Variation in Children's Understandings of Fractions: Preliminary Findings

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This paper was prepared for the *National Council of Teachers of Mathematics Research Conference*, Boston, MA, April 13-15, 2015.

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Acknowledgments

The authors acknowledge the contributions of Ryan Ziols in conducting and reporting this research. Support for this research was provided by the U.S. Dept. of Education-IES Research Training Programs in the Education Sciences under grant no. R305B130007, and as part of the Wisconsin Center on Education Research Postdoctoral Training Program in Mathematical Thinking, Learning, and Instruction at the University of Wisconsin-Madison, and the Hewlett Foundation under the Friday Institute MOOC-Ed grant.

Abstract

This research targets children's informal strategies and knowledge of fractions by examining their ability to create, interpret, and connect representations in doing and communicating mathematics when solving fractions tasks. Our research group followed a constant comparative method to analyze clinical interviews of children in grades 2-6 solving fraction tasks. Several iterations of coding yielded an emergent coding scheme that helps capture the nuances of children's reasoning with multiple types of fractions representations. Initial results from the interview analyses suggest variation in children's reasoning across four categories: (a) model types, (b) children use of representations and representational fluency, (c) connectedness of children's informal to formal reasoning type, and (d) meanings of fractions. We discuss the challenges of negotiating first-order versus second-order models of children's meaning of fractions, and the relationship between children's representational fluency and conceptions of fractions.

Keywords: fractions, children, representational fluency, meanings, clinical interviews

Fractions, ratios, and proportions play an important role in school curriculum that spans from elementary to university-level mathematics, and arguably they are the most challenging to learn and most difficult topic to teach (e.g., Lamon, 2007). Fractions are difficult because they have multiple meanings: part-whole, quotient, measure, ratio, and operator (Kieren, 1980). Research shows that students persistently hold limited conceptions of fractions, especially in the United States (Siegler, et al., 2010), which poses a barrier in their learning of later topics of rational numbers and algebraic reasoning. Building a solid understanding of fractions at the elementary level lays a foundation for other advanced mathematics.

Research has documented the importance of building from children's informal knowledge and experiences to develop a robust fractions sense before being introduced to operation rules (Empson & Levi, 2011; Lamon, 2007; Mack, 1990). Investigating students' representational fluency—the ability to create, interpret, and connect representations in doing and communicating mathematics—offers an important entry into student thinking and understanding of mathematical ideas. Research on fractions suggests that translating between multiple representation types and models helps make concepts more meaningful to learners (e.g., Behr, Lesh, Post, Silver, 1983), and in turn provides potential for researchers and teachers to build a second-order model of children's understanding (Steffe & Olive, 2010). Yet little is known about how children reason within and between fraction representation systems and what informal and prior knowledge children draw on when learning fractions. Drawing from Behr, Lesh, Post, & Silver's (1983) representation systems (i.e., pictures, manipulatives, spoken and written symbols, and "real-world" situations), we investigate the problem of how children drawn on their prior/informal knowledge when solving fractions tasks by looking at their representational fluency and meanings of fractions in solving such fraction tasks.

Purpose and Research Questions

The purpose of this study is to examine children's informal strategies and knowledge of fractions by looking at their creations, interpretations, and connections within and between multiple fractions representations. In particular, this study addresses an unanswered question about children's understanding of fractions as established by Lamon (2007): *What informal/intuitive strategies and prior experiences are drawn on as children reason about tasks for fractions concepts and operations with fractions?*

To address this question we conducted and analyzed clinical interviews with children on their solving of fraction tasks. We adopt a conceptual lens that privileges children's observable representational activities across various models of fractions and researcher-conjectured second order models of children's meanings of fractions (Steffe & Olive, 2010).

Conceptual Perspective

Three mutually supportive conceptual lenses were adopted to guide this research: (a) model types for fractions, (b) an interactive model for using representation types (Figure 1), and (c) multiple meanings of fractions. Each lens is discussed in turn.

First, we consider three model types for fractions as a single whole, fractions as multiple wholes, and fractions as a discrete collection (e.g., see Wilson et al. [2012] who distinguish between "collection" and "whole"). Second, the interactive model as proposed by Behr et al. (1983), distinguishes between five inter-related types of representations: written symbols (e.g., ⅝), spoken symbols (e.g., five-eighths), graphs/diagrams/pictures (e.g., •••••ooo), manipulatives (e.g., three-eighths cut out of a circle of paper with a five-eighths wedge remaining), and real-world situations (sharing one cake fairly among eight people, how much is five people's shares?). The representation types informed the design of tasks for clinical interviews and in our analysis of clinical interviews with children.

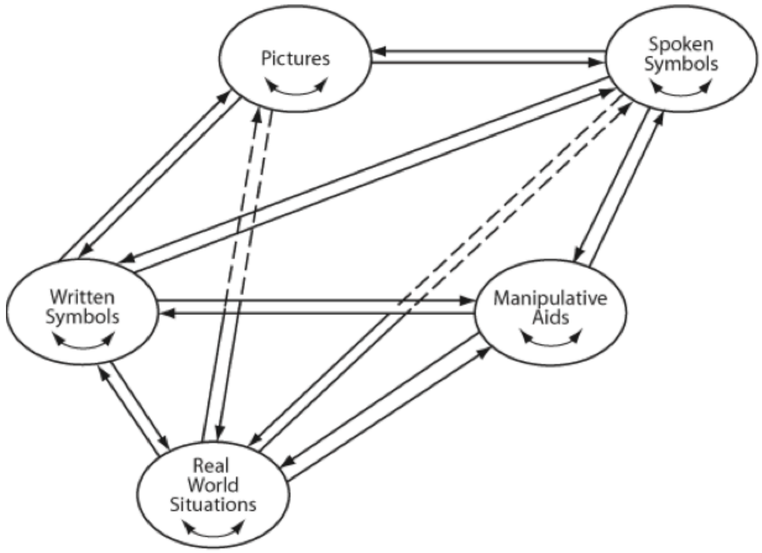

Figure 1. Representation types (Behr et al., 1983).

A third conceptual lens on children's understandings of fractions is described as multiple meanings of fractions (Behr et al., 1983; Charalambous & Pitta-Pantazi, 2007; Kieren, 1980). These five personalities of fractions include fraction as: part-whole (e.g., 3 out of 4), measure (e.g., ¾ on a number line), ratio (e.g., 3: 4), operator (e.g., 3/4 of something), and quotient (e.g., 3 divided by 4). As elaborated next in the Methodology section we used these three mutually supportive lenses on model types, representation types, and meanings of fractions to inform the design and selection of tasks and the analyses of children's activity and cognition.

Methodology

Clinical Interviews

Children across grades 2-6 at various schools within the southeast region of the U.S. volunteered to participate in an interview study. We followed a clinical interview method (Opper, 1977; Piaget, 1976) to discover the child's cognitive processes in the face of improvisations, to examine children's intuitive or informal knowledge, and to observe their construction of knowledge (Lamon, 2007). Probing questions were specific to each interview, following the path of the child, using evidence of their thinking to test and confirm or refute conjectures about their understandings based on initial hypotheses grounded in research literature and existing empirical evidence of children's thinking.

Tasks. Interview protocols were designed as a sequence of tasks that aimed to address various meanings of fractions (Table 1). Each protocol typically addressed more than one task type, selected based on best guesses of the prior experience and understandings of each child. In general, we sought to elicit a diversity of strategies in accessible contexts that shed light on children's understanding of fractions. We also sought to elicit children's informal strategies in accessible contexts thus often started from a fair-sharing real-world context (Empson & Levi, 2011). Congruous with the Common Core State Standards for Mathematics (National Governors Association Center for Best Practices, & Council of Chief State School Offices, 2010), fractions as measure was emphasized as a meaning of fraction.

Table 1
Task types for interview protocols

Meaning/Context	Type of Task
Fair-sharing	• Share things fairly • Name and justify the share
Measure	• Specify fractions on a number line • Given a fractional length, identify other fractions
Comparing and Operations	• Comparing fractions, equivalency • Addition/subtraction, estimation • Multiplication/division, estimation • Specify operations

Participants and data collection. A total of 24 40-60 minute interviews were conducted near the end of the 2013-2014 school-year with children in grades 1(1 child), 2(5), 3(5), 4(6), 5(7), and 6(2). Each interview was conducted by a single researcher trained in the method of clinical interviews; a second researcher assisted with equipment, recorded field notes and timestamps to inform coding, and sometimes gave additional prompts to elicit a child's understanding. Data include video and audio-recorded interviews, artifacts, and researcher field notes.

Data Analysis

In this paper we focus on analyses of two major phases: a task analysis, and initial interview analyses. Additional analyses for the larger corpus of data are in progress.

Task analysis. In an initial phase of analysis, all written tasks from interview protocols were compiled. A sample of 26 tasks were analyzed by the authors following a constant comparative method (Glaser, 1965) to develop a coding scheme for the types of understandings that were intended to be elicited in the task statements. Our conceptual lenses on model types, representation types, and meanings of fraction formed the basis of our analyses. We consider the process and results of this analysis to be first-order models of the intended mathematics based on the intent of the researcher (cf. Steffe & Olive, 2010).

Interview analysis. The analyses of clinical interview data ensued in several iterations. First, several rounds of open coding were conducted together in which three researchers (the authors of this report) viewed and memoed select clinical interviews together during research meetings. During these meetings we identified interesting segments and instances that seemed to contribute to our understanding of informal strategies children drew on while solving fraction tasks. We also summarized each clinical interview we coded by noting key reasoning patterns.

These memos helped establish the foundation of a coding scheme and general categories, three of which are relevant to this report: (a) representation types, (b) meanings of fractions, and (c) intuitive understandings/prior experience. The fourth category was "action," which was later decided to be too fine-grained to capture our overall research goal, yet informed our thinking when categorizing typologies of children's activity in various model types.

Once we had a relatively stable sense of coding categories, our second round of coding involved four researchers (the authors of this report, and another researcher) who each coded 1-3 different clinical interviews, capturing a diversity of student reasoning. Two major outcomes of this round of coding were: (a) the establishment of a common unit of analysis, and (b) elaborated coding categories. We denote a new *unit of analysis* by identifying a shift in the clinical interview such that a researcher uses a question that prompts beyond explanation or clarification such that task doesn't serve the same purpose (as prior activity) (i.e., it may potentially invoke a conceptual shift, cf. Steen, [1996]). Code categories were revised and updated to capture new insights gained from this round of coding, and two categories were added to the coding scheme: (a) model types, and (b) reasoning types (i.e., capturing several commonly observed explanations or strategies).

In the third and fourth rounds of coding, each researcher coded two clinical interviews, with two researchers overlapping on three of four children's interviews. Our focus in this round of coding was to identify formal and informal reasoning and how that relates to meaning(s) of fractions, representation types, models types, and types of objects in the task situation (e.g., cakes, candies). Each researcher also wrote a summary of the coded clinical interviews to inform discussions as a group. These discussions informed further refinement to the coding scheme, clarification of how to apply the unit of analysis in coding, and an updated coding template.

Reflection on this final round of coding in light of our research goal and conceptual frames informed the version of the coding scheme presented next as preliminary results with data exemplars. Future analyses are planned including validating the task codes, and a strategy analysis (*cf*. Empson et al., 1999).

Preliminary Results: Children's Informal Reasoning with Fraction Representations

This section reports progress toward addressing the research question: *What informal/intuitive strategies and prior experiences are drawn on as children reason about tasks for fractions concepts and operations with fractions?* The four major categories of our emergent coding scheme are: (a) model types, (b) children use of representations and representational fluency, (c) connectedness of children's informal to formal reasoning types, and (d) meanings of fractions.

Model Types

We classify children's use of model types according to their observed activity and discourse around the task(s) they worked on. From extant literature and supported by our emergent coding scheme, we identify three model types: single whole, multiple whole, and discrete collection. See Table 2 for descriptions and examples from clinical interviews.

Table 2
Model type evident in children's reasoning about fraction tasks

Model Type	Description	Example
Single Whole	One thing that can be partitioned or otherwise in some way divided	Oscar (grade 2) could partition a circle into thirds more accurately than a rectangle or linear strip of paper
Multiple Whole	From the child's perspective, something that can only be solved with fractional pieces	Oscar (grade 2) shared five tortillas with two people to get 2 ½ tortillas per person

Discrete Set	A collection of objects that cannot be cut or broken (e.g., gems, shells, balloons)	Asked to share 17 cookies fairly among 3 people, Ariel (grade 3) said it is not doable (cookies cannot be broken)

An emergent trend we observed in our data is that the model type seems to influence the strategy that children use. For example, Oscar's partitioning strategy was successful a manipulative of a circle but not with a rectangle (row 1 Table 2). In some situations, the model as intended was not the same as the model children used to make sense of the problem (for example the case of Ariel, row 3 of Table 2). Thus we argue that by coding children's treatment of model types we gain insight into their informal reasoning strategies.

Children's Use of Representations and Representational Fluency

Children's use of representation types. Recall that following the model proposed by Behr et al. (1983), there are five types of representations or representation systems: (a) spoken symbols, (b) written symbols, (c) manipulative aids, (d) pictures, and (e) real-world symbols. From a lens on constructing a second-order model of children's mathematics (ala. Steffe & Olive, 2010), a more nuanced view of fraction tasks is possible as evidenced in our interview data. Consider the representation types and examples provided in Table 3.

Table 3.
Representation use evident in children's activity in solving fraction tasks

Representation Use	Description	Example
Spoken Symbols	Audibly names a fraction that specifically deals with number (i.e., child qualifies or quantifies the size of a fraction or fractional amount)	"one out of three" "one third" "each person gets 2 pieces"

Written Symbols	Records a formal symbol, word, or operation string/equation using paper-and-pencil	"¼" (fraction symbol) "1 4th" (numeric/word symbols) "1 fourth" (word)
Manipulative Aids	Enacts a physical object that's not a drawn representation (but can be drawn on)	A child may systematically or randomly cut or fold a cutout circle or rectangle. Other materials may include a maker box, paper strip, or coin
Pictures	Draws or interprets a drawn shape or figure	Bar, number line, rectangle
Real World Situation	Engages with a context such as fair-sharing or measuring	Child describes a fair share or describes a fractional measurement (e.g., one-eighth of a bar)

In spoken symbols, there are different ways children name fractions, which could give a hint on what meaning of fractions they are holding. For example, most children were observed to verbally give a fraction name such as "three fourths" whereas some children, such as Caroline, used language such as "ten out of sixteen." These two children might hold different conceptions when naming fractions. In the former case, s/he may be regarding the fractions as a whole, one singleton. In contrast, Caroline's "ten out of sixteen" language may indicate that she tends to look at fractions in a part-whole relationship, treating the numerator and denominator as separate numbers.

In written symbols, we capture if children record their spoken symbols in words (i.e., one-fourth), in symbolic numbers (1/4), or as operations (1÷ 4). For example, at the beginning of a task Lucy (a second grader) expressed difficulty in writing down "one and a fourth" in a symbolic form (i.e., 1(¼)), but put the fractions in words. Likewise, Paul (a fourth grader) could reason that ⅝= 2 and ½ of ¼, but could not make this into an equation: ⅝ = 2(½) × 1/4. By coding for the nuanced nature in which children use written representations, it helps us to capture children's informal and formal understanding of fractions.

For manipulatives aids, we capture the action that children enact in the physical object such as folding and cutting. For example, one child folded a cut-out carefully before cutting into fourths. This action gave us a sense that this child had some systematic way of partitioning, which is different from a child who randomly breaks the cut-out into (generally unequal) fourths.

In coding for children's use of pictures, we capture the type of picture used in a child's reasoning such as a bar diagram, number line, or circular area model. Children may create a picture or reason from a given picture or diagram to make sense of the problem. In some instances, a child's use of pictures sheds light on additional nuances of their reasoning that would otherwise be hidden if only symbolic representations were considered.

Finally, for a real world situation, we capture if children refer to a context when reasoning with fractions or fractions operations. Children may also extract fractions out of a given context, or draw on a different context to make sense of the problem. An example of this was given above in Ariel's reasoning from the context of fair-sharing cookies to the context of fair-sharing cake.

Overall, coding for the ways in which children use representations within one representation type provides an importance lens on children's understanding of fraction concepts and operations. Initial results suggest that we should not overlook children's abilities to reason within one representation system, and for some children, they may benefit from opportunities to move forward from their sound informal understanding of fractions such as in spoken words or real-world contexts to the formal use of symbolic form.

Children's representational fluency. Recall that representational fluency is the ability to create, interpret, and connect multiple representations in doing and communicating about mathematics. This section of the coding scheme specifically captures children's interpretations of multiple representations either within the same representation type or across representation types. As demonstrated in Table 4, preliminary results highlight that children often express inconsistencies or mismatches when reasoning across representations types, sometimes posing a hindrance to their problem solving process (see Elliott, Tran, Fonger, & Ziols, under review).

Table 4.
Mismatches within and between representation types

Mismatch	Examples
Spoken-written	Lucy (a second grader) says "a whole and one fourth" and writes "1/1 4/4"
Spoken-manipulative	Oscar (a second grader) says "one-third" yet partitions a rectangle into three non-equal pieces
Picture-spoken	Sadie (a fourth grader) is able to use a diagram to identify ¼ and 1/8, yet cannot use spoken symbol to state that is 1/8 more
Manipulatives	Oscar (a second grader) is able to cut a circle into thirds, but not a rectangle

Per our emergent coding scheme, mismatches between different representation types are more prominent than correct connections in children's reasoning about fraction concepts and operations with fractions. Another level of abstraction of representational fluency involves talking about the nature of children's understanding of the connections between representations and their understandings/conceptions. Consistent with Fonger (2011), we consider a connection as an instance in which students are able to articulate invariant features across representations within and across representation types.

Reasoning Types

An important result of our emergent coding scheme is the classification of children's reasoning into three categories: (a) informal intuitive, (b) formal and disconnected, and (c) formal and connected. See Table 5, which is elaborated in Ziols, Fonger, Tran, Elliott (in review). These categories of reasoning type are largely related to children's representational fluency in solving fraction tasks with formal or disciplinary representations (i.e., written symbols per Behr et al., 1983) and their relation to informal representation types (i.e., real-world situations, spoken symbols, manipulatives, and pictures per Berh et al., 1983).

Table 5.
Connectedness of children's informal to formal reasoning type (Ziols et al., in review)

Reasoning	Description	Example
Informal/ Intuitive	Student may not have formal instruction yet (observing all informal strategies); often reasoning involving real-world situation and spoken symbols	• Sadie (a fourth grader) introduces brownies that can be partitioned instead of candies in a discrete collection in order to reason about fraction sizes. • Ariel (a third grader) recognizes that 2 cakes can be cut and shared equally with 3 people after saying it was impossible to equally share 2 cookies with 3 people.
Formal and Disconnected	Formal instruction of fractions is observed, can make sense of the problem using informal reasoning but could not connect the two; knowledge within one representation is not extended to the other	• Isaac (a third grader) argues that 2/3 is larger than 5/8 when using an area model to compare fractions but that 5/8 comes after 2/3 on a number line.
Formal and Connected	Formal instruction of fractions evident, can make sense of the problem using informal reasoning and make the connection between the two;	• Paul (a fourth grader) identifies a unit fraction from a bar described as 2/5 and is able to find the length of the whole. He generalizes his knowledge when given 13/15 as a numeric symbol only.

	knowledge is extended from one representation to another	

Informal/intuitive reasoning. Across all interviews results indicate that fair-sharing contexts are accessible for children even before formal instruction of fractions (i.e., grade 2), and children exhibit flexibly using their own strategies (typically informal working on either manipulatives, picture/diagram or in a verbal form). In the examples given in Row 1 of Table 5, both Sadie and Ariel relied on informal understandings of fractions in a contextual situation of fair-sharing to make the fraction task meaningful.

Formal and disconnected reasoning. A common reasoning strategy across children's problem solving was to exhibit a disconnection between their informal reasoning and a formal or rule-based procedure. This disconnection was often observed when children toggled back and forth between a real-world context, using a picture/diagram, manipulatives, describing verbally, and a non-contextual numeric calculation. For example, Thomas, a 3rd grader, was successful using a model to solve the task of sharing 1¾ submarine sandwiches among two people, naming each share as 7/8. However, typical of several children, Thomas repeatedly used cross multiplication to manipulate parts of the share (¾ and 1/8), which confused him and was in conflict with reasoning within other representation type.

Formal and Connected. In formal and connected reasoning a student demonstrates representational fluency in their creation and/or interpretation of symbolic representations by drawing meaning from other formal representations or informal representation types such as pictures, spoken symbols, real-world contexts, or manipulatives. For example, children could reason with symbolic forms and connect them to specific situations where they rely on picture/diagram or context to base the reasoning. For example, Paul identifies a unit fraction from a bar described as 2/5 and is able to find the length of the whole. He generalizes his knowledge when given 13/15 or any given fractions as a numeric symbol.

Meaning of Fractions

Following the framework of fractions meanings (Kieren, 1980), we coded the meaning that children elicit when reasoning with the tasks. See Table 6 for a summary of code names and descriptions.

Table 6.
Meanings of fractions

Meaning of Fraction	Description
Part-Whole	Counting number of parts of the total number of parts of a partitioned whole; does not require a unit to measure, involves fractions less that 1
Measure	Assignment of a number to a region done through an iteration of the process of counting the number of units used in covering a region (Kieren 1980); use one fraction to measure another such as on a number line
Quotient	A division relationship, "a/b" as in "a" divided by "b"
Ratio	A relationship between two quantities by comparison of whole quantities. This includes fraction as pieces compared to pieces as in "a" compared to "c" (in context of "a/b", "c/d").
Operator	Two fractions are mentioned in reasoning, related by an operation (explicitly or tacitly) "a/b of __"; a fraction operates on another fraction

Clear examples of meanings of researchers' second-order models of children's meaning of fractions are forthcoming. We can report, however, on two interesting nuances in our findings. First, sometimes there are discrepancies between the researchers' intended meaning when designing the tasks and the meaning from children's perspective when they reason work on the tasks. For example, in the interview with Paul, the task asked him to compare two fractions bars

$\frac{5}{8}$ and $\frac{1}{4}$ in a given rectangular area figure. In the task design, we intended for the child to expose to the measure meaning of fractions, especially using one fraction to measure the other. This meaning does come through in his reasoning. In addition, another meaning of fraction emerged: Paul seemed to use an operator meaning of 5/2 in the statement, "$\frac{5}{8}$ is 5/2 of $\frac{1}{4}$" (i.e., the fraction 5/2 operates on $\frac{1}{4}$ to make $\frac{5}{8}$). Also, as Paul worked with $\frac{5}{8}$ and $\frac{1}{4}$, sometimes it is not clear that he was drawing on the part-whole meaning (seeing $\frac{5}{8}$ as 5 out of 8) or as a measurement meaning (5 of $\frac{1}{8}$). The coding for fractions meanings is still challenging and in some situations we could not classify student reasoning into one of the five meanings. Mostly, these instances arise when children seem to talk about fractions as a number without attaching to any specific meaning. One speculation is to consider adding a category of "rational number" to capture the idea that from a researcher's view it seems as though the child is considering the size of the fraction as a number or perhaps as quantity.

Discussion

Research on children's informal and intuitive reasoning strategies in solving fractions tasks is needed (Lamon, 2007). Our conceptual lens combines three perspectives: model types, representation types, and meanings of fraction (Behr et al., 1983; Kieren, 1980). The data corpus for this research consists of clinical interviews with children in grades 2-6 who solved a series of fraction tasks. Our conceptual lens was useful in two main ways. First, this lens informed our task design and first-order models of intended task types for the clinical interviews. Second, this lens served as a foundation to an emergent coding scheme that developed over several cycles of coding the clinical interview data. The preliminary finding presented in this report is an emergent coding scheme that spans four categories: (a) model types, (b) children use of representations and representational fluency, (c) connectedness of children's informal to formal reasoning type, and (d) meanings of fractions.

In all interviews, our results confirm that contexts are accessible for children based on their prior experiences (Empson & Levi, 2011). Akin to task variations proposed by Behr and colleagues (1983), we found that models of the whole influence the common strategies used and the big ideas highlighted. The findings reported here combine these perspectives, extending research on multiple meanings of fractions across a diverse range of children. In this study we found that syntheses of research from various perspectives on meaningful fraction instruction can inform the purposeful design and selection of tasks to use with a variety of children at different levels.

We argue that our preliminary findings help address the issue of better understanding children's informal reasoning with fractions as it relates to their more formal strategies and representational fluency in solving fractions tasks. This is especially prominent in the results presented in Table 5 in the classification of the connectedness of children's reasoning. However, the relationship of children's reasoning types and mismatches in representations remains to be explored. Another future direction of this research is to further elaborate the role of representational fluency in relation to children's conceptions of fractions.

Finally, in addressing meaning of fractions in Table 6, we noted several difficulties in coding children's meanings of fractions. Our attempt to use these first-order logico-mathematical meanings in analyses of second-order models of children's mathematics is complex and interesting. Our future research is guided toward expanding this aspect of the coding scheme with the aim of constructing second-order models of children's understanding of fractions as it relates to their informal representational activity.

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