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An Integrative Theory of Numerical Development

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Abstract

Understanding of numerical development is growing rapidly, but the volume and diversity of findings can make it difficult to perceive any coherence in the process. The integrative theory of numerical development posits that a coherent theme is present, however – progressive broadening of the set of numbers whose magnitudes can be accurately represented—and that this theme unifies numerical development from infancy to adulthood. From this perspective, development of numerical representations involves four major acquisitions: 1) increasingly precise representations of magnitudes of numbers expressed nonsymbolically, 2) linking nonsymbolic to symbolic numerical representations, 3) extending understanding to increasingly large whole numbers, and 4) extending understanding to all rational numbers. Thus, the mental number line expands rightward to encompass larger whole numbers, leftward to encompass negatives, and interstitially to include fractions and decimals.

An Integrative Theory of Numerical Development

Introduction

Research on numerical development is expanding at a remarkable rate. Thriving literatures have arisen on numerical development in infancy, childhood, and adolescence; on development of subitizing, counting, estimation, and arithmetic; on knowledge of whole numbers, fractions, decimals, and negatives; on nonsymbolic and symbolic representations; on conceptual and procedural knowledge; on underpinnings of numerical development in evolutionary processes, neural processes, cognitive processes, and emotional processes; on longitudinal stability of individual differences; and on numerical competence in normal and special populations. The list does not end there; researchers have also examined relations to numerical knowledge of variations in economic status, culture, language, and instruction; relations among numerical, spatial, and temporal knowledge; relations of numerical knowledge to more advanced mathematics; and relations of interventions that improve numerical knowledge to subsequent learning; to name a subset of areas within the field (see Table 1 of the Supplemental Materials available online for references for each area).

Important and intriguing recent discoveries in all of these areas attest to the health of the field of numerical development. However, the sheer number of discoveries and areas can make it difficult to perceive any coherence in the developmental process. Are coherent themes present, or is numerical development just "one darn thing after another"?

The Integrated Theory of Numerical Development

The integrated theory of numerical development proposes that the continuing growth of understanding of numerical magnitudes provides a unifying theme for numerical development. Within this perspective, numerical development is a process of broadening the set of numbers whose magnitudes, individually or in arithmetic combination, can be accurately represented. The theory identifies four main trends in numerical development: 1) representing increasingly precisely the magnitudes of numbers expressed nonsymbolically, 2) linking nonsymbolic to symbolic representations of numerical magnitudes, 3) extending the range of whole numbers whose magnitudes can be represented accurately, and 4) representing accurately the magnitudes of numbers other than whole numbers, in particular fractions, decimals, and negatives.

(Insert Figure 1 about here.)

The integrative theory begins with the popular metaphor of the mental number line. However, it goes on to propose that this mental number line is a dynamic, continually changing, structure rather than a fixed, static one. Initially useful for organizing knowledge of nonsymbolic numbers and then of small, positive, symbolic whole numbers, the mental number line is progressively extended rightward to represent larger symbolic whole numbers, leftward to represent negative numbers, and interstitially to representing symbolic fractions and decimals (see Figures 1 and 2).

A variety of data, both correlational and causal, support the integrated theory's emphasis on the importance of accurately representing numerical magnitudes. Preschoolers' success in identifying the more numerous of two dot collections predicts their math achievement as much as two years later, even after controlling for other intellectual variables, (Libertus, Feigenson, & Halberda, 2011; Mazzocco, Feigenson, & Halberda, 2011). Accuracy of number line estimation correlates substantially with overall mathematics achievement from kindergarten through at least eighth grade (Booth &

Siegler, 2006; Siegler & Booth, 2004; Siegler, Thompson, & Schneider, 2011). The accuracy of first graders' location of symbolic whole numbers on number lines predicts the accuracy of their fraction number line estimation and fraction arithmetic in seventh and eighth grades, even after controlling for IQ, working memory, and socioeconomic status (SES; Bailey, Siegler, & Geary, in press). Moreover, manipulations that improve representations of whole number magnitudes improve subsequent learning of whole number arithmetic (Booth & Siegler, 2008; Siegler & Ramani, 2009), and manipulations aimed at improving fraction magnitude representations improve learning of fraction arithmetic (Fuchs et al., 2013; Fuchs et al., in press).

These studies demonstrate the importance of numerical magnitude representations from early childhood through adolescence. The acquisition of knowledge of nonsymbolic numerical magnitudes actually begins even earlier, in infancy.

Nonsymbolic Representations of Numerical Magnitudes

Long before children learn number words, they represent numerical magnitudes nonverbally. The mechanism by which people (and many other species) do so has been labeled the Approximate Number System (ANS) (Halberda, Mazzocco, & Feigenson, 2008). From early in infancy, the ANS allows discrimination between sets of objects in which the ratio of the larger to the smaller set is sufficiently large, independent of the area and perimeter of the objects, their luminance, and other potentially confounding variables. Discriminability between sets is a function of the ratio of their number of items, as described by Weber's Law. For example, discriminating between 8 and 12 objects and between 16 and 24 is equally difficult (De Smedt, Noel, Gilmore, & Ansari, 2013). A second magnitude representation mechanism, sometimes termed *object files*, also exists early in infancy and yields more precise discrimination between sets of 1-4 objects than the general ratio-based mechanism would produce (Feigenson, Dehaene, & Spelke, 2004). These patterns are sometimes described in terms of distance and magnitude effects; discrimination between set sizes is more accurate when the set sizes are more discrepant (distance effects) and involve fewer objects (magnitude effects).

The precision of nonsymbolic number discrimination increases considerably over the first few years. At six months, infants discriminate 2:1 ratios, but not until nine months do they discriminate 3:2 ratios (Wood & Spelke, 2005). The improvement continues well beyond infancy; 3-year-olds consistently discriminate dot displays that differ by 4:3 ratios, 6-year-olds discriminate displays that differ by 6:5 ratios, and some adults discriminate displays that differ by 11:10 ratios (Halberda & Feigenson, 2008; Piazza et al., 2010).

Discrmination of nonsymbolic numerical magnitudes might seem an isolated skill of little importance, but individual differences in the skill at six months are related to mathematics achievement on standard symbolic mathematics tasks at three years, even after statistically controlling for IQ (Starr, Libertus, & Brannon, 2013). Moreover, individual differences at age 3 are related to scores on standardized mathematics achievement tests concurrently and two years later (Mazzocco, Feigenson, & Halberda, 2011).

On the other hand, three recent literature reviews, two including meta-analyses, indicate that relations between ANS acuity and math achievement are considerably weaker and less consistently present than relations between representations of symbolic numerical magnitude representations and math achievement (Chen & Li, 2014; De

Smedt, Noël, Gilmore, & Ansari, 2013; Fazio, Bailey, Thompson, & Siegler, 2014). Furthermore, symbolic numerical knowledge has been found to fully mediate the relation between nonsymbolic numerical knowledge and mathematics achievement with both 4-year-olds (vanMarle, Chu, Li, & Geary, 2014) and 6-year-olds (Göbel, Watson, Lervag, & Hulme, 2014). Nonetheless, knowledge of nonsymbolic magnitude seems to provide a foundation for understanding the referents of at least small symbolic numbers.

(Insert Figure 2 about here.)

From Nonsymbolic to Symbolic Representations of Numerical Magnitudes

Behavioral and neural data show several striking parallels between representations of nonsymbolic and symbolic magnitude. Behaviorally, the same distance and magnitude effects that are shown with nonsymbolic magnitudes are shown with symbolic ones, and the mathematical functions relating solution times to problem characteristics also are similar (Moyer & Landauer, 1967). At the neural level, not only do brain areas involved with representations of symbolic and nonsymbolic magnitude correspond closely, but habituation of either symbolic or nonsymbolic representations produces habituation of the other, as measured by fMRI activations (Piazza, et al., 2007).

Despite these similarities, the process of connecting symbolic to nonsymbolic numerical magnitude representations is surprisingly slow and piecemeal. On a task in which 3- and 4-year-olds were asked to give the experimenter N objects, some children who could count to 10 knew only the number 1; others only the numbers 1 and 2; others only the numbers 1, 2, and 3; and others only the numbers 1, 2, 3, and 4 (Le Corre & Carey, 2007). Despite being able to count accurately sets of 5-10 objects, many of these children assigned the numbers 5-10 to sets of objects in ways uncorrelated with the set size. Not until age 4 ½ did most children respond to number words beyond 4 on the "give X task" in ways correlated with the sets' magnitudes. Other paradigms have yielded similar results (e.g., Schaeffer, Eggleston, & Scott, 1974), indicating that even with very small whole numbers, connecting symbolic numbers to their magnitudes develops slowly.

Representing an Increasing Range of Whole Number Magnitudes

Even after children know the relative magnitudes of the numbers 1-10, they continue to have limited knowledge of the magnitudes of larger numbers. The acquisition of knowledge of the magnitudes of two-, three-, and four-digit whole numbers, like the acquisition of knowledge of the magnitudes of single-digit ones, is slow and piecemeal. This is apparent in number line estimation. On this task, children are presented a series of lines with a constant pair of numbers at the two ends (e.g., 0 and 100) and asked to locate a series of other numbers on the line (one number per line). Afterward, alternative mathematical functions are fit to the estimates to establish the one that best describes the estimation pattern.

With symbolic as with nonsymbolic numbers, the psychological distance between numbers at the low end of the range is much larger than that between numbers of identical arithmetic distance at the high end of the range. Thus, when asked to locate symbolically expressed numbers on a 0-10 number line, 3- and 4-year-olds space their estimates of small numbers (e.g., 2 and 3) much farther apart than their estimates of large numbers (e.g., 7 and 8), whereas 5- and 6-year-olds space the two pairs of numbers equally (Bertelletti, et al., 2010). The younger children's estimates increase logarithmically with the sizes of the number being estimated, whereas the estimates of the older children increase linearly.

The developmental sequence repeats itself at older ages with larger numbers. Thus, in the 0-100 range, 5- and 6-year-olds generate logarithmically increasing patterns of estimates, whereas 7- and 8-year-olds' estimates increase linearly (Geary, et al., 2007; Siegler & Booth, 2004). In the 0-1000 range, 7- and 8-year-olds generate logarithmically increasing patterns of estimates, but 9- and 10-year-olds generate linearly increasing ones (Booth & Siegler, 2006; Siegler & Opfer, 2003). The same 7- and 8-year-olds who consistently produced linear estimation patterns on 0-100 number lines produced logarithmic patterns on 0-1000 lines (Siegler & Opfer, 2003), reflecting the gradual extension of numerical magnitude knowledge to larger whole numbers.

Performance on number line estimation and other measures of numerical magnitude knowledge with symbolically expressed numbers is related strongly to other aspects of mathematical knowledge. Accuracy and linearity of number line estimation of symbolic whole number magnitudes for both the 0-100 and 0-1000 ranges correlate strongly with arithmetic proficiency (Booth & Siegler, 2008; Ramani & Siegler, 2008) and overall mathematics achievement (Geary, et al., 2007). The relation between symbolic numerical magnitude estimation and mathematics achievement remains even after statistically controlling plausible third variables including arithmetic, reading achievement, and IQ (Bailey, et al., in press; Booth & Siegler, 2006; 2008; Geary, et al., 2007). Furthermore, having children randomly play a number board game designed to improve representations of symbolic magnitude yields gains not only in their magnitude knowledge but also in other tasks, such as learning novel arithmetic problems (Ramani & Siegler, 2008; Siegler & Ramani, 2009).

These findings suggest that arithmetic is far from the rote memorization process that it is often portrayed as being. The role of magnitude knowledge in arithmetic is evident on verification tasks, where both children and adults consistently take longer to reject incorrect answers that are close in magnitude (e.g., 6*8=46) than incorrect answers that are distant in magnitude (e.g., 6*8=26) (Ashcraft, 1982). This role of magnitude knowledge in arithmetic is also evident in spontaneous retrieval errors, which usually are close in magnitude to the correct answer (Lemaire & Siegler, 1995). Magnitude knowledge can lead to activation of plausible answers, detection of implausible answers, and substitution of correct procedures for flawed ones that produce implausible answers.

From Whole Numbers to Rational Numbers

Development of knowledge of nonsymbolic rational numbers shows clear similarities to acquisition of nonsymbolic whole numbers. For instance, 6-month-olds discriminate 2:1 but not 3:2 ratios, just as they do with whole numbers. Thus, when habituated to a 2:1 ratio of blue and yellow dots, they dishabituate when shown a 4:1 ratio but not when shown a 3:1 ratio (McCrink & Wynn, 2007). Moreover, just as specific neurons are tuned to respond maximally to specific whole numbers, specific neurons are tuned to respond maximally to specific ratios (Jacob & Nieder, 2009).

Development of symbolically expressed whole numbers and fractions also shows several striking differences. The most obvious differences are that acquisition of knowledge of symbolic fractions begins much later and never reaches as high a level. Thus, even adults attending community college are only 70% accurate in comparing magnitudes of fractions, whereas they are almost 100% accurate in comparing corresponding magnitudes of whole numbers (DeWolf, et al., 2014).

Within the integrated theory of numerical development, acquiring knowledge of symbolic fractions requires learning that several invariant properties of whole numbers are not invariant properties of all numbers. Whole numbers have unique successors, can be represented by a single symbol, never decrease with multiplication, never increase with division, and so on. In contrast, none of these qualities are invariant for rational numbers. However, whole and rational numbers are alike in having magnitudes that can be represented on number lines. Thus, understanding rational numbers requires learning that many invariant properties of whole numbers are not true for rational numbers and also learning the mapping between symbolically expressed rational numbers and the magnitudes they represent.

Consistent with this analysis, despite the obvious differences between development of understanding of whole number and fraction magnitudes, the two also show important similarities. One similarity is that brain regions associated with fraction magnitude representations greatly overlap with those associated with whole number magnitude representations (Ischebeck, Schocke, & Delazer, 2009). In addition, both show distance effects on magnitude comparison tasks: For fractions with unequal numerators and denominators, the closer the fraction magnitudes being compared, the longer the comparison takes (Meert, Grégoire, & Noël 2009; Schneider & Siegler, 2010). Furthermore, with fractions as with whole numbers, individual differences in magnitude knowledge are highly correlated with individual differences in arithmetic and overall math achievement (Bailey, Hoard, Nugent, & Geary, 2012; Siegler, Thompson, & Schneider, 2011), even when reading achievement and executive functioning are statistically controlled (Siegler & Pyke, 2013). Moreover, longitudinal data show that 6-

year-olds' knowledge of whole number magnitudes predicts 13-year-olds' knowledge of fraction magnitudes (Bailey, Siegler & Geary, in press), even after controlling for the common influence of IQ, working memory, and SES. Finally, the positive effects on subsequent arithmetic learning of training designed to increase knowledge of whole number magnitude extends to fractions: Not only does training aimed at improving fraction magnitude knowledge also improve fraction arithmetic, but gains in fraction magnitude knowledge fully mediate improvements in fraction arithmetic learning (Fuchs et al., 2013; Fuchs et al., in press).

Fewer studies have been conducted on representations of negative than positive numbers, and none of them appear to be with negative fractions. However, on the basis of the limited available data, magnitudes seem to play a similarly central role in representations of negative and positive numbers. Brain regions associated with representations of the magnitudes of negative numbers overlap considerably with those associated with representations of the magnitudes of positive numbers (Gullick, Wolford, & Temple, 2012). Developmental changes in brain activity associated with the two also show parallels; with negative as with positive numbers, activation of frontal areas on magnitude comparison tasks decreases from childhood to adulthood, whereas activation of parietal areas increases (Gullick & Wohlford, 2013). Behavioral evidence indicates that as with positive numbers, both 10- to 12-year-olds and adults show distance effects with negatives on magnitude comparison problems (Ganor-Stern, Pinhas, Kallai, & Tzelgov, 2010; Gullick & Wohlford, 2013). Moreover, the size of distance effects with positive numbers are related to the size of distance effects with negative numbers, at least for 10-year-olds (Gullick & Wohlford, 2013). Thus, although the research base is scanty, magnitude knowledge appears to play a similar role with negative and positive numbers.

Conclusions

A basic tenet of the integrated theory is that numerical development is largely a process of broadening the range and types of numbers whose magnitudes are well understood. The developmental process includes at least four trends: representing nonsymbolic numerical magnitudes increasingly precisely, linking nonsymbolic and symbolic representations of small whole numbers, extending the range of numbers whose magnitudes are accurately represented to larger whole numbers, and representing accurately the magnitudes of rational numbers, including fractions, decimals, percentages, and negatives. These trends begin at different ages, and the level of mastery reached by adulthood varies considerably among different types of numbers, but many commonalities, both neural and behavioral, are evident in the acquisition process. Individual differences in mastery of all types of numerical magnitudes are linked to individual differences in arithmetic proficiency and mathematics achievement, and experiences that improve magnitude representations also improve other numerical skills, such as arithmetic learning. Thus, accurate representations of numerical magnitude can be seen as the common core of numerical development.

Authors' Note

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Figure Captions

Figure 1. Improving precision of nonsymbolic number discrimination with age. The sets of black and white dots represent experimental stimuli whose numerosity can be discriminated reliably at the specified ages.

Figure 2. Knowledge of symbolic numerical magnitudes expands from small whole numbers to larger whole numbers to rational numbers, including common fractions, decimals, and negatives. The ages associated with the expansions indicate the period in which knowledge of each type of magnitude typically shows the greatest increases.

Figure 1: The development of knowledge of *nonsymbolic* numerical magnitudes

<u>Precision of Discrimination</u>

2:1 ratio (≈ 6 months)

3:2 ratio (≈ 9 months)

4:3 ratio (≈ 3 years)

6:5 ratio (≈ 6 years)

11:10 ratio (some adults)

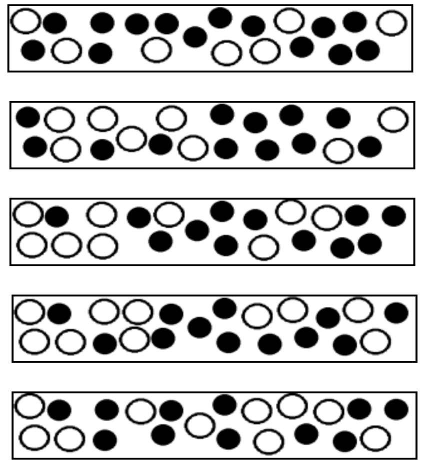
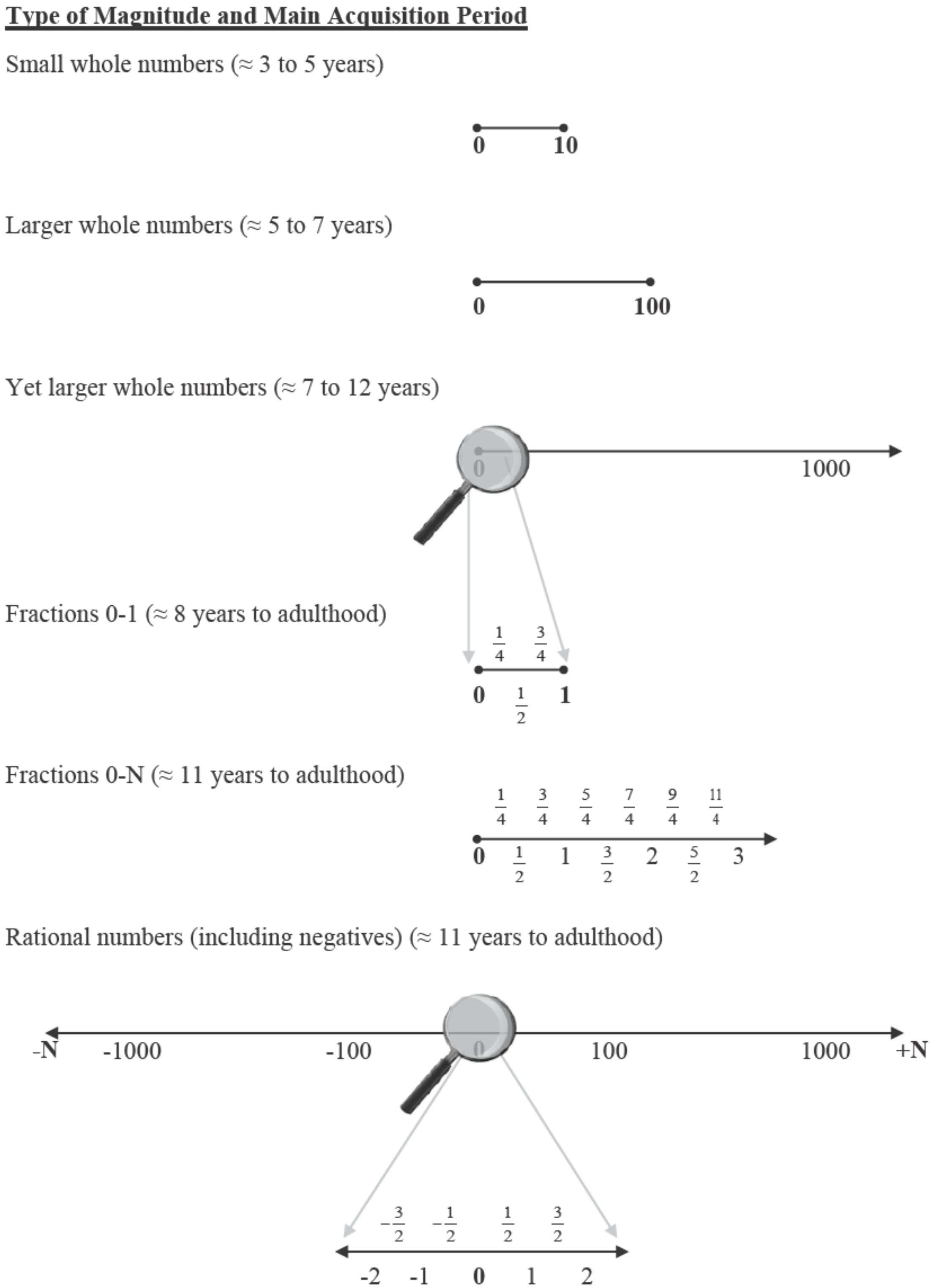


Figure 2: The development of knowledge of *symbolic* numerical magnitudes



Supplementary Material

Table 1
Example of research from different areas of numerical development

Area	Example of research
Numerical development in infancy	Spelke, E. S., & Kinzler, K. D. (2007). Core knowledge. Developmental Science, 10, 89-96. 10.1111/j.1467-7687.2007.00569.x.
Numerical development in childhood	Sophian, C. (2007). The origins of mathematical knowledge in childhood. New York, NY: Lawrence Erlbaum Associates.
Numerical development in adolescence	Case, R., & Okamoto, Y. (1996). The role of central conceptual structures in the development of children's thought. Monographs of the Society for Research in Child Development, 61(Nos. 1-2). http://dx.doi.org/10.2307/1166077
Development of subitizing	Halberda, J., Sires, S. F., & Feigenson, L. (2006). Multiple spatially overlapping sets can be enumerated in parallel. Psychological Science, 17, 572-576. 10.1111/j.1467-9280.2006.01746.x
Development of counting	Le Corre, M., Van de Walle, G., Brannon, E. M., & Carey, S. (2006). Re-visiting the competence/performance debate in the acquisition of the counting principles. Cognitive Psychology, 52, 130-169. 10.1016/j.cogpsych.2005.07.002
Development of estimation	Siegler, R. S., & Booth, J. L. (2005). Development of numerical estimation: A review. In J. I. D. Campbell (Ed.), Handbook of mathematical cognition (pp. 197-212). Boca Ratan, FL: CRC Press.
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Knowledge of whole numbers	Leslie, A. M., Gelman, R., & Gallistel, C. R. (2008). The generative basis of natural number concepts. Trends in Cognitive Sciences, 12, 213-218. http://dx.doi.org/10.1016/j.tics.2008.03.004
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