Posing Problems to Understand Children's Learning of Fractions

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In this study, ways in which problem posing activities aid our understanding of children's learning of addition of unlike fractions and product of proper fractions was examined. In particular, how a simple problem posing activity helps teachers take a second, deeper look at children's understanding of fraction concepts will be discussed. The problems posed by the students were explored and insights into the students' understanding of fractions were identified.

"Problem posing refers to both the generation of new problems and the re-formulation of given problems" (Silver, 1994, p. 19). There are three different kinds of problem posing activities which can occur either before, during or after the solution of a problem. The first kind involves reformulation or restating a problem in the "Planning" (Polya, 1957) phase of problem solving so that the problem can be solved. The second kind occurs prior to any problem solving and involves the "creation of a new problem from a situation or experience" (Silver, p. 2 0). The third kind occurs in the "Looking back" (Polya, 1957) phase of problem solving when the problem solver examines the conditions of the problem after solving a particular problem to generate related problems. In this study, the third kind of problem posing activity was used in a naturalistic setting as an extension activity for word problems.

Problem posing and problem solving have been identified to be central themes in mathematics education. Problem posing is considered as an important mathematical activity because the problems that students pose reflect their mathematical understandings, skills and beliefs. That is, teachers can gain insight into students' understanding of mathematical concepts from the problems students posed (Silver, 1994; Simon, 1993; Stoyanova, 2003; Van den Heuvel-Panhuizen, Middleton, & Streefland, 1995; Whitin, 2004). Studies by Kontorovich, Koichu, Leikin & Berman (2012) and Toluk-Uçar, (2009) show that problem posing activities not only help identify the knowledge base of the problem poser but also serve as indictors of their misunderstanding.

Research shows that children gain success in mathematics learning after problem posing instructions (English, Fox, & Watters, 2005). In Singapore, some research have been conducted in problem posing in mathematics learning (Yeap, 2000; Chua, 2011; Ong, 2003; Quek, 2002;Yeap & Kaur, 1998) but little is known about problem posing as a tool to help teachers understand children's learning in primary schools. This study examines what we can learn about children's understanding of addition of unlike fractions and product of proper fractions from problem posing activities. The domain of fractions is chosen because fraction is a challenging topic for children. "Students often have difficulty understanding fractions, in general, and understanding how to multiply fractions, in particular" (Wyberg, Whitney, Cramer and Monson, 2011, p. 289) . The domain of fractions is semantically rich because several conceptual meanings exists within the domain. Fractions can be treated as lengths, or more broadly as measurable extensive quantities (Schwartz, 1988). Students would have made significant achievement when they are able to conceive fractions as lengths, rather than solely as parts of wholes (Steffe, 2002; Steffe & Olive, 2010) because this means that the students have constructed a partitive fraction scheme (Steffe & Olive, 2010). Studies have shown that many students develop

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little meaning of fraction operations because they learn the operations with fractions through procedure oriented, memory-based instruction (Cramer & Bezuk, 1991; Kennedy & Steve, 1997).

Methods

This is a qualitative study conducted for an intact class of 35 Primary 5 from a mainstream government school. The mathematics teacher of the class and the researcher designed the 1 hour problem posing lesson. The tasks to be completed within the 1 hour lesson are shown in Figure 1. The mathematics teacher delivered the lesson as an extension to a word problem lesson for fractions. Neither the students nor the teacher had any formal experience with posing problems during mathematics lessons.

Task 1

Solve the following word problem.

Susan had a piece of ribbon $\frac{1}{2}$ m long. She used $\frac{3}{4}$ of it to tie a box. Find the length of ribbon left.

Task 2

(a) Pose a word problem on *adding unlike fractions*.

(b) Pose a word problem on *product of proper fractions*.

Provide a solution to your word problems.

Figure 1. Task administered to the students.

Implementation

The students were familiar with pair work and had designated partner when it comes to pair work. There were 17 pairs altogether with 1 student working alone. Prior to this lesson, the students had been taught the skill to add unlike fractions and they had solved word problems for addition of unlike fractions. The students had also been taught the skills of computing product of proper fractions and were moving into word problems for product of proper fractions in this lesson. The lesson began with Task 1. The teacher first helped the students understand Task 1 before modelling the solution for Task 1. Next, the children worked in pairs to complete the problem posing activity in Task 2. Students were encouraged to exercise their creativity as they engaged in the problem posing activity. For both of the problem posing tasks, the students were required not only to pose problems but also to provide the mathematical solutions of the constructed problems. The two tasks were administered to the students by their classroom teacher in a 1 hour mathematics lesson. The mathematics teacher expected the students to perform better at posing problems for addition of unlike fractions.

Data Collection and Analysis

All the samples of students' work were collected for analysis. Only the analysis was conducted for students' responses to Task 2 in this study. The students' responses were marked and categorized as correct responses, incorrect responses or blank responses.

Incomplete or partial responses were considered as blank responses. The problem was considered incorrect if an incorrect solution was given to the problem. Next, the problems that were posed correctly were further examined. To examine the range of problems posed correctly by the students for addition of unlike fractions, the problems were categorised into three main sematic categories - combine, change and compare (Hershkovitz & Nesher, 2003, p. 3). The three semantic categories were used as tools for analysis because problem solving ability in mathematics can be fostered by "enriching students' mathematical schemes as the building blocks of the students' cognition" (Hershkovitz & Nesher, 2003, p. 20). According to Patterson and Smith (1986), experts in a given area have rich and complex schemes that enable them to absorb new information in those areas and suggest the most efficient solution. Similarly, the knowledge of good solvers is organised by rich schemes (Lester & Garofalo, 1982; Lester, 1994).

The range of responses was examined for problem posing for product of fractions by looking at whether the students used a variety of real-life situations and models (e.g. measurement model). Unacceptable responses were also examined to further aid our understanding of students' learning of product of fractions and addition of unlike fractions. The incorrect responses were grouped according to the type of errors made. The errors were tagged as codes and regrouped into larger themes.

Findings

Our data showed that the students posed problems that mirrored school experiences for both problem posing tasks. That is, they tended to pose traditional word problems that were simply variations of those found in textbooks (English, 1997a; Lowrie, 2002). Table 1 shows a summary of students' responses. Of the 15 correct responses for addition of unlike fractions, 14 were 1-step word problem. One of the 15 problems is a 2-step problem. In the following sections, the types of word problems posed by students for addition of unlike fractions followed by product of proper fractions will be discussed.

Table 1
Summary of responses by students

	Correct response	Incorrect response	Blank	Total
Addition of unlike fractions	15	3	0	18
Product of fractions	5	6	7	18

Responses for Addition of Unlike Fractions

The data showed that the students posed a variety of problems using the 3 semantic structures. 9 problems were posed using combine structure, 5 problems were posed for change structure and 1 problem was posed using the change structure. Table 2 shows examples of problems posed by students for each of the semantic structures. Using the possessive verb 'has' or 'had' showed that the children were able to represent a set of items belonging to Toogol, or Ming Hua or Aron. In Toogol's problem, the children understood that 'gave' refers to an increase, and then increase the initial set by an appropriate number of books.

A wide range of real-life context was used, for example, cookies, books, apples, stickers, marbles, sweets, pens, seashells and stamps. However, the context may not be appropriate in many of the problems posed. For example, in Aaron's problem in Table 2, it does not make sense for Aaron to have $\frac{3}{5}$ stickers. Similarly, it does not make sense when students wrote $\frac{3}{10}$ pens or $\frac{3}{10}$ marbles. The wrong choice of context was also evident in the problems posed for product of fractions. Measurement model was used by two groups as $\frac{5}{6}\ell$ of apple juice and $\frac{7}{8}m$ of cloth.

Table 2

Types of correct problems posed by students categorised using the semantic structures

Word problems posed by students	Semantic structure
Toogol has $\frac{3}{10}$ books. Googol gave him $\frac{6}{20}$ books. How many books does Toogol have altogether?	Change model
Mary have $\frac{7}{8}$m of cloth. She then buy $\frac{7}{9}$m of cloth. How many metre of cloth are there?	Change model (measurement)
Aaron has $\frac{3}{5}$ stickers. Jack has $\frac{4}{6}$ more sticker than Aaron how many sticker does Jack have?	Compare model
Sally has $\frac{1}{2}$ sticker. Mary have $\frac{3}{8}$ sticker. How many do they have all together.	Combine model

In the 2-step word problem posed in Figure 2, a combination of compare and combine schematic structures were used.

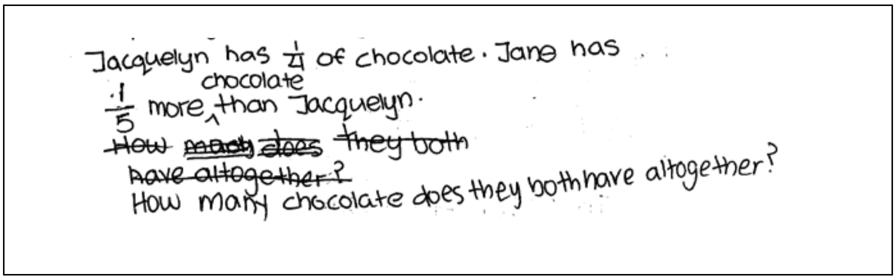
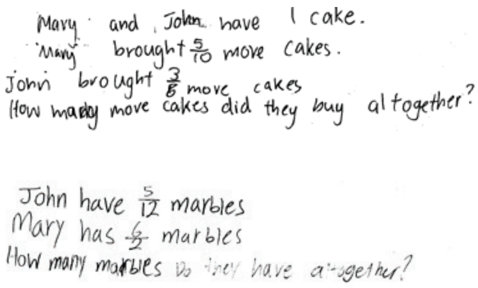


Figure 2. 2-step word problem posed by students.

Unacceptable responses found in the students' response were either due to children's difficulty in expressing fraction concepts (Figure 3a) or incorrect understanding of the term unlike fraction (Figure 3b).



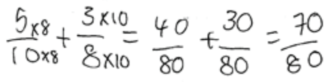


Figure 3(a). Difficulty in expressing fraction concepts

Figure 3(b). Unlike fractions.

Response for Product of Proper Fractions

A variety of incorrect responses were found for problems posed for product of proper fractions. The incorrect responses are summarized in Table 3.

Table 3
Types of incorrect problems posed by students for product of proper fractions

Word problems posed by students		Comments/Codes
Annie	Mary has $\frac{10}{12}$ pieces of pie. She gave $\frac{2}{3}$ of the remainder to her friend. How many pieces of pie did her friend receives? $\frac{2}{12} \times \frac{2}{3} = \frac{1}{3}$	Inappropriate use of the term remainder.
John	Bala bought $\frac{2}{30}$ of a cake. Bala ate $\frac{3}{20}$ of it. How many does he left? $\frac{3}{30} \times \frac{3}{20} = \frac{1}{20}$ Bala had $\frac{1}{20}$ left.	Wrong solution
Rose	Ben had $\frac{3}{8}$ of bicycle. Mark had $\frac{9}{8}$ of bicycle. Find the product of the two bicycle.	Question does not make any sense.
Mark	peter have $\frac{3}{7}$ of apples, chen sin have $\frac{5}{6}$ times as much as peter, ~~How much did Chensin have.~~ what is the product of apples.	Associating 'product' with 'times' from whole numbers.
Elias	Atom have 5 ropes. Each of the length is $\frac{7}{10}$m. What is the length of 5 ropes	Posed a problem for another fraction topic
Marie	Ben sold 5kg of sugar. It cost $35. How much is 18kg?	Posed a problem for another fraction topic

From an analysis of 6 incorrect responses, some observations can be made regarding the students' understanding of product of fraction. Clear, explicit language is needed to convey fractions concepts and this can be a challenge for students. For example, Annie used the 'term' remainder in her word problem to refer to $\frac{2}{12}$ piece of pie not given to Mary. The use of remainder can be confusing because 'remainder' is usually associated with the amount left over. In Annie's word problem, there is no sentence to suggest that anything is left over.

Our data also showed that the language used in whole number concepts influenced the students' construction of word problems in product of fractions. In Rose's case, she associated the term 'product' as the operation 'multiply' from whole numbers and applied this association directly to posing her fraction word problem.

Whole numbers : Find the product of 5 and 2.
Fraction : Find the product of the two bicycles.

Mark posed his word problem by using his knowledge of 'product' as 'multiplication as comparison' in whole numbers. 'Multiplication as comparison' refers one set involves multiple copies of the other.

Whole numbers : Peter has 30 apples. Chen Sin has twice as many apples as Peter. How many apples does Chen Sin have?

Fraction : Peter has $\frac{3}{7}$ apples. Chen Sin has $\frac{5}{6}$ as many apples as Peter. What is the product of apples?

Instead of asking *how many apples does Chen Sin have?* Mark committed the same error as Rose. In both Rose and Mark's response their word problems make no sense as they literally took 2 objects and multiply them together. Elias and Marie posed problem correctly for another fraction concept and provided correct solution to the problems that they posed.

Table 4
Types of correct problems posed by students for product of fractions

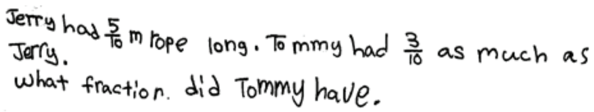
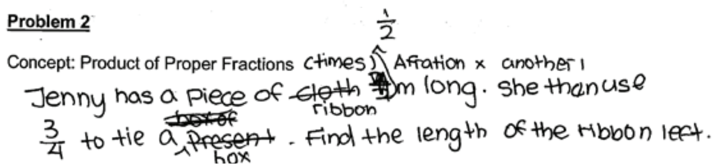
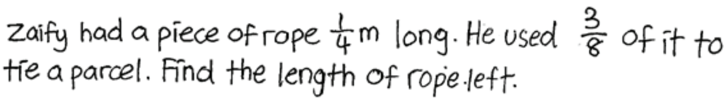
Word problems posed by students	Comments
	Associating 'product' with 'times' or 'as much as'.
	Creating new problems by changing only the name of the character in the story.
	Creating new problems by changing only numerical value and context. Same sentence structure as Task 1.

Table 4 s hows samples of the correct response for product of fractions. Correct responses were due largely to students modifying the word problem from Task 1 by numerical variation (Silver, Mamona-Downs, Leung & Kenney, 1996). That is, a new problem is created by substituting the given numerical values with the new ones (Lavy & Bershadsky, 2003). The children were also able to create problems by altering the contexts of the problem in Task 1. Of the 5 correct responses, 4 responses posed problems using measurement context. Context used were largely, rope, ribbon, a piece of paper.

Conclusion and Recommendations

This study investigated children's knowledge base for addition of unlike fractions and product of proper fractions from problem posing activities. The results showed that there were more correct problems posed for addition of fractions as compared to the problems for product of fractions. This result matched the mathematics teacher's expectations. There was a variety of problems posed for addition of unlike fractions. Examples of problems could be found for each of the semantic structure. Most of the problems posed for addition of unlike fractions used combine structure. In future studies, students could be asked to pose at least 3 different types of problems for addition of unlike fractions to add richness to their understanding of addition of unlike fractions. This recommendation is made to enrich the repertoire of schemes available to each student. Although a wide range of real-life context was used in the problems posed for addition of fractions, not every context is suitable for fractions. The choice and appropriateness of the context to be used for fraction topic could be discussed in the classroom to deepen students' understanding of fraction concepts.

Instrumental understanding of product of proper fractions was evident in the students' solutions to the problems posed. However, the word problems posed by some of the students suggested a lack of relational understanding towards this concept. The students' responses suggested that there was 'sense-less' direct transfer of language used in whole number word problems to fraction word problem.

The problem posing activities in this activity helped the teacher gain deeper insights into children's understanding of two fraction concepts. The findings suggested that the language and context associated with fractions is far more complex than whole numbers. Classroom instruction can be designed to help students be familiar with the problems in each of the semantic structures for whole numbers before moving into the semantic structures for fraction word problems. Effort should also be made to help students connect and organise the language used for fraction and whole number concepts.

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