Elementary Students' Generalization and Representation of Functional Relationships: A Learning Progressions Approach

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Abstract

In this paper, we describe our learning progressions approach to early algebra research that involves the coordination of a curricular framework, an instructional sequence, written assessments, and levels of sophistication describing the development of students' thinking. We focus in particular on what we have learning through this approach about the development of students' abilities to generalize and represent functional relationships in a grades 3-5 early algebra intervention by sharing the different levels of responses we observed in students' written work and the percent of students situated at each level across different tasks.

Keywords: functions, functional thinking, learning progression, early algebra, representations.

Purpose

The failure of school mathematics' traditional arithmetic-then-algebra approach to adequately prepare students for formal algebra has prompted calls for early algebra (Kaput, 1998; Kaput, Carraher, & Blanton, 2008). Early algebra is an approach whereby elementary students are provided the time and space necessary to develop an understanding of important algebraic concepts—such as generalized arithmetic, variable, and function—through their engagement in rich, age-appropriate tasks.

Our purpose in this paper is to share results from a three-year longitudinal study designed to measure the impact of an early algebra intervention on students' algebra understanding and readiness for middle grades. We focus in particular on the development of students' abilities to generalize and represent functional relationships and share the levels of sophistication we observed in students' thinking over time.

Theoretical Framework

This study is situated in the context of an *Early Algebra Learning Progression* [EALP] that integrates curriculum, instruction, assessment, and analyses of student learning. With a focus on generalizing and representing functional relationships, we describe the first three parts of our learning progression to frame the fourth, which will be elaborated on in the results section. For a more complete description of the learning progressions approach we employ in this project, see Fonger, Stephens, Blanton, and Knuth (2015).

Curricular framework

We define our *curricular framework* to include the big ideas, algebraic thinking practices, core concepts, and learning goals on which our intervention and assessments items were based. Big ideas are “key ideas that underlie numerous concepts and procedures across topics” (Baroody, Cibulskis, Lai, & Li, 2004, p. 24). Drawing from existing domains around which much of early algebra research has matured, the big ideas in our progression are (a) equivalence, expressions, equations, and inequalities, (b) generalized arithmetic, (c) functional thinking, (d) variable, and (e) proportional reasoning. The algebraic thinking practices cutting across these big ideas are based on Kaput’s (2008) early algebra framework and include generalizing, representing, justifying, and reasoning with mathematical relationships. Core concepts are underlying ideas that are critical to understanding a big idea. Under the big idea of functional thinking, for example, a core concept is *Recursive patterns describe variation in a single sequence of values. A recursive pattern indicates how to obtain a number in a sequence given the previous number or numbers*. Finally, a learning goal (Clements & Sarama, 2014) makes a statement about the nature of understanding or skills expected of students around a given concept. A learning goal under the big idea of functional thinking, for example, is to *Understand how to identify and describe correspondence relationships using words or variables*. The identification of learning goals was heavily informed by existing empirical research suggesting what students are capable of understanding at particular grade levels (e.g., Blanton, Brizuela, Gardiner, Sawrey, & Newman-Owens, 2015; Carraher, Martinez, & Schliemann, 2008; Lannin, Barker, & Townsend, 2006; Martinez & Brizuela, 2006; Warren, Cooper, & Lamb, 2006).

Instructional sequence

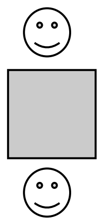
The curricular framework provided a starting point for the development of an instructional sequence for grades 3-5. The instructional sequence includes 17-18 lessons at each grade level built around the curricular framework's learning goals. These lessons include "Jumpstarts," or short tasks that review previously-discussed topics, and more extensive problem-solving tasks that provide the focus for student activity in the lessons. These tasks were often adapted from those used in previous research and generally allow for multiple points of entry so that students at varying levels of sophistication can demonstrate competence. The lesson plans additionally provide teacher supports, including anticipated student responses, potential student difficulties and misconceptions, and suggestions for questions to promote students' algebraic thinking.

Assessment items

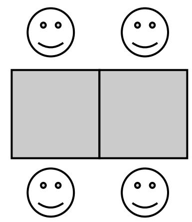
Written assessments for each of grades 3-5 were designed to align with the curricular framework and learning goals targeted by the instructional sequence. Items were piloted and revised if necessary prior to administration. Several items appeared at multiple grade levels to allow for the tracking of growth over time. Assessment items included a focus on the range of big ideas and algebraic thinking practices included in the curricular framework and, like the tasks used in the instructional sequence, were often adapted from those that had performed well in previous research and generally offered multiple points of entry. See Figures 1 and 2 for the assessment tasks related to the big idea of functional thinking that will be discussed in the results.

Brady is celebrating his birthday at school. He wants to make sure he has a seat for everyone. He has square desks.

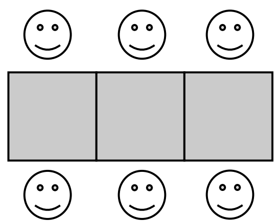
He can seat 2 people at one desk in the following way:



If he joins another desk to the first one, he can seat 4 people:



If he joins another desk to the second one, he can seat 6 people:



a) Fill in the table below to show how many people Brady can seat at different numbers of desks.

Number of desks	**Number of people**
1	2
2	4
3	
4	
5	
6	
7	

b) Do you see any patterns in the table from part a? If so, describe them.

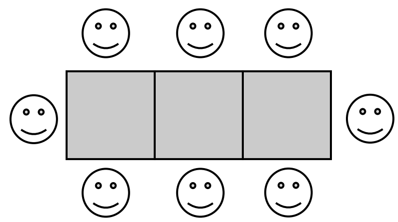
c) Think about the relationship between the number of desks and the number of people.

Use words to write the rule that describes this relationship.

Use variables (letters) to write the rule that describes this relationship.

d) If Brady has 100 desks, how many people can he seat? Show how you got your answer.

e) Brady figured out he could seat more people if two people sat on the ends of the row of desks. For example, if Brady had 3 desks, he could seat 8 people.



How does this new information affect the rule you wrote in part c?

Use words to write your new rule:

Use variables (letters) to write your new rule:

Figure 1: The *Brady* assessment task.

The following pattern is growing so that each picture is made up of more and more stars.

Picture 1 Picture 2 Picture 3

The following table shows the picture number and the number of stars in that picture:

Picture	Number of stars
1	1
2	4
3	9
4	16
5	25
6	36

a) Think about the relationship between the picture number and the number of stars in that picture.

Use words to write the rule that describes this relationship.

Use variables (letters) to write the rule that describes this relationship.

b) Use your rule to predict how many stars will be in the 100^{th} picture. Show how you got your answer.

Figure 2: The *Growing Stars* assessment task.

Levels of sophistication

In the context of the curricular framework, instructional sequence, and assessment items, levels of sophistication describing students' understanding over time were posited based on findings from existing research and formed the starting point for the coding schemes (described below) used to evaluate student responses to the assessment items. The coding scheme used to describe students' generalization and representation of functional relationships, for example, was based in part on Blanton et al.'s (2015) learning trajectory describing the development of first-grade students' functional thinking (see Table 1).

Table 1: Blanton et al.'s (2015) levels in first-grade students' understandings of functional relationships.

Levels	Characteristics
Pre-Structural	Does not describe or use (even implicitly) any mathematical relationship in talking about problem data. May notice a non-mathematical regularity in the inscriptions.
Recursive-Particular	Conceptualizes a recursive pattern as a sequence of particular instances.
Recursive-General	Conceptualizes a recursive pattern as a generalized rule between arbitrary successive values.
Functional-Particular	Conceptualizes a functional relationship as a sequence of particular relationships between specific corresponding values "quasi-generalization" (Cooper & Warren, 2011).
Primitive Functional-General	Conceptualizes a functional relationship as a general relationship between two quantities, but cannot describe a mathematical transformation on two arbitrary quantities.
Emergent Functional-General	Conceptualization of functional relationship reflects emergence of key attributes (e.g., characterizing the generalized quantities or mathematical transformation).
Condensed Functional-General	Conceptualizes a functional relationship as a generalized relationship between two arbitrary and explicitly-noted quantities.
Function as object	Perceives boundaries in the generality of the functional relationship; conceptualizes functional relationship as object on which operations could be performed.

In the results section, we present our findings—in the form of levels of sophistication—describing students' abilities to generalize and represent functional relationships over time. Consistent with a learning progressions approach, we emphasize that these levels must be considered in the context of the curricular framework, the instructional sequence, and the assessment items, and that the four components together make up the EALP.

Method and Data Sources

Participants and Intervention

Participants in our early algebra intervention were 104 students from one school in the Northeastern United States. These students were taught 17-18 lessons (as described in *instructional sequence* above) in each of grades 3, 4, and 5 that focused on the big ideas and algebraic thinking practices identified in the curricular framework. A member of our research team—a former third-grade teacher—taught the lessons to all students in all three years of the study.

Data collection

Students completed a one-hour written assessment (described in *assessment items* above) as a pre-test at the beginning of grade 3 (prior to the instructional intervention), then again at the end of grades 3, 4, and 5.

Data analysis

Responses to the assessment items were coded for correctness as well as for strategy use. The development of the strategy codes began with the identification of strategies from existing research on students' algebraic thinking and continued with the identification of patterns of responses in the data collected. The strategy codes ultimately became the "levels of sophistication" that will be shared in the results below.

Results and Discussion

We now present the levels of sophistication observed in students' written work on the *Brady* task (see Figure 1) administered at the beginning of grade 3, the end of grade 3, and the end of grade 4 and the *Growing Stars* task (see Figure 2) administered at the end of grade 4. We focus only on the parts of these items that involve generalizing and representing generalizations. The final paper and presentation will include results from the end of grade 5 as well. See Table 2 for the levels of sophistication we use to describe students' abilities to generalize and represent generalizations and Tables 3, 4, and 5 for the percent of student work falling into the various levels in response to specific assessment prompts.

As illustrated in Table 2, we identified a range of responses students provided when asked to identify or represent a generalized relationship. The ordering of the levels was informed by existing research (e.g., Blanton et al., 2015) and our observations of student work in previous studies. Like Blanton and colleagues, we considered what types of thinking might be viewed mathematically as more sophisticated and did not use students' thinking alone as the means for ordering the progression. Note one difference between our levels and Blanton et al.'s is the separation of the *Emergent Functional* and *Condensed Functional* levels into representations using words and representations using variables. This separation allows us to examine which of these representations emerge first for students across different types of tasks. We order the levels as we do because we unexpectedly found that students were generally more successful representing generalizations using variables than using words. This ordering (i.e., L6 before L7 and L8 before L9) is supported by the data displayed in Tables 3, 4, and 5.

Table 2: Levels of sophistication describing students' generalizing and representing of functional relationships.

Levels of sophistication	Description of Levels
No response	Student does not provide a response.
L0: Restatement	Student restates the given information.
L1: Recursive pattern-particular	Student identifies a recursive pattern in either variable by referring to particular numbers only.
L2: Recursive pattern-general	Student identifies a correct recursive pattern in either variable.
L3: Covariational relationship	Student identifies a correct covariational relationship. The two variables are coordinated rather than mentioned separately.
L4: Functional-particular	Student identifies a functional relationship using particular numbers but does not make a general statement relating the variables.
L5: Functional-basic	Student identifies a general relationship between the two variables but does not identify the transformation between them.
L6: Functional-emergent in variables	Student identifies an incomplete function rule using variables, often describing a transformation on one variable but not explicitly relating it to the other. Student might set the expression equal to a specific number of to the same variable rather than a new variable.
L7: Functional-emergent in words	Student identifies an incomplete function rule in words, often describing a transformation on one variable but not explicitly relating it to the other or not clearly identifying one of the variables.
L8: Functional-condensed in variables	Student identifies a function rule using variables in an equation that describes a generalized relationship between the two variables, including the transformation of one that would produce the second.
L9: Functional-condensed in words	Student identifies a function rule in words that describes a generalized relationship between the two variables, including the transformation of one that would produce the second.

Table 3 displays the percent of students whose written work fell into each of the levels in response to part b of the *Brady* task.[1] Note that students are simply asked to describe any relationships they notice, so a *Recursive pattern-general* response (e.g., "the number of people is going up by 2s") is acceptable. We noticed over time, however, that some students began to choose to describe their observations using function rules.

Table 3: Percent of student responses falling into each level on part b of *Brady* task

Level of sophistication	*Brady* task part b: Describing patterns in a table		
	Grade 3 pre	Grade 3 post	Grade 4
No response	**35**	0	1
L0: Restatement	1	0	0
L1: Recursive pattern-particular	2	1	0
L2: Recursive pattern-general	**43**	**70**	**38**
L3: Covariational relationship	2	6	6
L4: Functional-particular	1	7	1
L5: Functional-basic	4	7	**20**
L6: Functional-emergent in variables	0	0	0
L7: Functional-emergent in words	1	1	4
L8: Functional-condensed in variables	0	1	4
L9: Functional-condensed in words	0	4	**20**
(Other responses)	8	4	5

Table 4 shows a progression in students' thinking on tasks asking them to describe a functional relationship in words. The downward sloping arrow indicates a trend over time from no response, to L2 and L5, to L9. Table 5 shows a progression in students' thinking on tasks asking them to describe a functional relationship using variables. The downward sloping arrow indicates a trend over time from no response to success stating a condensed function rule in variables (L8).

[1] Percentages $\geq$ 10% are bolded for emphasis in Tables 3, 4, and 5.

Table 4: Percent of student responses falling into each level on parts c1and e1 of *Brady* task and part a1 of *Growing Stars* task.

Level of sophistication	*Brady's party* part c1: Writing a function rule in words			*Brady's party* part e1: Writing a new function rule in words	*Growing Stars* part a1: Writing a function rule in words (quadratic)
	Grade 3 pre	Grade 3 post	Grade 4	Grade 4	Grade 4
No response	**77**	3	5	3	3
L0: Restatement	7	5	2	0	0
L1: Recursive pattern-particular	0	2	0	1	0
L2: Recursive pattern-general	3	**21**	7	4	1
L3: Covariational relationship	0	6	5	2	0
L4: Functional-particular	1	8	4	1	1
L5: Functional-basic	3	**25**	**22**	**12**	**36**
L6: Functional-emergent in variables	0	0	0	0	0
L7: Functional-emergent in words	0	1	3	1	7
L8: Functional-condensed in variables	0	0	1	1	3
L9: Functional-condensed in words	0	**19**	**41**	**19**	**32**
(Other responses)	10	12	8	56	17

Table 5: Percent of student responses falling into each level on parts c2 and e2 of *Brady* task and part a2 of *Growing Stars* task.

Level of sophistication	*Brady's party* part c2: Writing a function rule in variables			*Brady's party* part e2: Writing a new function rule in variables	*Growing Stars* part a2: Writing a function rule in variables (quadratic)
	Grade 3 pre	Grade 3 post	Grade 4	Grade 4	Grade 4
No response	**90**	**10**	1	2	3
L0: Restatement	0	0	1	0	0
L1: Recursive pattern-particular	0	0	0	0	0
L2: Recursive pattern-general	0	6	0	0	0
L3: Covariational relationship	0	0	0	0	0
L4: Functional-particular	1	3	1	1	0
L5: Functional-basic	1	0	0	0	1
L6: Functional-emergent in variables	0	5	6	5	6
L7: Functional-emergent in words	0	0	0	0	0
L8: Functional-condensed in variables	0	**37**	**64**	**39**	**67**
L9: Functional-condensed in words	0	0	1	0	1
(Other responses)	8	39	25	53	21

Note that not all parts of these items were given across grade levels. These tables display an initial inability to engage with the tasks and a rather dramatic shift towards being able to describe a functional relationship in words or variables. Table 4 illustrates that a significant percent of students spent some time at the *Recursive-General* and *Functional-Basic* levels as well.

Looking across Tables 3, 4, and 5 we see that, prior to instruction, students had great difficulty engaging with the tasks discussed in this paper. This is particularly true of the tasks included in Tables 4 and 5. We also see that, with just 18 third-grade early algebra lessons (only seven of which focused on functional thinking), many students could respond with some level of competence. Across the two years, we see the elimination of *No response* and an increase in students' abilities to identify general recursive rules and express correspondence rules in both words and variables.

As mentioned above, we were initially surprised to see that more students were able to reach the *Functional-condensed in variables* than the *Functional-condensed in words* level. This was true across both linear (the *Brady* task) and quadratic (the *Growing Stars* task) items. Note that the sum of the percent of responses at the *Functional-basic* and *Functional-condensed in words* levels approximately equal that at the *Functional-condensed in variables* level for each item at the end of grade 4. It thus appears that while students may understand the general underlying relationship, they tend towards being able to fully describe it in symbols first.

We also note that in the general shift from *No response* to *Recursive pattern-general* or the *Functional-condensed* responses, it appears that levels are often skipped. This is consistent with others' work on learning progressions and trajectories (e.g., Clements & Sarama, 2014), where it is often noted that students may skip levels or operate at different levels depending on the context. Given the course grain size of our work, on the other hand, it may be the case that levels were not skipped but rather simply not observed.

Finally, we acknowledge that a very large "Other" category exists for some of these tasks; in particular, parts e1 and e2 of the *Brady* task. We are beginning to review these responses to determine if they might fall into other coherent categories. Our early findings suggest that while these are incorrect responses, they often demonstrate some understanding related to functional thinking.

Significance

The work presented is part of a comprehensive effort to coordinate curriculum, instruction, assessment, and analyses of student learning. We believe it is important work to share with both the research community and, ultimately, with teachers as "knowledge of developmental progressions enables high quality teaching based on understanding both mathematics and students' thinking and learning" (Clements & Sarama, 2014, p. 13).

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