
Kieran, C. (2007). Learning and teaching algebra at the middle school through college levels: Building meaning for symbols and their manipulation. In F. K. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (Vol. 2, pp. 707-762). Charlotte, NC: Information Age.
Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning and teaching. *Review of Educational Research*, *60*(1), 1-64.
Piaget, J. (2001). *Studies in reflecting abstraction* Philadelphia: Taylor Francis.
Simon, M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, *26*(2), 160-162.
Smith, E. (2003). Stasis and change: Integrating patterns, functions, and algebra throughout the K-12 curriculum. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A Research Companion to Principles and Standards for School Mathematics* (pp. 136-150). New York: Erlbaum.
Smith, J., & Thompson, P. W. (2007). Quantitative reasoning and the development of algebraic reasoning. In J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 95-132). New York: Erlbaum.
Steffe, L., & Olive, J. (2010). *Children's fractional knowledge*. New York: Springer.
Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In A. Kelly & R. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 267-306). Mahwah, NJ: Lawrence Earlbaum Associates.
Strauss, A. L. (1987). *Qualitative analysis for social scientists*. New York, NY: Cambridge University Press.
Strauss, A. L., & Corbin, J. M. (1990). *Basics of qualitative research: Grounded theory procedures and techniques* (Vol. 270). Newbury Park, CA: Sage.
Thompson, P. W. (1994a). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 181-234). Albany, NY: SUNY Press.
Thompson, P. W. (1994b). Students, functions, and the undergraduate curriculum. In E. Dubinsky, A. H. Schoenfeld & J. J. Kaput (Eds.), *Research in collegiate mathematics education* (pp. 21-44). Providence, RI: American Mathematical Society.
Thompson, P. W., & Carlson, M. P. (in press). Variation, covariation and functions: Foundational ways of mathematical thinking. In J. Cai (Ed.), *Third Handbook of Research in Mathematics Education*. Reston, VA: National Council of Teachers of Mathematics.
Zbiek, R. M., Heid, M. K., Blume, G. W., & Dick, T. P. (2007). Research on technology in mathematics education. In F. K. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 1169-1207). Greenwich, CT: Information Age Publishing.

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.
Articles published in the Proceedings are copyrighted by the authors.