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Numerical Development

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Abstract

In this review, we attempt to integrate two crucial aspects of numerical development: learning the magnitudes of individual numbers and learning arithmetic. Numerical magnitude development involves gaining increasingly precise knowledge of increasing ranges and types of numbers: from non-symbolic to small symbolic numbers, from smaller to larger whole numbers, and from whole to rational numbers. One reason why this development is important is that precision of numerical magnitude knowledge is correlated with, predictive of, and causally related to both whole and rational number arithmetic. Rational number arithmetic, however, also poses challenges beyond understanding the magnitudes of the individual numbers. Some of these challenges are inherent; they are present for all learners. Other challenges are culturally contingent; they vary from country to country and classroom to classroom. Generating theories and data that help surmount the challenges of rational number arithmetic is a promising and important goal for future numerical development research.

Keywords: numerical magnitudes; logarithmic-to-linear shift; rational numbers; arithmetic; mathematics achievement; conceptual understanding

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Numerical Development

Numerical knowledge is of great and increasing importance for success in modern society. Reflecting this pervasive importance, numerical knowledge at age 7 predicts socio-economic status (SES) at age 42, even after controlling for the rearing family's SES and the child's IQ, working memory, reading skill, years of education, academic motivation, and other variables (Ritchie & Bates 2013).

Consequential individual differences in numerical knowledge are present even before children reach first grade. The numerical knowledge of kindergartners from low-income backgrounds already lags, on average, at least a year behind that of peers from middle-income families (Jordan et al. 2006). These early differences have long-term consequences: A meta-analysis of six large longitudinal studies in the U.S., U.K., and Canada revealed that in all six, numerical knowledge in kindergarten predicted numerical knowledge in fifth grade, even after statistically controlling for a wide range of relevant variables (Duncan et al. 2007). Even more striking, numerical knowledge at age four predicts mathematics achievement at age fifteen (Watts et al. 2014), again after controlling for numerous relevant variables. Many aspects of cognitive development are fairly stable over time, but the relations between early and later mathematical knowledge are stronger than for other important competencies measured in the same studies, including reading, control of attention, and regulation of emotions (Duncan et al. 2007).

Numerical knowledge is also central to theories of cognitive development. Kant (1781) argued that number is an a priori concept, an idea that must be present from birth for people and other animals to function in the world. Evidence collected more than 200 years later supports this insight. Animals as varied as guppies, lions, rats, newborn chickens, and human infants mentally represent the approximate number of objects and events they encounter (Dehaene 2011, Piffer et al. 2013, Rugani et al. 2015).

Piaget (e.g., 1952) also assigned numerical development a prominent place in his theory, devoting his classic book "The Child's Concept of Number," as well as parts of many other books, to it. Contemporary theories, such as neo-Piagetian (Case & Okamoto 1996), core knowledge (Feigenson et al. 2004), information processing (e.g., Siegler 2006), sociocultural (Goncu & Gauvain 2012), evolutionary (Geary et al. 2015), and dynamic systems (Cantrell & Smith 2013), also emphasize the growth of numerical understanding as central to cognitive development.

Consistent with this theoretical emphasis, numerical development is a thriving research area, one with at least 15 active sub-areas (Siegler 2016). The focus of the present review will be on four especially notable themes that have emerged in recent years and that transcend the boundaries of these sub-areas:

1) Acquisition of increasingly precise knowledge of the magnitudes of increasing ranges and types of numbers

provides a unifying theme for numerical development from infancy to adulthood;

2) Understanding numerical development requires understanding acquisition of knowledge about rational numbers (i.e., fractions, decimals, percentages, and negatives) as well as whole numbers (i.e. 0, 1, 2, 3, etc.);

3) Knowledge of the magnitudes of individual numbers is essential to learning arithmetic;

4) Major goals for future research should include clarifying why understanding rational number arithmetic is so hard for so many people and generating effective means for improving it.

The integrated theory of numerical development provides a structure within which to discuss these themes.

The Integrated Theory of Numerical Development

Underlying the integrated theory of numerical development is the assumption that increasing understanding of numerical magnitudes is the common core of numerical development, one that affects arithmetic and more advanced mathematics as well (Siegler et al. 2011). The theory can be summarized as follows:

1) People, like a wide variety of other animals, represent the magnitudes of numbers on a mental number line, a dynamic structure that is first used to represent small whole numbers

and then is progressively extended rightward to include larger whole numbers, leftward to include negative numbers, and interstitially to include fractions and decimals. The approximate age ranges during which these changes occur are shown in Figure 1.

Insert Figure 1 Here

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2) Whole number magnitude representations progress from a compressive, approximately logarithmic distribution to an approximately linear one. Transitions occur earlier for smaller than for larger ranges of whole numbers, corresponding both to the complexity of the numbers and to the ages when children gain experience with them.

3) Development of numerical understanding also involves learning that many properties of whole numbers do not characterize other types of numbers, but that all real numbers have magnitudes that can be represented and ordered on number lines.

4) Knowledge of the magnitudes of both whole and rational numbers is correlated with, and predictive of, learning of arithmetic and more advanced aspects of mathematics.

5) Knowledge of numerical magnitudes is also causally related to learning of arithmetic; interventions that improve knowledge of numerical magnitudes can have positive effects on learning arithmetic.

The integrated theory, like many other approaches, assumes that numerical magnitudes are represented along a mental number line (for a review of evidence for this assumption, see Hubbard et al. 2005). At least in Western and Far Eastern cultures, the mental number line is usually horizontally oriented, with smaller numbers on the left and larger ones on the right. People are only one of many species that use this organization. For example, after newborn chicks are repeatedly presented a constant number of dots (e.g., 4), they spontaneously associate smaller sets of dots (e.g., 2) with the left side of space and larger sets of dots (e.g., 8) with the right (Rugani et al. 2015).

One type of evidence for the mental number line is the distance effect: identification of the larger of two numbers is faster the farther apart the numbers are (Moyer & Landauer 1967). The effect known as SNARC (Spatial-Numerical Association of Response Codes) provides another type of evidence. When asked which of two numbers is larger, people are faster to indicate “smaller” by pressing a button on their left and to indicate “larger” by pressing a button on their right than with the opposite pairings of side of the body and relative magnitude (e.g., Dehaene et al. 1990). SNARC effects emerge between 5 and 9 years of age, at the lower end of the range when magnitudes are relevant to achieving the goal of the task and at the higher end when they are irrelevant to the goal (Berch et al. 1999, Hoffmann et al. 2013).

The hypothesis that the mental number line is a dynamic structure that is gradually extended to represent all numerical magnitudes expands the range of ages and types of numbers that can be integrated within a single theory of numerical development. In particular, the integrated theory allows analyses of development to extend from infants' knowledge of non-symbolic numerical magnitudes (i.e., numbers represented as sets of dots, sequences of tones, etc.) to young children's knowledge of small symbolic whole number magnitudes (i.e., numbers represented by number words or numerals) to older children's knowledge of increasingly large symbolic whole number magnitudes to older children's and adults' representations of the magnitudes of symbolic fractions, decimals, and negatives.

One consequence of the focus of the integrated theory on rational as well as whole numbers is to reveal a basic challenge that numerical development poses to children: learning which properties of whole numbers apply to all numbers and which do not. Many properties that are true for all whole numbers do not hold for other types of numbers. Each whole number is represented by a unique symbol within a given symbol system (e.g., "4"), but each fraction can be expressed in infinitely many ways (e.g., "1/4, 2/8, 3/12,

…”). All natural numbers (whole numbers other than zero) have unique predecessors and successors, but no decimal does. Multiplying natural numbers never generates a product smaller than either multiplicand, but multiplying rational numbers between 0 and 1 always does. Adding whole numbers never yields an answer smaller than either addend, but adding negatives always does. However, all real numbers share the property that they can be located and ordered on number lines.

A great deal of evidence indicates that numerical magnitude knowledge is related to important mathematical outcomes. For example, such magnitude knowledge correlates positively, and often quite strongly, with children’s mathematics achievement in both cross sectional studies (Ashcraft & Moore 2012, Booth & Siegler 2006, Siegler & Booth 2004) and longitudinal ones (de Smedt et al. 2009, Östergren & Träff 2013). This relation remains present even after controlling for other variables related to mathematics achievement, including IQ, language, non-verbal reasoning, attention, working memory, calculation skill, and reading fluency (Bailey et al. 2012, Jordan et al. 2013).

A basic prediction of the integrated theory is that precision of numerical magnitude knowledge should be correlated with, and predictive of, arithmetic proficiency. This relation is not logically necessary. Children could memorize arithmetic facts and procedures without understanding the magnitudes of the operands (the numbers in problems) and answers. Indeed, many mathematics educators have lamented that rote memorization of arithmetic procedures is exactly what most students do (Cramer et al. 2002, Mack 1995).

In contrast, we hypothesize that many children do use numerical magnitudes to understand arithmetic and that knowledge of numerical magnitudes is much of what makes arithmetic, as well as algebra and other aspects of mathematics, meaningful. For example, when the task is to remember answers to specific arithmetic problems (e.g., 6 × 4 = 24), more precise representations of the operands and answer may facilitate recall of the answer, in part by making alternatives implausible (e.g., a child with good magnitude understanding might think of "36" when asked to answer "6 × 4," because 36 is a multiple of both operands, but not state it because it seems too big.) When the task is to learn arithmetic procedures, accurate magnitude representations make it possible for students to reject procedures that produce implausible answers, leading them to generate and test alternative procedures. For example, when presented 1/2 + 1/3, a child with good knowledge of the magnitudes of individual fractions might initially add the numerators and denominators separately, but reject the procedure after seeing that it resulted in an answer, 2/5, that is smaller than one of the addends.

Thus, the integrated theory predicts that arithmetic facts and procedures will be learned more quickly and completely by children with greater knowledge of the magnitudes of the numbers used in the computations. It also predicts that instruction and other manipulations that improve magnitude understanding will improve arithmetic learning. Due to the pervasiveness of both whole and rational number arithmetic in algebra, trigonometry, and other more advanced areas of mathematics, the theory predicts similar relations between numerical magnitude knowledge and overall math achievement.

Development of Numerical Magnitude Knowledge

The development of numerical magnitude knowledge includes four main changes: generating increasingly precise representations of non-symbolic magnitudes, connecting symbolic to non-symbolic magnitudes for small whole numbers, accurately representing increasingly large whole numbers, and accurately representing the magnitudes of rational numbers. Siegler (2016) provides an in-depth review of development of numerical magnitude knowledge; here, we summarize major findings about it.

Development of Whole Number Magnitude Knowledge

Non-symbolic magnitudes. Infants and many non-human animals can discriminate sets of dots or tones that differ in their number but not in dimensions correlated with number, such as summed surface area of dots or duration of tones. With number as with many other dimensions, the larger the ratio of number of entities in the sets being compared, the easier the discrimination. This property of numerical discrimination is known as ratio dependence. Six-month-olds can discriminate 2:1 but not 1.5:1 ratios, and 9-month-olds can discriminate 1.5:1 but not 1.3:1 ratios (Cordes & Brannon 2008). Discrimination between non-symbolic numerical magnitudes continues to become more precise well beyond infancy, with many adults able to reliably discriminate 1.14:1 ratios (Halberda & Feigenson 2008).

One exception to the ratio dependence of non-symbolic number discrimination is that discrimination of sets of 1-4 objects is more accurate, faster, and less variable than would be expected from the ratios alone (Piazza 2011). Interestingly, guppies show the same superiority in discriminating between sets of 1-4 objects, relative to their usual ratio dependence (Agrillo et al. 2012). This superiority may reflect subitizing, a process that produces quick and automatic perception of small sets of objects or events.

Although precision of discrimination increases with age, the brain areas that most actively process non-symbolic numerical magnitudes are similar over the course of development. Parts of the intraparietal sulcus (IPS) and dorsolateral prefrontal cortex (DLPFC) play major roles in processing non-symbolic numerical magnitudes from infancy to adulthood (Dehaene 2011).

Small symbolic whole number magnitudes. Non-symbolic numerical magnitude knowledge provides useful referents for learning the magnitudes represented by symbolic numbers. For example, the spoken word “three” and the Arabic numeral “3” can be mapped onto three dots or fingers. A number of investigators have hypothesized that individual differences in symbolic representations of numerical magnitudes grow out of individual differences in non-symbolic numerical discrimination (e.g., Dehaene 2008; Halberda et al. 2008).

Several types of evidence have been cited to support this hypothesis. Ratio dependence like that observed with discrimination of non-symbolic numerical magnitudes is also evident with discrimination of symbolic numbers (e.g., "Which is bigger, 6 or 8"; Dehaene 2011). Individual differences in the precision of non-symbolic magnitude discrimination are predictive of concurrent and future individual differences in discrimination between symbolic magnitudes and are also predictive of symbolic arithmetic and math achievement test scores (e.g., Libertus et al. 2011). Habituation to non-symbolic numbers generates habituation to equivalent symbolic numbers, as measured by fMRI (functional magnetic resonance imaging) activations (Piazza et al. 2007). Brain areas used to process non-symbolic and symbolic numbers overlap considerably (Nieder & Dehaene 2009).

Other data, however, have called into question the strength and specificity of the relation between non-symbolic and symbolic numerical magnitude knowledge. Ratio and distance effects are present not just with non-symbolic and symbolic numerical stimuli but also with totally non-numerical tasks, including odor discrimination (Parnas et al. 2013). Relations between the precision of non-symbolic and symbolic number representations, and between non-symbolic representations and overall math achievement, have proved far weaker than suggested by early studies (Halberda et al. 2008). Two meta-analyses (Chen & Li 2014, Fazio et al. 2014) found that the average weighted correlation between non-symbolic magnitude discrimination and overall math achievement was $r = .20$ and $r = .22$, respectively. An internet study with more than 10,000 adult participants yielded a relation of $r = .21$ (Halberda et al. 2012), and a carefully controlled study with 200 children in each year from first through sixth grade yielded a relation of $r = .21$ (Lyons et al. 2014). All of these relations were significant, but all also were weak.

Neural data also have proved more complex than they originally appeared. Symbolic and non-symbolic numbers are processed in the same general areas of the brain, but processing activity within those areas elicited by corresponding non-symbolic and symbolic numbers are weakly related (Bulthé et al. 2014, Lyons et al. 2014).

More generally, it is unclear how approximate non-symbolic representations could contribute to creating precise symbolic understanding of large numbers. No one can consistently discriminate 158 from 159 dots, but everyone who understands the decimal system knows with absolute certainty that “159” is larger than “158.”

These data do not imply that knowledge of non-symbolic numerical magnitudes plays no role in learning the magnitudes of symbolic numbers. It seems likely that toddlers and preschoolers acquire the meaning of small symbolic single-digit numbers by associating them with non-symbolic representations of the corresponding sets. Consistent with this view, Le Corre & Carey (2007) found that 2- and 3-year-olds learn the magnitudes denoted by the symbolic number words 1-4 through a slow process in which they associate first the word “1,” then “2,” then “3,” and then “4,” with the corresponding non-symbolic quantities produced by subitizing.

The process might well extend beyond the subitizing range. Young children frequently put up fingers and count them. The auditory, visual, kinesthetic, and temporal cues that accompany counting of fingers and other objects provide data for associating non-symbolic quantities with the symbolic numbers that children most often use in counting and adding, that is, the numbers 1-10. Consistent with this hypothesis, behavioral data from 4- to 6-year-olds indicate that putting up fingers activates symbolic numbers and answers to addition problems with sums of 10 or less (Siegler & Shrager 1984). Neuro-imaging studies provide converging evidence; fMRI data indicate a common neural substrate for finger representations and mental addition and subtraction with the numbers 1-10 for both children and adults (Andres et al. 2012, Berteletti & Booth 2015). Thus, non-symbolic numerical representations seem to help symbolic numerical magnitude knowledge get off the ground, through providing concrete referents for small symbolic whole numbers. Whether they contribute to individual differences in understanding larger symbolic numerical magnitudes remains an open question.

Large symbolic whole number magnitudes. Children learn about the magnitudes of symbolic whole numbers surprisingly slowly. After children count flawlessly from 1-10, they still take a year or more to understand the relative magnitudes of the numbers they are counting (Le Corre & Carey 2007).

Beyond the ages when children have mastered ordinal relations among the symbolic numbers 1-10, they represent their magnitudes as increasing in a non-linear, approximately logarithmic, pattern (Figure 2). Illustratively, when 3- and 4-year-olds are asked to perform the number line estimation task by placing numbers on a line with 0 at the left end, 10 at the right end, and nothing in between, they locate small numbers, such as 2 and 3, much farther apart than larger numbers, such as 8 and 9. In contrast, 5- and 6-year-olds space the two pairs of numbers equally (Berteletti et al. 2010). Thus, between ages 3 and 6 years, children progress from an approximately logarithmic relation to an approximately linear relation between the numbers 1-10 and estimates of their magnitudes.

Insert Figure 2 Here

The same developmental sequence recurs at older ages with larger numbers. In the 0-100 range, 5- and 6-year-olds generate logarithmically increasing estimates, whereas 7- and 8-year-olds generate linearly increasing ones (Geary et al. 2007, Siegler & Booth 2004). In the 0-1000 range, 7- and 8-year-olds generate logarithmically increasing estimates but 9- and 10-year-olds generate linearly increasing ones (Booth & Siegler 2006, Thompson & Opfer 2010). Chinese children generate the same developmental progression for each numerical range at younger ages (Siegler & Mu 2008, Xu et al. 2013); children with math learning difficulties generate the same progression at older ages (Reeve et al. 2015). Even in adulthood, the logarithmic representation continues to be seen in transitory processes during number line estimation, though the final estimates increase linearly (Dotan & Dehaene 2013).

Analogies between smaller and larger orders of magnitude seem to contribute to broadening the range of whole numbers whose magnitudes children represent linearly. Encountering problems presented in ways that highlighted the analogy between smaller and larger numerical ranges – for example, the analogy between the location of 15 on a 0-100 number line and 1500 on a 0-10,000 number line – led second graders to extend the linear estimation pattern from 0-100 to 0-10,000 and 0-100,000 lines (Thompson & Opfer 2010). Analogies involving smaller numbers, such as the analogy from 0-100 to 0-1000 number lines, also have been found to promote generalization of accurate magnitude representations (Opfer & Siegler 2007, Sullivan & Barner 2014).

Development of Rational Number Magnitude Knowledge

Non-symbolic rational numbers. Development of understanding of non-symbolic rational number magnitudes, like that of whole number magnitudes, begins in infancy. Six-month-olds can accurately discriminate between two ratios that differ by at least a factor of two. For example, they dishabituate (i.e., look longer) when, after repeated presentations of 2:1 ratios of yellow to blue dots, the ratio of yellow to blue dots switches to 4:1 (McCrink & Wynn 2007). This level of precision matches age peers' discrimination abilities with non-symbolic whole number displays.

Neuroscience research also points to similarities in whole and rational number processing. Both non-symbolic and symbolic rational number magnitudes are processed by a fronto-parietal network closely resembling that used to process whole-number magnitudes (e.g., Jacob et al. 2012). Neural activations show similar distance effects with fractions as with whole numbers on non-symbolic numerical magnitude comparison tasks (Ischebeck et al. 2009).

Symbolic rational numbers. The process of acquiring understanding of the magnitudes of symbolic rational numbers starts later, proceeds more slowly, and asymptotes at a lower level than the corresponding process with symbolic whole numbers. Although written fraction notation is usually introduced in early elementary school, connecting written fractions with the magnitudes that they represent remains challenging even for many adults. In several studies, adult community college students correctly answered only 70%-80% of magnitude comparison problems for fractions with unequal numerators and denominators, where chance was 50% (e.g., Givvin et al. 2011; Schneider & Siegler 2010). This weak performance does not appear to be due to forgetting after years of not studying fractions; middle school students' fraction magnitude comparison accuracy is in the same range (Bailey et al. 2014, Siegler & Pyke 2013).

Acquisition of magnitude knowledge is faster and reaches a higher asymptotic level with fractions from 0 to 1 than for larger fractions, such as fractions from 0 to 5. For example, in Torbeyns et al. (2015), sixth graders in Belgium had a mean percent absolute error of 9% for fractions between 0 and 1 but 20% for fractions between 0 and 5; the corresponding numbers for peers in the U.S. were 17% and 26% and in China 10% and 14%. Strikingly, second graders' whole number estimates on 0-100 number lines are often more accurate than eighth graders' estimates of fractions on 0-5 number lines (Siegler & Booth 2004, Siegler et al. 2011).

Understanding the magnitudes of symbolically expressed decimals also poses problems for learners. Many fourth to eighth graders predict that, as with whole numbers, longer sequences of digits imply larger numbers (e.g., they think that .123 > .45; (Lortie-Forgues & Siegler submitted; Resnick et al.

1989). Nonetheless, accuracy and speed of numerical magnitude comparison and number line estimation tend to be greater for decimals than for common fractions (Iuculano & Butterworth 2011).

Alongside the striking differences between development of whole and rational numbers are notable similarities. When comparing the magnitudes of pairs of fractions or decimals, distance effects are present among both children (Fazio et al. 2014, Iuculano & Butterworth 2011) and adults (Meert et al. 2009, Schneider & Siegler 2010). Moreover, for both symbolic rational and whole numbers, relations are present between precision of magnitude knowledge and arithmetic, as well as between magnitude knowledge and overall mathematics achievement (e.g., Bailey et al. 2015; Torbeyns et al. 2015).

Early whole number magnitude knowledge is also predictive of later rational number magnitude knowledge. For example, in Bailey, Siegler, & Geary (2014), 6-year-olds' number line estimation accuracy with whole numbers predicted their accuracy with fractions at age 13, even after statistically controlling for many relevant variables. Lending discriminant validity to the findings, number line estimation accuracy at age 6 did not predict reading achievement at age 13. Similar predictive relations have emerged in other age ranges (Hecht & Vagi 2010; Jordan et al. 2013; Resnick et al. 2016). Causal relations between earlier symbolic whole number magnitude knowledge and later rational number magnitude knowledge remain to be established, but may well be present.

Development of Arithmetic

Non-symbolic Whole Number Arithmetic

In addition to representing non-symbolic numerical magnitudes, infants also can perform approximate arithmetic on these representations. Four- to five-month-olds dishabituate when it appears (through trickery) that adding one or two objects to an initial one or two has produced more or fewer objects than the correct number; infants of the same age also dishabituate when shown unexpected subtractive outcomes with similarly small sets (Wynn 1992). Somewhat older infants dishabituate to surprising addition and subtraction outcomes on sets of 5-10 objects (McCrink & Wynn 2004).

The neural substrates subserving non-symbolic arithmetic overlap considerably with those involved in non-symbolic numerical representations of single sets of objects. In particular, non-symbolic addition elicits activation in IPS (Venkatraman et al. 2005), and non-symbolic addition and subtraction are associated with increased coordinated activity in functionally connected left and right parietal areas, including IPS (Park et al. 2013).

Symbolic Whole Number Arithmetic

There is impressive agreement on descriptive features of the development of symbolic whole number arithmetic procedures and concepts (Geary et al. 2016; Verschaffel et al. 2007). The process typically begins around age three with counting-based procedures, such as first putting up fingers to represent each addend and then counting from one (the sum strategy) and retrieval from memory of answers to a few small addend problems. Individual preschoolers typically use several distinct counting-based strategies, as well as retrieval, rather than relying on a single approach. This variable strategy use continues into childhood and adulthood and is present on all four arithmetic operations. Even students attending high quality universities use strategies other than retrieval to generate answers on 15% to 30% of single-digit problems on all four arithmetic operations (Campbell & Xue 2001, LeFevre et al. 2003).

From early in development, choices among arithmetic strategies are highly adaptive, in the sense of promoting desirable combinations of accuracy and speed (Siegler 1996). In choosing whether to state a retrieved answer or to use a strategy other than retrieval, preschoolers, older children, and adults predominantly use retrieval, the fastest strategy, when they can execute it accurately. They predominantly use slower strategies, such as counting from one or counting-on from the larger addend, when they cannot retrieve accurately. In choosing among strategies other than retrieval, children most often use each backup strategy on the problems where it yields the fastest and most accurate performance relative to available alternatives. For example, 8- to 10-year-olds often subtract by counting up or down, depending on which can be done with fewer counts. They usually count down on 13 – 2

("12 is 1, 11 is 2, answer is 11") but usually count up on 13 − 11 ("12 is 1, 13 is 2, answer is 2"; Siegler 1989).

Along with the commonalities over age of variable strategy use and adaptive strategy choice are large developmental changes. At a general level, arithmetic becomes faster and more accurate. Four sources of this greater speed and accuracy are discovery of new, more effective, strategies; increasing use of the more effective strategies that are already known; faster and more accurate execution of strategies; and increasingly adaptive choices among strategies (Siegler 1996). For example, between ages 5 and 8, children 1) discover the addition strategies of counting-on from the larger addend and decomposition (e.g., adding 3 + 9 by thinking "10 + 3 = 13, 13 – 1 = 12"); 2) increasingly use those relatively efficient strategies and decreasingly use the less efficient strategies of counting from one and guessing; 3) execute all strategies more quickly and accurately; and 4) generate increasingly adaptive strategy choices.

Beyond these changes in whole number arithmetic procedures, young children also gain considerable conceptual understanding of whole number arithmetic. They discriminate between the validity of strategies that they do not themselves use but that are legitimate (counting-on) and the validity of illegitimate strategies that they also do not use (counting the first addend twice; Siegler & Crowley 1994). They also gain understanding of the commutative and associative principles (Baroody & Tiilikainen 2003).

Moreover, observations of strategy discoveries indicate that children rarely generate conceptually flawed addition strategies under conditions in which they do discover novel correct strategies (Siegler & Jenkins 1989).

One reason for whole number arithmetic strategies being conceptually grounded is that knowledge of earlier learned operations is used to build knowledge of later ones. Knowledge of counting provides a basis for learning and understanding addition, and knowledge of addition provides a base for learning and understanding subtraction (as in the previous example of solving $13 - 11$). Addition and subtraction provide a base for understanding multiplication (as when solving 4×3 by adding three fours or four threes), and division (as when solving $12 \div 4$ by adding or subtracting 4's, while keeping track of the number of 4's.) Learning of division also benefits from knowledge of multiplication (as when solving $12 \div 4$ by reasoning that if $3 \times 4 = 12$, then $12 \div 4 = 3$).

Another likely reason for children's whole number arithmetic strategies being conceptually grounded is that magnitude knowledge provides a way to check whether answers, and the strategies that yielded them, are reasonable. When asked to verify whether a given answer to a single-digit arithmetic problem is correct, both children and adults are faster to reply "no" when the incorrect answer is far from the correct magnitude (e.g., $5 + 7 = 18$) than when it is close (e.g., $5 + 7 = 14$; Ashcraft 1995; Campbell & Fugelsang 2001). Similarly, errors generated while solving single-digit arithmetic problems are usually close in magnitude to the correct answer (Siegler 1988).

Understanding does take longer to develop for some arithmetic principles, such as mathematical equality (e.g., 3 + 4 + 5 = ____ + 5; Alibali & Goldin-Meadow 1993; McNeil 2014) and inversion relations between multiplication and division (e.g., 9 × 17 ÷ 17 = ____; Robinson 2016). Moreover, understanding of multi-digit arithmetic often is less good than understanding of single-digit procedures, as indicated by the existence of buggy subtraction approaches (Brown & VanLehn 1982) among elementary school students (e.g., answering 308 – 145 by writing 243, on the logic that the smaller digit in a given column should be subtracted from the larger one, regardless of which number includes the smaller digit in the column). Nonetheless, most children develop substantial understanding of whole number arithmetic.

The cognitive processes that generate both procedural and conceptual development of whole number arithmetic have been modeled in computer simulations (e.g., Siegler & Araya 2005; Shrager & Siegler 1998). These simulations produce changes in thinking that closely resemble those of children along many dimensions. Within the models, development is produced by associating problems with answers and with strategies that produce fast and accurate performance on the problems, freeing of working memory capacity with more efficient strategy execution, and use of goal sketches, a cognitive structure used to evaluate the plausibility of potential new strategies by whether they meet the goals viewed as essential for a reasonable strategy for a given task. Together, these processes lead to increasing frequency of retrieval of correct answers, greater use of the more advanced strategies from among existing approaches, increasing knowledge of the types of problems on which each strategy is most effective, and discovery of useful new arithmetic strategies without attempting flawed ones.

Fraction Arithmetic

Knowledge of fraction arithmetic is indispensable for many purposes; learning physics, chemistry, biology, psychology, economics, statistics, engineering, and many other areas depends on it. Reflecting this importance, fraction arithmetic was part of more than half of the equations on the reference sheets for the most recent U.S. advanced placement exams in physics and chemistry (College Board 2015). Knowledge of fraction arithmetic also is essential in many occupations, not only in STEM areas but also in such positions as nurse and pharmacist (e.g., for calculating and evaluating prescribed drug doses), automotive technician, stone mason, carpenter, and tool and die maker (e.g., for adjusting the angles of precision cutting tools; (Davidson 2012, McCloskey 2007, Sformo 2008).

Unfortunately, learning fraction arithmetic is often difficult. Many children do not master it even after the prolonged instruction in it that students typically receive from fourth through eighth grade. Averaging across three recent studies that examined knowledge of all four arithmetic operations, U.S. sixth and eighth graders solved only 59% of fraction arithmetic items (Siegler & Lortie-Forgues 2015, Siegler & Pyke 2013, Siegler et al. 2011). Fraction arithmetic accuracy has been found to improve with age and experience in these and other studies (e.g., Byrnes & Wasik 1991), but even college students err on roughly 20% of problems (e.g., Siegler & Lortie-Forgues 2015; Stigler et al. 2010).

The problem is widely recognized by educators. A nationwide sample of 1,000 U.S. Algebra 1 teachers who were asked to rate 15 types of knowledge limitations that interfered with their students' algebra learning rated lack of knowledge of fractions and fraction arithmetic as the second greatest problem (Hoffer et al. 2007). The only knowledge limitation rated more serious was of the amorphous category "word problems."

Also reflecting widespread recognition of this difficulty were the recommendations of the Common Core State Standards Initiative (CCSSI 2010). These standards, which have been adopted as instructional policy in more than 80% of U.S. states, influence the teaching, textbooks, and standardized achievement tests presented to students (Davis et al. 2013). The CCSSI recommends that rational number arithmetic, and the closely related topics of ratios, rates, and proportions, be emphasized from fourth through eighth grades. However, children in the U.S. have continued to struggle with rational number arithmetic after implementation of the CCSSI, just as they did before.

Most fraction arithmetic errors are of one of two types. *Independent whole number errors* (Ni & Zhou 2005) involve performing the arithmetic operation independently on numerators and denominators (e.g., 1/2 + 1/3 = 2/5). *Wrong fraction operation errors* involve using sub-procedures that are correct within another fraction arithmetic operation but incorrect within the operation in the problem. For example, people often maintain common denominators on fraction multiplication problems (e.g., 3/5 × 3/5 = 9/5), an approach that parallels correct fraction addition and subtraction (3/5 + 3/5 = 6/5) but is incorrect for multiplication. Both types of errors are common among community college students as well as children (Stigler et al. 2010).

These errors reflect lack of understanding of, or attention to, the magnitudes produced by fraction arithmetic operations. Claiming that 1/2 + 1/3 = 2/5 implies that adding positive numbers can produce answers less than one of the addends. Claiming that 3/5 × 3/5 = 9/5 indicates a belief that multiplying two numbers below 1 can result in an answer larger than 1.

This lack of understanding also is apparent in the variability of fraction arithmetic strategies. Within a single session, children often use both correct and incorrect procedures on virtually identical fraction arithmetic problems, for example 3/5 + 1/4 and 3/5 + 2/3 (Siegler & Pyke 2013, Siegler et al. 2011). Children who used both correct and incorrect procedures were more confident when they used the correct procedures, but not overwhelmingly so (mean confidence ratings on a 1-5 scale of 4.2 for correct procedures and 3.5 for incorrect procedures).

These findings indicate that many children's fraction arithmetic knowledge includes a mix of correct procedures, incorrect procedures based on incorrect application of whole number knowledge, and incorrect procedures involving sub-procedures detached from the fraction arithmetic operation on which they are appropriate. The incorrect procedures often yield answers that violate basic principles, such as when adding two positive numbers yields an answer less than one of them. These children's fraction arithmetic seems to be unconstrained by magnitude knowledge that could be used to eschew procedures that yield implausible answers.

Decimal Arithmetic

Development of decimal arithmetic follows a similar path to that of fraction arithmetic. The most comprehensive data on this development come from Hiebert and Wearne (1985; 1986). Between the first semester of fifth grade and the second semester of ninth grade, accuracy of the 670 children in this study increased from 20% to 80% for addition, 21% to 82% for subtraction, and 30% to 75% for multiplication. Although a great deal of effort has been devoted in the past 30 years to improving rational number arithmetic instruction, students' performance has not changed much. Stigler, et al. (2010) reported that community college students correctly answered only around 80% of decimal arithmetic problems, and Lortie-Forgues & Siegler (2016) found similar levels of accuracy with sixth and eighth graders.

The largest source of errors in decimal arithmetic is misplacement of the decimal point. This error is often seen on decimal addition and subtraction problems when the operands have unequal numbers of digits to the right of the decimal point. For example, the ninth graders in Hiebert and Wearne (1985) erred on 36% of trials when subtracting .86 − .3 but on only 10% of trials when subtracting .60 − .36.

Students' difficulty in placing the decimal point in answers also is seen in multiplication and division. For example, in Lortie-Forgues and Siegler (2016), 73% of errors, and 34% of answers, to decimal multiplication problems of sixth and eighth graders involved misplacing the decimal point. The most common type of error was of the form "1.23 × 2.34 = 287.82," an answer akin to claiming that multiplying two numbers below three yields an answer above 200. Like the correct addition procedure "1.23 +2.34 = 3.57," the answer 287.82 leaves the decimal point in the answer two digits to the left of the rightmost digit. As with fraction arithmetic errors, use of such procedures suggests that many students ignore the plausibility of the magnitudes of the answers to decimal arithmetic problems.

Relations Between Knowledge of Magnitudes and Arithmetic

Whole Numbers

Individual differences in children's whole number arithmetic competence correlate with individual differences in the precision of their symbolic whole number magnitude knowledge, even after controlling for differences in IQ, working memory, executive function, language, and spatial ability (Booth & Siegler 2008, Fuchs et al. 2010, Linsen et al. 2015). These correlations remain significant into adulthood (Castronovo & Göbel 2012) and are robust compared to correlations between arithmetic and accuracy of non-symbolic magnitude knowledge (Holloway & Ansari 2009, Skagerlund & Träff 2016). For example, in a study of almost 1400 first to sixth graders, symbolic magnitude knowledge predicted unique variance in arithmetic proficiency, but non-symbolic comparison acuity did not (Lyons et al. 2014). Consistent with the possibility that numerical magnitude knowledge could facilitate arithmetic performance by allowing accurate estimation of correct answers, accuracy of children's symbolic magnitude knowledge predicts their computational estimation (approximate mental arithmetic) performance (Dowker 1997; Gunderson et al. 2012).

The relation between symbolic arithmetic competence and magnitude knowledge also is evident in neural data. Sensitivity of IPS activation during symbolic magnitude comparison to the ratio of the numbers being compared correlates with children's arithmetic fluency, after controlling for general intelligence (Bugden et al. 2012). Similarly, children who represent numerical magnitudes more precisely show greater sensitivity of parietal response to the sizes of operands and answers in arithmetic problems (Berteletti et al. 2015).

Rational Numbers

As with whole numbers, individual differences in children's knowledge of fraction magnitudes correlate with individual differences in the children's fraction arithmetic competence (Byrnes & Wasik 1991, Siegler et al. 2011). These correlations persist across national differences in educational practices (Torbeyns et al. 2015) and after statistically controlling for whole number arithmetic ability, language ability, executive function, working memory, and attentive behavior (Hecht et al. 2003, Siegler & Pyke 2013).

If, as the integrated theory of numerical development asserts, there is developmental continuity in understanding of whole and rational number magnitudes, then whole number magnitude knowledge should also correlate with fraction arithmetic ability. It does (Bailey et al. 2014, Jordan et al. 2013). Strikingly, accuracy of number line estimation with whole numbers in first grade predicts fraction arithmetic proficiency in seventh grade – a relation fully mediated by fraction magnitude understanding in middle school (Bailey et al. 2014).

Effects of Interventions Emphasizing Magnitudes on Arithmetic Learning

In addition to these individual difference correlations, experimental evidence indicates that numerical magnitude knowledge is causally related to arithmetic competence.

Whole Number Interventions

If numerical magnitude knowledge is causally related to arithmetic learning, then interventions that produce improvements in numerical magnitude knowledge should also improve arithmetic learning. Ramani and Siegler (e.g., 2008; Siegler & Ramani 2009) produced evidence consistent with this prediction. Their studies examined effects of playing a numerical board game called "The Great Race" on the numerical knowledge of preschoolers from low-income families. The board in this game has 10 squares, labeled with the numbers 1-10. Players take turns spinning a spinner that can land on "1" or "2", moving their token forward by the number of spaces indicated on the spinner, and saying the numbers in the squares aloud as they proceed. For example, children whose token is on the square labeled "4" and who spin a "2" need to say "5, 6" as they move their token forward. Children who either say a wrong number or do not know what number to say are helped by the experimenter to state the correct number(s). The player whose token first reaches "10" wins.

Playing this game improved children's numerical magnitude knowledge, as well as their counting and identification of printed numbers (Ramani & Siegler 2008). Siegler and Ramani (2009) replicated those findings and found that although the game involves no explicit arithmetic problems, children who had played it earlier subsequently learned more from feedback on simple addition problems than peers who had earlier engaged in other numerical activities, such as counting objects and identifying printed numbers. The improvements in arithmetic performance included both a higher percent correct and answers closer in magnitude to the correct answer.

Other interventions based on the same ideas have yielded similar effects. A 0-100 version of “The Great Race” improved children’s magnitude knowledge, as well as their understanding of the base 10 system (Laski & Siegler 2014). An 8-week intervention focusing on building number sense and including “The Great Race” as a component improved arithmetic accuracy among kindergartners from low-income communities (Jordan et al. 2012). Similar but distinct games, such as “The Number Race” (Wilson et al. 2006) and “Rescue Calcularis” (Kucian et al. 2011), also have shown positive effects on arithmetic performance among typically developing children and children with developmental dyscalculia, a condition characterized by unusual difficulty with numbers and arithmetic.

Manipulations that activate magnitude knowledge in the context of arithmetic also improve arithmetic learning. In one such study, first graders studied single- and multi-digit addition facts either alone or accompanied by analog number-line representations of the magnitudes of addends and sums (Booth & Siegler 2008). On a subsequent posttest, children who had earlier seen the analog magnitude representations recalled the arithmetic answers more accurately. In another study, first graders who practiced an approximate addition task using non-symbolic numerical magnitudes (dot arrays) subsequently showed superior accuracy and speed at exact symbolic arithmetic, relative to children who practiced non-numerical tasks (Hyde et al. 2014). Thus, activating knowledge of numerical magnitudes increases addition performance and learning.

Rational Number Interventions

Causal evidence linking rational number magnitude knowledge and rational number arithmetic skill comes from several studies comparing effects of instruction emphasizing fraction magnitudes to effects of instruction emphasizing the part-whole interpretation of fractions. This latter interpretation, which is heavily emphasized in U.S. schools, construes fractions as parts of objects or subsets of sets (Kieren 1976). Instruction emphasizing the part-whole interpretation provides children practice with tasks that can be solved by counting a part and the whole, such as choosing whether 3/4, 4/3, 3/7, or 4/7 best represents the fraction of red segments in a circle with three red and four blue segments. Instruction emphasizing magnitudes provides practice with such tasks as fraction number line estimation and magnitude comparison.

Interventions that focus on fraction magnitudes have yielded greater improvement not only in fraction magnitude knowledge but also in fraction arithmetic (Fuchs et al. 2013; 2014; 2016; Moss & Case 1999). For example, after 12 weeks of participation in Fraction Face-off!, an intervention emphasizing fraction magnitude knowledge, fourth graders' accuracy at fraction addition and subtraction exceeded that of control students who received, over the same period, a curriculum emphasizing the part-whole interpretation. This finding was especially striking because the curriculum that emphasized magnitudes devoted less explicit instruction to fraction arithmetic procedures (Fuchs et al. 2013; 2014; 2016).

Although improved understanding of fraction magnitudes often facilitates learning of fraction arithmetic, increasing children's magnitude knowledge is sometimes insufficient to produce such learning. In one such case, an intervention that included instruction in fraction magnitudes and approximate arithmetic showed no advantage over a traditional curriculum with respect to learning arithmetic procedures (Gabriel et al. 2012). Moreover, even after the quite successful Fraction Face-off! intervention, students scored on average only 18 of 41 points on a posttest assessment of fraction arithmetic (Fuchs et al. 2014). This was considerably higher than they had scored on the pretest and considerably higher than a control group, but below 50% in absolute terms. The question is why learning this material is so challenging for so many students.

Why Is Rational Number Arithmetic So Difficult?

At a general level, the cause of the problems that many children and adults experience in learning fraction and decimal arithmetic is well known: they do not understand the rational number arithmetic operations that they are asked to learn. This lack of understanding of rational number arithmetic is evident in many ways. When asked to estimate the closest whole number to the sum of 12/13 +7/8, only 24% of the more than 20,000 eighth graders who took a standardized test of math achievement (the National Assessment of Educational Progress or NAEP) chose “2” (Carpenter et al. 1980). (The response alternatives were 1, 2, 19, 21, and “I don’t know”.) Presenting the same problem in 2014 to eighth graders taking an Algebra 1 course in a fairly affluent suburban school district showed that little had changed in the subsequent 30 years; 27% of the eighth graders in the recent sample correctly answered the problem (Siegler & Lortie-Forgues 2015).

The problem is not unique to fractions. When the NAEP presented the decimal multiplication problem 3.04 × 5.3, and asked whether 1.6, 16, 160, or 1600, was closest to the answer, only 21% of eighth graders chose the correct answer, 16; their most common answer was 1600 (Carpenter et al. 1983). Many high school and college students, pre-service teachers, and practicing teachers also lack basic conceptual understanding of rational number arithmetic (Fischbein et al. 1985, Hanson & Hogan 2000, Ma 1999, Siegler & Lortie-Forgues 2015).

Difficulty in learning rational number arithmetic might be thought of as a problem that falls within the field of education more than psychology. We believe, however, that psychological theories, concepts, and analytic techniques are essential for addressing children's difficulties with acquiring this foundational body of knowledge. We also believe that applying knowledge from psychology to challenging real world problems, including ones in education, provides a valuable test of the utility of our theories and analytic techniques.

As a first step toward the goal of applying psychological theories, concepts, and analytic techniques to the goal of improving children's understanding of rational number arithmetic, we attempt in the remainder of this chapter to specify sources of difficulty in learning fraction and decimal arithmetic. These sources of difficulty are divided into two categories: inherent and culturally contingent. *Inherent sources of difficulty* are ones that make understanding rational number arithmetic difficult regardless of the particulars of the society and educational system. Examples include the complex relations among different fraction arithmetic operations and between whole number and fraction arithmetic operations. *Culturally contingent sources of difficulty* are ones that vary among societies and educational systems. They include teachers' understanding of the material they are teaching, textbooks' accuracy and clarity in presenting the material, and students' prior knowledge of prerequisites. A more extensive discussion of these issues can be found in Lortie-Forgues et al. 2015; here we summarize the main points in the analysis.

Inherent Sources of Difficulty of Rational Number Arithmetic

Inaccessibility of rational number magnitudes. For both whole and rational numbers, arithmetic learning is correlated with, and causally related to, knowledge of numerical magnitudes. However, fraction and decimal notations make accessing fraction and decimal magnitudes more difficult than accessing whole number magnitudes. Whole number magnitudes are expressed by a single number, and after age 8 or 9, children access them automatically (Berch et al. 1999, White et al. 2012). In contrast, fraction magnitudes have to be derived from the ratio of two numbers, and even in adulthood, they are usually not accessed automatically (DeWolf et al. 2014).

Accessing most decimal magnitudes is similar to accessing whole number magnitudes (DeWolf et al. 2014). However, this is not the case for decimals with one or more zeros immediately to the right of the decimal point (e.g., .012). Indicative of this difficulty, Putt (1995) found that only about 50% of U.S. and Australian pre-service teachers correctly ordered the magnitudes of five decimals, four of which included one or more zeroes immediately to the right of the decimal. Hiebert and Wearne (1985) reported similar results regarding decimal magnitude knowledge among U.S. ninth graders. The difficulty of accessing these decimal magnitudes poses an inherent challenge in learning decimal arithmetic.

Opacity of rational number arithmetic procedures. The conceptual basis of standard fraction arithmetic procedures is often far from obvious. Why are equal denominators needed for adding and subtracting but not for multiplying and dividing? Why is the denominator inverted and multiplied when dividing fractions? The most straightforward explanations require knowledge of algebra, a subject that usually is taught after fractions, so that even strong students usually lack such knowledge when they are learning fractions.

Superficially, the conceptual basis of decimal arithmetic is more transparent than that of fraction arithmetic, because the decimal computations resemble those with the corresponding whole number operations. However, considering issues related to placement of the decimal point reveals that understanding decimal arithmetic procedures is more challenging than it first appears. To understand the complexity, try to explain to someone who does not already understand why $.123 \times .45$ yields an answer with five digits to the right of the decimal point. The challenge of providing a clear explanation without recourse to concepts that students lack when they learn about decimal arithmetic, such as negative exponents, seems likely to contribute to difficulty understanding decimal arithmetic.

Complex relations between whole and rational number arithmetic operations. Whole number arithmetic knowledge is often incorrectly generalized to rational number arithmetic. This problem was mentioned earlier in the context of independent whole number errors with fractions (e.g., $2/3 + 2/3 = 4/6$).

Similarly tempting generalizations from whole number arithmetic lead to incorrect judgments about the direction of effects of multiplying and dividing fractions. Multiplying two natural numbers (whole numbers other than 0) never results in an answer less than either multiplicand, but multiplying two fractions between 0 and 1 always does. Similarly, dividing by a natural number never results in an answer greater than the number being divided, but dividing by a fraction between 0 and 1 always does. However, even after years of experience with fraction multiplication and division, both eighth graders and adults predict that all four fraction arithmetic operations will produce the same direction of effects as with whole numbers, regardless of whether the operands are above or below one (Siegler & Lortie-Forgues 2015).

Similar overgeneralizations from whole number experience affect decimal arithmetic. With whole number addition and subtraction, aligning the rightmost digits of the numbers in the problem preserves the correspondence of their place values. In contrast, with decimal addition and subtraction, such alignment yields incorrect answers when the numbers of digits to the right of the decimal point differ for the operands. The locations of the decimal point in the operands, rather than the rightmost digits, need to be aligned to correctly add and subtract decimals (i.e., with .82 − .6, the "6" needs to be aligned with the "8" rather than with the "2".) Correct mapping is probably made more difficult by the fact that the strategy of aligning the rightmost digit does work when the decimals being added or subtracted have the same number of digits (e.g., .82 − .64).

Variable relations among rational number operations. Partial overlaps in sub-procedures within different rational number arithmetic operations add to the difficulty of learning rational number arithmetic. For example, when adding or subtracting fractions with the same denominator, the operation is applied only to the numerator, but when multiplying or dividing fractions with the same denominator, the operation must be applied to both numerator and denominator. Perhaps because children generally learn fraction addition and subtraction before fraction multiplication and division, errors of the form 4/5 × 4/5 = 16/5 are very common. In one recent study of sixth and eighth graders' fraction arithmetic, 46% of fraction multiplication answers and 55% of fraction division answers involved such errors (Siegler & Pyke 2013). These errors again are easy to understand; if a child does not understand that 4/5 × 4/5 means 4/5 of 4/5, and that the answer must therefore be less than 4/5, why not proceed as in fraction addition?

Overgeneralization of decimal addition and subtraction procedures to decimal multiplication and division occur for similar reasons. On addition and subtraction problems with equal numbers of digits to the right of the decimal point (e.g., .5 + .7 = 1.2), the placement of the decimal point relative to the rightmost digit of the addends is maintained. However, with decimal multiplication, the location of the decimal point is determined by the sum of the number of digits in the multiplicands to the right of the decimal point (e.g., .5 × .7 = .35, rather than 3.5). Overgeneralization of the rule from addition for placing the decimal point accounted for 76% of sixth graders' multiplication answers in Hiebert and Wearne (1985). Again, the error is understandable; if a child does not know that .5 × .7 means half of .7, why shouldn't the answer be 3.5?

Culturally contingent sources of difficulty

Inherent sources of difficulty make learning rational number arithmetic challenging for learners in all cultures. However, the degree to which learners surmount these inherent difficulties depends on several culturally contingent factors.

Teachers' understanding. Understanding a topic is no guarantee of ability to teach it well, but without such understanding, high quality instruction is unlikely. Consistent with this view, teachers' understanding of rational number arithmetic is strongly related to their knowledge of how to teach it effectively (Depaepe et al. 2015).

Unfortunately, many teachers in Western countries have limited understanding of rational number arithmetic. When asked to illustrate the meaning of a fraction division problem, such as 1 3/4 ÷ 1/2, only a minority of U.S. and Belgian teachers provided an explanation other than stating the invert-and-multiply algorithm (Depaepe et al. 2015). In contrast, roughly 90% of East Asian teachers provided coherent explanations for the same problems

(Luo et al. 2011, Ma 1999). Similar national differences were found in those studies in teachers' ability to judge whether multiplication or division was the way to solve a rational number story problem.

Teachers' emphasis on rote memorization. For many years, researchers, associations of math teachers, and national commissions have argued that typical mathematics instruction overemphasizes rote memorization of procedures (e.g., Brownell 1947; National Mathematics Advisory Panel 2008). Lack of understanding of procedures limits children's initial learning, and the deleterious effects become larger with the passage of time following initial learning (e.g., Reyna & Brainerd 1991). The degree to which teachers emphasize rote memorization of rational number arithmetic procedures may be related to the quality of their understanding of the procedures. Emphasizing rote memorization is one way to avoid the embarrassment of being unable to answer questions about mathematics that some students in the class can answer.

*Textbook explanations of limited generality***.** U.S. textbooks typically explain whole number multiplication as repeated addition (CCSSI 2010). The problem 3 × 2, for example, could be explained as adding two three times (2 + 2 + 2). This approach has the advantage of grounding understanding of multiplication in understanding of addition, but it also has two limitations. One is that the repeated addition interpretation cannot be straightforwardly applied to multiplication of fractions and decimals. For instance, how to interpret $1/3 \times 3/4$ in terms of repeated addition is not straightforward.

The other limitation is that the repeated addition explanation often leads students to infer that because adding positive numbers always yields an answer larger than either addend, multiplying positive numbers has the same effect. However, this inference is mistaken: Multiplying two numbers between 0 and 1 never yields a product greater than either multiplicand.

The present theoretical emphasis on numerical magnitudes, together with empirical findings about the effectiveness of interventions that emphasize number lines in teaching fractions, suggest alternative instructional approaches that might be more effective. For example, multiplication could be presented as “N of the M’s” with whole numbers (e.g., 4 of the 3’s) and “N of the M” with fractions (1/3 of the 3/4). Using a number line to represent (e.g.) the 3/4, and then demonstrating that 1/3 of the 3/4 means dividing the segment from 0 to 3/4 into three smaller, equal sized segments, with the answer being one of them, could help students understand why multiplying fractions less than one yields an answer less than either operand.

A similar point can be made about division. In U.S. textbooks, division is usually introduced as fair sharing (dividing objects equally among people). For example, a teacher or textbook might explain $12 \div 4$ as 12 cookies shared equally among 4 friends. This interpretation is straightforward with natural numbers but not with rational numbers, at least when the divisor is not a whole number (e.g., what does it mean to share 3/4 of a cookie among 3/8 of a friend?).

Again, the standard explanation is not the only possible one. At least when a larger number is divided by a smaller one, both whole number and fraction division can be explained straightforwardly as indicating how many times the divisor can go into the dividend (e.g., how many times can 8 go into 32, how many times can 3/8 go into 3/4). Moreover, number lines can be used to illustrate the process in both whole and rational number contexts, which can promote understanding of the shared meaning of multiplication and division with the two types of numbers.

Students' limited prerequisite knowledge. Rational number arithmetic requires mastery of whole number arithmetic. Even adding 1/3 + 2/5 requires five correct single-digit multiplications and additions; fraction arithmetic problems involving multi-digit numerators and denominators require far more.

Although much better than their knowledge of rational number arithmetic procedures, many U.S. students' knowledge of whole number arithmetic procedures falls short of the ideal of consistent fast and accurate performance. This leads to a fair number of rational number arithmetic errors. For example, in Siegler and Pyke (2013), sixth and eighth graders made whole number arithmetic errors on 21% of fraction arithmetic problems. Such computational errors seem likely to interfere with learning of correct rational number procedures, because they obscure whether implausible answers are attributable to flawed procedures or to flawed execution of correct procedures.

Summary and Conclusions

Prior to the past decade, the field of numerical development focused almost exclusively on infants' and young children's understanding of whole numbers. In recent years, however, the field has expanded greatly to include rational as well as whole numbers and the entire developmental period from infancy to adulthood. Development of numerical magnitude knowledge has provided a unifying theme for this broader field. With age and experience, children gain knowledge of the magnitudes of non-symbolic numbers; small symbolic whole numbers; larger symbolic whole numbers; and fractions, decimals, and other rational numbers.

A major goal of the present chapter had been to integrate the literature on development of knowledge of numerical magnitudes with the literature on development of arithmetic. We found that magnitude knowledge was correlated with, predictive of, and causally related to, arithmetic competence for both whole and rational numbers. Most individuals eventually achieve relatively good understanding of both whole number magnitudes and whole number arithmetic, but even adults with excellent knowledge of the magnitudes of individual fractions and decimals often have limited understanding of rational number arithmetic. Poor understanding of rational number arithmetic is common not only among students but also among teachers. This problem is especially serious, because knowledge of rational number arithmetic is foundational not only for further mathematics and science learning but also for a wide range of occupations.

The reasons for many children's and adults' poor understanding of rational number arithmetic include both inherent and culturally contingent sources. One inherent source is the greater difficulty of accessing magnitudes of numbers in fraction and decimal arithmetic problems, relative to accessing the magnitudes in whole number arithmetic problems. This greater difficulty reflects the magnitudes of fractions and decimals depending on relations between components (explicitly with fractions, implicitly with decimals) rather than on any single number. Weak knowledge of the magnitudes of rational number operands and answers limits the ability of many children and adults to evaluate the plausibility of answers and the procedures that produce them.

However, the difficulty of learning rational number arithmetic cannot be reduced to poor understanding of rational number magnitudes. Other inherent features of arithmetic with decimals and fractions add to the difficulty of learning. One is that rational number arithmetic procedures are often opaque. For example, it is far from immediately apparent why the invert-and-multiply procedure yields correct answers for fraction division, or why multiplying two decimals that each have two digits to the right of the decimal point must lead to an answer with the decimal point four digits to the left of the rightmost digit. Understanding why these procedures yield correct answers is unlikely without knowledge of algebra, which students rarely have when they are learning fraction and decimal arithmetic. The complex relations between corresponding whole number and rational number arithmetic operations, and of different rational number arithmetic operations to each other, add to the inherent difficulty of learning rational number arithmetic. These complex relations lead to tempting but incorrect analogies and obscure correct connections between the same operation with whole and rational numbers.

In the U.S. and many other Western countries, culturally contingent variables further increase the difficulty of learning rational number arithmetic. Teachers' understanding of rational number arithmetic is often limited, and their instruction often emphasizes rote memorization rather than understanding. Textbook explanations often fail to convey to students the shared meaning of corresponding arithmetic operations with whole and rational numbers. Some students have not fully mastered whole number arithmetic, leading to errors in rational number arithmetic even when they choose the correct procedure.

In sum, investigators have gained considerable knowledge of what develops and how development occurs in numerical development. These gains include increased understanding of the development of both numerical magnitudes and how those magnitudes combine in arithmetic operations. An important current challenge is to apply this knowledge to facilitate understanding of rational number arithmetic. Attaining this goal will require not only methods for improving learners' knowledge of rational number magnitudes but also effective means for increasing their understanding of the logic underlying rational number arithmetic procedures. Pursuing this agenda will provide tests of theories of numerical development, as well as useful data for expanding and refining the theories. It will also help surmount the artificial distinction between developmental and educational psychology, to the likely benefit of both fields.

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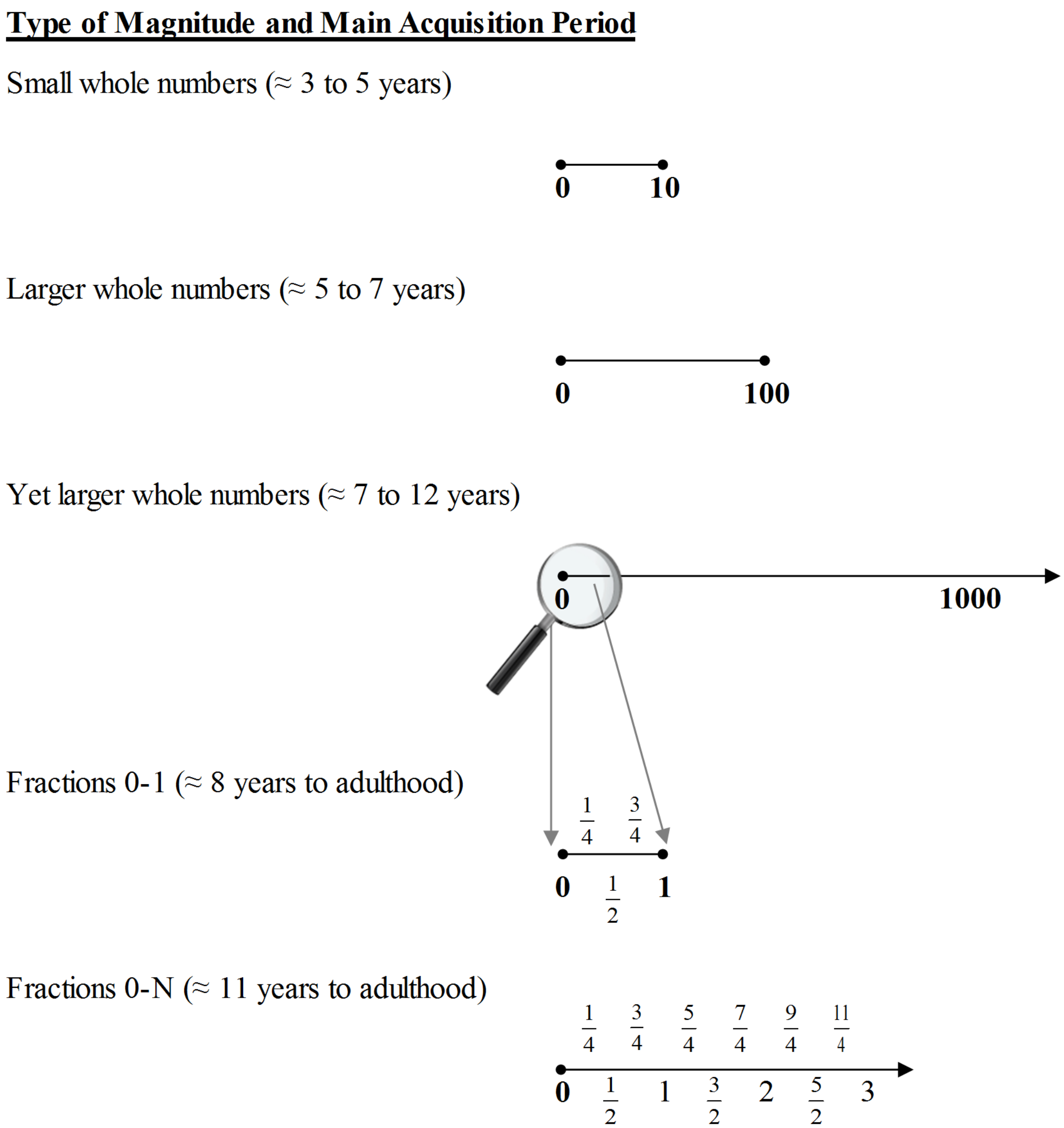

Figure 1. Approximate age ranges of major changes to the sizes and types of numbers whose magnitudes individuals can represent.

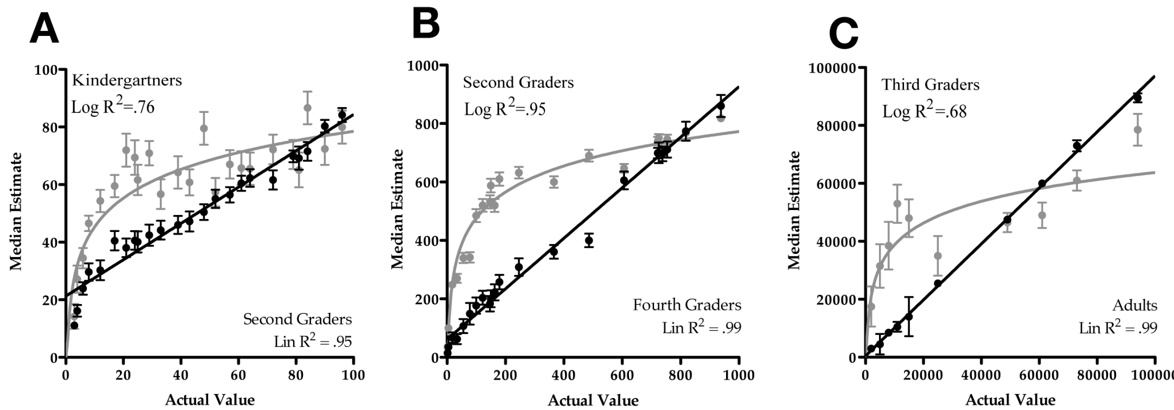

Figure 2. Relations between presented and estimated whole number magnitudes for different ranges of numbers and ages of participants. (A) When estimating numbers in the range 0-100, kindergartners generate a logarithmic relation, while second graders generate a linear one. (B) With numbers in the range 0-1000, second graders generate a logarithmic relation and fourth graders a linear one. (C) With numbers in the 0-100,000 range, third graders generate a logarithmic relation and adults a linear one.