A STUDY OF PRE-SERVICE TEACHERS USE OF REPRESENTATIONS IN THEIR PROPORTIONAL REASONING

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Proportional reasoning is important to the field of mathematics education because it lies at the crossroads of additive reasoning in the elementary school and multiplicative reasoning needed for more advanced mathematics. This research reports on the representations used by pre-service teachers (PSTs) as they responded to tasks involving proportional reasoning. The findings highlight three common difficulties that were prevalent among participants' responses. An analysis of the representations used by participants revealed that the representations that PSTs created in their effort to solve the problems often enabled them to overcome these difficulties. Prior research is used to hypothesize explanations of the extent to which different forms of representations were useful and productive for the participants. Implications include ways that use of these multiple representations could aid in the teaching of proportional reasoning.

Keywords: Mathematical Knowledge for Teaching, Rational Numbers, Modeling

Proportional reasoning is important to the field of mathematics education because it lies at the crossroads of transitioning from additive reasoning in the elementary school to multiplicative reasoning necessary for proportional reasoning and more advanced mathematics. Lesh, Post, and Behr (1988) describe the importance of proportional reasoning, saying that it is "widely recognized as a capability which ushers in a significant conceptual shift from concrete operational levels of thought to formal operational levels of thought" (p. 101). This shift in understanding can lead to advanced mathematical thinking and is paramount in achieving success in higher level mathematics courses.

Pre-service teachers (PSTs) enter college with prior assumptions about mathematics and mathematical concepts. Often PSTs have many deep-rooted misconceptions about the multiplicative relationships involved in proportional reasoning and struggle with solving tasks that involve these concepts (Simon & Blume, 1994; Smith, Silver, Leinhardt, & Hillen, 2003; Sowder, Armstrong, Lamon, Simon, Sowder & Thompson, 1998). The question becomes: What mathematical knowledge do PSTs have in relation to proportional reasoning? Understanding this knowledge is important in helping them develop the specialized content knowledge necessary for teaching. And how do PSTs use representations to deepen their understanding of proportional relationships? This report focuses on particular tasks that were used to elicit proportional reasoning of PSTs, the misconceptions that surfaced and how PSTs used representations in their problem solving to overcome these obstacles. These findings can help improve mathematics teacher education, as we can gain a better understanding of how PSTs think about proportional reasoning.

Conceptual Framework

While the definition of representation in mathematics education vary, most researchers differentiate between external and internal representations where external representations are embodiments of ideas or concepts such as charts, tables, graphs, diagrams, etc., and internal representations are cognitive models that a person has (e.g., Janvier, Girardon, & Morand, 1993). In this study, representations are external mathematical embodiments of ideas and concepts that provide the same information in a drawing, picture, table or graph.

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According to Dufour-lanvier et al. (1987) the role of representations in mathematics education has several characteristics: (1) Representations are an inherent part of mathematics, (2) Representations provide a concrete example of a concept, (3) Representations are used locally to mitigate certain difficulties and (4) representations are intended to make mathematics more interesting. It is this third role that this report will illustrate in terms of PSTs proportional reasoning. In particular this study focuses on the use of representations to overcome difficulties and misconceptions.

The use of multiple representations is advocated by many mathematics educators and supported by the National Council of Teachers of Mathematics (NCTM) Standards (NCTM, 2000). It is suggested that multiple representations provide an environment for students to abstract and understand major mathematical concepts. Constructivist theory also suggests that we need to understand students' thinking processes in order to facilitate their learning in more empowering ways (Steffe, 1991). Therefore, it is necessary for mathematics teacher educators to understand how PSTs use representations, not only to understand their thinking, but to develop a repertoire of useful representations for teaching and discussing proportional reasoning. The results of this report will provide mathematics teacher educators with multiple representations that are productive in the teaching and learning of ratio and proportion.

Methodology

Twenty-five elementary and secondary math education PSTs were selected for this study at the beginning of their first mathematics methods courses at a large research university. A nine-problem questionnaire was developed and used to ascertain each PST's current level of understanding about proportional reasoning. (See Johnson, 2013 for more details on questionnaire). The responses were coded and participants were divided into four groups based on the analysis of their responses. Group 1 was distinguished by having a high level of proportional reasoning while Groups 2 and 3 showed moderate levels of proportional reasoning and group 4 gave evidence of little to no proportional reasoning. Eleven individualized interview schedules were created in order to challenge the PST understandings and misconceptions about proportional reasoning that surfaced from the questionnaires; the interviews were implemented, videotaped, transcribed and annotated. Individual interview data was coded and analyzed to create descriptions of the nature of the participants' understanding of proportional reasoning. A group of trained graduate students also coded the data and these codes were then discussed and revised to provide a higher degree of validity and reliability (Johnson, 2013). Another pass through the analysis showed patterns that emerged within each of the four groups in terms of their use of representations and it was noted that there were stark differences between those students in Group 1 and the students in the other groups. This report discusses and illustrates how PSTs in Group 1 utilized representations in solving these proportional problems during the interview and why these representations were important for them in overcoming certain challenges. Additionally, I will contrast these representations with those created by participants who were less successful in reasoning proportionally about these problems.

How Did Pre-Service Teachers Use Representations When Given Tasks Focused on Proportional Reasoning?

For this study tasks were designed to address distinct aspects of PSTs' difficulties with proportional reasoning that surfaced from the questionnaire. Three of these misunderstandings were: (1) The epistemological obstacle of linearity (Brousseau, 1997), (2) confusion between ratio and fraction (Karplus, Pulos, & Stage, 1983), and (3) inability to reason quantitatively (Thompson, 1994). Below, I illustrate how these particular tasks were designed to challenge PSTs' prior assumptions and how PSTs used representations to reason proportionally.

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Difficulty #1: Epistemological Obstacle of Linearity.

Modestou and Gagatsis (2007) studied students' improper proportional reasoning as an epistemological obstacle of linearity. An epistemological obstacle is NOT one in which there is a lack of knowledge, but one in which a piece of knowledge is appropriate only within particular contexts. The epistemological obstacle often generates false responses outside that context (Brousseau, 1997). These responses are recurrent, universal, and resistant to a variety of forms of support aimed at overcoming the problem (De Bock, Verschaffel, Janssens, Van Dooren, & Claes, 2003). For example, problems involving proportionality are often characterized as an epistemological obstacle in linearity. Missing value problems often include the basic structure of four quantities (a, b, c and d) of which, in many cases, three are known and one is unknown. Additionally, many proportional problems involve the context of speed. The Bike problem is this type of scenario; it involves students riding their bikes to school at the same speed. It provides PSTs with three numbers and asks them to find the fourth (see Figure 1).

Ben and John both ride their bikes to school at the same rate. Ben leaves his house first and meets John after riding 10 blocks at which time John has only rode 7 blocks. If John's trip to school is 14 blocks, how many blocks does Ben ride his bike to school? (Adapted from Cramer, Post and Currier [1993, p. 159])

Figure 1. Bike problem designed to elicit the obstacle of linearity.

Despite the structure of a missing value problem and context of speed, this problem does not involve a proportional relationship between the quantities. Research has found that this structure and context evoke a strong tendency of students to use direct proportions even if it does not fit the problem (DeBock, et al., 2002; Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2005). 64% of the PSTs answered the bike problem incorrectly which demonstrates that students are drawn to the illusion of linearity in this problem and desire to solve it by setting up a proportion and cross multiplying even though there is not a proportional relationship. Verschaffel, Greer and DeCorte (2000) claim it takes a radical conceptual shift to move from the uncritical application of this simple neat mathematical formula to the modeling perspective that takes into account the reality of the situation being described. It is not surprising that the 36% of the PSTs interviewed who correctly used additive reasoning to solve this problem all created a diagram as part of their reasoning. The diagrams all illustrated the context of the problem (see figure 2).

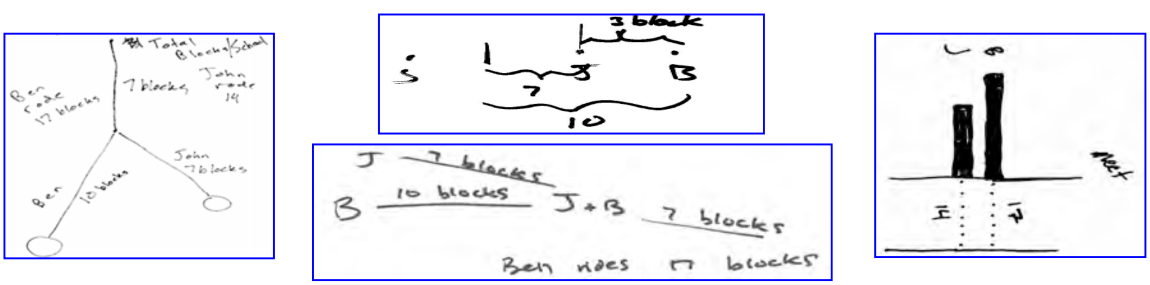

Figure 2. Representations created by PST to solve the Bike problem and overcome the obstacle of linearity.

It was the use of the diagram that helped them to situate and solve the problem as well as overcome the obstacle of linearity. Their representations modeled the additive reasoning necessary to solve the task despite its context and structure that led most PSTs to overextend the concept of proportions. These PSTs used the representation to illustrate that the two boys ride at the same speed

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then meet and travel together meaning that Ben rides his bike 3 more blocks than John. It is through modeling the context of the problem that meaning is achieved.

Difficulty #2: Confusion between Ratio and Fraction.

A link between fractions and ratio is often not made explicit in mathematics textbooks or classrooms. The difficulties surface if ratio and fraction are understood as equivalent mathematical terms when they are fundamentally different. Even though there are similarities in representations of ratio and fraction (i.e., fraction 2/3; ratio 2/3), the interpretations of those representations differ in important ways. In the case of ratio both the numerator and the denominator can represent parts (i.e., 2 parts to 3 parts); this is not the case with fractions. The Dog/Cat problem (see figure 3) was meant to elicit this type of difficulty in reasoning by PSTs. The correct interpretation of the situation is a part-to-part relationship. The numbers were chosen so that regardless of whether the participant interprets the ratio as a part-whole relationship or a part-part relationship, the solution will be an integer.

When animal lover Mr. Henry died he left 240 thousand dollars to be divided amongst two animal shelters using a 2:3 ratio between the amounts that Cat Best Home and Dog Lover Home gets. How much money should each shelter get? (Adapted from Peled [2007, p. 21])

Figure 3. Dog/Cat problem to elicit understanding of part to part ratios.

This problem was given on the initial questionnaire and a third of the PSTs interpreted 2:3 as 2/3rds and arrived at an incorrect solution of $80,000 to the Cat home and $160,000 to the Dog home. When interviewed these students who were asked to explain their reasoning and some were able to recognize that 2:3 is a part to part relationship, not a part whole relationship. In order to explain this relationship, these PSTs utilized representations to find the solution, either in a table or a model. For example, Eve started by doing an easier problem of 100 thousand dollars and created a pie chart to show the distribution of money. She then used a similar pie chart to determine the distribution for the 240 thousand dollars in the problem (see figure 4).

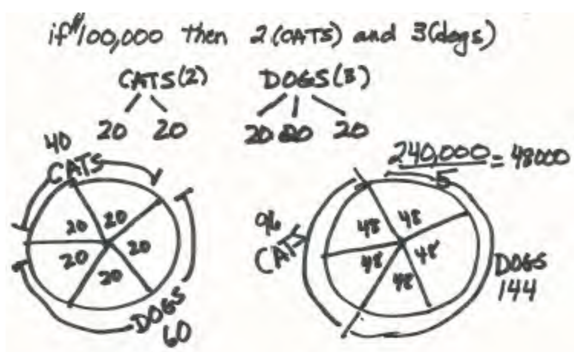

Figure 4. Eve's representation of her solution to the Dog/Cat problem.

Her representations illustrate a deep understanding of the part-to-part relationship presented in this problem. However, many of the PSTs who did answer this problem correctly on the questionnaire were unable to explain the procedure they used to find the solution. When asked how they solved the problem they would re-iterate the steps in the procedure (i.e. you add the two numbers given in the ratio, then you create two fractions 2/5 and 3/5 and multiply by them by the 240) but when asked to explain why it makes sense, replied, "I don't know, this is what I was taught

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to do when solving this type of problem, I don't know what it means or why it works." In order for these students to develop the specialized content knowledge needed for teaching proportional reasoning a discussion of illustrations such as Eve's would be beneficial. The pie chart can provide an understanding of what part-to-part relationships represent and why their procedure finds the solution. The slices of the pie show the number of pieces in the whole created by adding the two parts together and provide a visual representation as to why 2/5 times 240 (i.e. it is 2/5ths of the whole) results in the amount of money the Dog home receives.

Difficulty #3: Inability to Reason Quantitatively

PSTs in this study struggled with proportional reasoning situations that involved the distinction between quantitative reasoning and computation. Quantitative reasoning is making sense of relationships among measureable attributes of objects in a situation (Thompson, 1994) while computation is the result of an arithmetic operation to evaluate quantities. In general, reasoning about quantitative situations involves conceiving of circumstances in terms of quantities by constructing networks of quantitative relationships. For example, PSTs often set up proportions but do not understand what the ratios represent in the context of the situation.

The Lemon/Lime task was used to challenge the PSTs' misconception about quantitative reasoning and computation (see figure 5). In this task, participants were asked to compare two different lemon/lime mixtures (3 lemon:2 lime and 4 lemon:3 lime) to determine which was more lemony, **without doing ANY calculations** but by representing the mixtures with green and yellow unifix cubes. The request to not use calculations posed a high degree of difficulty for most of the PSTs interviewed, because it forced them to reason quantitatively and conceptually rather than computationally.

Jen and Alice are making lemonade. Jen mixes 3 cups of lemon juice with 2 cups of lime juice. Alice mixes 4 cups of lemon juice with 3 cups of lime juice. Whose mixture is more lemonier? Justify your solution without using calculations. You may rearrange the cubes in such a way that they demonstrate which one's more lemony tasting or that they are the same. (Heinz [2000, p. 70])

Figure 5. Lemon/Lime Problem.

60% of the PSTs interviewed used an additive relationship when they reasoned without calculations, claiming that there was "one more cup of lemon in each mixture so the mixtures were the same." However, when allowed to utilize calculations, these same PSTs created a multiplicative relationship (i.e. 3/2 = 1.5 and 4/3 = 1.333) by dividing the quantities in order to compare the decimal representations of the mixtures. Their calculation of the relationship caused them to reevaluate the original statement that the mixtures were the same and state that the 3 lemon:2 lime mixture had more lemon taste than the 4 lemon:3 lime mixture. But why did the PSTs not recognize the multiplicative relationship when reasoning quantitatively (without calculations)? What is surprising is how the PSTs used the cubes when they initially reasoned about the situation. The 40% of the PSTs who utilized multiplicative reasoning ALL created models where the green and yellow cubes were separated (see figure 6), while the 60% who reasoned additively all created models with the green and yellow cubes attached (see figure 7).

Separating the lemon from the lime allowed the PSTs to recognize the multiplicative relationship between the two quantities and not focus on the fact that there is one-cup difference between the two mixtures. In contrast, those representations created by the PSTs where the lemon and lime remained attached seemed to force the PSTs' focus on the fact that there was one more cup in each mixture. The reason may be because the attached cubes resemble lines that would have length. Kaput and West (1994) found that there was a strong tendency to adopt additive reasoning when problems

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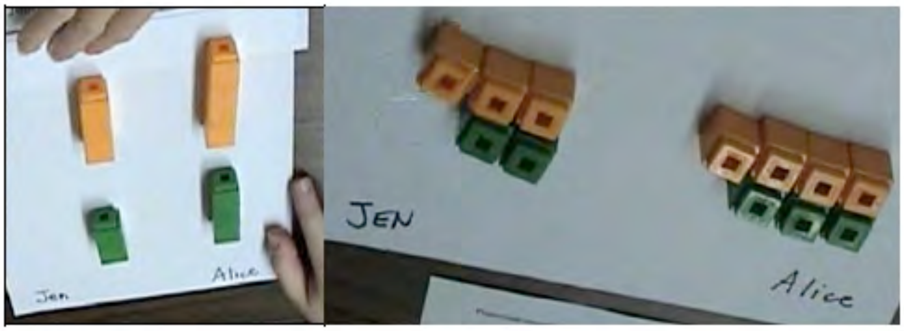
involved linear measurements. So by, separating the cubes by color the PSTs were able to attend to both quantities multiplicatively because their prior assumptions about length would not have been brought forward as strongly by the representation. Knowing this distinction we can create powerful discussions about the quantitative reasoning necessary for solving proportional problems.



(a) (b)

Figure 6. Examples of PSTs representation of the Lemon/Lime problem that utilized multiplicative reasoning.

(a) (b)

Figure 7. Examples of PSTs representatoin of the Lemon/Lime problem that utilized additive reasoning.

Conclusion

In all of the cases presented in this report, PSTs used representations to clarify and explain their proportional reasoning. Whether they used tables, drawings, pie charts, or unifix cubes, the models that represented the context and particular situation of the tasks led to reasoning that had deep meaning. Lo (2004) suggests that providing pre-service teachers with mathematical tasks that are rich in context and encouraging them to develop drawings and representations that convey the meaning of their solution methods to other students deepens their mathematical reasoning. The use of representations when teaching proportional reasoning can provide opportunities to distinguish between proportional and non-proportional situations, explain part to part relationships involved in ratios, and support students' multiplicative reasoning necessary for the development of deep proportional reasoning. Lobato and Ellis (2010) discuss the use of representations in many of their proposed essential understandings of ratio and proportion; however, the role of representation in developing and modeling reasoning is not given the priority it warrants.

This study suggests that greater importance should be given to representations that students produce while solving proportional problems and that the use of multiple representations while

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teaching about ratio will lead to deeper understanding of the concept of proportion. Representations allow individuals to attend to important aspects of their reasoning, including the two quantities involved in the ratios, the context of the problem, and the multiplicative relationship needed in proportional reasoning. Yetkiner and Capraro's (2009) research summary for National Middle School Association stated that until teachers can develop specialized content knowledge in multiplicative and proportional reasoning, they would struggle to provide students with multiple representations that can address the different learning styles found in their classroom. As mathematics teacher educators we should begin to address the difficulties PSTs have with proportional reasoning by providing multiple representations in our own classrooms and discussing their benefits. This report illustrates several representations of proportional reasoning that proved to be useful.

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