Strategies for Solving Fraction Tasks and Their Link to Algebraic Thinking

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Many researchers argue that a deep understanding of fractions is important for a successful transition to algebra. Teaching, especially in the middle years, needs to focus specifically on those areas of fraction knowledge and operations that support subsequent solution processes for algebraic equations. This paper focuses on the results of Year 6 students from three tasks from a Fraction Screening Test that demonstrate clear links between algebraic thinking and students' solutions to fractional tasks involving reverse processes.

The National Mathematics Advisory Panel (NMAP, 2008) stated that the conceptual understanding of fractions and fluency in using procedures to solve fractions problems are central goals of students' mathematical development and are the critical foundations for algebra learning. Teaching, especially in the primary and middle years, needs to be informed by a clear awareness of what these links are before introducing students to formal algebraic notation.

Sixty-seven Year 6 students from an eastern suburban metropolitan school in Melbourne were tested using our Fraction Screening Test (Pearn & Stephens, 2014). This paper aims to identify and examine students' responses to three tasks from the test that demonstrate clear links between algebraic thinking and students' solutions to fractional tasks involving reverse processes.

Previous Research

According to Wu (2001) the ability to efficiently manipulate fractions is "vital to a dynamic understanding of algebra" (p. 17). Many researchers believe that much of the basis for algebraic thought rests on a clear understanding of rational number concepts (Kieren, 1980; Lamon, 1999; Wu, 2001) and the ability to manipulate common fractions. There is also research documenting the link between multiplicative thinking and rational number ideas (Harel & Confrey, 1994).

Siegler and colleagues (2012) used longitudinal data from both the United States and United Kingdom, to show that, when other factors were controlled, competence with fractions and division in fifth or sixth grade is a uniquely accurate predictor of students' attainment in algebra and overall mathematics performance five or six years later. They controlled for factors such as whole number arithmetic, intelligence, working memory, and family background. We need to extend these important findings to highlight for teachers those specific areas of fractional knowledge that impact directly on algebraic thinking.

Lee and Hackenburg (Lee, 2012; Lee & Hackenburg, 2013) conducted research with 18 middle school and high school students. Their research showed that fractional knowledge appeared to be closely related to establishing algebra knowledge in the domains of writing and solving linear equations and concluded: "Teaching fraction and equation writing together can create synergy in developing students' fractional knowledge and algebra ideas" (p. 9). Their research used both a Fraction based interview and an Algebra based interview. The two interview protocols were designed so that the reasoning involved in the Fraction based interview provided a foundation for solving problems in the Algebra

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Interview. In both Interviews students were asked to draw a picture as part of the solution. For the Fraction tasks they were also asked to find the answer whereas in the Algebra tasks they were asked to write an appropriate equation but not solve it. Examples of one of each of the Fraction and Algebra Tasks are shown in Table 1 below.

Table 1
Examples of tasks used by Lee and Hackenburg

Fraction Task	*Algebra Task*
Tanya has \$84, which is $\frac{4}{7}$ of David's money. Could you draw a picture of this situation? How much does David have?	Theo has a stack of CDs some number of cm tall. Sam's stack is $\frac{2}{5}$ of that height. Can you draw a picture of this situation? Can you write an equation?

After analysing the data, Lee (2012) constructed models to determine the fraction schemes used by students and their reasoning about unknowns and writing equations. However, the important point that these authors make is that the thinking required to solve this type of fraction task is very similar to the kind of thinking required to "solve for *x*" in a corresponding algebra equation. Both the Fraction Task and the Algebra Task from the Lee and Hackenburg study (2013) shown in Table 1 require multiplicative thinking to move from a given fraction to the whole, and relating these actions to the corresponding quantities. They cannot be solved additively, for example, by saying "I have to add another three-sevenths". We notice that in the Fraction Task above students are not asked to explain their thinking or what the picture represents. Moreover Lee and Hackenburg do not discuss the range of possible methods that students might use to solve the fraction task, presenting instead an example of a picture and associated comments by one student. Students are not required to solve the algebra equation ($S = \frac{2}{5}T$ where S and T represent the number of CDs that Sam and Theo have).

Stephens and Pearn (2003) identified Year 8 proficient fractional thinkers as students who demonstrated a capacity to represent fractions in various ways, and to use reverse thinking with fractions to solve problems. This research also showed that effective reverse thinking depends on a capacity to apply multiplicative operations to transform a known fraction to the whole. This capacity will later be fundamental to the solution of algebraic equations. In this study we identify algebraic thinking in terms of students' capacity to identify an equivalence relationship between a given collection of objects and the fraction this collection represents of an unknown whole, and then to operate multiplicatively on both in order to find the whole. Jacobs, Franke, Carpenter, Levi, and Battey (2007) also emphasise the need to "facilitate students' transition to the formal study of algebra in the later grades (of the elementary school) so that no distinct boundary exists between arithmetic and algebra" (p. 261). Three distinct aspects of algebraic thinking identified by Jacobs et al. (2007) and by Stephens and Ribeiro (2012) are important for this study. They are students' understanding of equivalence, transformation using equivalence, and the use of generalisable methods.

This Study

Unlike the Lee and Hackenburg study (2013) which used both a Fraction Interview and a separate Algebra Interview, our study is based on analyses of students' performances in a single paper and pencil test of fractional thinking. Previously Pearn and Stephens (2007) used a Fraction Screening Test and Fraction Interview using number lines to probe students' understanding of fractions as numbers. Results from these showed that successful students demonstrated easily accessible and correct whole number knowledge and knew relationships between whole and parts.

The current version of the Fraction Screening Test (Pearn & Stephens, 2014) includes items that require students to use reverse or reciprocal thinking. The Fraction Screening Test was divided into three parts. Part A included 12 tasks, 11 tasks had been trialled in previous work (Pearn &Stephens, 2007). Part A tasks included routine fraction tasks such as equivalent fractions, ordering fractions and recognising simple representations. Part A also included a simple reverse thinking task showing a collection of four lollies and saying: "This is one-half of the lollies I started with. How many lollies did I start with?" This task was correctly answered by the majority of students and was one of the easiest questions in Section A. Part B included five number line tasks with four tasks trialled in previous work. One number line task involving reverse thinking gave a number line showing "where the number $\frac{1}{3}$ is. Put a cross (x) where you think the number 1 would be on the number line." Part C included three questions which required students to use reverse thinking using less familiar fractions (see Figure 1).

Our Sample

Sixty-seven Year 6 students from an eastern suburban metropolitan school in Melbourne were tested using our Fraction Screening Test (Pearn & Stephens, 2014). Students completed the tests in approximately 30 minutes. After analysis of the 67 sets of responses, 19 students were chosen for closer analysis. These 19 students had correctly solved each of the three questions shown in Figure 1 and provided adequate explanations of their thinking. They were asked to provide a more detailed written explanation of their solution to one question only in order to confirm their thinking.

Our Three Key Questions

The analysis for this paper is based on these three items from Part C which specifically required students to use reverse or reciprocal thinking in which their task is to find a whole collection when given a part of a collection and its fractional relationship to the whole.

We devised these three items to offer students opportunities to use more explicit algebraic thinking which was not needed in the earlier task relating to one-half. Each of the three questions was marked out of three. Only one mark was given if there was some evidence of correct diagram or of an initial representation which the student did not take further (starting point). Two marks were given for a correct answer but without explanation and three marks were given for an adequate explanation.

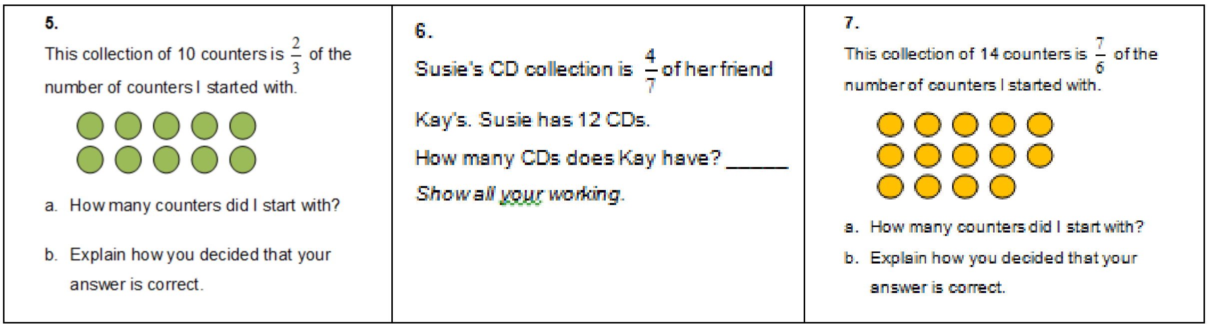

Figure 1. Questions 5 – 7

Like Kieran (1981) and Jacobs et al. (2007) we do not restrict correct algebraic thinking to students' ability to use pro-numerals or unknowns or necessarily to set up formal algebraic equations. We expect that these Year 6 students who have not necessarily been exposed to formal algebra will employ a variety of successful representations to solve these problems. We also expect that some students may use a routine algorithm to solve the problem. Simply using a routine without an appropriate explanation may not be convincing evidence of algebraic thinking. However, we also expect that some students may choose to solve the same problems in non-algebraic ways.

Results

Among the 67 students, five groups were identified: Group A (19 students) who correctly answered and adequately explained each of the three questions (scoring 9 marks out of a possible 9). Group B (9 students) answered the three questions correctly but gave an incomplete explanation or no explanation for one of their correct answers (scoring 7 or 8). All Group B students scored a 3 for Question 5. Group C (14 students) all had correct answers to Question 5, with 12 providing adequate explanations (scoring between 4 and 6). Group D (11 students) scored between one and three marks on the same three items. All 11 students omitted to answer at least one of the questions. Four students had correct answers to Question 5, with three providing adequate explanations. No student in this group correctly answered Question 7. Group E, (14 students) scored 0 on all three questions, providing insufficient evidence of performance.

Forty-six of the 67 students (69%) gave correct answers and 43 gave adequate explanations to Question 5. The diagram accompanying this problem may have assisted students to solve the problem. Some students' explanations used reverse thinking to show that one-third was equivalent to 5 dots and therefore the whole needed to be 15. Other students' explanations involved additive one-step thinking saying that one more row was needed to make the whole. Either explanation is suitable for this question.

Question 6 was correctly answered by 38 students (57%). Not being supported by a diagram, it appeared more difficult. Using one-step additive thinking is not helpful in solving this problem. It was necessary for students to calculate the number of CDs represented by $\frac{1}{7}$ and to scale up that quantity to make a whole. More difficult was Question 7 involving an improper fraction $\frac{7}{6}$ even though it was supported by a quantitative representation. Question 7 was correctly answered by 32 of the 67 students (48%). Some successful explanations applied a fractional lens to decode the 14 dots shown, arguing that each pair of dots represents $\frac{1}{6}$ and that the whole can be found by subtracting two dots. This solution, as is Emily's solution to Question 6 (see Figure 2), involves similar two-step thinking as those students who first divide the 14 counters by 7 to find how many counters are represented by $\frac{1}{6}$ and then to multiply (scale up) by 6 to find a whole. These two questions, even with a diagram provided for Question 7, were more difficult than Question 5.

Analysis

In this section, we focus on the 19 students (Group A) who gave completely correct responses and adequate explanations to all three questions. Some explanations were briefly written leaving some thinking unstated and raising a question of whether these students may have been using a routine. Each of the 19 students was asked to provide a short written elaboration of their initial explanation to one question selected by the researchers. In looking at their initial responses and their subsequent elaborations our goal was to identify those features that could be confidently taken to indicate evidence of algebraic thinking. Our focus was to identify instances of student thinking that could be clearly classified as algebraic; namely, understanding of equivalence, transformation using equivalence, and use of generalisable methods. Students in Group A offered the best chance to show this.

Confident Reverse Thinkers

Responses of Group A students show that confident reverse thinkers are able to step back from a visual representation, and to relate the fraction to the numerical quantity it represents. These students know how to scale down and scale up fractions and the quantities they represent to obtain a measure for the whole. Scaling down and scaling up is a reliable two-step procedure for finding the whole. It may even be compacted into one-step. These students are not dependent on using additive strategies which may be appropriate for simple fraction problems like the one-half task in Part A.

From the 19 fully correct responses four different types of responses were evident: *Response Type 1*. Eleven students employed equivalent operations using fractions and whole number quantities in parallel. See for example Emily's response to Question 6 in Figure 2 where she wrote $\frac{4}{7} \div 4 = \frac{1}{7} \times 7 = \frac{7}{7} = 1$ on one line and $12 \div 4 = 3$, $3 \times 7 = 21$ on the one underneath tracking both fractional and whole number computation in parallel.

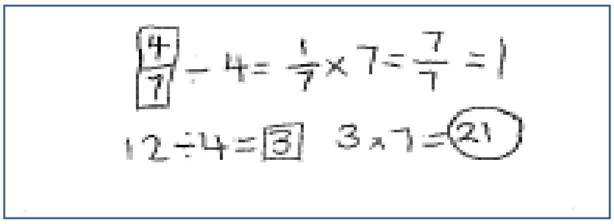

Figure 2. Emily's response to Question 6

Emily's response can be directly compared to a two-step solution for $\frac{4}{7}\,x = 12$. Like some other students, Emily uses an equal sign idiosyncratically to connect her steps as in the first line of her response. However, Emily clearly understands the need for equivalent operations to relate the two lines of her solution. Other students write "equivalence relationships" involving fractions and whole numbers together. For example, in Question 6 some students wrote: $\frac{4}{7} = 12$, $\frac{1}{7} = 3$, $3 \times 7 = 21$

Sometimes a two-step reverse operation is compacted into one step as Kenneth's response to Question 5 as shown in Figure 3.

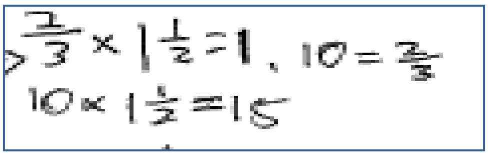

Figure 3. Kenneth's response to Question 5

Kenneth's response mirrors very closely the kind of transformational thinking needed to solve the algebraic equation $\frac{2}{3}\,x = 10 \;\rightarrow x = 10 \times 1\frac{1}{2}$.

Response Type 2. Six students left the fraction unstated and operated directly on the whole number quantity. While scaling up the fraction is left invisible, this transformation clearly guides the operations on the associated whole numbers using equivalence: For example, one two-step response to Question 6 was $12 \div 4 = 3, 3 \;\times 7 = 21$; or by another student on the same question: $12 \times 7 = 84$, $84 \div 4 = 21$ or in one compacted step by another student for Question 7 was $14 \div \frac{7}{6}$. These strategies explicitly show the kind of generalisable algebraic thinking needed to solve the equation $\frac{7}{6}\,x = 14$

Response Type 3. Symbolic representation using an unknown was used by one student only. Figure 4 shows Julie's response to Question 6: 12 is $\frac{4}{7}$ of *x*, meaning that $x = 12 \div \frac{4}{7}$ which is now $\frac{12}{1} \times \frac{7}{4}$

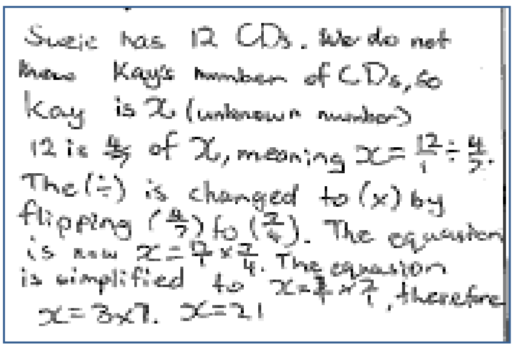

Figure 4. Julie's response to Question 7

Julie's response shows a clear understanding of equivalence and transformation. It is also generalizable unlike Response Type 4 which relies on written descriptions involving continued adding. This was used by one student for Question 6 who stated: " $\frac{4}{7}$ of Kay's CD collection is 12. That means that $\frac{1}{7}$ is 3. I started adding 3 onto 12 until it reached $\frac{7}{7}$. That number is 21". Multi-step responses like this correctly establishing that one-seventh is equivalent to 3 then rely on additive strategies to achieve the whole. This is a more limiting strategy than shown in the preceding Response Types which demonstrate reciprocal thinking.

Mixed Methods

Julie, who used a pro-numeral expression for Question 6, used the second and also generalisable method to solve other questions (e.g. $10 \div \frac{2}{3}$ to solve Question 5). While some Group A students tended to use either the first or second method consistently, most used a mix of methods. We wondered, for example, if the student who wrote $14 \div 7 = 2 \times 6 = 12$ might be using a routine, but this student later explained that "14 was split into seven numerator groups". Adding, "I could have taken one group away".

A similar range of methods, excluding symbolic representation, was evident among students in Groups B and C. However, among students in Groups C and D additive processes became more evident, like this Type 4 explanation from a student in Group C for Question 5: "I had to halve 10 because $\frac{2}{3}$ is 10, halve 2 to get 1, and so I did this to get 5. I just added it (5) on after (to get 15)."

Among students in Group D explanations begin to show less evidence of multiplicative (reverse) thinking: "Started with 10 to get 15"; or "Every 5 is $\frac{1}{3}$"; or "Because there are 5 in each row and 10 is $\frac{2}{3}$ of 15"; or "$\frac{1}{3} = 5$, $\frac{2}{3} = 10$, $1 = 15$". There is clear evidence of equivalence but these additive strategies have less algebraic potential compared to the efficient multiplicative (reverse) strategies shown by those using Response Types 1, 2, and 3. Algebraic thinking, as we have defined it, requires more than use of equivalence. It needs to be reflected in confident and appropriate transformations of the fractional entities involved.

Conclusion and Implications

Confident reverse thinkers are able to scale down and scale up (or scale up and then scale down) based on the meaning of the particular fractional relationship. This is exactly what is required in "solving for *x*" in corresponding algebraic representations. Their working shows that scaling down and scaling up of fractional quantities must be accompanied by equivalent changes in the quantities represented by a particular fraction. These methods and their resulting mathematical relationships are indicative of algebraic thinking, by which students demonstrate that they can manipulate the fractional and numerical quantities independently of any diagram or visual representation.

The algebraic significance of these findings is that they draw attention to three quite specific aspects of fractional operations that are not sufficiently emphasised in earlier studies. The first is being able to transform (operate on) a given fraction in order to return it to a whole, regardless of whether the fraction is expressed in proper or improper form. The second is students' understanding of equivalence, meaning that the operations that are required to restore a fraction to a whole need to be applied to the corresponding numerical quantities represented by the fraction. The third is to utilise efficient and generalisable multiplicative methods to achieve this goal; in contrast to other methods, usually additive, which may work only with simple fractions. All three aspects are essential for the subsequent solution of algebraic equations. Teachers especially need to help students identify and use these efficient and generalisable strategies.

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