ELEMENTARY STUDENTS' REASONING ABOUT ANGLE CONSTRUCTIONS

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In this report, we discuss the findings from 2 pilot studies investigating the effects of interventions designed to provide students in Grades 3–5 with opportunities to work with dynamic and static models of angles in a dynamic geometry environment. We discuss the effects of the interventions on the children's development of quantitative reasoning about angle measure.

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Geometric measurement is a branch of mathematics that integrates number and space and includes length, area, volume, and angle measurement. Based on Barrett and Smith's review of the literature in the measurement chapter of the research compendium, Smith (2016) asserted that they found markedly less research on angle measurement in comparison to the other three forms of geometric measurement (length, area, and volume). The present research on angle measurement focuses on student difficulties with several measurement concepts, including identifying and attending to the correct attribute of angle as well as angle unit, unit iteration, and origin. Specifically, students struggle to identify what is being measured when referring to size of angles and what is one degree (Keiser, 2004). Often elementary and middle school students attend to ray lengths (e.g., Clements, 2003; Keiser, 2004), and elementary, middle, and high school students are distracted by angle orientation (Mitchelmore, 1998; Noss, 1987; Fyhn, 2008). Mitchelmore (1998) argued that students need opportunities to work with both dynamic (the motion of an angle opening) and static (the resultant figure after the opening) angle models to confront their misconceptions.

Previous research has incorporated both models. Several previous studies utilized the LOGO environment (e.g., Clements & Burns, 2000; Noss, 1987; Simmons & Cope, 1993) or used a sequence of static models to indicate motion to varying degrees of success.[i] In one study, Clements, Battista, Sarama, and Swaminathan (1996) explored third grade students' understanding of angle measurement in a modified LOGO environment. These researchers found that immediate feedback helped students reflect on their turn commands and thus angle measurement (cf. Simmons & Cope, 1993) and that using benchmarks helped students assign numbers to turns. However, because there was no record of the turn during the turn in Clements et al.'s modified LOGO environment, turn commands were less salient to their students than forward or backward commands, which could be because the forward and backward commands leave a line segment as a trace, whereas there is not a similar record with turns.

Mathematics education has yet to fully determine how to address the misconceptions present in the literature since the 1980s. The authors of the Common Core State Standards in Mathematics (CCSSM, National Governor's Association for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010) have renewed mathematics educators' interest in elementary students' conceptions of angle and angle measurement through their definition of angle and mandate for how angle should be understood by fourth grade:

> An angle is measured with reference to a circle with its center at the common endpoint of the rays, by considering the fraction of the circular arc between the points where the two rays intersect the circle. An angle that turns through 1/360 of a circle is called a "one-degree angle," and can be used to measure angles. (p. 31)

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To promote this interpretation of angle and extend the literature on angle, we designed interventions. Both were enacted in a dynamic geometry environment utilizing the computer software, Geometer's Sketchpad, to provide children with opportunities to work with movable angle situations as well as reflect on dynamic (the motion of an angle sweeping open) and static (the resulting image of an angle after sweeping open) angle models. In this report, we discuss the results from our testing of these interventions. We posed the research question: *In what ways does interacting with dynamic and static angle models affect students' reasoning about angle constructions in Grades 3, 4, and 5?*

Theoretical Perspective

We approached this study from a quantitative reasoning approach. According to Thompson (1990), "a quantity is a quality of something that one has conceived as admitting some measurement process. Part of conceiving a quality as a quantity is to explicitly or implicitly conceive of an appropriate unit" (p. 5). In the case of angle measure, quantifying involves reasoning about the unit (e.g., a degree) in terms of a quantitative relationship (i.e., a multiplicative relationship) between a fraction of the circular arc of a circle and the circle's circumference, which is consistent with the CCSSM authors' (NGA & CCSSO, 2010) recommendations for how fourth grade students should understand angle measure.

There are three types of quantity—gross, extensive, and intensive (Piaget, 1965). Gross quantity depends on perception—one object is bigger, smaller, more, less, or the same as another in terms of some attribute. Extensive quantity is additive (Piaget, 1965) and the result of unitizing activity (Steffe, 1991), whereas intensive quantity is not additive (Piaget, 1965). Instead, it requires proportional reasoning. For example, if Person A is traveling 20 mph and Person B is traveling 40 mph, we are not traveling 60 mph. We used a quantitative reasoning approach in our design of the interventions as well as in our interpretation of the findings.

Method

To investigate dynamic and static angle models affected students' reasoning about angle constructions, we wanted to observe and document changes. Thus, we utilized the microgenetic method (Siegler & Svetina, 2006). The microgenetic method has three main tenets:

> (1) observations span the whole period of rapidly changing competence; (2) the density of observation within this period is high, relative to the rate of change; and (3) observations of changing performance are analyzed intensively to indicate the processes that give rise to them. (Siegler & Svetina, 2006, p. 1000)

The data presented in this report were collected during the 2014–2015 school year at a suburban public school in the Midwestern region of the United States. In the first pilot study, we interviewed 18 students in Grades 3–5 (ages 8–12), six students per grade. For the second pilot study, we interviewed 19 students in Grade 3 (ages 8–9). Each student participated in three 4 to 18 minute individual interviews with one of the three authors of this report during the normal school day. We used a structured interview protocol and recorded the interviews using screen-capturing software, Screencast-o-matic, which also records audio. Consistent with the microgenetic method, observations were dense. The three interviews occurred on three separate days, and the mean elapsed time between first and third session was 2.9 school days (max of 5 school days). Prior to the first interview and after the third, children took a written survey. On this survey, children were asked to give a definition of angle, estimate the measure of a given angle, and construct an angle. On six items, children were asked to select one out of three angles that (a) had a specified measure, (b) had the largest measure, or (c) had the smallest measure.

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Pilot 1

During the first interview, we guided each student through a tutorial on how to use four sliders in a dynamic geometry environment (i.e., Geometer's Sketchpad). Each of the sliders had a different effect. One slider opened and closed the angle, one translated the image left and right, one rotated the image of the angle, and one lengthened and shorted the rays.

During the second interview, the student went through eight trials, which we define to be a task-intervention pair (Siegler & Crowley, 1991). Specifically, the student was asked to construct an angle of a specified measure and to let the interviewer know when he or she was ready to check. During the check, the researcher clicked a check button and a ray swept from the initial ray to the angle of the terminal ray of the desired angle. In addition, benchmark rays appeared at each 30-degree interval until the ray stopped at the terminal side of the angle (see Figure 1). Note that the ray left a fading trace as it swept across the screen to provide a record of the turn (cf. Clements et al., 1996). The measure of the angle the students constructed was also briefly displayed, providing them with feedback. This feedback allowed them to compare the size of two angles (the created and desired) to their associated measure as well as to compare the difference in the created and desired angle. For example, in the sequence below a student would have had the opportunity to see a 120-degree angle, a 128-degree angle, and to see that small difference between the two was about eight degrees. During the third interview, the student went through nine trials with the same design principles.

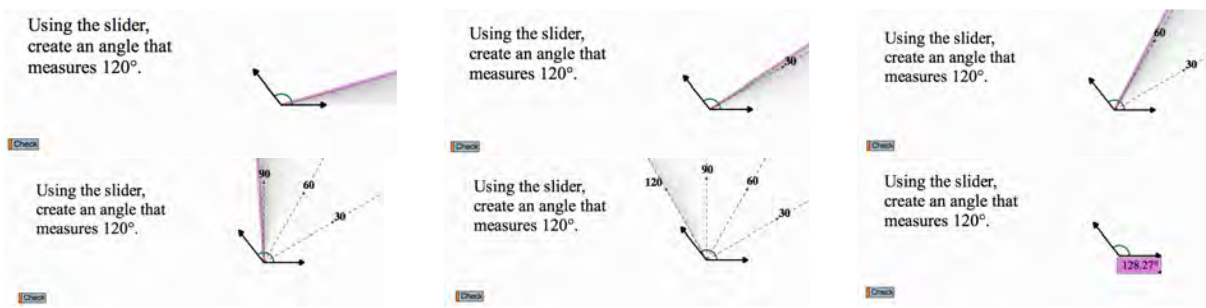

Figure 1. A sequence of screen shots displaying the labeled benchmark rays (e.g., 30, 60, 90) appearing as the student checked her attempt at a 120-degree angle.

Pilot 2

During the first interview, we asked the student to use one slider to create an angle of a specified measure on eight trials. This process was repeated for nine trials during the second interview and nine trials during the third interview. For the construction of all 26 angles, the student started with one initial horizontal ray and one slider that opened and closed the angle. For each angle, the interviewer told the student to use the slider to create an angle of a specified measure.

The 19 participants were divided between the two intervention groups: 10 were in Intervention Group 1 (IG1) and nine were in Intervention Group 2 (IG2). For children in IG1, during alternating trials in Interviews 1 and 2, unlabeled benchmark rays of 30 degrees would appear as the child used the slider to open the angle (see Figure 2). During the trials in Interview 3, the benchmark rays did not appear during any of the trials. When the child indicated that he or she was ready to check, the check button was clicked, and a ray swept from the initial ray of the angle to the terminal ray of the angle the child had been instructed to create. The main components of Pilot 2 IG1 were a subset of those in Pilot 1: Benchmark rays appeared at each 30-degree interval until the ray stopped at the terminal side of the desired angle. The measure of the angle the child constructed was briefly displayed for the interviewer to record (see Figure 1). In contrast, for children in IG2, benchmark

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rays appeared neither during construction nor during the check. Instead, when the check button was clicked, only the measure of the angle the child constructed was displayed. Figure 3 summaries the key components of the two pilot studies.

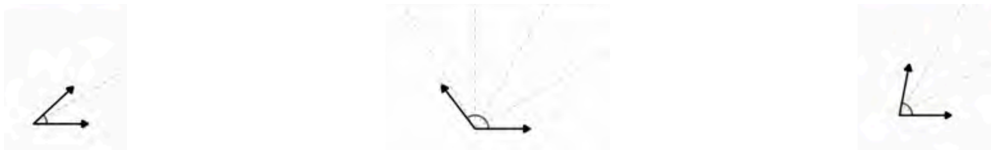

Figure 2. A sequence of screen shots displaying the 30-degree benchmark rays appearing for students in Pilot 1 and Pilot 2 IG1 as the terminal ray of the angle swept open.

Pilot 1	Pilot 2, Group 1	Pilot 2, Group 2
Grades 3–5	Grade 3	Grade 3
18 participants (6 per grade)	10 participants	9 participants
3 interviews	3 interviews	3 interviews
17 trials	26 trials	26 trials
4 sliders	1 slider	1 slider
Alternating trials, unlabeled benchmark rays of 30 appeared as created angle (*)	Alternating trials, unlabeled benchmark rays of 30 appeared as created angle (*)	NA
Trial 9, 90 Trial 10, 80* Trial 11, 30 Trial 12, 40* Trial 13, 60 Trial 14, 70* Trial 15, 120 Trial 16, 110*	Trial 9, 90 Trial 10, 80* Trial 11, 30 Trial 12, 40* Trial 13, 60 Trial 14, 70* Trial 15, 120 Trial 16, 110*	Trial 9, 90 Trial 10, 80 Trial 11, 30 Trial 12, 40 Trial 13, 60 Trial 14, 70 Trial 15, 120 Trial 16, 110
After check button clicked, a ray swept from the initial ray of the angle to the terminal ray of the desired angle, and that ray left a trace.	After check button clicked, a ray swept from the initial ray of the angle to the terminal ray of the desired angle, and that ray left a trace.	NA
After ray swept from initial ray to the terminal ray of the desired angle, the measure of constructed angle appeared.	After ray swept from initial ray to the terminal ray of the desired angle, the measure of constructed angle appeared.	After check button clicked, the measure of constructed angle appeared.

Figure 3. Comparison of the key components of the two pilot studies.

In the design of both pilots (including Pilot 2 IG2), we privileged approximations of 10 degrees. During Interview 2 (Trials 9–16), we sequenced trials to pair benchmark angles with near benchmark angles (i.e., angles measuring 10 degrees more or less than one of the bench mark angles). There were four sets of paired trials (see Figure 4) to provide children with experiences that would support

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their development of a sense of 10 degrees. (It was our conjecture that developing a sense of 1 degree would be more difficult than developing a sense of 10 degrees.)

Trial 9, 90 degrees	Trial 10, 80 degrees
Trial 11, 30 degrees	Trial 12, 40 degrees
Trial 13, 60 degrees	Trial 14, 70 degrees
Trial 15, 120 degrees	Trial 16, 110 degrees

Figure 4. Four sets of paired trials.

We also privileged the integration of number and space. At the end of each trial, we displayed the measure of the constructed angle, providing students with the opportunity to pair the image of the constructed angle with its measure. For students in Pilot 1 and Pilot 2 IG1, this was taken further because of how the terminal ray of the angle swept open, providing students with the opportunity to pair the image of the intended angle with its measure.

In the design of Pilot 1 and Pilot 2 IG1, we privileged 30-degree benchmarks by displaying unlabeled benchmark rays at 30-degree intervals during construction on alternating trials. Additionally, the check on every trial displayed 30-degree benchmark rays with labels (e.g., 30, 60, 90). Our purpose for including these supports was to encourage students to reason about, use, and operate on specific benchmark angles as well as to encourage the integration of number and space.

In the design of Pilot 2 IG2, we privileged only feedback. Upon clicking the check button, the measure of the angle the child constructed on a given trial was displayed. This provided the children with the opportunity to reflect upon how the measure of the angle they constructed compared to the desired angle measure (cf. Jaehnig & Miller, 2007).

Discussion of Findings

Our findings indicate that the interventions had an effect on students' angle constructions. In this section we provide a representative sample of quotes to illustrate each of our three main findings.

First, we found that interacting with dynamic and static angle models increased third and fourth grade students' recognition and use of 30-degree benchmarks. Most of the third and fourth grade students did not mention the 30-degree benchmarks during the first eight trials, but by Trial 9, many of these young students in Pilot 1 and Pilot 2 IG1 were referring to them when they described what they were thinking about during or after angle construction.

- Trial 10, 80: "[While creating] 30, 60, that's about 80. [After check] I noticed that I was right, because since the 90 degree angle is um, is up I kind of noticed that if its 80 you might need to make it a little more slanted." (Bob, Grade 4, Pilot 1)
- Trial 10, 80: "I was, um, pretty close. [What were you thinking about when you made that?] Uh, well, it showed us the lines. So I knew it- one was 30, one was 60, and I knew it wouldn't be all the way– all the way to 90 but it's be a little less than 90 but a little more than 60." (Erin, Grade 3, Pilot 2 IG1)
- Trial 15, 120: "I was counting by 30s by like 30, 60, 90." (Charlie, Grade 3, Pilot 2 IG1)

Second, we found that by asking students what they noticed after the check button was clicked as part of our interview protocol, we prompted students to compare the desired and constructed angles numerically and spatially. Although this occurred for students in Pilot 2 IG2, it was more pronounced for students in Pilot 1 and Pilot 2 IG1. To illustrate, we provide four quotes below. We found the first

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two (from students in Pilot 1 and Pilot 2 IG1) to be qualitatively different from the last two (from student in Pilot 2 IG2).

- Trial 10, 80: “Well, I remember how last time, um, if I- I moved a lot but it was only just this much to get ta- just um like a tiny bit to get to like 74 from 70, [Mmm so 70] like 71 or something, so I didn’t want to go too far from 60 because I know that’s about 20 away.” (Genny, Grade 3, Pilot 2 IG1)
- Trial 14, 70: “I had to make it just a little bit smaller in order to make it 70 degrees and like down here where I tried to make a 40 degree angle and I gave it too much space.” (Frank, Grade 5, Pilot 1)
- Trial 11, 30: “28 degrees. [What do you think?] Good.”(Pam, Grade 3, Pilot 2 IG2)
- Trial 12, 40: “35. 5 off. That’s good still.” (Billy, Grade 3, Pilot 2 IG2)

Third, we found that most students appeared to benefit from paired trials. Some students reasoned about a unit of 10 degrees and fractional parts of the benchmark created wedge. These students’ quantitative reasoning was a bit more advanced than the students who could only reason about a little more or a little less than the benchmark rays, as illustrated by their quotes below in which they utilized specific numeric relationships (e.g., difference between the measure of the benchmark angle and the measure of the desired angle).

- Trial 14, 70: “So like 90, so that like is 60,…okay. Um, ‘cuz 70 is, well obviously, uh 10 degrees larger than 60.” (Amber, Grade 5, Pilot 1, created 71.97 degree angle)
- Trial 17, 150: “That, um, like the, um, the arrow, it went a little bit past it because, um, 120 is only 30 away and it seems like 30’s long but it’s not.” (Eric, Grade 3, Pilot 1, created 139.01 degree angle)

One benefit of the microgenetic method is to have dense observations during a change to document that change. Hence, after we documented some shifts, we dug deeper to explore how one student’s explanations and actions changed from trial to trial. Oscar, a fourth grade student in Pilot 1, exhibited improvement in his approximations for 10 degrees across the four sets of paired trials and clearly articulated how he learned from the previous trials. After constructing an angle that measured 86.36 degrees (for a desired angle measure of 80 degrees), Oscar explained, “I needed to go a little further back. I put, I was trying to measure about a little before the 90-degree angle.” On this trial (Trial 10), Oscar correctly reasoned quantitatively about the benchmark angle measure and the desired angle measure (i.e., that 80 degrees is less than 90 degrees); however, he did not identify how much less (i.e., difference between the measure of the benchmark angle and the measure of the desired angle). On the next off-benchmark trial (Trial 12), Oscar constructed an angle that measured 47.84 degrees (for a desired angle measure of 40 degrees). When asked what he was thinking about when constructed his angle, he said, “It goes just a few shades away from the 30 degree angle, mmm, not that much.” Although Oscar’s constructed angle and explanation indicate he knew that a 40-degree angle was larger than a 30-degree angle, his reference to “not that much” supports an inference that he realized his attempt to create an angle that was 10 degrees more than the benchmark angle was too large. We take his explanation as evidence that Oscar had identified a 10-degree angle as a mental unit, and we predicted that his next attempt to create an angle 10 degrees more than the benchmark angle would be improved.

Based on these prior experiences, Oscar made an adjustment to his mental unit of 10-degrees on the next two off-benchmark trials (Trials 14 and 16). On Trial 14, Oscar constructed a 69.85-degree angle for a desired angle measure of 70 degrees. After checking the measure of his constructed angle, he said,

> It's like the other question...When it was asking the 40-degree, um, I went a little far; I was right there so then I kinda realized on a few more problems it was about like 80 and stuff like that—that I didn't need to go as far as I was going.

We take this as evidence that Oscar is reflecting on relating number to space—that is, his image of what a 10-degree angle looks like was too large in previous trials (Trials 10 and 12), so he had to make it smaller on this trial.

On the last off-benchmark trial (Trial 16), Oscar constructed a 110.91-degree angle for a desired angle of 110 degrees. When asked to explain his thinking before getting feedback, he said,

> I think this one is going to be right 'cuz…90 degrees. Because um, kinda of like the 60 degree and stuff, I knew that I had to go a little short, shorter because it was only 10 degrees higher than, I mean the 40 degrees… I only need it to go about right there. So then on the 90 degrees it was 20 degrees um higher than the 90 degrees so I went probably about another half of the way, right there [points to location on the screen with the cursor].

Oscar's explanation indicates that he was again reflecting on what a 10-degree angle looks like. However, how he used that 10-degree angle as a unit can be interpreted in two different ways. One interpretation is that he thought about a 20-degree angle as composed of two 10-degree angles—10 more than 90 degrees and then 10 more than that. Another interpretation is that he thought about a 30-degree angle as 10-degrees and 20-degrees and then a 10-degree angle as half of a 20-degree angle—10 more than 90 degrees (100 degrees) and then half way between 120 and 100 is 110 (i.e., "about another half of the way"). Although we expected to see Oscar use the benchmark angle of 120 degrees and think about 10 degrees less than the 120-degree angle, Oscar showed flexibility in his thinking by starting with the benchmark angle of 90 degrees to approximate a 110-degree angle. Regardless, Oscar exhibited improvement in his approximations for 10 degrees and his reasoning about angles.

Educational Importance of the Research

The experimental interventions enacted in Geometer's Sketchpad were designed to provide opportunities for students to engage with dynamic and static angle models. The results from the two pilot studies extend the previous literature on children's reasoning about angles and angle measurement. Specifically, our results suggest providing children with opportunities to reason about angles as multiples of 30 (e.g., a 120-degrees angle as four 30-degree angles) and as partitionings of 30s (e.g., a 40-degree angle as a 30-degree angle plus one-third of another 30-degree angle) has the potential to support children's recognition and use of 30-degree benchmarks. Our study also indicates that asking students what they noticed after receiving feedback on their angle constructions (after the check button was clicked) prompted students to compare the desired and constructed angles numerically and spatially.

Although we do not have evidence that the participants in this study were able to reason about angle measure by comparing the fraction of the circular arc and the circle's circumference, we do have evidence that the students were constructing their own knowledge and adapting their thinking. As illustrated by Oscar's explanations discussed in the results section, Oscar and others were adapting their thinking based on their experiences with the interventions. Specifically, Oscar's repeated attempts to approximate 10 degrees when constructing angles that were designed to be 10 more or 10 less than a benchmark angle helped him adapt this thinking and connect his numerical reasoning (i.e., 10 more degrees) to his spatial reasoning (i.e., what 10 more degrees looks like). Hence, his perception and interpretation of his experiences on these trials allowed him to describe an angle as something that can be quantified. The paired trials appeared to be an important component of the interventions in that it gave students opportunities to receive feedback on their quantitative reasoning. Thus we recommend future studies include more paired trials of 10 more or 10 less than a benchmark angle, interviewing more students from each grade level, including more trials, and parsing out the differences between groups of students with the benchmark rays and without the benchmark rays.

Endnotes

[i] Although other researchers have considered the sequencing of static images to be dynamic angle situations (e.g., Clements, Battista, Sarama, & Swaminathan, 1996; Devichi & Munier, 2013), we argue that to be truly dynamic, the sequencing of these static images needs to be more continuous.

References

Clements, D. H. (2003). Teaching and learning geometry. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 151–178). Reston, VA: National Council of Teachers of Mathematics.

Clements, D. H., Battista, M. T., Sarama, J., & Swaminathan, S. (1996). Development of turn and turn measurement concepts in a computer-based instructional unit. *Educational Studies in Mathematics, 30*(4), 313–337. doi:10.1007/BF00570828

Clements, D. H., & Burns, B. A. (2000). Students' development of strategies for turn and angle measure. *Educational Studies In Mathematics*, *41*(1), 31–45. doi:10.1023/A:1003938415559

Devichi, C., & Munier, V. (2013). About the concept of angle in elementary school: Misconceptions and teaching sequences. *Journal of Mathematical Behavior*, *32*(1), 1–19. doi:10.1016/j.jmathb.2012.10.001

Fyhn, A. B. (2008). A climbing class' reinvention of angles. *Educational Studies in Mathematics, 67*(1), 19–35. doi:10.1007/s10649-007-9087-z

Keiser, J. M. (2004). Struggles with developing the concept of angle: Comparing sixth-grade students' discourse to the history of the angle concept. *Mathematical Thinking and Learning, 6*(3), 285–306. doi:10.1207/s15327833mtl0603_2

Mitchelmore, M. C. (1998). Young students' concepts of turning and angle. *Cognition and Instruction*, *16*(3), 265–284. doi:10.1207/s1532690xci1603_2

National Governor's Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common core state standards for mathematics.* Washington, DC: Authors. Retrieved from http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf

Noss, R. (1987). Children's learning of geometrical concepts through Logo. *Journal for Research in Mathematics Education*, *18*(5), 343–362. doi:10.2307/749084

Piaget, J. (1965). *The child's conception of number.* London, England: Routledge.

Siegler, R. S. & Crowley, K. (1991). The microgenetic method: A direct means for studying cognitive development. *American Psychologist, 46*(6)*,* 606–620. doi:10.1037/0003-066X.46.6.606

Siegler, R. S., & Svetina, M. (2006). What leads children to adopt new strategies?: A microgenetic/cross-sectional study of class inclusion. *Child Development*, *77*(4), 997–1015. doi:10.1111/j.1467-8624.2006.00915.x

Simmons, M., & Cope, P. (1993). Angle and rotation: Effects of different types of feedback on the quality of response. *Educational Studies In Mathematics*, *24*(2), 163–176. doi:10.1007/BF01273690

Smith, J., III. (2016, April 12). *Measurement, discourse, and technology: Three research compendium chapters intact.* Research symposium presented at the 2016 Research Conference of the National Council of Teachers of Mathematics, San Francisco, CA.

Steffe, L. P. (1991). Operations that generate quantity. *Learning and IndividualDifferences, 3*(1), 61–82. doi:10.1016/1041-6080(91)90004-K

Thompson, P. W. (1990). *A theoretical model of quantity-based reasoning in arithmetic and algebra.* San Diego, CA: Center for Research in Mathematics & Science Education, San Diego State University. Retrieved from http://pat-thompson.net/PDFversions/1990TheoryQuant.pdf

Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.