Examples in the Teaching of Mathematics: Teachers' Perceptions

Lay Keow Ng
National Institute of Education, Singapore
< ng_lay_keow@moe.edu.sg>

Jaguthsing Dindyal
National Institute of Education, Singapore
<jaguthsing.dindyal@nie.edu.sg>

As part of a study examining how teachers in Singapore select and use examples for teaching mathematics, 121 teachers from 24 secondary schools responded to three open-ended questions about the use of examples in teaching. The results show that students' abilities and the difficulty level of the examples were among the topmost considerations teachers have when introducing mathematical ideas or when selecting homework tasks. This paper also reports on teachers' perceptions of a *good* example.

The use of examples by teachers in the mathematics classroom is a well-established practice. While researchers have attended to the roles of sub-categories of examples, research into how teachers integrate examples into their teaching remains scarce (Zodik & Zaslavsky, 2008). Research has also shown that the use of examples, or exemplification in short, is neither arbitrary nor straightforward, where prospective teachers (Huntley, 2013) and experienced teachers (Zodik & Zaslavsky, 2008) both face problems, hence summoning the need for research in this area.

Literature has also revealed a strong connection between teachers' knowledge and their use of examples in teaching. Rowland, Huckstep, and Thwaites (2005) found that teachers' ability in selecting suitable mathematical examples was strongly related to their mathematics content knowledge for teaching. Also, Chick (2010) stressed that the capacity of teachers in crafting effective examples relies heavily on their pedagogical content knowledge too.

Teachers use examples in various ways, often to introduce an idea or illustrate a concept. Also, examples are used by teachers in the assignment of specific tasks, such as homework, which in Singapore is a common practice. Several factors may affect the choice of specific examples by teachers. This paper focuses on the following three questions.

1. What factors do secondary mathematics teachers consider when choosing examples for introducing new mathematical ideas?
2. What factors do secondary mathematics teachers consider when selecting examples for homework tasks?
3. What are the characteristics of a *good* example used for teaching mathematics in the eyes of secondary teachers?

Examples in the Teaching of Mathematics

The significance of examples is summarised by Watson and Mason (2002): "learning mathematics can be seen as a process of generalizing from specific examples" (p. 39). Examples are therefore paramount in mathematical teaching and learning.

The definition of examples used by researchers generally refers to an example as an illustration of a larger class. This broad definition can include geometrical figures, demonstrations of solving problems, tasks, and worked examples, as long as the mathematical object is offered or perceived as an example of something. In this study, a task can be an exercise, problem, or assessment assigned to students for completion during or beyond curriculum time. The same task may differ in operation and learning outcomes,

2015. In M. Marshman, V. Geiger, & A. Bennison (Eds.). *Mathematics education in the margins* (Proceedings of the 38th annual conference of the Mathematics Education Research Group of Australasia), pp. 461–468. Sunshine Coast: MERGA.

depending on the intentions of the author, the aims and knowledge of the teacher, the goals, knowledge, and experiences of the students, and on the learning environment. The role of teachers therefore lies in setting up appropriate tasks.

Example selection is, however, not merely choosing or implementing good examples, but entails leveraging on coherent example sets to build students' understanding in order to attain instructional goals. Watson and Mason (2006) claimed, "the starting point of making sense of any data is the discernment of variations within it' (p. 92). They proposed to systematically change certain aspects of a task while keeping others invariant, to help learners better perceive the mathematical structure. In addition, Skemp (1971) advised educators to reduce the *noise* in examples during concept formation so as to draw learners' attention to the key characteristics of the concept.

Empirical findings from work with teachers have also revealed principles that guide teachers in making their example choices. One common approach was the use of simple first examples (Bills & Bills, 2005) that include keeping the numbers small and ordering examples in increasing complexity. To scaffold students' learning, teachers have also proposed using examples that build on students' prior knowledge (Bills & Bills, 2005) and keeping unnecessary work to a minimum (Zodik & Zaslavsky, 2008). Sometimes, teachers tend to craft and use examples that allow them to attend to common errors and misconceptions to forewarn their students (Zodik & Zaslavsky, 2008) or to include uncommon cases to increase students' exposure.

Teacher Knowledge and the Use of Examples in Teaching Mathematics

A closer scan of the literature on mathematical examples highlights the close connection between teachers' examples and their knowledge. In particular, content knowledge and pedagogical content knowledge (PCK) have been identified to directly influence teachers' exemplification abilities. Content knowledge is the knowledge of the subject matter content. PCK is the "blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction" (Shulman, 1987, p. 8). Ball, Thames, and Phelps (2008) sub-divided PCK into knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of content and curriculum (KCC). KCS includes an awareness of topics that students will find easy or difficult and their common conceptions and misconceptions. KCT comprises of knowledge on the sequencing of examples and the use of appropriate representations. Finally, KCC encompasses knowledge of educational goals, assessments, and the sequencing of topics across grade levels.

Rowland et al. (2005) observed how content knowledge and PCK contributed to the decisions and actions of their participants. Of the four units of their Knowledge Quartet framework, transformation or knowledge-in-action was strongly tied to teachers' example choice. Variables, sequencing, representations, and learning objectives were also identified as related to teachers' awareness in exemplification.

Noticing the lack of research between teachers' PCK and their exemplification practices, Chick and her colleagues (see Chick, 2007) studied the instructional practices of Australian elementary teachers and were successful in surfacing moments where aspects of PCK were enacted through the teachers' examples. Chick (2007) also noted that most of the examples that the teachers used were planned and selected based on the examples' structures and qualities. The selection process was much guided by the teachers' PCK, especially on what affordances they perceived the examples could offer. Even when teachers have to come up with an example on the spot, their ability to do so is greatly influenced by their PCK (Chick & Pierce, 2008). Similarly, Zodik and Zaslavsky (2008) who carried out an in-depth study with five secondary teachers concluded that content knowledge, PCK, and knowledge of students' learning, a sub-category of PCK, shape teachers' examples.

Methodology

This study surveyed the exemplification practices of secondary mathematics teachers in Singapore for which a purposeful sample of experienced teachers was used to provide richer information. Participants were chosen from teachers who had taught mathematics for at least five consecutive years and had some experience in teaching at the upper secondary level. A questionnaire was then constructed and distributed to teachers who fit the criteria.

The questionnaire was pilot-tested with 16 teachers from two schools and thereafter refined. Of the 128 questionnaire returns from 24 secondary schools, seven were invalid as three had only lower secondary (grade 7 and grade 8) teaching experience and four had taught for less than five years. The remaining 121 teachers had a mean of 12 years of teaching and 89 of them had experience in teaching Additional Mathematics: an advanced level of mathematics that is offered to more mathematically able students in upper secondary and includes topics like plane geometry proofs and introductory Calculus. Of these 121 teachers, 44 teachers taught one other subject and the rest taught mathematics only. All respondents had a first degree and a teaching qualification. 25 of the teachers had a masters degree of which 19 were masters in mathematics or mathematics education. The gender composition was almost 50:50 (57 females). 119 indicated their age group and the age distribution is shown in Table 1.

Table 1
Age Group of 119 Teacher Respondents

Age	Under 30	30-39	40-49	50-59	60+
Number of teachers	7	58	32	17	5

The purpose of the questionnaire was to explore teachers' opinions on mathematical examples, their mathematical knowledge of teaching, and their mathematical beliefs. For this paper, the focus is on the three questions that surveyed the teachers' exemplification considerations. The first question read "list down two factors you consider when selecting examples to *introduce* a new concept/procedure/rule/principle". Research has shown that teachers like to begin with a simple or familiar first example and order examples in increasing degree of difficulty (Rowland et al., 2005). Teachers also reported to be conscious of the importance to reduce the *noise* in examples so as to focus learners' attention on the critical aspects (Skemp, 1971). Hence, the objective of this question was to elicit teachers' decisions in selecting their first few examples in order to focus on those teachers who can better justify their choice of mathematical examples.

The second question asked teachers to list down two factors they considered when selecting homework tasks. Hiebert et al. (1996) proposed that teachers look for tasks that can offer situations that students will perceive as problematic and that provide platforms for students to think about important mathematics. Tasks should also connect to some part of the students' knowledge so that they are attainable by students. Hence, it is worthwhile to investigate how teachers decide on homework tasks.

Finally, teachers were asked to write down three characteristics of what they think a *good* example would have. Zaslavsky and Lavie (2005) defined a good example as one "that conveys to the target audience the essence of what it is meant to exemplify or explain" (p. 2). They described good examples as transparent, can foster generalisation, and aid in explaining and resolving mathematical subtleties. Thus, the third question was to elicit what teachers believed that a good example would entail.

Results and Discussion

The data collected for this study involved teachers' responses to the three questions. Teachers' responses for each question were categorised and 13 category codes were created to facilitate the analysis and discussion both within and between the questions. In all 13 categories, some were common. Table 2 presents the percentage category frequencies for each question, ordered in decreasing frequencies for question one.

Table 2
Categories of 121 Teachers' Exemplification Considerations

Category Code	Category Description	Teach Mathematics Idea (%)	Select Homework (%)	Good Example (%)
SA	Students' Abilities	25.5	17.4	13.1
DL	Difficulty Level	21.3	23.0	16.1
FC	Familiar Context	18.3	-	8.36
LO	Learning Objectives	8.09	8.12	5.97
EC	Exemplify Content	8.09	-	10.7
VE	Variety of Examples	6.81	19.2	10.1
CE	Clarity of Examples	5.11	-	15.8
TI	Thinking and Interesting	3.83	-	9.25
CM	Common Misconceptions	2.13	0.855	4.18
CH	Classwork and Homework	0.851	5.98	-
NE	Number of Examples	-	9.83	-
RL	Reinforce Learning	-	8.94	4.78
AU	Assess Understanding	-	6.41	1.49

RQ1. What Factors do Secondary Mathematics Teachers Consider when Choosing Examples for Introducing New Mathematical Ideas?

A total of 235 teachers' considerations, when they teach new mathematical ideas, were gathered in which the first three categories surfaced more often. From Table 2, *Student Abilities* (SA) was reported as the major concern teachers have when introducing new content (60 counts). SA consisted of responses on students' abilities, prior knowledge, and the need to scaffold students' learning. The comments included "must suit students' ability" and examples should be able to "link to prior knowledge". Some teachers, like the mentors in Bills and Bills' (2005) study, also advocated instructional scaffolding via examples like "easy ones first, then progressively more challenging ones".

The second most common category was *Difficulty Level* (DL) which pertains to whether the examples were easy or hard (50 counts). Many teachers echoed that they would take note of the difficulty level of examples. Others proposed to use an example that is "easy to understand" and this resembles the key theme in another study, which was keeping things simple (Bills & Bills, 2005). A related category was to use *Familiar Context* (FC) that students can easily relate to by linking to the "personal experience of students" or "real-world situations", of which there were 43 counts. In a way, SA, DL, and FC encompassed one of the guiding principles teachers in Zodik and Zaslavsky's (2008) study demonstrated which was to start with a simple or familiar case.

Of the participating teachers, 19 were concerned if examples used could "address the instructional objectives" and prepare students for examinations (LO). This factor was also identified by Rowland (2008) in his study. Teachers were also mindful when selecting the first few examples that could exemplify a new content (EC), so as reduce the noise (Skemp, 1971) by selecting only those that were able to "highlight the key points".

There were 16 comments on using different examples, *Variety of Examples* (VE), when presenting a new mathematical idea whereas some included examples that "show the application of the new concept". 12 wrote about the *Clarity of Examples* (CE) that examples should be clear, "should not be overly tedious to solve", and should involve "small numbers, positive integers if possible". This partially reflected the approach by teachers in another research to draw attention to relevant features (Zodik & Zaslavsky, 2008). Arousing interest and stimulating thought processes, *Thinking and Interesting* (TI) was also raised (9 counts). Fewer (5 counts) attended to the need to address *Common Misconceptions* (CM) and only two teachers selected examples that "can help them [students] to solve questions given for homework later" (*Classwork and Homework*-CH). Since the teaching of a new mathematical idea was the focus of this question, it was logical that the following categories: *Number of Examples* (NE), *Reinforce Learning* (RL), and *Assess Understanding* (AU), were not part of the teachers' considerations.

RQ2. What Factors do Secondary Mathematics Teachers Consider when Selecting Examples for Homework Tasks?

There were 234 written factors where the top three categories, DL, VE, and SA were more frequently cited. Similar to teachers' choice of the first few examples, when they plan homework, DL (54 counts) and SA (41 counts) were important too. What differs in DL was teachers were more prone to choose challenging over simple homework tasks. "Tasks should be reasonable within ability of students" so that "students can manage the homework". Hiebert et al. (1996,) considered SA as vital too as teachers should select tasks that "students can see the relevance of the ideas and skills they already possess" (p. 16).

A key approach by many (45 counts) was to expose students to varied examples (VE), as a limited range of examples might lead to an incomplete or erroneous understanding. "Direct application of concepts, challenging questions, and integrated mathematics and real-life situations" should be tasked for a "comprehensive coverage of exercise".

The next three codes, NE (23 counts), RL (21 counts), and LO (19 counts) had comparable ratings. Some teachers carefully considered the "time taken to complete homework questions" by reminding themselves to give "manageable number of questions" (NE). However, this category was absent in the teachers' exemplification considerations when they introduced new concepts or when they identified good examples.

Some teachers were concerned whether homework could "reinforce classroom teaching" (RL). The "purpose of the homework task" (LO) to cover the school's scheme of work or to "prepare students for examinations" was also raised. 15 teachers suggested that the role of the homework is "to assess students' understanding" (AU) and that "tasks should give feedback on students' learning". Slightly fewer (14 counts) shared that their homework selection was based on the classwork and that for the homework they "will give questions similar to the work done in class". Only two stated that they would include "questions that can surface common mistakes or misconceptions".

It was noticeable that the teachers did not consider FC, EC, CE, and TI when they set homework tasks. Since homework served mainly for students to develop their skills, teachers reported that they tended to expose students to different types of problems rather than focus on context familiar to them (FC). The same can be said for EC and CE, which were more relevant to mathematical understanding. What was more conspicuous was the absence of thinking and interesting aspect in homework tasks, as this is fundamental in Singapore mathematics framework (Ministry of Education, 2012).

RQ3. What are the Characteristics of a "Good" Example used for Teaching Mathematics in the Eyes of Secondary Teachers?

The respondents gave 335 written descriptions of their concept of good examples. Likewise, when teachers look for critical attributes in examples, DL (54 counts) and SA (44 counts) were pivotal. Interestingly, over 75% were more likely to pick an "easy to understand" example over one that "can stretch their thinking". A good example should also be "pitched at the right level for the class" and be able to "link with prior knowledge". Unlike the previous two questions, there were five teachers who favoured the use of "illustrations and diagrams" to "assist in the conceptualisation", which Rowland et al. (2005) found to be tied to teachers' exemplification practices.

A substantial number of teachers (53 counts) felt that good examples are "clear" (CE) and "well-crafted", where they "test students on the concept but not on the English". "Ease in calculation" and having "no complicated equations" reflected the *keep unnecessary work to a minimum strategy,* discussed earlier in Zodik and Zaslavsky (2008).

Teachers (36 counts) also characterised those that "highlight the salient points" (EC) and enable one "to generalise ideas or rules" as good examples. Hence, good examples are transparent and promote generalisation (Zaslavsky & Lavie, 2005). Others (34 counts) see examples as a set of "varied examples" (VE) to provide "sufficient coverage", to "link concepts together", and to allow the "application of concepts across topics".

Another desirable attribute of an example is if it is "able to provoke thinking" and "arouse students' interest" (TI). Of this type, 31 counts were identified and we can draw a parallel between TI and what Hiebert et al. (1996) meant by tasks that problematised the subject, so that they will "pique the interests of students and engage them in mathematics" (p. 18). Following next, is teachers' preference (28 counts) for examples "related to everyday experiences of students" (FC) or "has real-life application".

Twenty teachers indicated that a good example "delivers the lesson objectives" (LO) and some felt that it should be "similar to the examination syllabus type of questions". Fewer comments (16 counts) highlighted examples that "reinforce concepts or skills taught in class" (RL). 14 felt that good examples offer "opportunities to sieve out misconceptions in students" (CM) so as to attend to students' errors (Zodik & Zaslavsky, 2008). There were only five comments on choosing examples that can "assess students' understanding" (AU).

Connections to Teacher Knowledge

The three questions discussed in this paper were not based on any specific mathematical content. However, another section of the questionnaire examined teachers' mathematical knowledge. The data suggested that there were obvious connections between teachers' PCK and their use of examples. When teachers present new content, KCS is exhibited in how they considered students' prior knowledge (SA) and the difficulty level (DL) of the topic. As such the teachers try to choose examples that students can relate to (FC) and find interesting (TI) to make learning more manageable and meaningful for the students. Furthermore, knowledge of students' conceptions and misconceptions (CM) means that teachers prefer examples "that should not be clouded by other concepts or difficult algebra manipulation" (CE) so as not to confuse their students (Ball et al., 2008). Each of the above-mentioned categories requires teachers' knowledge of how students learn the mathematical content or KCS in short.

Teachers' example choice is influenced by their KCT too. They select examples that are able to exemplify the mathematical idea (EC) and also provide students with sufficient contact with the mathematical content through varied examples (VE). Teachers' KCT guide them in the sequencing of homework tasks in "ascending difficulty" (SA) in order to scaffold students' learning. In addition, teachers tend to pick those tasks that are able to reinforce what has been taught (RL) or by relating homework tasks to what have been covered in class (CM), in order to help students retain knowledge and gain fluency in their mathematical competency (Rowland, 2008). Furthermore, challenging tasks (DL) are also utilised to bring students deeper into the topic.

Finally, teachers' knowledge of the curriculum (KCC) sensitises them to those examples that are able to address and deliver learning objectives stipulated in the mathematics syllabus, as well as prepare students for assessment (LO) by making available to them examples that are similar to those tested in examinations. At the same time, teachers leverage on examples that "provide good feedback about students' understanding" (AU) in order to improve students' learning.

Conclusion

Teachers will continue to use examples in teaching their students, for whom examples may be a primary means for learning mathematical concepts. The use of certain examples for teaching a particular topic may not be universal, which implies that the survey of the teachers from Singapore who participated in this study may be very context-specific. It is important to be aware of the limitations in using questionnaire findings to study teachers' pedagogical practices since what is written may not be used in actual lessons. Nevertheless, this study brings us some insights into the exemplification perceptions of experienced mathematics teachers in Singapore. Teachers are most concerned over students' abilities and the difficulty level of examples when choosing examples. However, when selecting examples for different purposes, the considerations differ to some extent. For instance, when introducing new content, teachers favoured examples that connect with students' experiences whereas for homework, they are more concerned with providing students with varied exposure.

Finally this research reveals the potential direction for further research into the reasons teachers considered as critical factors in their choice of examples and points to a connection between teacher knowledge and beliefs about what constitutes effective teaching and learning of mathematics through the use of mathematical examples.

References

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389–407.

Bills, C., & Bills, L. (2005). Experienced and novice teachers' choice of examples. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce, & A. Roche (Eds.), Building connections: research, theory, and practice (Proceedings of the 28th Annual Conference of the Mathematics Education Research Group of Australasia, Vol. 1, pp. 146–153). Melbourne, Australia: MERGA.

Chick, H. L. (2007). Teaching and learning by example. In J. Watson & K. Beswick (Eds.), Mathematics: Essential research, essential practice. (Proceedings of the 30th Annual Conference of the Mathematics Education Research Group of Australasia, pp. 3–21). Sydney, Australia: MERGA.

Chick, H. L. (2010). Aspects of teachers' knowledge for helping students learn about ratio. In L. Sparrow, B. Kissane, & C. Hurst (Eds.), *Shaping the future of mathematics education*. (Proceedings of the 33rd Annual Conference of the Mathematics Education Research Group of Australasia, pp. 145–152). Fremantle, Australia: MERGA.

Chick, H. L., & Pierce, R. (2008). Issues associated with using examples in teaching statistics. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano, & A. Sepúlveda (Eds.), *Proceedings of the Joint Meeting of PME 32 and PME-NA XXX* (Vol. 2, pp. 49–55). Mexico: Cinvestav–UMSNH.

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K., Human, P., Murray, H., Olivier, A., & Wearne, D. (1996). Problem solving as a basis for reform in curriculum and instruction: The case of mathematics. *Educational Researcher*, *25*(4), 12–21.

Huntley, R. (2013). Pre-service primary teachers' choice of mathematical examples: Formative analysis of lesson plan data. In V. Steinle, L. Ball, & C. Bardini (Eds.), *Mathematics education: Yesterday, today and tomorrow*. (Proceedings of the 36th annual conference of the Mathematics Education Research Group of Australasia, pp. 394–401). Melbourne, Australia: MERGA.

Ministry of Education (2012). *O– & N(A)–Level mathematics teaching and learning syllabus*. Singapore: Curriculum Planning and Development Division.

Rowland, T. (2008). The purpose, design and use of examples in the teaching of elementary mathematics. *Educational Studies in Mathematics*, *69*(2), 149–163.

Rowland, T., Huckstep, P., & Thwaites, A. (2005). Elementary teachers' mathematics subject knowledge: The knowledge quartet and the case of Naomi. *Journal of Mathematics Teacher Education*, *8*(3), 255-281.

Shulman, L. S. (1987). Knowledge and teaching: foundations of the new reform. *Harvard Educational Review*, *57*(1), 1–22.

Skemp, R. R. (1971). *The psychology of learning mathematics.* Harmondsworth, UK: Penguin.

Watson, A., & Mason, J. (2002). Extending example spaces as a learning/teaching strategy in mathematics. In A. Cockburn & E. Nardi (Eds.), *Proceedings of the 26th Conference of the international Group for the Psychology of Mathematics Education* (Vol. 4, pp. 378–385). Norwich: PME.

Watson, A., & Mason, J. (2006). Seeing an exercise as a single mathematical object: Using variation to structure sense-making. *Mathematical Thinking and Learning, 8*(2), 91–111.

Zaslavsky, O., & Lavie, O. (2005). *Teachers' use of instructional examples*. Paper presented at the 15th ICMI Study Conference, Águas de Lindóia, Brazil.

Zodik, I., & Zaslavsky, O. (2008). Characteristics of teachers' choice of examples in and for the mathematics classroom. *Educational Studies in Mathematics, 69*(2)*,* 165–182.