Profiling Students' Capacities to Link Number and Algebra in Years 5, 6 and 7 in Nanjing, China

Wenbin Xu
Nanjing Normal University
<xuwenbin@njnu.edu.cn>

Max Stephens
The University of Melbourne
<m.stephens@unimelb.edu.au>

Qinqiong Zhang
Wenzhou University China
<zhang922@yahoo.com>

This study investigates how 217 students in Years 5, 6 and 7 from three schools in Nanjing, China, link number and algebra (called relational thinking in this study). It categorizes their performances in terms of five levels, and uses these levels to create profiles of algebraic thinking across Years 5, 6, and 7. The study examines the interrelationships between the different types of questions used, and highlights the importance of connecting students' arithmetic learning and the development of their algebraic understanding.

Rationale for the Research and Relevance to the Curriculum

In China, "Number and Algebra" is one out of four content strands in the National Mathematics Curriculum Standards (Ministry of Education, 2001; 2011). China's official curriculum endorses the building of closer relationships between the study of number in the primary school and the development of algebraic thinking. This is very different to what in past curriculum documents was an almost exclusive focus on arithmetic methods in the primary school years (Stephens and Zhang, 2011).

Some Chinese researchers, such as Xu (2003), suggest that curriculum reform requires a closer alignment between the study of number and number relationships in the primary school and the study of algebra in the secondary school. Jacobs, Franke, Carpenter, Levi and Battey (2007) also argue that a more coherent treatment of number and algebra requires changed emphases in and treatment of the way number and number relations are taught and understood by students. While admitting that there is still room for debate and whether *relational thinking* in arithmetic represents a way of thinking about arithmetic that provides a foundation for learning algebra, *or is itself a form of algebraic reasoning*, Jacobs et al. (2007) emphasise that "one fundamental goal of integrating relational thinking into the elementary curriculum is to facilitate students' transition to the formal study of algebra in the later grades so that no distinct boundary exists between arithmetic and algebra" (p. 261).

Five key ideas underpin our theoretical perspective on relational thinking and these constitute a bridge between number and number operations and early algebra thinking. These ideas are all now prominent in the research literature on early algebra:

- Structure of number sentences (Cai, Ng and Moyer, 2011; Jacobs, Franke, Carpenter, Levi and Battey, 2007);
- Equivalence (Kaput, Carraher and Blanton, 2008; Lins and Kaput, 2004);
- Compensation using equivalence according to specific operations (Britt and Irwin, 2011; Stephens and Wang, 2009; Irwin and Britt, 2005);
- Numbers that can vary (Cooper and Warren, 2011; Fujii and Stephens, 2001); and
- Generalization (Cooper and Warren, 2011; Mason, Stephens and Watson, 2009).

In J. Dindyal, L. P. Cheng & S. F. Ng (Eds.), Mathematics education: Expanding horizons (Proceedings of the 35th annual conference of the Mathematics Education Research Group of Australasia). Singapore: MERGA.

All the above ideas rest on deep understanding of number sentences and are often left implicit in text book treatment of algebra in junior secondary school, where it is introduced as the generalization of arithmetic and formal use of letters in equations. As Cooper and Warren (2011) argue, "quasi-generalisation in an elementary (school) context appears to be a necessary precursor to expressing the generalisation in natural language and algebraic notation" (p. 193). Moreover, assessment frequently emphasises procedural fluency assuming that procedural success carries with it conceptual understanding. Unless students experience these five key ideas in the context of number sentences and number operations in the primary school, they will usually have a difficult transition to learning algebra in the junior secondary school.

Methodological Position

The Sample

Three schools in Nanjing were involved in this study: two of them were primary schools (respectively named as Pn and Pj in this study), and one was a junior secondary school (named as Sh). All three participating schools are considered to be among the middle group of achievement. In Pn, the same questionnaire was given to 83 students, 45 from a Year 5 class and 38 students from a Year 6 class. In Pj, it was given to 69 students, 35 from a Year 5 class and 34 from a Year 6 class. In the junior secondary school Sh, there were 65 students from two Year 7 classes; one class had 32 students and another had 33. In summary, 217 students from six classes across three Year levels in three Nanjing schools completed the same questionnaire.

Design of Questionnaire

The questionnaire followed that used by Stephens and Wang (2009), and contained three types of mathematical sentences for each of the four operations. In Type 1 number sentences (single unknown), students were given four questions based on one of the four operations and asked to find the value of a missing number and to explain their thinking. The following addition sentences are provided as examples showing how the placement of the unknown number was deliberately varied.

For each of the following number sentences, write a number in the box to make a true statement. Explain your working briefly.

23 + 15 = 26 + □	43 + □ = 48 + 76
73 + 49 = □ + 47	□ + 17 = 15 + 24

Following each set of four Type 1 number sentences, the subsequent page for each operation consisted of related Type 2 and Type 3 sentences. Taking addition as an example, the questions for Type 2 sentences all followed a similar format:

Can you think about the following mathematical sentence?

18 + □ = 20 + □
Box A *Box B*

a) *In each of three sentences below, can you put numbers in Box A and Box B to make each sentence correct?*

18 + □ = 20 + □
Box A *Box B* [2 more templates given]

b) *When you make a correct sentence, what is the relationship between the numbers in*

Box A and Box B?

c) *If instead of 18 and 20, the first number was 226 and the second number was 231 what would be the relationship between the numbers in Box A and Box B?*
d) *If any number was put in Box A, could the number sentence still be true? Please give your explanation.*

Part e then followed presenting students with a Type 3 sentence, using literal symbols *c* and *d* in place of Box A and Box B (see Table 1). Following Hart (1981), students were asked: What can you say about *c* and *d* in this mathematical sentence? As can be seen, the Type 3 sentence for addition, $c + 2 = d + 10$, is structurally similar to its corresponding Type 2 Number Sentence. Students needed to describe the relationship between *c* and *d* that permits this mathematical sentence to be true. The total length of the questionnaire was eight pages.

Table 1
Type 2 and Type 3 Sentences for the Other Three Operations

Operation	Type 2 sentences	Type 3 sentences
Subtraction	72 – □ = 75 – □ *Box A* *Box B*	$c - 7 = d - 10$
Multiplication	5 × □ = 20 × □ *Box A* *Box B*	$c \times 2 = d \times 14$
Division	3 ÷ □ = 15 ÷ □ *Box A* *Box B*	$c \div 8 = d \div 24$

Qualitative Categorization of Students' Responses

Students' Responses to Type 1 Sentences

Students' responses to Type 1 sentences were classified as Computational, Computational/Relational Mixed or Relational. Computational thinking was clearly shown where students had calculated the result of the two given numbers. In other responses, students showed relational thinking clearly using words or by means of arrows or diagram (see figure 1). Incorrect relational thinking was sometimes evident when some students solved a subtraction sentence, such as 104 – 45 = □ – 46, giving 103 as the missing number, by applying an incorrect direction of compensation; that is, using the same direction of compensation as in addition. When correct answers were given with no explanation or other evidence of a student's thinking, responses were classified as Computational.

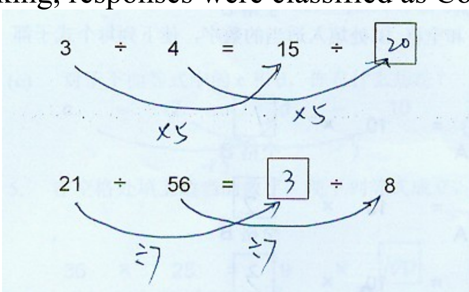

Figure 1: Arrows used by students to solve Type 1 sentences.

Because some students may have been more confident or comfortable with computation, Type 1 number sentences may not fully disclose the capacity for relational thinking. For that reason, the questionnaire needed to include additional Type 2 and Type 3 questions in order to distinguish between students who preferred to solve Type 1 questions by computation from those students whose thinking was restricted to computation.

Students' Responses to Type 2 and Type 3 Sentences

To discuss students' responses to Type 2 sentences, three questions were considered: (1) Were correct numerical exemplifications given in Part a? (2) How did students describe the relationship between the numbers in Box A and Box B in Parts b and c? (3) Could they generalise the relationship between numbers in Box A and Box B and could they utilise that generalisation in answering Part d? Following Stephens and Wang (2009), the following categorisations were used to classify responses to Type 2 and Type 3 sentences. Table 2 illustrates how some students described the relationship between the numbers in Box A and Box B and between *c* and *d*?

Table 2
Categories of Students' Responses to Type 2 and Type 3 Questions

Response type	Examples for Type 2 and Type 3 questions
Incorrect Relation	Students use 'difference' to describe multiplication and division the difference (between *c* and *d*) is always 16, in $c \div 8 = d \div 24$
Non-directed Relation	they would always be 5 apart, as in 3 ÷ (Box A) = 15 ÷ (Box B) there is always a 3 difference, as in 72 – (Box A) = 75 – (Box B) the numbers have a distance of 2, as in 18 + (Box A) = 20 + (Box B)
Directed Relation: Non-referenced* or No magnitude	so long as the number in Box B is larger, as in 72 – (Box A) = 75 – (Box B); or as in 3 ÷ (Box A) = 15 ÷ (Box B) *d* will be more than *c*, as in $c - 7 = d - 10$, *or in* $c \div 8 = d \div 24$ one number is always 2 higher than the other number, as in 18 + (Box A) = 20 + (Box B) one is 4 times more than the other, as in 5 × (Box A) = 20 × (Box B)
(Fully)Referenced and Directed Relation	one is 3 more than the other, Box B is bigger (subtraction); or B – 3 = A *c* is 8 more than *d,* as in $c + 2 = d + 10$; or $c = d + 8$ A is 5 times less than B, as in 3 ÷ (Box A) = 15 ÷ (Box B); or a difference of 2 with A larger, as in 18 + (Box A) = 20 + (Box B); or A – 2 = B

*"Non-referenced" means not to point out the relational objects, such as Box A and Box B, or *c* and *d*

Categorising Students' Relational Thinking on Type 2 and Type 3 Sentences

According to their responses on Type 2 and 3 questions, five categories of students' relational thinking were created: Established; Highly Consolidating; Consolidating; Emerging; and Non-Relational. Two categories, Highly Consolidating and Non-relational were not included in the Stephens and Wang (2009) study.

Established Relational thinkers gave responses to Type 2 and Type 3 sentences across all four operations that showed fully referenced and directed relations. These students were almost always able to: (1) specify the relationship between the numbers in Box A and the numbers in Box B with clear references to the magnitude and direction of the difference between them; (2) employ the form of words used to describe this relationship as a part of the condition that describes how *any* number can be used in Box A and still make a true sentence; (3) explain clearly how *c* and *d* are related for the Type 3 sentence to be true, treating *c* and *d* as general numbers.

Highly Consolidating Relational thinkers' responses to Type 2 and Type 3 sentences across all four operations demonstrated clear relational thinking in Parts a, b and c. They *occasionally* had difficulty with one of Part d and/or e using some form of incomplete relational thinking. In some cases their responses to three of the four operations were fully relational but the questionnaire was not entirely completed.

The third category, Consolidating Relational, included students who, in their responses to Type 2 and Type 3 sentences across the four operations demonstrated clear relational thinking in Parts a, b and c, but *almost always* gave incomplete responses to one or both of Parts d and e. These students: (1) were *almost always* able to specify the relationship between the numbers in Box A and the numbers in Box B with clear references to the magnitude and direction of the difference between them; (2) *rarely* gave a complete explanation as to how any number might be used in Box A and still have a true sentence; and (3) *usually* referred to some feature of the relationship between *c* and *d*, typically giving only a specific pair of values for *c* and *d*.

Emerging Relational thinkers were *unable* to give a complete relational description in Parts b and c to the relationship between the numbers in Box A and in Box B. Their descriptions were *typically* non-directed or non-referenced or written without specifying the magnitude of the relation. *Invariably*, their incomplete description of the relationship between the numbers used in Box A and Box B determined their explanation in Part d of how any number can be used in Box A and still have a true sentence. In Part e, they either attempted to give a correct pair of values for *c* and *d*, or they omitted this question altogether.

Non-Relational responses typically gave correct numerical results for Part a, but could not relate or see connections between the numbers involved in Parts b, c, d and e. This is illustrated by a Year 5 student who answered Part b by saying: "*Meets the relation that the sum is the same*"; and Part d by saying: "*Cannot, because if any number is filled in, it's very easy to have incorrect (numbers) filled*."

Coding Procedures for Student Questionnaires

Two groups of graduate students independently coded questionnaire responses from each grade using the above classifications and their indicators. The coders then compared results to identify any differences. For Type 1 number sentences, complete agreement was almost always achieved. However, where students had not completed all questions or where they had given some incorrect answers, an on-balance judgement was necessary. Students' responses to Type 2 and Type 3 questions across all four operations were used to decide which category best described their performance. Differences in coding occurred in approximately 10% of questionnaire responses, where an overall response might be classified one level higher or one level lower among the five categories. For these discrepancies, researchers and graduate scorers were almost always able to reach an agreed classification.

Quantitative Analysis of Students' Responses

Summary Profiles of Years 5, 6 and 7

Table 3 presents the summary statistics for each Year level. It first shows how students from each Year level responded to the Type 1 number sentences using the three classifications: Computational, Computational/Relational Mixed, and Relational. It then shows how these different responses to Type 1 number sentences were spread across the five categories of responses to Type 2 and Type 3 sentences: Established Relational (EsR), Highly Consolidating Relational (HCR), Consolidating Relational (ConR), Emerging Relational (EmR) and Non-Relational (NonR).

Table 3
Summary Statistics for Type 1, Type 2 and Type 3 Responses

Year level	Type 1 Responses		Type 2 and Type 3 responses				
			NonR	EmR	ConR	HCR	EsR
Year 5 (77 valid)	Computational	43	11	16	15	1	0
	Comp/Rel Mixed	19	0	10	7	2	0
	Relational	15	0	5	4	5	1
Year 6 (72)	Computational	22	7	9	4	2	0
	Comp/Rel Mixed	24	2	6	7	5	4
	Relational	26	3	7	8	4	4
Year 7 (65)	Computational	16	0	4	5	6	1
	Comp/Rel Mixed	20	0	2	6	8	4
	Relational	29	0	1	5	17	6

In Year 5, of the 77 valid responses out of 80 from the two classes in schools Pn and Pj, more than half of the students (43 out of 77) were still using computational thinking for Type 1 number sentences. All 11 Non-Relational responses on Type 2 and Type 3 sentences and 16 out of 31 Emerging Relational responses on Type 2 and Type 3 sentences – 27 out of 43 students – came from the Computational group on Type 1 sentences. Of the 34 students who used either Relational or a mix of Computational and Relational thinking on Type 1 number sentences, most (26) were Emerging or Consolidating Relational thinkers on Type 2 and Type 3 sentences. However, nine Year 5 students were either Established or Highly Consolidating, with only one using Computational thinking on Type 1 sentences. Reliance on computation to solve Type 1 sentences appears to be linked to limited capacity for Relational thinking on Type 2 and Type 3 sentences in the Year 5 group.

Year 6 included 72 samples from two classes in schools Pn and Pj. In Year 6, nearly one third of the students (22 out of 72) were still using computational thinking for Type 1 number sentences, showing a trend towards relational thinking on Type 1 sentences. Seven of the 12 Non-Relational students on Type 2 and Type 3 sentences and 9 of the 22 Emerging Relational thinkers on Type 2 and Type 3 sentences – 16 out of 34 students – used Computational thinking on Type 1 sentences, suggesting that reliance on computation on Type 1 sentences is still associated with limited capacity to deal with Type 2 and Type 3 sentences. Of the 50 students who used either Relational or a mix of Computational and Relational thinking on Type 1 number sentences, 5 were Non-Relational, 13 were Emerging, 15 were Consolidating, 9 were Highly Consolidating. All 8 students who were Established Relational thinkers on Type 2 and Type 3 sentences used either Relational or a mix of Computational and Relational thinking on Type 1 sentences.

All 65 samples from two Year 7 classes came from one junior secondary school Sh. No Year 7 students were Non-Relational on Type 2 and Type 3 sentences. The 16 students out of 65 who used computational thinking for Type 1 number sentences were spread across

Emerging (4), Consolidating (5), Highly Consolidating (6) and Established (1) on Type 2 and Type 3 sentences. Reliance on Computational thinking to solve Type 1 sentences here appears to be more of a choice. Of the 49 students who used either Relational or a mix of Computational and Relational thinking on Type 1 number sentences, 3 were Emerging, 11 were Consolidating, 25 were Highly Consolidating and 10 were Established. Ten of the 11 students who were Established Relational thinkers on Type 2 and Type 3 sentences used either Relational or a mix of Computational and Relational thinking on Type 1. Forty two of the 65 Year 7 students were classified as Highly Consolidating or Established Relational thinkers. While this may be explained, in part, by exposure to literal symbolic algebra in Year 7, it seems to have had no major benefit for the other 23 students.

Quantifying Students' Responses

We used the following weighting system to give a numerical profile of relational thinking for each class and grade level: 0 for Non-Relational, 1 for Emerging, 2 for Consolidating, 3 for Highly Consolidating, and 4 for Established. We applied these numbers to responses at each Year level and divided by the number of students to obtain a mean score for each Year level and for each class.

Table 4
Mean Measures of Students' Relational Thinking

Year	5		6		7	
Class	Pj-5	Pn-5	Pj-6	Pn-6	Sh-7a	Sh-7b
Class Mean	1.424	1.455	1.735	1.737	2.875	2.545
Year Mean	1.442		1.736		2.708	

From Table 4, it can be seen that the Year means of students' relational thinking – 1.442, 1.736 and 2.708 respectively – show an increase from Year 5 to Year 6, and an even greater increase from Year 6 to Year 7. There was no statistically significant difference between the two participating classes at each Year level.

Research Findings and Implications for Teaching

In each class, the profile of students' performances was spread across 4 to 5 levels. While there was evident growth between Year 6 and Year 7, there remained for each Year level a wide range of ability to express mathematical thinking. Many Year 7 students were still thinking mathematically well below the capacity of some students in Year 5 and 6. This is a challenge for teaching at every Year level.

The results of this study confirm the various affordances and impediments that, as discussed by Stephens and Wang (2009), connect different responses to Parts b and c for Type 2 sentences and students' likely success in answering the associated Part d question, and its related Part e question. If students used Non-directed, or Directed (no magnitude), or Directed (non-referenced) relational thinking in their responses to Parts b and c, they were almost always unable to give a complete mathematical generalisation in Part d of how any number can be used in Box A and still have a true sentence, *and* they were also unable to describe the mathematical relationships between *c* and *d* in Type 3 sentences. Students who successfully completed both Part d and Part e appeared to recognise the structural similarity between the two questions. Some students who successfully completed Part e were unable to give a successful generalisation needed for Part d. Teachers need to help their students articulate referenced and directed relational descriptions by highlighting to students the disadvantages and advantages that different descriptions offer.

All three Types of sentences appear to have a high teaching potential for developing and refining mathematical thinking with respect to: equivalence and compensation; attention to structure and operations; attending to the range of possible variation (which numbers vary, which numbers stay the same); and generalisation. Relying on Type 1 number sentences alone to achieve to achieve these goals is probably insufficient. Strategies, such as teaching with variation, can help students to see that, for both Type 2 and 3 sentences, the permissible range of variation can include rational numbers and negative numbers. This can help to build a stronger foundation for the subsequent idea of variable in high school mathematics.

References

Britt, M., & Irwin, K. (2011). Algebraic thinking with and without algebraic representation: A pathway for learning. In J. Cai & E. Knuth (Eds.), *Early Algebraization: A global dialogue from multiple perspectives*, pp. 137-159. Heidelberg: Springer.

Cai, J., Ng, S., & Moyer, J. (2011). Developing students' algebraic thinking in earlier grades: Lessons from China and Singapore. In J. Cai & E. Knuth (Eds.), *Early Algebraization: A global dialogue from multiple perspectives* (pp. 25-41). Heidelberg: Springer.

Cooper, T., and Warren, E. (2011). Year 2 to Year 6 students' ability to generalize: Models, representations and theory for teaching and learning. In J. Cai & E. Knuth (eds.), Early *Algebraization: A global dialogue from multiple perspectives*, (pp. 187-214). Heidelberg: Springer.

Fujii, T., & Stephens, M. (2001). Fostering understanding of algebraic generalisation through numerical expressions: The role of quasi-variables. In H. Chick, K. Stacey, J. Vincent & J. Vincent (Eds.), *The Future of the teaching and learning of algebra. Proceedings of the 12th ICMI study Conference* (Vol. 1, pp. 258-264). Melbourne, Australia.

Hart, K. M. (1981). *Children's understanding of Mathematics*. London: John Murray.

Irwin, K., and Britt, M. (2005). The algebraic nature of students' numerical manipulation in the New Zealand Numeracy Project. *Educational Studies in Mathematics, 58*, 169-188.

Jacobs, V., Franke, M., Carpenter, T., Levi, L. and Battey, D. (2007). Developing children's algebraic reasoning. *Journal for Research in Mathematics Education*, *38*(3), 258–288.

Kaput, J., Carraher, D., & Blanton, M. (Eds.), *Algebra in the early years*. New York: Lawrence Erlbaum Associates and National Council of Teachers of Mathematics.

Lins, R., and Kaput, J. (2004). The early development of algebraic reasoning: The current state of the field. In K. Stacey, H. Chick, & M. Kendal (Eds.), *The future of the teaching and learning of algebra: the 12th ICMI Study* (pp.47-70). Boston: Kluwer Academic Publishers.

Mason, J., Stephens, M., & Watson, A. (2009). Appreciating mathematical structure for all. *Mathematics Education Research Journal*, *21*(2), 10-32.

Ministry of Education of PRC (2001). *Mathematics curriculum standards for compulsory education* (experimental) (in Chinese). Beijing: Beijing Normal University Press.

Ministry of Education of PRC (2011). *Mathematics curriculum standards for compulsory education* (revised) (in Chinese). Beijing: author.

Stephens, M., & Wang, X. (2009). Some key junctures in relational thinking: A study of Year 6 and Year 7 students from Australia and China. In R. Hunter, B. Bricknell, & T. Burgess (Eds.), *Crossing Divides: Proceeding of the 32nd annual conference of the Mathematics Education Group of Australasia* (pp. 499-506). Wellington, New Zealand: MERGA.

Stephens, M., & Zhang, Q. (2011).Teacher capacity as a key element of national curriculum reform in mathematics: a comparative study between Australia and China. In Clark, J., Kissane, B., Mousley, J., Spencer, T. & Thornton, S (Eds.), *Mathematics: traditions and (new) practices. Proceedings of the 23rd biennial conference of the Australian Association of Mathematics Teachers & 34th annual conference of the Mathematics Education Group of Australasia* (pp. 702 - 710). Alice Springs: AAMT & MERGA.

Xu, W. (2003). Algebraic thinking in arithmetic: Quasi-variable expressions (in Chinese). *Xueke Jiaoyu (Journal of Subject Education)* 11, 6–10, 24.