THE INTERPLAY BETWEEN STUDENTS' UNDERSTANDINGS OF PROPORTIONAL AND FUNCTIONAL RELATIONSHIPS

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This research explores the interplay between students' understandings of proportional and functional relationships. Approximately 90 students participated in an early algebra intervention in Grades 3–5. Before the intervention and after each year of the intervention, we evaluated their understandings of proportional and functional relationships. Data revealed that among Grades 4 and 5 students who identified a correct function rule, a higher percentage were unsuccessful solving a proportional reasoning problem than those who were not able to identify a correct function rule. Namely, the data suggest that students' development of functional thinking may interfere with their development of understanding proportional relationships.

Keywords: Algebra and Algebraic Thinking, Elementary School Education

Decades of reform initiatives in teaching and learning algebra (e.g., National Council of Teachers of Mathematics, 2000, 2006) have brought about the "algebrafication" of elementary grades mathematics in which a number of core algebraic concepts are introduced into classroom curriculum and instruction (Kaput & Blanton, 2001). While research has documented the development of students' understanding of these concepts, what is less well understood is the interplay between concepts that, at face value, seem to be developmentally complementary to one another (e.g., functional thinking and proportional thinking). In what ways does learning particular algebraic concepts support or hinder the learning of other algebraic concepts? This question lies at the core of our study in which we examine the ways in which children's understandings of two concepts appear to interact and potentially constrain the development of their algebraic thinking. Specifically, we investigate the interplay between students' understanding of proportional relationships and students' understanding of relationships between quantities that co-vary in a non-proportional way (e.g., the functional relationship $y = 2x + 2$).

We chose to study these concepts because recent findings have shown that elementary students can reason about and describe relationships between co-varying and corresponding quantities (e.g., Blanton & Kaput, 2004; Schliemann et al., 2003) and, in fact, that even students in kindergarten and first grade can engage in this kind of thinking about co-varying and corresponding quantities (e.g., Blanton, Brizuela, Gardiner, Sawrey, & Newman-Owens, 2015; Brizuela, Blanton, Sawrey, Newman-Owens, & Gardiner, 2015).

Our Early Algebra Intervention

This research is part of a three-year longitudinal study (viz., Blanton et al., 2017) whose overarching goal is to design, implement, and evaluate a Grades 3 – 5 early algebra intervention. We based the intervention on a synthesis of Kaput's (2008) analysis of algebra in terms of content strands and thinking practices (see Blanton, Stephens, et al., 2015 for an elaboration of the intervention). In particular, using Kaput's content analysis of algebra we frame the content of our

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intervention in terms of four fundamental thinking practices that characterize algebraic thinking: (1) *generalizing*, (2) *representing*, (3) *justifying,* and (4) *reasoning with* mathematical structure and relationships. We also identified several "big ideas" of algebra, that is, principles in a domain that are essential to developing an integrated understanding in that domain (Shin, Stevens, Short, & Krajcik, 2009) and that reflect content spaces in which the core practices of algebraic thinking (e.g., generalizing) can occur. The big ideas of algebra that comprised the early algebra intervention are as follows: (a) equivalence, expressions, equations, and inequalities; (b) generalized arithmetic; (c) functional thinking; and (d) variable.

One of the areas that becomes increasingly important as students transition into middle grades is proportional reasoning (NGA & CCSSO, 2010). Because of the connections between proportional reasoning and functional thinking, particularly as it relates to issues of rate of change and slope, we were interested in the interplay between students' functional thinking, developed as part of our Grades 3 – 5 early algebra intervention, and their early notions of proportional reasoning addressed in the regular curriculum. At face value, these two conceptual areas seem developmentally complementary to one another. Proportional reasoning involves generalizing two related quantities in which "the ratio of one quantity to the other is invariant" (Blanton, Stephens, et al., 2015, p. 43). Functional thinking involves "generalizing relationships between (two) covarying quantities and representing" those generalizations "using natural language, algebraic notation, tables, and graphs" (Blanton, Stephens, et al., 2015, p. 43). We view proportional reasoning as a subset of functional thinking because all proportional relationships can be described as functions, but not all functional relationships are proportional relationships. The purpose of this study is to investigate the interplay between students' functional thinking, developed as part of our Grades 3 – 5 early algebra intervention, and their early notions of proportional reasoning addressed in the regular curriculum.

Methods

We share data collected from a three-year longitudinal study in which we implemented and evaluated our early algebra intervention. To evaluate our early algebra intervention we assessed the algebraic thinking of students who participated in our intervention at several time points using a pretest and posttests.

Participants

At the pretest, participants included 103 Grade 3 students from a school in southeastern Massachusetts. The school's district is 8% non-white, 5% ELL students, and 20% low SES students. Due to attrition, 90 students participated in the entire early algebra intervention (i.e., participated in Grades 3 – 5).

Intervention

The Grades 3 – 5 intervention consisted of approximately 18 lessons per year and engaged students in the aforementioned algebraic thinking practices of generalizing, representing, justifying, and reasoning and the targeted big algebraic ideas. One member of our project team served as the classroom instructor for the intervention, beginning with the Grade 3 cohort and continuing with this cohort through the completion of Grade 5. The intervention was taught as part of students' regular mathematics instruction. The sequence of 18 lessons in each of Grades 3 – 5 included 6 lessons focusd on functional thinking. Functional thinking lessons were designed to get students to generate data, use function tables to organize data, identify functional relationships and represent in words and variables, and use these relationships to make inferences about function behavior. Lessons also included developing graphs to represent functions and interpreting functional behavior in graphs through quantitative and qualitative means. Functional thinking tasks focused primarily on linear

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functions, but also included quadratic and exponential functions. Proportional reasoning concepts were not explicitly taught in the intervention.

The instructional sequence was organized into Grades 3, 4, and 5. For each year of our intervention, we listed learning goals and organized them according to the associated big idea. The lessons were designed to address these learning goals. Each lesson began with a small-group discussion regarding a previously addressed learning goal, so that learning goals were revisited throughout the lessons. Then, a new learning goal was addressed through small-group problem solving and a whole-class discussion. Associated assessments were designed to test the effectiveness of the intervention by evaluating students' understandings of the big ideas and administered at the beginning of the intervention (Grade 3 pretest) and after each year of the intervention (Grade 3 posttest, Grade 4, and Grade 5).

Data Collection

Students who participated in the intervention were assessed at the beginning of Grade 3, and then again at the end of each year in Grades 3 – 5 using grade-level assessments designed by the project team. The same Grade 3 assessment was used as a pre/post measure in Grade 3, while the Grades 4 and 5 assessments included some identical items and some new items. Each assessment consisted of about 12 items (10 were multi-part open response, 2 were multiple choice). Here we focus on students' responses to two items that appeared on the assessments at each grade level.

The first item (see Figure 1), the *Caterpillar* task (adapted from NAEP, 2003), is designed to evaluate students' ability to reason proportionally. The second item, the *Brady* task (see Figure 2), was designed to evaluate students' understandings of functional relationships. Here, we focus in particular on part c2, which was designed to assess students' ability to generalize and represent a functional relationship using variables. Both of these items appeared on all four assessments given across Grades 3 – 5.

Data Analysis

Responses were scored using a coding scheme developed by the project team to capture both correctness of student responses as well as the types of strategies students used (Blanton, Stephens, et al., 2015). For the response to the *Caterpillar* task to be coded as correct, students must have provided a response of 30. If students also provided an explanation that demonstrated proportional reasoning, coders further identified the way that the student reasoned proportionally (i.e., as using calculations, tables, pictures, a unit rate or repeated addition). If students provided an incorrect answer, coders labeled the response with one of the incorrect strategies or "other." If no explanation or indication of strategy was provided, coders labeled the response "answer only." Here we focus on a specific incorrect strategy, incorrect linear relationship. Students who demonstrated this strategy wrote a response of "25" and typically explained that they used the linear relationship "$2x + 1$" to find their solution. Students found this solution because the relationship "$2x + 1$" results in the example provided, 5 caterpillars and 2 leaves.

A fourth-grade class needs 5 leaves each day to feed its 2 caterpillars. How many leaves would they need each day for 12 caterpillars? Explain how you got your answer.

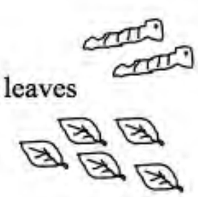

Figure 1. Proportional reasoning item (*Caterpillar* task).

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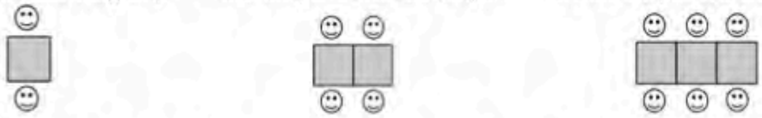
Brady is celebrating his birthday at school. He wants to make sure he has a seat for everyone. He has square desks.

He can seat 2 people at one desk in the following way: If he joins another desk to the first one, he can seat 4 people: If he joins another desk to the second one, he can seat 6 people:



a) Fill in the table below to show how many people Brady can seat at different numbers of desks.

Number of desks	Number of people
1	2
2	4
3	
4	
5	
6	
7	

b) Do you see any patterns in the table from part a? If so, describe them.

c) Think about the relationship between the number of desks and the number of people.
1. Use words to write the rule that describes this relationship.
2. Use variables (letters) to write the rule that describes this relationship.

Figure 2. Functional thinking item (*Brady* task).

When refining the coding scheme, we conducted iterative analyses of students' responses to these items. First, we identified strategies already documented in the research literature on children's algebraic thinking. For example, research shows that children may begin generalizing functional relationships by focusing on particular instances, demonstrating a recursive strategy (Blanton, Brizuela, et al., 2015). These external strategies served as a starting point for developing our coding scheme. If a response did not fit an external strategy, it was grouped with similar responses. We then identified patterns in these responses and developed new codes to capture these responses.

For the response to part c2 of the *Brady* task to be coded as correct, students must have written a function using variables to represent the relationship at hand (e.g., $y = 2x$). If students provided an incomplete (e.g., $2x$) or incorrect answer, coders labeled the response with the appropriate incorrect strategy. If a student's response could not be categorized using our coding scheme, coders labeled the response as "other."

Inter-rater reliability scores were computed for 20% of the items and at least 80% agreement was achieved between the coders. When coders disagreed, they discussed codes until agreement was obtained.

Results

In this section, we share results from the two written assessment items and focus on relationships we observed between students' understandings of functional and proportional relationships. For the *Caterpillar* task, we focus on one strategy in particular, the incorrect linear relationship strategy because we found an unexpected relationship between this strategy and another strategy. Students who used the incorrect linear relationship strategy incorrectly generalized a linear relationship between the number of caterpillars and the number of leaves (e.g., $2x + 1= y$) were coded as using

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this strategy. For the *Brady* task, we also focus on one strategy, the correct function rule. Students who demonstrated this strategy correctly identified a function rule and represented it using variables in an equation.

We observed a trend in the ways that students who generalized functional relationships on part c2 of the *Brady* task reason about the *Caterpillar* task. In particular, the data revealed that among the students in Grades 4 and 5 who identified a correct function rule using variables to describe a generalized relationship between the two covarying quantities, a higher percentage of those students demonstrated the incorrect linear relationship strategy on the *Caterpillar* task than did the overall population of students. That is, the data suggest that in the context of the intervention, students' development of functional thinking and the development of their understandings of proportional relationships may be related. Although we are not certain of the nature of this relationship, these data suggest that some students' development of functional thinking may impede the development of their understandings of proportional relationships in the context of the intervention.

Table 1 shows the percentage (and number) of students who identified the correct function rule in response to the *Brady* task in Grades 3, 4, and 5. The number of students who identified the correct function rule is listed in parentheses. The data reveal that as students progressed through the intervention, they were better able to write the correct function rule using variables.

Table 1: Overall Student Performance on *Brady* Task

	Gr 3 Pre	Gr 3 Post	Gr 4	Gr 5
Correct Function Rule	0.00% (0)	35% (36)	64% (61)	67% (60)

The percentages in Tables 2 – 4 were calculated using the number of students who used the correct function rule for each grade (as shown in Table 1). In other words, the denominator for each percentage in Tables 2 – 4 is the number of students (in parentheses) for the respective grade in Table 1. Table 2 shows how the subgroup of students—the 36 students—who identified the correct function rule at the Grade 3 posttest performed on the *Caterpillar* task at each time point. Table 3 shows how the subgroup of students—the 61 students—who identified the correct function rule at the Grade 4 test performed on the *Caterpillar* task at each time point. Table 4 shows how the subgroup of students—the 60 students—who identified the correct function rule at the Grade 5 test performed on the *Caterpillar* task at each time point.

Table 2: Percentage of Gr 3 Post Students who Provided a Correct Function Rule (*Brady* Task) and Used the Incorrect Linear Relationship Strategy (*Caterpillar* Task)

	Gr 3 Pre	Gr 3 Post	Gr 4	Gr 5
Incorrect Linear Relationship	0%	3%	7%	12%

Table 3: Percentage of Gr 4 Students who Provided a Correct Function Rule (*Brady* Task) and Used the Incorrect Linear Relationship Strategy (*Caterpillar* Task)

	Gr 3 Pre	Gr 3 Post	Gr 4	Gr 5
Incorrect Linear Relationship	0%	7%	13%	30%

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Table 4: Percentage of Gr 5 Students who Provided a Correct Function Rule (*Brady* Task) and Used the Incorrect Linear Relationship Strategy (*Caterpillar* Task)

	Gr 3 Pre	Gr 3 Post	Gr 4	Gr 5
Incorrect Linear Relationship	0%	12%	30%	30%

Table 5 shows the percentage of all students who demonstrated the incorrect linear relationship strategy in Grades 3, 4, and 5. By comparing the performance of the subgroups on the *Caterpillar* task (Tables 2 – 4) to the overall performance of students on the *Caterpillar* task we noticed that students who identified the correct function rule were more likely to also demonstrate the incorrect linear relationship strategy in Grades 4 and 5 than was the general population of students.

The percentage of students identifying the correct function rule in response to the *Brady* task who demonstrated the incorrect linear relationship strategy in response to the *Caterpillar* task is less than the total percentage of students who demonstrated the incorrect linear relationship strategy in response to the *Caterpillar* task in Grade 3. We do not believe we can draw many conclusions from this due to the low overall performance on the *Brady* task in this grade. However, as success on the *Brady* task increases into Grades 4 and 5, we feel that more can be said about the interaction between students' strategy use on these items.

Table 5: Percent of Students who Used the Incorrect Linear Relationship Strategy (*Caterpillar* Task)

	Gr 3 Pre	Gr 3 Post	Gr 4	Gr 5
Incorrect Linear Relationship	2%	5%	9%	20%

Of the subgroup of students who identified the correct function rule in response to the *Brady* task in Grades 4 and 5, the percentage that also demonstrated the incorrect linear relationship strategy in response to the *Caterpillar* task is greater than the overall percentage of students who demonstrated the incorrect linear relationship strategy. Interestingly, the reverse relationship holds true for Grades 4 and 5 as well. That is, of the subgroup of students who demonstrated the incorrect linear relationship strategy in Grades 4 and 5, the percentage who also identified the correct function rule is greater than the overall percentage of students who identified the correct function rule.

Table 6: A Representative Student's Responses in Grades 4 and 5

	Incorrect Linear Relationship Resonse to *Caterpillar* Task	Correct Function Rule Response to *Brady* Task
Grade 4	X Y / L c / 5 2 / 25 12 — the rule: Well I saw it was 2×Y+1=X So I did 12×2+1=25	D×2=P D=Desks P=People
Grade 5	Cater. X / leaves Y / 2 5 / 12 25 — Rule: X·2+1=Y	X·2=Y

Table 6 shows one student's responses to both tasks at the Grades 4 and 5 assessments. This student was selected because his strategy gives an example of the combination we focus on in this

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paper. We chose to show the students' responses in both Grades 4 and 5 because we observed that several students demonstrated both strategies in both Grades 4 and 5. Specifically, 8 students demonstrated both these strategies in Grade 4. Of those 8 students, 6 of them also demonstrated both strategies in Grade 5. An additional 11 students demonstrated both strategies in Grade 5, totaling to 17 students.

Discussion

The results presented here highlight the complexity of the interrelated concepts involved in studying corresponding relationships in the elementary grades. It leads us to question the role of proportional reasoning in the regular curriculum and how that of functional thinking in our intervention (or even the regular curriculum, to the extent that functions are addressed in elementary grades) coincide.

Moreover, the findings reveal that the characteristics of corresponding relationships that are salient to students are not what we anticipated when designing the intervention. That is, the findings highlight that educators need to be cautious when drawing conclusions about what children know and how it is they come to know it. The findings show that while children may have knowledge of a particular concept, functional thinking in this case, they might misappropriate the concepts and tools in other situations. We can infer from the student's responses shown in Table 6 that this student, and based on the percentages shown in Tables 2 – 4 likely many students, chose to use one of the tools they were taught (e.g., a function table) to use when interpreting functions to represent the *Caterpillar* task. Students are taught function tables as a tool for interpreting functional relationships. Therefore, the fact that this student correctly responded to the *Brady* task and used a function table to interpret the *Caterpillar* task makes sense in the context of our intervention. The reason we did not observe students incorrectly responding to the *Brady* task and using the incorrect linear relationship strategy is because they did not have the tools (e.g., a function table) for interpreting functional relationships.

Lastly, we believe the data displayed in Table 6 highlight that when children come to know a concept in a certain way, they struggle to change the way they know that concept, especially in different contexts. This observation may indicate that students' thinking is entrenched from year to year because the context is relatively consistent. McNeil and Alibali (2004) noted that students resist adapting their understandings of the equal sign in different contexts and we view this finding as relevant to our interpretation of our findings. Similarly, our prior research (e.g., Strachota et al., 2016) on students' understanding of the equal sign and functional thinking have led us to consider how different contexts and co-developing big ideas might impede or support the development of children's algebraic reasoning. We believe these studies are a small slice of an increasingly important area of research in early algebra. In order to move forward in supporting students in developing understandings of algebraic concepts, we must better understand the interplay between concepts.

Due to the nature of our data (i.e., written assessments) we do not know with certainty what students might have been thinking when they demonstrated the incorrect linear relationship strategy. However, we can infer that students who demonstrated this strategy associated some aspect of the proportional relationship with the process of writing a function rule using variables. Moving forward, we hope to investigate what aspects of the task are salient to the students who use the incorrect linear relationship strategy and use these data to refine our instruction.

We acknowledge the limitations of this study specifically the small sample size of the subgroups and the fact that only two tasks are considered, but hope the findings will serve as the premise for future research that takes the same line of inquiry. Researchers have long advocated that algebra be developed as a longitudinal curricular strand. We agree with this perspective, and believe that our findings reveal the importance of continuously supporting students in developing understanding of core algebraic concepts associated with functional thinking and proportional reasoning. Further, we view the conceptual areas of proportional reasoning and functional thinking as interrelated, and recommend that early algebra curricula be designed to synthesize these core concepts of algebra, as well as all core concepts of algebra.

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