The Concept of Generalised Number: Valuable Lessons from the History of Algebra

Mala S. Nataraj
The University of Auckland
<snat016@aucklanduni.ac.nz>

Michael O. J. Thomas
The University of Auckland
<moj.thomas@math.auckland.ac.nz>

The importance of understanding the various uses of the literal symbol in algebra, and in particular the idea of generalised number, is well documented in the literature. Many research findings have also reported student difficulties with this vital and central concept. This research study examines the use of a combination of historical and educational research ideas as a way of enhancing students' understanding of generalised number. The results suggest that this approach helped some students to make some generalisations and to understand the difference between specific unknown and generalised number.

Background

Students know that algebra is 'to do with letters' but research has clearly documented that many students have little understanding of what the letters mean and the reason that they are used (Graham & Thomas, 2000; Kieran, 1992; Küchemann, 1981; MacGregor & Stacey, 1997). The concept of variable, in particular, continues to be poorly understood even though it is fundamental in the transition from arithmetic to algebra (Schoenfeld & Arcavi, 1988) and is central to all higher mathematics. Hence, while students need to understand the multi-faceted meaning of the literal symbol as specific unknown, generalised number, and variable, many studies have stressed student difficulties and errors in this area. A view put forward by Radford (1996), and others, is that generalised number is a pre-concept for variable, and he suggests that the ways of thinking associated with generalisation (involving generalised number and variable) and problem solving (involving specific unknown) are "independent and essentially irreducible, structured forms of algebraic thinking" (*ibid*, p. 111). Thus, both generalisation and problem-solving approaches appear to be mutually complementary fields in the didactics of algebra. However, there is often a prevalence of use of letter as specific unknown rather than as a generalised number in schools, possibly due to a focus on examples such as substitution and equation solving in which the letters represent a single specific value. An additional factor that may confuse students is that the term 'variable' is used to refer to letters regardless of whether the actual usage of the letter is as a variable, generalised number, specific unknown or place holder, or has some other use. From the CSMS study, Küchemann (1981) concluded that many students did not progress beyond viewing letters as numerical placeholders, and of the 30-40% who did, the majority interpreted letters as specific unknowns rather than generalised numbers or variables. Another algebraic error that students are prone to make, reported by MacGregor and Stacey (1997) is the misuse of exponential notation; such as writing x^3 instead of $3x$.

In the light of the many difficulties that students face in mathematics, in recent years some researchers have attempted to analyse the history of mathematics in order to inform teaching practice. Educators have asserted (Fauvel & van Maanen, 2000) that the history of mathematics is an excellent resource for teaching and can be of great benefit in enhancing the understanding of mathematics. There are different ways of incorporating lessons from history in the classroom, both explicit and implicit (Harper, 1987) and either way it can bring about a major change in the teacher's approach. A review of historical texts (Datta & Singh, 2001) reveals that the decimal number system with place value and zero, as well as many algebraic ideas, originated in India (Cajori, 1919; Joseph, 2011). For example, Puig

In J. Dindyal, L. P. Cheng & S. F. Ng (Eds.), Mathematics education: Expanding horizons (Proceedings of the 35th annual conference of the Mathematics Education Research Group of Australasia). Singapore: MERGA.

and Rojano (2004) cite how a Mathematical Sign System (MSS), in which the different unknown quantities and their powers are differentiated, was an important step in the development of the algebraic symbolism that had developed by the time of Bhaskara II in the 12th century (or possibly even earlier) in India, and, by Viete, in the 16th century in Europe. This enabled the construction of *general methods* of solutions to equations. One particularly interesting feature of the sign system developed in India was that different *colours* were used to represent various unknowns. Bhaskara II (1150) says: "*yavat-tavat* (so much as), *kalaka* (black), *nilaka* (blue), *pitaka (*yellow), *lobita* (red) and other colours have been taken by the venerable professors as notations for the measures of the unknowns, for the purpose of calculating with them" (Datta & Singh, 2001, Vol. 2, p. 18). Thus Bhaskara II employed abbreviations of the names of the unknown quantities in order to represent them in an equation, such as *ka* for *kalaka* (black) and *ni* for *nilaka* (blue). Although *yavat-tavat* (quoted above) is not a colour its inclusion shows the persistence of an ancient symbol employed long before colours were introduced to denote unknowns.

According to the theory of the structure of attention proposed by Mason (2004) we may focus our attention on the whole, the details, the relationships between the parts, the properties of the whole or the parts, and deductions, becoming more aware of what we notice. He also states (Mason, Graham, & Johnston-Wilder, 2005) that classification is a form of generalisation and asserts that children have an inherent ability to classify objects. Thus in order to detect generalities in arithmetic patterns, he suggests guiding students' attention towards number patterns by asking questions such as "what is the same about each row?", "what is different and how is it changing"? This same method of guiding attention was also described by Srinivasan (1989) who advocated the use of '*pattern language*' for number patterns and '*design language'* for shape arrangements to concentrate students thinking on variation and invariants in number and geometrical patterns. His recommended vocabulary centres around *changing*, *not changing*, *changing in the same way* and *changing in different ways* in order to elicit an algebraic expression from students in the form of pattern language, and this was used in this study. Given the above, this research sought to use a combination of historical and mathematics education research ideas to address the question of whether explicitly directing students' attention to classification, and getting them to write colours in the place of the changing numbers, as in Indian history, would help improve students' understanding of the concept of generalised number. A framework (see Figure 1) was developed and implemented combining the ideas from the two domains of history and pedagogical research to teach algebraic generalisation.

Method

The research presented here comprised a case study of a class of 26 Year 9 (age 13 years) students at a decile 5 (middle socio-economic level) secondary school in Auckland, New Zealand. The class used in the research represented a wide variety of cultural backgrounds. However, most of the students had their intermediate schooling in New Zealand and hence were proficient in English. The exceptions to this were a Chinese student and a Burmese student who had only recently arrived in the country and hence were taking ESOL classes.

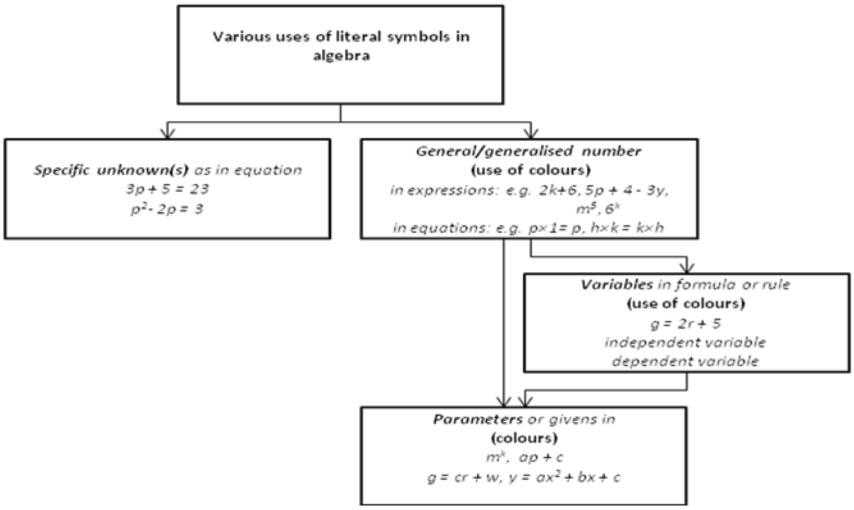

Figure 1. The teaching sequence involving algebraic generalisation.

2. $(\text{Eight})^3$, $(\text{six})^3$, $(\text{four})^3$, $(\text{seven})^3$ ....Can you generalise this pattern?
3. 4×7, 4×9, 4×12, 4×6............Generalise the given pattern.
4. Observe what is changing and not changing for the following patterns and then make a generalisation

20+1×15	5×1–2=3
20+2×15	5×2–2=8
20+3×15	5×3–2=13
20+4×15	5×4–2=18
20+5×15	5×5–2=23

5. i) Write down what you understand by $P + 5$ and what are the possible values for *P*?

ii) Write down what you understand by *G* – 2 + *B*. What are possible values that *G* and *B* can take?

6. a) Compare and then explain what you understand by

i) $3 \times y$ and ii) $3 \times y = 12$.

b) Explain the meaning that you would give *y* in 3 × *y* and also in 3 × *y* = 12

7. John has a certain number of sweets in his pocket and to these he adds 3 more. How many does he have altogether now? Can you write this down?
8. Observe what is changing and not changing. Then make a generalisation and simplify.

9×1+8+7×1	16 × 1 + 8
9×2+8+7×2	16 × 2 + 8
9×3+8+7×3	16 × 3 + 8
9×4+8+7×4	16 × 4 + 8

What do you notice about the generalisation in the above columns?

10. What is the meaning of 5×6 ?
11. What is the meaning of 3×3×3×3?
12. What is the meaning of 5^6?
13. Write what you understand by a^7.
14. What does 2^{96} mean to you?
15. What is the meaning of 2^y?
16. Write what you understand by a^y.

Figure 2. A representative selection of the algebra test problems.

The teacher, who was the first-named researcher, explained to the students at the beginning of the year and during some of the subsequent lessons what was going to be taught, including what algebra was about and why it was important to their learning. The teacher and students also discussed why it was important to have a clear understanding of the different uses of letters. Students had intermittent guidance in observation of variation and invariants in number patterns put up on the board, including whole class discussion using vocabulary previously presented (see Figure 3). This vocabulary included words such as changing, not changing, changing in the same way, changing in a different way, expression, equation, specific unknown number, general number, variable, exponent, power, repeated multiplication, symbol, and solving. The students' attention was also guided to variation and invariants in simple number patterns as in r+6 and then to other number patterns, such as 9+b–r with the use of *colours*. As presented in Figure 3, the changing number was initially shown with a patch of *colour* (for example, red), then via discussion, denoted with the *word* (red) and then shortened to the letter r. After the whole group work, students had to record conclusions in their books and then attempt similar examples on their own. The meaning and notation of exponential forms such as 10^3, 4^5, m^4, 4^g, and r^w were presented too. The students experienced substitution, expansion, collecting terms and simplifying, solving and checking equations, order of operations, pattern generalisation and formulas as part of their normal curriculum. After approximately sixteen one-hour periods on the work, the class spent two such periods answering the 33 algebra questions, not all of which are considered here (see Figure 2 for a representative selection of questions presented in this paper).

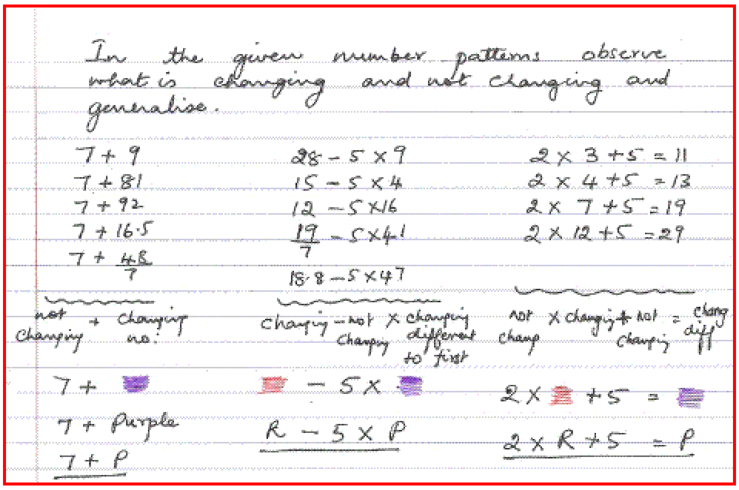

Figure 3. An example of the way in which colours were employed to represent changing numbers.

Results

Questions 1, 2, 3, 4 and 8 were designed to see whether the students could recognise quantities that are changing (ie variation) and distinguish them from those that are not (invariant). Table 1 gives the numbers (and %) of the 26 students that were able to do so for each question (Q1 was similar to Q's 4 and 8).

Table 1
Facilities for Questions on Recognition of Variation and Invariance (N = 26)

Question	1a	1b	2	3	4a	4b	8a	8b	8c
Number Correct	24	19	18	18	22	23	6	24	1
%	92.3	73.1	69.2	69.2	84.6	88.5	23.1	92.3	3.8
Number Wrong	1	2	4	1	2	1	15	1	11
%	3.8	7.7	15.4	3.8	7.7	3.8	57.7	3.8	42.3

With the exception of Q8a), which proved difficult, around 70% or more of the students were able to distinguish variation from invariance and could symbolise the variation using a symbolic literal. In Q8a), most students were able to observe what was changing and what was not changing, but only a few were able to notice that the second general number was changing in the same way as the first general number and so most used a different colour or letter for each rather than the same colour or letter. In Figure 2 we see that S12 has used the same colour for each but has not gone on to symbolise it, while S10 has used the word symbol ‘Blue’. However, S2, S3, S4, S7, S11, S13, S19 and S24 were all able to write a correct expression such as 9**y*+8+7**y* and S19 and S24 were able to simplify it, with S24 writing 16*B*+8 (see Figure 4), by removing the explicit multiplication signs and adding terms.

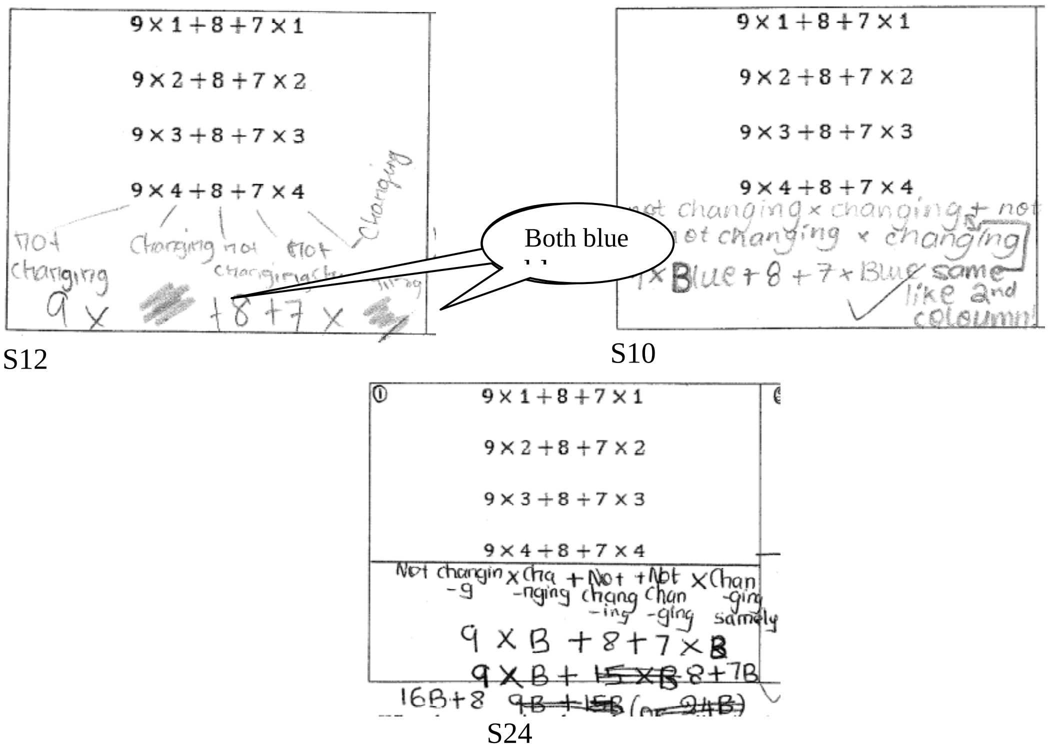

Figure 4. Students’ work showing an appreciation of variation and invariance.

Questions 5 and 7 addressed whether students could see or use letter as specific unknown or generalised number. In the event 21 (80.8%) of them did in Q5a) and 69.2% in 5b), with just 2 (7.7%) unable to do so in 5a) and none in 5b). Many of the responses in 5a) showed that they had the idea of a general number and some students also gave possible values for the number (some only gave possible values): S18 “P is a general number it can mean any number like 600 so it could be 600+5”; S10 “any number +5, unknown number +5”; S13 “*P* +5 is a generalisation, 1+5 3+5 5+5 8+5...”; S22 “*P* +5 is an expression, *P* can be any number so is called a general number”. For 5b) student answers demonstrated they understood that the generalised numbers *G* and *B* can vary differently, writing: S24 “It is an expression. *G* and *B* are changing differently”; and S1 “*G* can equal any number along with *B*”. However, this did lead S17 to the common wrong conclusion that *G* and *B* can not take the same value: “unknown number – 2 + unknown number. *G* and *B* can be any number. But *G* and *B* can’t be the same number.”

In Q7, 69.2% were correct and 19.2% wrong, with most correct answers of the form *S*+3, or similar. Questions 10 through 16 targeted students' understanding of repeated addition and multiplication, and their respective notations. The teaching sequence had involved some discussion on, and practice with matchsticks for the diagrammatic representations of numbers such as 3×4 and 3^4. Most students were successful in writing the correct answers. As Table 2 shows all the facilities were above 69.2%, with five over 88%. Most correct answers were of the form: a^y is *a*×*a*×*a*…*y* times. However, despite the practice, there were some students who were still confused, which shows that this is not an easy concept. For example, S6 and S14 wrote 2^y as *y* times *y*, and although S16 was successful in writing the correct answer to Q12 where he had to give the meaning of 5^6, he was unable to transfer the same understanding to 2^y and made the classic error of giving the answer as 2 times *y*.

Table 2
Question Facilities on Notation for Repeated Addition and Multiplication (N = 26)

Question	9	10	11	12	13	14	15	16
Number Correct	24	21	25	24	24	23	18	18
%	92.3	80.8	96.2	92.3	92.3	88.5	69.2	69.2
Number Wrong	2	4	1	1	1	2	6	6
%	7.7	15.4	3.8	3.8	3.8	7.7	23.1	23.1

This shows that the transition to algebra is not always easy for students and they continue to need extended discussion and practice for some concepts. Two students, S8 and S18, struggled with these questions, with S18 attempting the questions but getting most of them incorrect, and S8 not attempting to answer questions that involved large numbers and symbolic literals. S4's answers to Q15 and Q16 were interesting. For 2^y in Q15 he wrote "2 multiplied by itself an unknown number of times", and for question 16 (What do you understand by a^y?) he wrote "an unknown number multiplied an unknown number of times". In Q6a(i) and (ii) 15 (57.7%) of the students were able to distinguish between an expression and an equation. S22 justified her answers here as: "3 × *y* is an expression as it doesn't have an answer. 3 × *y* = 12 is an equation as it has an answer and therefore *y* is a specific unknown number". In a similar vein S21 wrote: i) is an expression and ii) is an expression and an answer. This seems to support the well-known idea that for many students the equals sign signals an answer rather than the notion of equality. Some of the students showed a process-oriented preference and hence tried to solve the equation rather than describing it.

In Q6b) 14 (52.9%) of the students successfully described *y* in 3 × *y* as a general number and *y* in 3 × *y* = 12 as a specific unknown, using varying descriptive terms. Some of their responses included: S24 "in i) *y* is a general number and it can take any values. In 3×*y*=12…it can only take a particular number 4" (see Figure 5); S4 "In the expression 3×*y* *y* could be anything like 3 × 2 but in 3 × *y* = 12 it HAS to equal 4 or it will make no sense. In ii) *y* is a specific unknown number and it can only take a particular number 4". S3, S7, S11, (see Figure 5), S14 and S20 also saw *y* in i) as an unknown or a general number and *y* in ii) as a specific unknown (meaning a particular number). Of the remaining students, S6, S10, S12 and S16 ignored i) and solved ii) successfully. S2, S5, S8, S23 and S25 simply ignored the question altogether. S23 was one of the students who was attending ESOL classes and so it is possible that he was hindered by language difficulties.

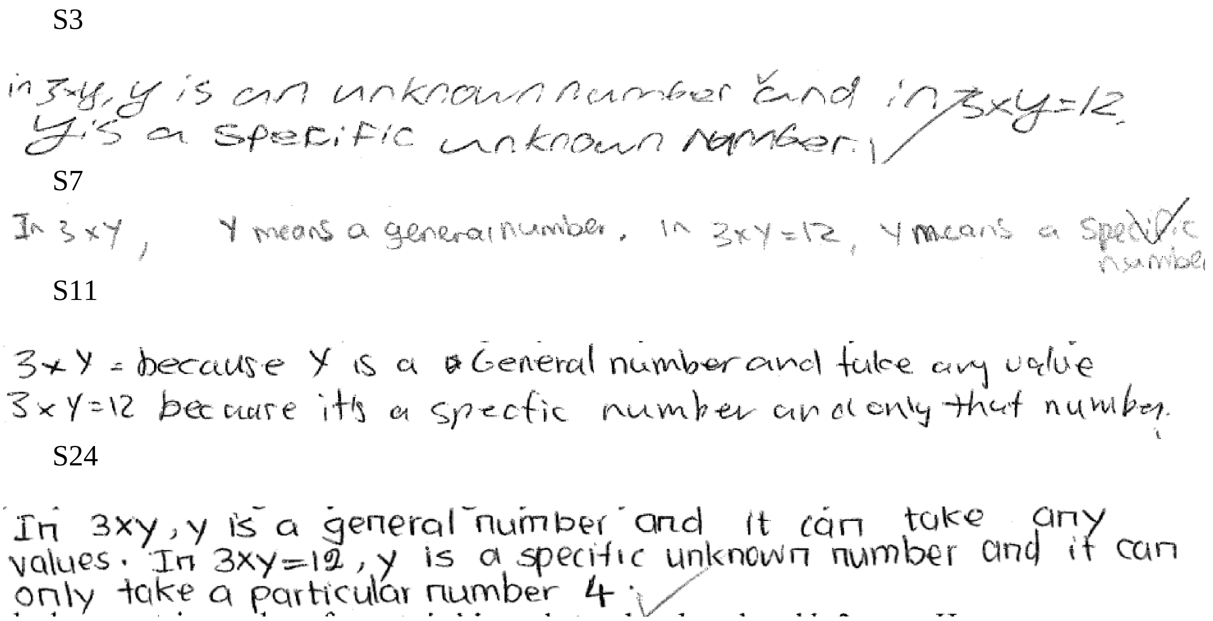

Figure 5. Students' show appreciation of generalised number and specific unknown.Conclusion

Sfard (1995) maintains that a study of history provides us with opportunities to understand student difficulties and also ways to overcome those difficulties. One such lesson we learn from history is that for the students of today a clear understanding of the notation and concept of letter as variable is difficult, but is nonetheless vital for progress in algebra. In their paper, Puig and Rojano (2004) propose that a study of history of algebra reveals how, historically, substantial progress in algebra (developing general methods of solutions of equations) was only made once a clear terminology for different variables and powers of letters was developed. In Europe this terminology was constructed by Viete (16^{th} century) who developed general methods for solutions of equations, using letters to refer to coefficients in polynomials instead of numbers. However, Indian mathematicians were able to develop general methods for solutions of equations by the time of Brahmagupta (7^{th} century) when a clear notation for different variables and powers of variables existed. It is possible that the decimal number system and notation that arrived in Europe from India via the Arabs played a significant part in that process, since its firm foundation in India, including general methods of operations on zero and integers, seems to have paved the way for a transition to algebra there.

The results of this study show that a method of observing patterns and using colours as symbols to note the variation and invariants was accessible to most students and they were able to classify them with varying degrees of success. This supports Mason's (Mason et al., 2005) contention that observing patterns and classifying/generalising may be inherent in children. Even students who struggled with many of the concepts were able to experience a certain measure of success with questions 1 to 4. In spite of this students need their attention guided to assist with identifying variation. Responses to questions 5, 6 and 7 revealed students conceptual understanding of specific unknown and generalised number, and the ability to distinguish the two, in expression and equation contexts. The fact that around 69% of the students were able recognise them and show understanding of the difference between the two is notable, especially given that Küchemann's (1981) large-scale study reported only 17% of 13 year-old students able to use letter as specific unknown and less than 2% could deal with letter as generalised number. These two views of the literal symbol, as stated by Radford (1996), are critical for progress in algebra. If students are to make sense of the many uses of letters in their high school algebra learning then they have first to make sense of these fundamental uses as generalised number and specific unknown. This is in agreement with the findings of Puig and Rojano (2004), whose study of the history of algebra identified conceptual understanding of names/letters for different unknowns and powers of unknowns as two vital categories for progress in algebra. In summary, it appears that combining historical ideas with modern didactics may reveal novel approaches to understanding of specific unknown, generalised number, and notation for variables and powers in algebra.

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