Use of Practical Worksheet in Teacher Education at the Undergraduate and Postgraduate Levels

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We have applied the 'practical paradigm' in teaching problem solving to secondary school students. The key feature of the practical paradigm is the use of a practical worksheet to guide the students' processes in problem solving. In this paper, we report the diffusion of the practical paradigm to university level courses for prospective and practising teachers. The higher level of mathematics content would demand higher order thinking skills. Learners without a model of problem solving would often revert to solving by referring to many examples of the same 'type' of problem. Polya-type problem solving skills framed by the practical worksheet was used as an attempt to elicit more effective problem solving behaviour from them. Preliminary findings show that they were able to use the practical worksheet to model their solution of problems in the courses.

Since the 1980s, problem solving has been at the heart of the Singapore mathematics curriculum. Some recent studies in Singapore schools (e.g., Foong, 2009, Teong et al., 2009), suggest that problem solving, especially that of non-routine problems, is done mostly as enrichment activities and not part of the core mathematics curriculum.

To address the issue of problem solving being regarded as peripheral, rather than central to the curriculum, the Mathematical Problem Solving for Everyone (MProSE) project was formed. The theoretical underpinnings for the MProSE design experiment can be found in Quek, Dindyal, Toh, Leong, & Tay (2011). Based on these design parameters, a package of problems, lessons and materials suitable for implementation into the mathematics curriculum was developed (Toh, Quek, Leong, Dindyal, & Tay, 2011). An account of the implementation of this problem solving module into a local independent Secondary school in 2009 is described in Dindyal, Tay, Toh, Leong & Quek (2012). Since then, the problem solving module has been incorporated in another three Secondary schools (Leong et al, 2012).

One distinct feature of the MProSE project is the practical paradigm through the use of the Practical Worksheet for the problem solver to work out his/her solutions. This worksheet contains sections that guides the problem solver through the four stages of Polya's model (Polya, 1945), and also incorporates Schoenfeld's framework (Schoenfeld, 1985), highlighting the cognitive resources, use of heuristics, control, and belief systems of the problem solver. A condensed form of the worksheet together with a suggested assessment rubric can be found in Toh, Quek, Leong, Dindyal, & Tay (2009).

We believe that a critical success factor for bringing problem solving back to the centre of the curriculum lies in having well-prepared teachers. In the MProSE project, teacher preparation is an important component (Leong, Dindyal, Toh, Quek, Tay, & Lou, 2011). That teachers spend a substantial amount of time in actual problem solving is a critical part of the teacher preparation process.

In J. Dindyal, L. P. Cheng & S. F. Ng (Eds.), Mathematics education: Expanding horizons (Proceedings of the 35th annual conference of the Mathematics Education Research Group of Australasia). Singapore: MERGA.

A prior effort to use the practical worksheet in a postgraduate course on "Discrete Mathematics and Problem Solving" for 21 practising teachers was reported in Tay, Quek, Dindyal, Leong & Toh (2011). In that paper, the reflections of the teachers as they embarked on their learning journey were reported and analysed, and it appeared that the practical worksheet was useful in bringing about the awareness of the problem solving process. In this paper, we report on how problem solving using the practical worksheet was incorporated into an undergraduate mathematics content course for prospective teachers, a postgraduate mathematics content course for practising teachers, as well as a postgraduate prospective teacher education course.

Undergraduate Mathematics Content Course

One of the authors—henceforth known as the lecturer— taught a 36-hour course introducing number theory to 59 undergraduate students. The content of the course was typical of similar courses taught elsewhere and includes divisibility, congruences, Diophantine equations, Euler's generalization of Fermat's little theorem etc. Most of the students were in the first year of their B.A. (Ed.) or B.Sc. (Ed.) programme, and had so far been mainly learning content mathematics. About 85% of them had not undergone the "teaching of mathematics" component that would have introduced to them Polya's problem solving framework.

The lecturer having previously taught calculus to the same group of students recognised that many of them would face difficulties in number theory because it was atypical of the mathematics that they were used to in their pre-university education. When faced with a problem like "prove that if m is a composite number, then 2^m-1 is also composite," most of them would simply freeze and not know how to proceed. Unless one has seen the solution before, a possible approach would be to use heuristics like "substitute numbers (for m)" and "search for patterns". Probing further, one could try to "solve a simpler problem". Simple examples of composite numbers are even numbers (greater than 2) and, with a little experimenting, students would generally be able to see that 2^{2k}-1 seems to be always divisible by 3. If they could then prove this conjecture, they would have partially solved the problem[1].

To help students overcome these difficulties, the lecturer tried to teach *through* problem solving (Shroeder & Lester, 1989) using Polya's model. He began with the first three stages of (1) understanding the problem, (2) devising a plan and (3) carrying out the plan, without explicit mention of Polya as he wanted to convey the idea that these are natural processes that mathematicians use to solve mathematics problems. During the lectures, the lecturer would demonstrate how he understood the problem, what kind of heuristics he would use, as well as possible plans for solving the problem, before finally carrying out the plan. This departs from the usual theorem–proof, theorem–proof type of exposition that is commonly used in teaching advanced mathematics. Gradually, the job of solving the problem was passed on to the students and the lecture notes would only have the names of the Polya stages, followed by spaces for students to work on. About a third of the way through the course, Polya's model, including Stage 4 which we renamed as "Check and Expand", was introduced to the students. They were also given the practical worksheet to be used for their problem solving assignments.

1 The problem solving attempt that was described actually took place during a consultation session with three students.

Data from one of the problem solving assignments is presented in Table 1. The problem was

Problem: *Find a million consecutive composite numbers.*

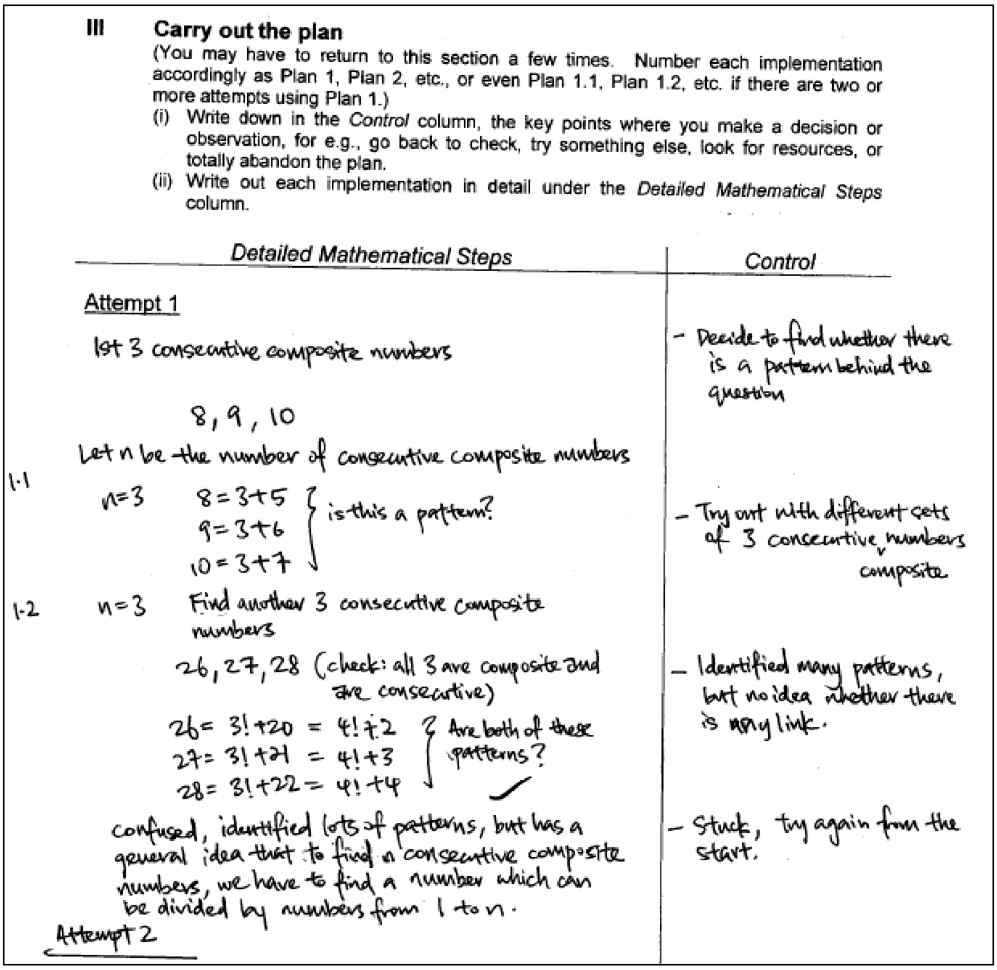
This was “non-routine” for the students as there were no similar examples in the lecture notes and the prescribed textbook. Among the 57 assignments received, 51 students managed to solve the problem, 50 of them displayed evidence of using the Polya stages in their solution, and slightly less than half (26 students) used the heuristic of working with a smaller number of consecutive composites first before going on to solve the problem for a million. The lecturer was pleasantly surprised that so many of the students were able to successfully use Polya’s framework to solve the problem. It was also encouraging to see that 37 of the students went on to Stage 4 and attempted to generalize the problem. Some students also demonstrated higher order thinking skills. For example in *Figure 1*, Student X’s working indicated that he started by searching for consecutive composites in the hope of finding a pattern. His metacognition process was clearly spelt out in the control column of the practical worksheet. Due to the lack of space, we do not reproduce how he successfully solved the problem in his second attempt.

Table 1

Data from Undergraduate Mathematics Content Course

Description	Number of Students
Practical Worksheets Received[1]	57
Solved the problem	51
Displayed Evidence of Using the Polya Stages	50
Use of at least one Heuristic	26
Attempted to Check and Expand	37

[1] 2 students did not submit their assignments.



III Carry out the plan

(You may have to return to this section a few times. Number each implementation accordingly as Plan 1, Plan 2, etc., or even Plan 1.1, Plan 1.2, etc. if there are two or more attempts using Plan 1.)

(i) Write down in the *Control* column, the key points where you make a decision or observation, for e.g., go back to check, try something else, look for resources, or totally abandon the plan.

(ii) Write out each implementation in detail under the *Detailed Mathematical Steps* column.

Detailed Mathematical Steps	*Control*
Attempt 1	
1st 3 consecutive composite numbers 8, 9, 10 Let n be the number of consecutive composite numbers	- Decide to find whether there is a pattern behind the question
1.1 n=3 8=3+5, 9=3+6, 10=3+7 } is this a pattern?	- Try out with different sets of 3 consecutive composite numbers
1.2 n=3 Find another 3 consecutive composite numbers 26, 27, 28 (check: all 3 are composite and are consecutive) 26= 3!+20 = 4!+2 27= 3!+21 = 4!+3 28= 3!+22 = 4!+4 } Are both of these patterns? ✓	- Identified many patterns, but no idea whether there is any link.
confused, identified lots of patterns, but has a general idea that to find n consecutive composite numbers, we have to find a number which can be divided by numbers from 1 to n.	- Stuck, try again from the start.
Attempt 2	

Figure 1. Student X's working and his control statements.

Postgraduate Mathematics Content Course

The practical paradigm of mathematical problem solving was also introduced to practicing teachers attending a Masters Degree course on Theory and Applications of Differential Equations. The students (8 in total) were introduced to the practical paradigm during the first introductory lesson of the module when the tutor (one of the authors) introduced the qualitative analysis, and finding the solution of an ordinary differential equation analytically and numerically. The tutor introduced the use of the practical worksheet alongside Polya's problem solving stages in the context of differential equations. During the first four lessons, practical worksheets were used by the participants. In the second lesson, the tutor modeled the use of the practical worksheet in solving a non-routine differential equation with the participants.

It was observed during the third lesson that generally the students were reluctant to use the practical worksheets, especially for Polya Stage 1 (Understand the Problem) and Stage 2 (Devising a Plan). Most of them attempted to solve the problem directly. It was also evident that they were not comfortable to proceed to Stage 4 (Check and Expand) of the worksheet.

A test on the content material of the first four lessons was conducted in the fifth lesson. The students were informed in the first lesson of this test, of which a significant portion would consist of a mathematics practical test. The test problem was

Problem: *Draw the direction field for* $\frac{dx}{dt} = x^2 t$.

This is non-routine problem for the students, as most of the examples considered in qualitative analysis in this course were autonomous equations of the form $\frac{dx}{dt} = f(x)$. Moreover, lecture examples used x and y to represent the independent and the dependent variables respectively. Furthermore, the shape of the direction field for this differential equation is not as easily obtained as for those in the lecture examples.

A pleasant surprise turned out in the analysis of the students' scripts. All the eight students filled up all the four pages corresponding to the four stages, even though in the first four lessons the students had expressed their reservations in completing Stages 1 and 2 of the worksheet. In communicating their feelings about this problem (in Stage 1), some of the students' responses were:

- Rather daunting as RHS of DE involves both the independent and dependent variables.
- Problems seem challenging since there are two variables on the RHS of the expression.
- Looks ok. However, RHS involves x and t.
- The problem scares me.
- This problem can be rather challenging as we are used to $\frac{dy}{dx}$ and not $\frac{dx}{dt}$.
- Seems manageable. Need to identify the variables involved.
- Need to construct a slope field & observe pattern.
- (One student began with substituting values into the right hand side of the equation to obtain values of the slopes.)

Further, it was heartening to observe that all the students attempted Stage 4 (Check and Expand) of the practical worksheet. We use student Y's script (Figures 2 and 3) as illustration. At Stage 3 (Carry out the plan), the student had only plotted values for various values of (t, x) as, according to him, "the direction field looks odd" (Figure 2). He progressed to Stage 4 and solved the differential equation analytically to obtain the general solution (Figure 3). This gave him a clearer picture of the slope field, and he reverted to Stage 3 to complete the sketch of the slope field.

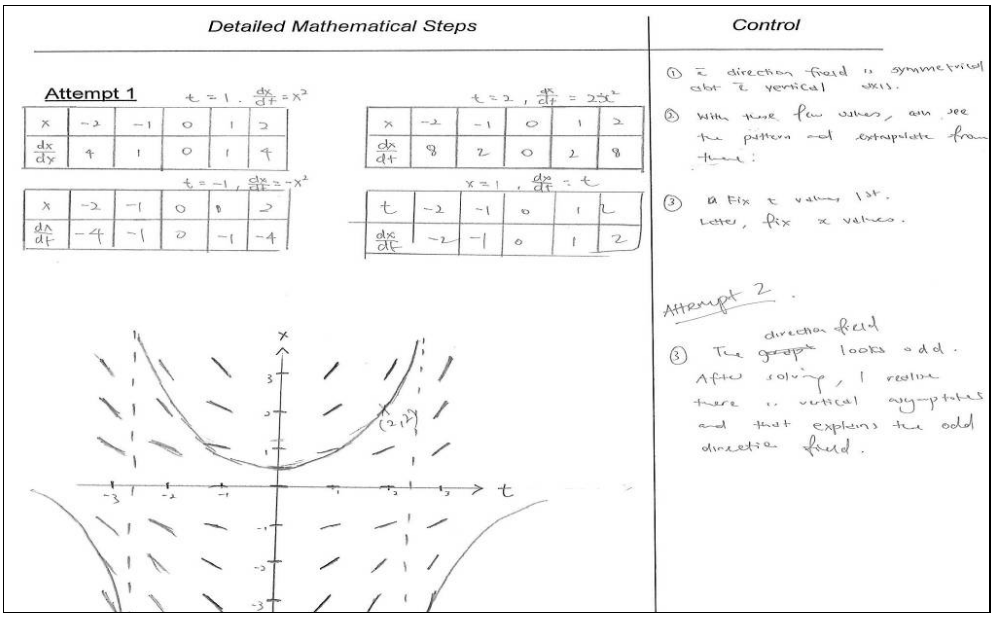


Figure 2. Student Y's completed Stage 3 of the worksheet.

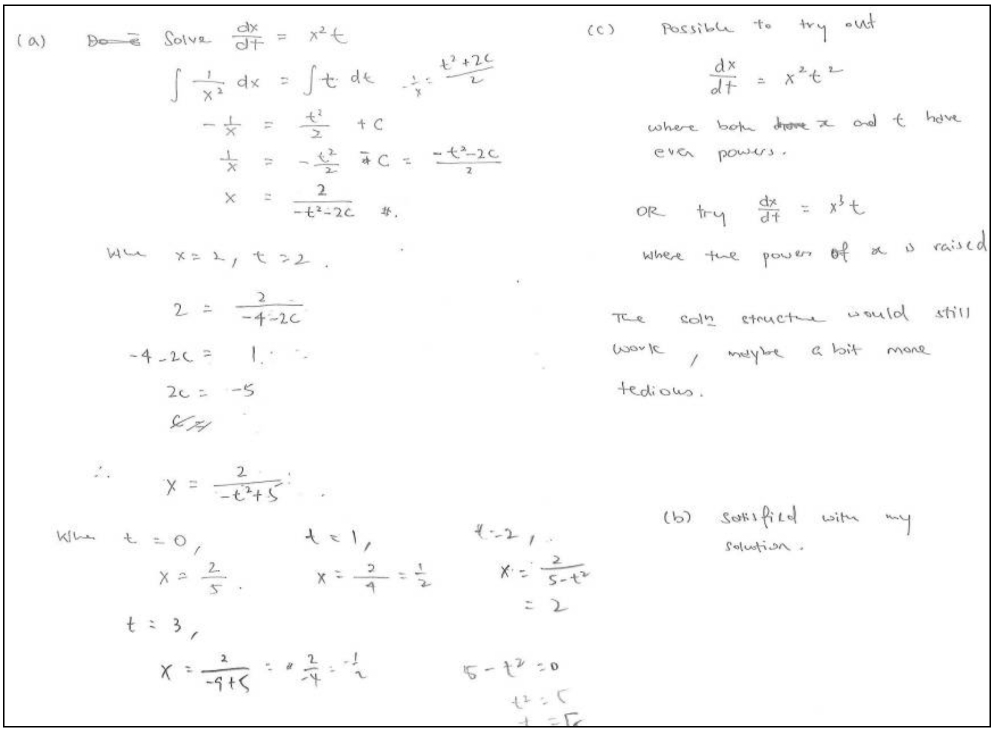


Figure 3. Student Y's completed Stage 4 of the worksheet.

Teacher Education Course for Prospective Teachers

One of the authors—henceforth known as the tutor—taught the mathematics methods course in the Postgraduate Diploma in Education (PGDE) programme which aims to prepare university graduates in Mathematics or in a Mathematics-related discipline such as Engineering to become Secondary school mathematics teachers. About six hours of the instructional time of the module was devoted to equipping prospective teachers with the necessary skills and pedagogy for teaching problem-solving and teaching mathematics through problem solving. In the past, the tutor conducted these problem-solving lessons without using the practical worksheet approach. To help them organise their problem solving processes, the tutor used the practical paradigm for a class of 24 prospective teachers in the 2011 cohort. The rest of this section is a brief description of the middle portion of this module on problem solving.

The tutor started by explaining what a mathematical problem is, emphasizing that it is different from a routine exercise and that it requires time and effort to solve. Both Polya's model and Schoenfeld's problem solving framework were highlighted so that prospective teachers understood that problem solving requires not only subject matter knowledge, but also heuristics, helpful beliefs, and control. The tutor demonstrated and discussed how different heuristics were used to solve some problems before he focused on how the practical worksheet should be used by going through a specific problem.

Problem : *ABC is an equilateral triangle. P is a point inside the triangle such that the distances from its three sides are 4, 5 and 6 cm. Find the length of one side of the triangle.*

The tutor noticed that in their first attempt to use the practical worksheet, many students did not follow the instructions on the worksheet and proceeded to solve the problem directly. In particular, there were hardly any comments on the thinking process in the "control column" of the practical worksheet. This may be due to the fact that most of them could solve the problem in about 15 minutes. Thus, they may think that control was irrelevant and unnecessary for them. During the class discussion, the tutor deliberately emphasised the importance of Polya's Stage 4 (Check and Expand section in the practical worksheet) for developing a deeper understanding of the problem. Indeed, the class suggested the following extensions:

- If a point P is inside a given equilateral triangle, the sum of the distances from each of the three lengths to the point is always a constant and equal to $\frac{\sqrt{3}}{2}$ times of the side of the equilateral triangle.
- If the point P is inside a given square (or rectangle), it is obviously true that the sum of the distances from each of the four sides to the point is a constant—equal to the perimeter of the rectangle.
- The same observation is true when the point P is inside a given pentagon with equal sides but not true if the sides of a pentagon are not all equal (the tutor used the *Geometers' Sketchpad* to verify these two observations in class).

At the end of the segment, the class came up with the following conjecture: "In a regular *n*-sided convex polygon, the sum of the distances from a point P inside the polygon to each of the *n* sides of the polygon is always a constant[2]."

Despite their initial reluctance at filling up the earlier sections of the practical worksheet, one encouraging sign was that the fourth page of the worksheet points to the need to push forward to expand the solution instead of stopping at the solution. One prospective teacher in particular commented: "I didn't see the usefulness of the template [referring to the practical worksheet] until the last bit. It is very rewarding when we extend the problem. It was really fun and exciting".

Upon reflection, the tutor thinks that the practical worksheet approach to problem-solving creates a platform for students to focus on Check and Expand. As the "looking-back" stage is part of the worksheet, they will likely sense that their attempts are not

[2] One of them subsequently managed to prove that the constant is equal to $\frac{n}{2\tan\frac{180^0}{n}}$ times the length of the sides of the *n*-gon.

complete when they stop at Stage 3. There is thus a greater impetus to re-examine the solution process and look beyond the problem—the process of expanding would be useful in helping students to make generalisations and achieve a deeper understanding of the problem.

Summary

We reported on the use of the practical worksheet to help prospective and practising teachers experience problem solving through the Polya's model and Schoenfeld's framework. The lecturer/tutors are encouraged to see some signs of success. The emphasis on the problem solving model as embodied in the practical worksheet has shown that the learners generally exhibited behaviour which we think (from our experience teaching these courses in the past) they would usually not show (e.g. Check and Expand). We believe that this is one of many *starting* steps that need to be taken in bringing problem solving back to the heart of the curriculum.

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