How Do Adults Perceive, Analyse and Measure Slope?

Bruce Duncan
University of Tasmania
<Bruce.Duncan@utas.edu.au>

Helen Chick
University of Tasmania
<Helen.Chick@utas.edu.au>

Slope is a mathematical concept that is both fundamental to the study of advanced calculus and commonly perceived in everyday life. The measurement of steepness of terrain as a ratio is an example of an everyday application of the concept of slope. In this study, a group of pre-service teachers were tested for their capacity to mathematize the measurement of steepness. Findings suggest that accuracy in the measurement of steepness may be related to success in linear algebra as they are both related to the perception of slope as a ratio.

The phenomenon of slope is experienced in daily life, in walking over hilly terrain, climbing stairs, and driving down a steep street. It is encountered more formally during the study of linear functions and coordinate geometry, where the "rise over run" formulation is introduced. The high-school mathematics textbooks and tests that tend to direct students' attention towards the algebraic contexts of slope usually pay less attention to other contexts such as the measurement of steepness of terrain. Indeed, the applications of slope measurement are often limited to interpreting the slope parameter in equations of straight lines, and rates of change between variables, rather than measuring the steepness of a hill.

Calls for connectedness between high-school mathematics and real-world contexts to be made more prominent (McIntosh, 2002) are reflected in the *Australian Curriculum: Mathematics*, which states "Mathematics is comprised of … concepts … which students apply beyond the mathematics classroom" (ACARA, 2012, p. 4). The extent to which this intention is fulfilled in relationship to slope is part of the focus of this paper.

This study examined pre-service teachers' understanding of slope and their attempts to measure steepness. Research by Stump (2001) suggests an emphasis on procedural applications of slope in algebraic contexts limits conceptual understanding and capacity to transfer understanding among different contexts. Studies that have examined students' and teachers' understanding of slope (e.g., Cheng, 2010; Postelnicu, 2011; Stump, 2001) suggest that conceptual understanding of slope as a ratio is a foundational requirement for continued study of mathematics. This paper considers whether measurement of steepness of terrain provides a context for learners' conceptual understanding of slope as a ratio.

Measurement, Mathematization, and Slope

Steepness, as an attribute of terrain, was described to participants in this study as *the extent to which the ground varies from being flat.* Mathematics curricula include two separate methods by which the steepness of terrain can be measured: the angle method or the ratio method. These methods require slope to be perceived differently and involve different systems of quantification (Figure 1). The angle method involves perceiving the surface of the ground as a line (ray) and the extent to which this line varies (turn, rotation) from horizontal, and is commonly measured in degrees. The ratio method involves perceiving the slope as a change in height between two horizontally displaced points. The quantities that affect steepness as a ratio are height and horizontal distance. The ratio is constructed as the change in height divided by the horizontal distance, interpreted as the change in height per unit of horizontal distance. As each quantity in the ratio is a distance, the resultant ratio is a measurement without units. As a ratio, steepness may be written as a

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fraction, a decimal, a percentage, or in other standard forms (e.g., 1 in 7). These two methods are combined in the tangent ratio.

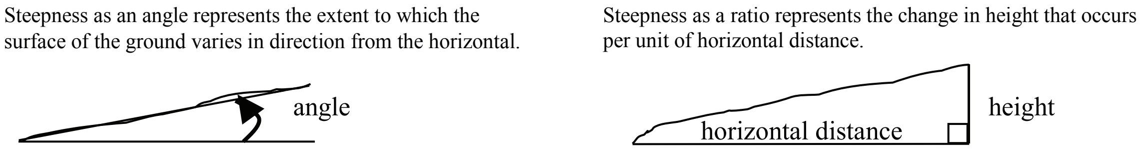

Figure 1. Two methods for measuring steepness.

A "research-based teaching sequence for measurement" (p. 171) that was proposed by Wilson and Rowland reflects the development of understanding of measurement concepts. This understanding develops through a series of stages that begins with the identification of the attribute (stage 1), the ability to accurately compare instances of the attribute (stage 2), the adoption of informal units to quantify measurement of the attribute (stage 3), the use of formal units (stage 4), and, finally, the use of formulae to aid measurement (stage 5) (Wilson & Rowland, 1993). Progression through these stages occurs for most types of attributes and their measurement, and is dependent on experience.

These five stages of Wilson and Rowland (1993) certainly apply to the case of slope, whether in a real-world context or in more abstract coordinate geometry contexts. The measurement of steepness requires first that the attribute is understood. Evidence of the second stage comes when instances of varying steepness can be compared. The remaining three stages require the understanding of an effective method for measuring steepness and an ability to apply that method appropriately to quantify the result. It is conceivable that in the case of coordinate geometry students might learn the formula for calculating slope without having a complete sense of the attribute; this is not the direct focus of this study.

Modern mathematics curricula mention a wide variety of measurement skills that students should gain, and a variety of contexts in which these skills should be applied (ACARA, 2012). More complex measurements—such as the derived measurements of speed or volume—are introduced in later years. Along with procedures, such as using equipment or manipulating an equation, effective measurement requires the capacity to select and apply the appropriate method. The measurement of steepness offers opportunities to develop and test students' understanding by requiring them to make choices, both in applying one method, and in selecting from two different methods.

The understanding required to appropriately select and apply mathematical concepts is described in the proficiency strands of the *Australian Curriculum: Mathematics* (ACARA, 2012). The ability to recognise and apply mathematical concepts appropriately to solve problems is called mathematization. Mitchelmore and White (2000) studied upper primary students' conception of angle, and how readily they perceived angles in a range of contexts such as opening doors, the hands of a clock, and the slope of a hill. While they report that "most children have formed contextual angle concepts of slope" (p. 215) by the age of 9 years, they also note that many students up to grade eight struggled to quantify the angle of a slope because of the lack of a fixed (horizontal) reference line. The mathematization of steepness as a ratio has been studied by Simon and Blume (1994) and later by Lobato and Thanheiser (2002). Simon and Blume studied pre-service teachers' ability to identify ratio as an appropriate method for measuring steepness in a number of contexts, including ski-slopes and wheelchair ramps. They concluded that understanding ratio-as-measure required understanding multiplicative relationships as well as mathematical modelling, or mathematization. Lobato and Thanheiser extended the work done by Simon and Blume in a study of high-school students. By examining students' attempts to mathematize steepness, they identified four components essential for understanding ratio-as-measure. These four components combine with the stages identified by Wilson and Rowland (1993) to form the description of understanding the measurement of steepness as a ratio used in this study.

1. Isolating the attribute that is being measured. Some students had difficulty isolating steepness from the work required to climb the hill, which increases with the climb.
2. Determining which quantities affect the attribute and how. Some students had difficulty recognising that an increase in height over a fixed horizontal distance would increase the steepness. Others struggled to see that a decrease in horizontal distance for a fixed height would also increase the steepness.
3. Understanding the reproducibility of a measure. Students need to understand that two hills with the same steepness may vary in their overall height or length.
4. Constructing a ratio. Once the quantities of height and horizontal distance have been found, they must be combined in the relationship of rise over run.

A further study by Stump (2001) examined high-school students' ability to perceive slope as a ratio in different contexts. Students were shown a range of slope contexts, including graphs of linear functions, a road sign showing steepness as a percentage, and the wheelchair ramp diagram used by Simon and Blume (1994). Stump found that students who could successfully apply understanding of slope as a ratio to problems in some contexts did not transfer that understanding to other contexts. Students who were successful in tasks involving linear graphs were unable to perceive slope as a ratio to measure the wheelchair ramp's steepness, or interpret the road sign percentage as a ratio.

With this background in mind, the present study sought to investigate the techniques that pre-service teachers use to measure slope in a variety of contexts. It also endeavoured to determine whether they understood slope as an attribute. The specific research questions asked were:

RQ1. What do pre-service teachers understand about the attribute of slope and the measurement of steepness?

RQ2. How does their understanding fit within the Wilson and Rowland (1993) stages?

RQ3. What high school mathematics content do they draw from to measure steepness?

Method

Survey activity

An initial group of 25 pre-service teachers took part in a survey activity which involved a variety of questions requiring them to measure slope. The participants were first asked to quantify the steepness of some sloping lines that had been drawn on paper, using whatever method of quantification they preferred. They were then invited to measure the steepness of four sections of path in a convenient outside location. They were instructed to choose their own method for measuring steepness but to use the same method that they had used on the paper diagram questions. Participants were asked to represent the steepness of each outdoor path in two ways: as a drawn line and as a number. Participants were not supplied with any formal measuring equipment nor were they given instruction about what kind of measurements to make. It was expected that their approaches would be chosen from the angle method, the ratio method, or some informal method. To ensure that the earlier stages of slope measurement understanding were mastered, (RQ2) participants were also asked to compare the paths and rank them according to their perceived steepness.

Interview

Ten participants were selected from the survey group on the basis of their responses and accuracy, ensuring a mixed representation of accuracy levels and methods used. These participants took part in individual semi-structured interviews with the first author. Questions elicited their understanding of the measurement of steepness as a ratio. Their responses were used to answer RQ1 and RQ3. The sections of the interview are described briefly in what follows.

The block task. A series of structures using blocks and rulers was shown to participants who were asked to comment on the steepness of the slope in each case and how it changed from one structure to the next (see Figure 2). This task was based on the study done by Lobato and Thanheiser (2002) that explored students' mathematization of steepness and their perception of the four components of ratio-as-measure and relates to RQ1 and RQ3. The structures, and the variation among the different cases, allowed clear illustration of the vertical and horizontal quantities of a slope which, in turn, facilitated an examination of participants' perception of these quantities and their effect on steepness. The task was intended to explore participants' recognition of slope as an attribute, so that their results could be cross-referenced against their actual measurement approaches.

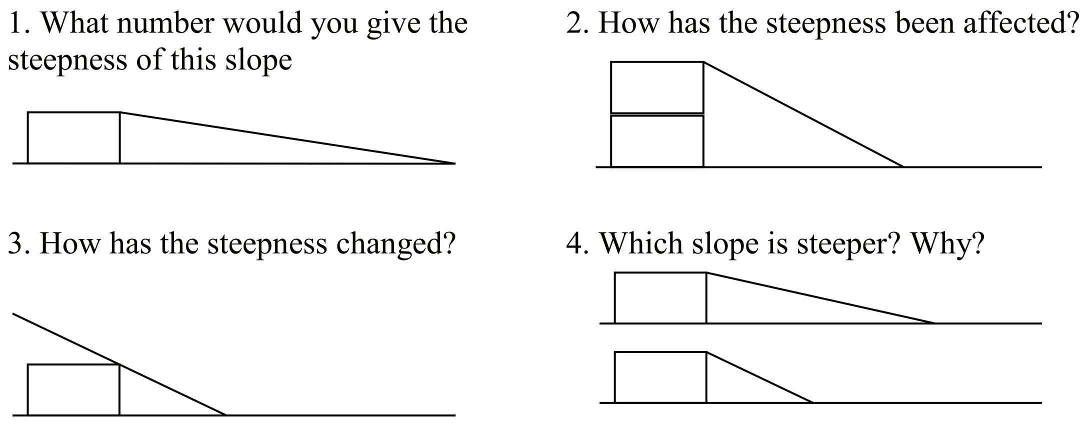

Figure 2. The block task structures made from blocks with a ruler ramp, used in the interview.

Measurement strategies. Participants were asked to describe their method for measuring the outdoor paths in the survey activity. Their responses clarified evidence from the survey about how they had determined the paths' steepness, and whether they had used the angle method, the ratio method, or some informal method.

Grid task. Participants were shown a rectangular grid that measured 8cm by 14 cm, marked in centimetre grid lines. The rectangle was marked with a heavy, black, diagonal line from one corner to the opposite. The participants were asked to explain how they would find the steepness of the line and to estimate the steepness as an angle. This use of a grid resembled the examples used in the study of coordinate geometry as seen in textbooks and likely studied in classrooms around Grade 9. The grid had no numerical values recorded on it, and the interviewer made no suggestions for approaching the task.

NAPLAN question. The final question of the interview was taken from a recent year 9 NAPLAN test. The task showed the graph of a linear relationship and required participants to select, from a choice of four, the linear expression that should be used to determine the value of the dependent variable. These last two tasks represent typical high school contexts of slope and responses were used to address RQ3.

Results and Discussion

These results focus on participants' apprehension of the slope attribute and factors affecting it, as well as the methods that they used to measure slope. Their accuracy is discussed only briefly due to space constraints.

Survey

Participants were markedly more successful at comparing the steepness of the four separate outdoor paths than they were at actually quantifying their slopes accurately. The slopes of the four paths were approximately 0°, 5°, 10°, and 15°, and their locations were well-separated physically so that their slopes could not be compared directly. Of the 25 participants, 21 correctly ordered the four paths according to steepness, indicating that they were able to accurately perceive the differences among them. Only three participants, however, were able to accurately draw and measure the slopes of all four paths to within 5° of their actual values (whether by estimating angle or ratio). Most of the measured values significantly overestimated the steepness (a report of this analysis is in preparation). The actual approaches taken by participants will be discussed further based on their interviews.

The difference in performance between comparing steepness accurately and being able to quantify it accurately reflects the developmental stages of understanding measurement described by Wilson and Rowland (1993) and relates to RQ2. Most participants operated effectively at stage 2, as indicated by their comparisons of instances of the attribute. Their varied success at quantification reflects development through stages 3, 4, and 5.

Block Task

Responses to the block task varied among participants, reflecting the observations made by Lobato and Thanheiser (2002) of the four components of understanding ratio as measure. While all participants were able to state that steepness increased or decreased appropriately as the block structures were changed, when asked to explain why they said that one structure was steeper than another, participants drew variously on s ound mathematical reasoning, simple observations, and, in some cases, misunderstandings.

The participants who made the least accurate measurements in the survey provided reasons in terms of angle size ("The angle is more sharp") or based solely on their perception ("It looks steeper"). Ian, who was least accurate in the survey, offered an observation that included informal language and a misunderstanding of the relationship between steepness and angle: "It's slightly more bent … Extremely steeper but, in terms of the degree, I'm not sure that there's that much difference." On a number of occasions throughout the survey and interview, Ian demonstrated a poor capacity to accurately estimate the size of angles in degrees. Another misunderstanding was observed when Ian was asked to comment on the block structure with a single block and a short base length for the ruler ramp. He suggested that, although a straight ruler was used, the steepness changed, saying "it gets steeper as it goes up." Lobato and Thanheiser (2002) also observed this misconception and associated it with the inability to isolate the attribute to be measured. Although Ian ordered the slopes of the outdoor paths correctly suggesting at least some comprehension of slope as an attribute, his results on the block task suggest that this understanding is incomplete.

Another participant demonstrated confusion when trying to reason which of the two structures with the same height block but different length rulers was steeper. Initially, Bill said the two models were the same because the blocks were the same height. He continued, "I was thinking I was in a car and these were the roads and thinking that, it doesn't matter where I started from, I have reached the same height." This is an example of an inability to determine which quantities affect the attribute and how (Lobato & Thanheiser, 2002). Bill was, however, able to correct his own misunderstanding, concluding that the structure with the shorter ruler was the steeper of the two.

Participants who were more accurate in the survey activity also showed a greater capacity to perceive the steepness of the ruler ramps as a ratio of the horizontal and vertical components. That is, those participants who made the most accurate measurements of the outdoor paths generally described their reasons for determining the change in steepness for the ruler ramps in terms of the quantities—vertical and horizontal distance—that affect the attribute. The reasoning in these cases often combined appropriate use of the terms "rise" and "run", and the relationship between them, with the related change in angle, as seen in the following quote: "So, for a given run, the higher the rise, the steeper the angle."

This informal correlation between accuracy in measuring the steepness of the paths and the perception of steepness as a ratio in the block task held for all but two participants. The participants who ranked second and third for accuracy in the survey activity offered explanations during the block task that appropriately described the relationship between steepness and the quantities that affect it but did not use formal mathematical language. Rather than relating the steepness to rise and run, reasons such as "It makes it higher, which makes the steepness steeper" were offered. One participant, who used the angle method exclusively throughout the survey and interview, recognised that, by doubling the height of the blocks, the steepness also was doubled, although it is not clear if he thought that the angle measure doubled, and he did not clearly discuss ratio in a formal sense.

Real-world methods

Participants described the method by which they judged and represented the steepness of the outdoor paths in the survey activity. In some cases, the methods that were adopted by participants revealed more about their understanding of the measurement of steepness.

Most participants chose the angle method and described a technique that involved imagining vertical and horizontal lines that form a right angle and then imagining half of that angle, which would be 45°. This imagined angle could then be divided further to aid in approximating the angle of the path. Eight of the ten interview participants said they used this approach. The remaining two participants, whose accuracy results were at the extremes, used different approaches. Ian, who was least accurate overall, admitted to simply guessing the measurement, and Julie, the most accurate participant throughout the study, explained that she chose the ratio method because "I'm not confident with angles."

In order to estimate the measurement of the steepness of the path, some sought a side-on perspective from which to make the judgement. Viewed from the side, a sloping path appears similar to the two-dimensional representation participants were shown in question 1 and to the diagram of slope that participants were asked to produce as part of the survey activity. While this perspective had the potential to improve the perception of the steepness, it was not used by the participants who made the most accurate measurements.

Another strategy that was used to assist in measuring the steepness of the paths was to use the clipboard that participants were given for the survey as a horizontal reference by resting one end on the upper part of the path and raising the opposite end above the lower part of the path until the clipboard appeared to be level. Two participants who used this strategy did so differently. One used the perception to estimate the angle and the other to construct a ratio. The use of a clipboard in this way reduces the problem when perceiving steepness as an angle, noted by Mitchelmore and White (2000), that is caused by the absence of a horizontal reference line. Although this method was used to improve perception, most of the quantifications were greater than the actual values, suggesting that this participant does not have a good understanding of actual angle magnitudes.

Grid task

Responses to the grid task also revealed a range of understanding of steepness as a ratio as well as some misunderstandings about angle. Ian, who had demonstrated a poor ability to estimate angle, recognised immediately that the angle of the diagonal of a rectangle had to be less than 45° but suggested that an appropriate tool for measuring it accurately would be a ruler. Two other participants reasoned that the angle was exactly 45° because the diagonal divides the rectangle exactly in half. Two participants also suggested that the angle would be 45° if the rise was exactly half the run. Seven participants used the grid to find the two quantities necessary to determine slope as a ratio but only five of these successfully constructed the ratio. One of the exceptions, Zoran, stated that, because he had the opposite and adjacent values, with a scientific calculator he could use tangent to find the angle. He did not offer the ratio itself as a measure of the steepness.

NAPLAN question

In explaining how they answered the NAPLAN question, participants revealed their reasoning and understanding of slope in a common high-school context. Half the participants answered this question correctly. Of the five that answered it incorrectly, only one stated simply that she had no idea and would have to guess. Of the four participants who reasoned incorrect answers, two stated that the correct answer of 16×gas units+20 was wrong because the number 16 was not apparent in data supplied in the question.

Success in this question was closely related to participants' perception of slope as a ratio in the grid task. Four of the five participants who constructed a ratio as a measure of steepness in the grid task answered the NAPLAN task correctly. The participant who answered the NAPLAN question correctly but did not construct a ratio in the grid task was Zoran, who said that, in the grid task, he could use tangent to find the angle. Of the five participants who were unable to answer the NAPLAN task correctly, only one attempted to describe the steepness of the diagonal in the grid task as a ratio and used the language of "about one in two" being the approximate relationship of the vertical to the horizontal distance. This participant explained that his grandfather, a builder, had taught him this method but he did not recall learning about steepness at school.

Conclusion

The pre-service teachers clearly perceived differences in steepness but varied in their capacity to quantify the measurement of slope (RQ1, RQ2). Some were able to use high school techniques for typical school slope contexts, but most struggled to apply these techniques successfully to real terrain (RQ3). A relationship was observed between the accuracy of interviewees in the survey activity and their understanding of the components of steepness as a ratio as described by Lobato and Thanheiser (2002). Those participants who identified the quantities that affect steepness as a ratio in the block task and the grid task were more accurate than those that referred to steepness simply in terms of angle. This result suggests that the ratio method for measuring steepness, while possibly more complex, may be more reliable than the angle method. Alternatively, since those two quantities are linear measures and length is generally better understood and estimated than angle, this may explain its relative accuracy, while the complexity of this method explains the rareness of its application.

The observation that participants who understood slope as a ratio were successful in the NAPLAN question also suggests that teachers should aim to develop this understanding in all students. The fact that steepness is measured as a ratio in the building and transport industries provides a reason to see that this understanding is not simply of benefit to those students who continue with the study of mathematics into tertiary courses. It is also of benefit to students who, once leaving high school, pursue careers in these industries. It also appears that the ability to measure steepness as a ratio may be related to a broader understanding of slope, and that students who intend to study mathematics at a tertiary level will benefit from understanding the measurement of steepness as a ratio. It follows then that measurement of steepness activities in middle school mathematics classes can provide valuable experiences for all students. The same learning activities can be of benefit to students who continue to study mathematics as well as those who do not.

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