DEVELOPING PRESERVICE TEACHERS' UNDERSTANDING OF FUNCTION USING A VENDING MACHINE METAPHOR APPLET

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The purpose of this study is to examine the use of a Vending Machine applet as a cognitive root for the development of preservice teachers understanding of function. The applet was designed to purposefully problematize common misconceptions associated with the algebraic nature of typical function machines. Findings indicated affordances and limitations of the applet as a cognitive root, motivating revisions to the applet for further study.

Keywords: Algebra and Algebraic Thinking, Instructional Activities and Practices, Technology

Functions are a critical base for mathematical understanding of STEM disciplines and are often regarded as the unifying element of much of secondary mathematics. This is recognized in the CCSSM where the study of functions is given its own domain, separate from Algebra, in grades 9–12 (National Governors Association Center for Best Practice & Council of Chief State School Officers, 2010). The conceptual obstacles with respect to putting function front and center have been well documented in the literature (e.g., Even, 1990, Tall et al., 2000) and have proven particularly hard to overcome. Rather than constructing their own definition of function based on tasks, students are often presented with a highly theoretical definition, resulting in a disconnect between their concept definition and their concept image (Even, 1990). The same is true for undergraduate students (Carlson, 1998), including preservice mathematics teachers (PSTs). Thus it is important that PSTs have an opportunity to develop a deep conceptual understanding of functions that remedies any existing misconceptions and understand how to engage students in tasks to develop and test their own definitions.

Theoretical Framework

There is evidence that PSTs often have a view of function that is limited to algebraic expressions and their associated graphs (e.g., Carlson 1998; Even 1990). Such understandings typically result in a "vertical line test" related definition of function (e.g., Carlson, 1998). Furthermore, those working with a vertical line test definition of function or an equation view of function often have the misconception that constant functions (horizontal lines) are not functions (Bakar & Tall, 1991). One suggested strategy for mitigating these common misunderstandings is the use of a function machine as a cognitive root.

The idea of a *cognitive root* was introduced by Tall as an "anchoring concept which the learner finds easy to comprehend, yet forms a basis on which a theory may be built" as he was developing a cognitive approach to calculus (Tall et al., 2000, p.497). One common example is the use of a *function machine* (sometimes referred to as a function box) as a cognitive root for the development of a definition for function as well as for building understanding of independent/ dependent variables and domain/range. The machine metaphor suggested by Tall and colleagues was typically a "guess my rule" activity. In such activities the inputs and associated outputs are provided, and students are challenged to determine what happened in the function machine (i.e., determine the function rule). While students are presented with a machine to embody the function concept, the rules used by the machine are algebraic in nature. Using such machines proved quite promising, yet some students still struggled with connecting representations and determining what is and is not a function (McGowan et al., 2000).

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Given the potential of the machine metaphor as a cognitive root for function, in our work designing learning experiences for PSTs that incorporate technology, we originally followed Tall and colleagues' recommendations and created a function machine using GeoGebra to engage PSTs in "guess my rule" activities, including designing their own function machines. Our PSTs demonstrated misconceptions that were similar to those in the literature, namely, that all functions needed to be represented by a formula, difficulty differentiating between the terms unique and exactly one output, and identifying constant functions as non-functions.

As a result, we set forth to design a set of machines that would not only be a familiar anchoring concept, but would hopefully also push and probe PSTs current understandings about functions in ways to intentionally problematize these misconceptions. To do this we considered Carlson's (1998) key aspects of function as they related to the function machine metaphor: interpreting and characterizing independent and dependent variables, ability to identify and describe the domain and range, using one representation of function to make sense of another, and distinguishing between functions and non-functions. The function machine as a cognitive root and these aspects of function as they related to common misconceptions framed both the design of our applet and our analysis of PSTs' work with the applet.

Design of the Applet

Unlike previous machine metaphors PSTs had experience with, the function machine applet we designed contains no numerical or algebraic expressions, rather the applet was built on the metaphor of a vending machine. The Vending Machine applet (https://ggbm.at/LdtLR0ex) is a GeoGebra file that contains five vending machines each with buttons for: Red Cola, Diet Blue, Silver Mist, and Green Dew. The directions state to explore the five machines and determine which are functions (Figure 1; McCulloch et al., 2015). When the user presses a button (input), one or more cans appear in the bottom of the machine (output). To remove the can(s) from the machine, the user clicks a reset button. The functionality of each machine was designed to address misconceptions from the literature on distinguishing functions and non-functions that our PSTs had previously demonstrated. Machine A is the identity function; each button produces a can of the corresponding color.

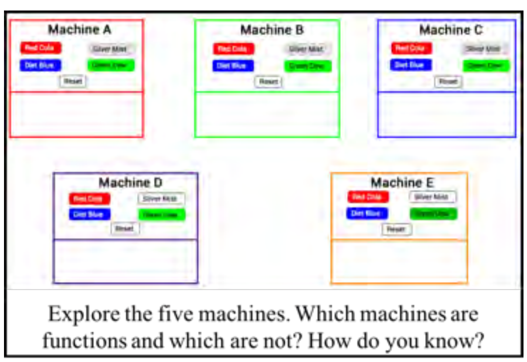

Figure 1. Screenshot of function machine applet.

Machine B is the same as A except when Silver Mist is selected, it produces two silver cans. This machine requires students to wrestle with the notion of what represents an element in the range. For Machine C, every button results in a single green can. The purpose of which is to present PSTs with a constant function to consider (i.e., the same number of cans of the same color for each button). For each button on Machine D a single can is produced, but the color is different from the color of the button pressed. This machine was designed to problematize their occasional use of the term "unique" when thinking about outputs. Finally, Machine E is similar to D, except the Silver Mist button

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randomly produces cans of different colors each time it is pressed. The purpose of Machine E is to provide a context in which testing the buttons on the machine once is not sufficient for determining whether or not the object is a function. Thus machines B, C, and E explicitly address the misconceptions we found when using function machines with associated algebraic expressions or graphs.

The purpose of this study was to examine the effectiveness of the Vending Machine applet as a cognitive root for function. In this paper we specifically address the following research question: What understandings of function do PSTs develop from engaging in a task using a vending machine metaphor applet?

Methods

To answer our research question, we engaged PSTs enrolled in a content-focused methods course in a task exploring the Vending Machine applet. Nine PSTs (referred to as S1-S9) engaged in the task; of the nine, seven were undergraduate secondary mathematics education majors (three of the seven were also dual mathematics majors), and two were enrolled in the Master of Arts in Teaching (initial licensure) program.

Data Collection and Analysis

The study began by asking PSTs to individually write a definition of a function, including examples and non-examples. After doing this independently, a whole class discussion was facilitated using their definitions through which the class agreed upon the following definition: *A function is a mathematical relationship such that each input has exactly one output.* Then, they were asked to engage in the Vending Machine applet task as a homework assignment; they were to explore the machines to determine which were functions and which were non-functions. Each PST captured a screencast of their work as they followed a "think aloud" protocol while working on the task. Simultaneously, they completed a worksheet to provide written documentation of their thinking. Following the task, PSTs completed a written reflection where they were asked to revise the agreed upon definition of function based on their experience with the task, to reflect on the different representations of functions presented in the task, and to discuss aspects of function highlighted by different machines with which they engaged (including possible uses with students). The PSTs uploaded their screencasts to a shared, secure online folder and submitted their written work during the following class session.

We began by examining the data to develop a code book based on four key aspects of function (Carlson, 1998) we previously identified that informed our theoretical framework. We chose two students' data (screencast, written worksheet, and reflection) and used open coding to identify themes related to each key aspect. We used a constant comparison method (Strauss & Corbin, 1998), which allowed for emerging categories within each key aspect and the refinement of these categories as they were contrasted with new data. Since we were particularly interested in how PSTs made sense of the applet, we also looked for evidence of the affordances and limitations of the applet, as well as misconceptions/errors, and an "other" category to capture any unanticipated themes. Then, to check for reliability, each author individually coded the two original students' data, plus an additional student's data, using the code book.

For the screencasts, we recorded the code and machine the PST referenced. For example, when working on Machine E, one PST commented "Machine E is a non-function because the silver button points to different colored cans" This was coded as "Distinguishing between functions and non-functions" and "Machine E" since the PST was justifying why that particular machine was a non-function. For the written artifacts, we coded their written explanations and drawings. For example, on the worksheet, several PSTs drew mappings that described each vending machine. These drawings

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were coded as “Interpret and characterize independent and dependent variables” and labeled with the letter of the corresponding machine that the drawing was in reference to. The researchers met to discuss the codebook and emerging themes and to clarify the codebook.

Once the codebook was finalized and reliability achieved, we coded all of the remaining data. We then looked within each code to understand the ways in which the PSTs engaged with the function machines to make sense of each key aspect. Finally, we examined the data by each function machine (for example, we looked at all codes associated with Machine A) to understand how each particular machine supported (or conflated) PSTs understanding of function, in particular the key aspects, as well as the affordances and limitations of the applet.

Results

Of the nine PSTs that completed the function machine task, five of them provided correct responses for all five machines. The other four PSTs’ errors were related to Machine B and/or Machine E (see Table 1). Their interpretations and characterizations of the independent and dependent variables (referred to as “inputs” and “outputs” by most PSTs) were central to their determination of whether or not each machine was a function. Here we focus on the ways in which the PSTs made sense of two specific machines, B and E. These machines are the only two for which PSTs provided incorrect answers and as such they provide insight to the aspects of function that the machines were designed to elicit. For the purposes of this paper we discuss two specific code categories, *distinguishing between functions and non-functions* and *interpreting and characterizing independent and dependent variables*, and the ways they provide insight to student sense making.

Table 1: PSTs’ Worksheet Responses: Function or Non-Function

Machine	S1	S2	S3	S4	S5	S6	S7	S8	S9
A	F	F	F	F	F	F	F	F	F
B	F	N	F	N	N	N	F	F	F
C	F	F	F	F	F	F	F	F	F
D	F	F	F	F	F	F	F	F	F
E	N	N	N	F	F	F	N	N	N

The Case of Machine B

As a reminder, pressing the Silver Mist button on Machine B results in the production of two cans of Silver Mist. The other three buttons act as the PSTs expected, producing one can of the soda noted on the button. Noting that Machine B is not a function was the most commonly made error in this task (n=4). The characterizations of the independent and dependent variables were quite different for those students that determined B was a function and those that determined it was not.

When examining the machines, the PSTs that described the independent and dependent variables (called inputs and outputs from here on out) only attending to how many elements there were in the output for each input determined that the machine was not a function. For example, S4 explained, “Red cola has one output, diet blue has one output, silver mist has two outputs, and green dew has one output...since silver mist has two outputs, I would say that is not a function.” Similarly, S5 noted “So you put in an input and you get two outputs, in a function you are not supposed to get two outputs.” Notice that both PSTs were focused on the number of outputs for each input.

In contrast, other PSTs were attending to not only the number of elements in the output, but also what they were and if the mapping from input to output was “predictable”. S3 stated, “Exactly one output for each input, 2 silver mist cans is the output for silver mist.” S9 went further and explained the importance of the output being “predictable” when he said, “Even though silver mist comes out with two cans versus one can but it always comes out with two silver cans, instead of three silver cans one time and two silver cans another time.” While the idea of two cans being the output was not

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problematic for some students, there were a few that grappled with it as they made sense of the situation. For example, S1 noted

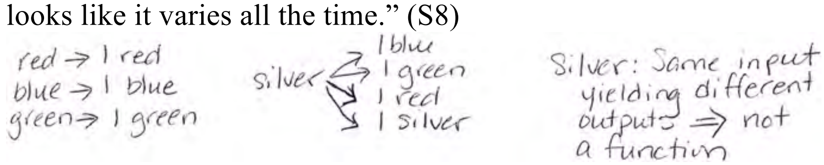
> So silver mist comes out with two sodas...hmmm...[keeps pressing]...that's interesting...cuz it depends on what you consider the output. The output could be how many different kinds of sodas there are...so it could be like you're putting in red you're getting out one, you're putting in blue you're getting out one, you're putting in silver you're getting out two, it doesn't necessarily mean it's not a function...it's just a different value.

The difference between these responses and those from the PSTs that focused on only the number of elements was that these students identified a mapping.

The Case of Machine E

Like we saw for Machine B, attending to only how many elements in each output was also problematic when trying to determine whether or not Machine E represented a function. As a reminder, Machine E included a button for Silver Mist for which the output was random, so it was not a function. S4 explained, "Red cola has an output, diet blue has an output, silver mist has an output, and green dew has one output...again [presses buttons]...they each have one output, silver mist has one output but it is a blue soda as well...each choice has one output...so it is a function." Even though she clearly identified that the Silver Mist button did not always produce the same output, she determined it was a function because it always produces one output. S5 only tried each button once, so he did not see that the Silver Mist button was unpredictable. He said, "For Red Cola you get a Red Cola, for Diet Blue you get you a Diet Blue, for Silver Mist you get a Green Dew, for Green Dew you get a Green Dew. I still think it's a function...you can get two of the same outputs because you are still getting one output for each input." While he did not have the opportunity to see the different outputs, the fact that he only tested the button once suggests that he does not understand what it means for an input to map to exactly one output.

The PSTs that interpreted outputs to have characteristics beyond just "how many" elements, correctly identified Machine E as a non-function. We have taken to describing this coordination between inputs and attending to multiple characteristics of the outputs as "mapping thinking". The following explanations are examples of what we are referring to as mapping thinking:

- "Silver mist can give you any soda. Yeah this one is not going to be a function because the silver mist button can give you all kinds of sodas. Machine E is not a function because it has different outputs for the same input." (S9)
- "Every input is supposed to have one output. It's not supposed to change all the time...The assignment of inputs to outputs can't change all the time. It has to be a unique prescription. It looks like it varies all the time." (S8)
- 

(S7)

These explanations not only include an interpretation of independent and dependent variables beyond simply how many elements in an output are associated with a given input, but also show evidence of a deeper understanding of the definition of function. This understanding goes beyond "one input gives one output" and is not attached to an equation or graph.

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Revising Definitions Based on Machines B and E

In a reflection assignment the PSTs were asked if they would suggest any changes to the agreed upon definition: *A function is a mathematical relationship such that each input has exactly one output*. The PSTs that misidentified both B and E did not suggest any changes to the definition. S1, who misidentified only B, wrote, "I think that to be a function, it had to map colas to one other cola, not change the colas it was giving out or producing more. So, I think that our definition is still a good one." Notice that he is still focused on the number of elements in the output. In contrast, all of the PSTs that correctly identified all 5 machines suggested some changes to the definition to address the misconceptions that might be associated with working in the context of vending machines. For example, S3 suggests,

> I would like to change it to a "function is a mathematical relationship such that each input produces the same output at all times." [...] I noticed that in Machine B, each input produced one can except Silver Mist, which produced two cans, and this made me initially say it wasn't a function. However, I know that you can have a function that is not uniform, so producing two cans each time is an appropriate thing for Silver Mist to do without violating the function rules. Thus, the "exactly one" language is misleading… Then, Machine E threw me off because each input produces exactly one output, but Silver Mist produces a different output each time...Because of that discovery, I realized that the language "exactly one output" could be very confusing to students and does not exactly explain what is going on in a function, which is that no matter when you place an input into the function, you will always get the exact same output that you got the last time you put that particular input into a function. That is what we mean by "exactly one output," but the language restricts students from really understanding what the function is doing with that input.

It is clear from these suggested revisions that the context of a vending machine was not only a powerful cognitive root for the PSTs' own understanding, but it also resulted in an understanding of ways to mitigate their future students' possible misunderstandings.

Discussion

As this was our first use of the Vending Machine applet as a cognitive root for function with PSTs, it is important to note affordances and limitations of its design. Machines B and E, the two most commonly misidentified machines, are nice examples of how these PSTs make sense of the definition of function within the context of the Vending Machine applet being used as a cognitive root and provide insight to the ways that they interpret independent/dependent variables and apply them to their existing definition of function. Results showed that through these experiences, PSTs recognized that the use of the commonly used phrase "exactly one output" in the definition of function can be problematic. Furthermore, the fact that the applet used non-expression objects seems to have been a helpful context for this realization.

The need to press the reset button to remove the output was first seen as a limitation as PSTs often forgot to do so and ended up with multiple outputs showing at once. However, this feature was also an affordance in that it offered insight to their thinking and, for those that did it intentionally, provided a representation of the range. Another limitation was the number of machines that were included in the applet. There were not enough examples of non-functions or functions with more than one element as an output. In more than one instance PSTs got to the final machine and assumed it was a non-function simply because "at least one of these must be". Finally, it is possible that agreeing upon a class definition of function prior to engaging with the task might have changed the way that PSTs interacted with the applet. It was not clear if all PSTs were actually in agreement with the adopted definition. If one did not understand the agreed upon definition, it would be difficult to apply it to an analysis of the machines.

From our research on the PSTs' engagement with the applet and the affordances and limitations identified, a revision to the applet has been made. The new version has been created to address PSTs' interpretations of Machine B and Machine E and the limitations noted above. It consists of four pages (Figure 2). The first two pages contain two vending machines, on each page one machine is labeled as a function and the other is labeled as not a function. The two non-function machines each have at least one button that produces a random can when pressed. The new applet provides the opportunity for PSTs to make a conjecture, after page two, on why Machines 1 and 3 are functions and Machines 2 and 4 are non-functions. They then test their conjecture on the pages three and four, that each contain five machines. The five original machines are included, along with five new machines. These new machines provide additional opportunities to examine machines that are not one-to-one, produce random pairs of cans as an output, and have random outputs for all four inputs.

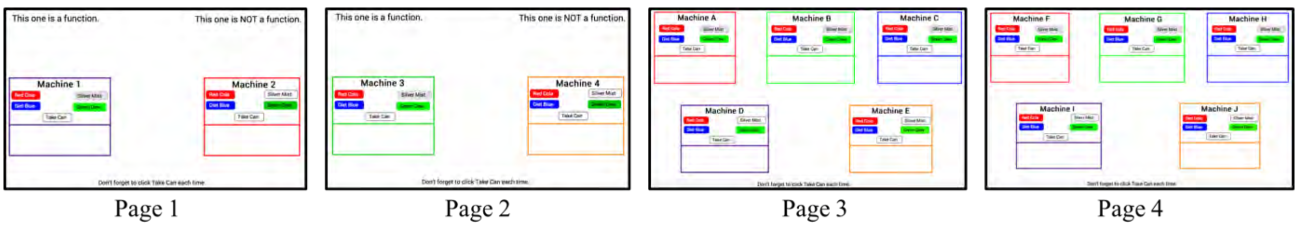

Figure 2. New version of the applet (https://ggbm.at/MAEdhkH6)**.**

Conclusion

Our findings suggest that the Vending Machine applet might be a powerful tool (cognitive root) for building understanding of key aspects of function as identified by Carlson (1998). In thinking about the theme of this conference, this work suggests a potential change in route in the ways we might consider engaging PSTs with concept of function. However, to determine whether or not these findings are generalizable the use of the applet needs to be studied on a larger scale. We are currently conducting a larger study using the new version of the applet with approximately 40 PSTs from five universities across the country. Our hope is that through further study and revision, this work results in a solid cognitive root that remedies existing misconceptions and on which deep conceptual understanding of function can be built.

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