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Coordinating Multiple Representations in a Reform Calculus Textbook

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Abstract Coordination of multiple representations (CMR) is widely recognized as a critical skill in mathematics and is frequently demanded in reform calculus textbooks. However, little is known about the prevalence of coordination tasks in such textbooks. We coded 707 instances of CMR in a widely used reform calculus textbook and analyzed the distributions of coordination tasks by chapter and for the type of task demanded (perception vs. construction). Results suggest that different coordination tasks are used earlier and later in learning and for different topics, as well as for specific pedagogical and scaffolding purposes. For example, the algebra-to-text coordination task was more prevalent in the first chapter, suggesting that students are being eased into calculus content. By contrast, requests to construct graphs from algebraic expressions were emphasized in later chapters, suggesting that students are being pushed to think more conceptually about functions. Our nuanced look at coordination tasks in a reform textbook has implications for research in teaching and learning calculus.

Keywords Calculus · Conceptual understanding · Functions · Multiple representations · Textbooks

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Introduction

The use of external representations is central in the learning of mathematics. As De Bock, Van Dooren & Verschaffel (2015) note, since mathematical concepts are abstract by nature, representations are a point through which to access their meaning. It is even argued that external representations or "signs, words or symbols, expressions or drawings" are *the only* means through which the nature of abstract mathematical objects is communicated (Duval, 2000, p. 61). Since no one representation can fully describe a single mathematical concept, the use of multiple external representations allows learners to utilize the different advantages each representation offers (Duval, 2006). In addition, multiple representations (MR) facilitates gains in student understanding beyond the use of single representations (Elia, Panaoura, Gagatsis, Gravvani & Spyrou, 2008), promotes conceptual understanding of mathematics (Elia, Gagatsis, Panaoura, Zachariades & Zoulinaki, 2009), and supports comprehension of advanced topics (e.g., multivariate calculus, McGee & Moore-Russo, 2014).

Still, sheer use of MR is not enough to confer these benefits; students must understand how different representations of a single concept relate to each other (Ainsworth, 2006). Research demonstrates that many students are not able to sufficiently interact with/interpret external representations (Waisman, Leikin, Shaul & Leikin, 2014) let alone coordinate or translate between representations of a single concept (Even, 1998). In other words, students often lack the "representational flexibility" needed for proficiency. The ability to "coordinate the translation and switching between representations within the same domain" is vital to students' representational flexibility (Acevedo Nistal, Van Dooren, Clarebout, Elen & Verschaffel, 2009, p. 628) and is characteristic of students who excel in mathematics (Waisman et al., 2014).

The coordination of mathematical representations is particularly important in the domain of *functions* (Nyikahadzoyi, 2015). Scholars posit that the concept of function is one of the most important aspects of mathematical understanding as it conveys relationships between variables in problem solving (Eisenberg, 1991), is the basic unit for learning algebra (Yerushalmy & Shavarts, 1993), and is a crucial foundational knowledge in calculus comprehension (Dreyfus, 1990). The function is also a central concept in students' education from primary years through graduate school (Dubinsky & Harel, 1992) and is particularly vital throughout the secondary and undergraduate years (Thompson, 1994). Aside from the utility of understanding the concept of function from an educational standpoint, Laughbaum (1999) notes that the function is ultimately a means for all to examine relationships that play out in our everyday lives.

Given the importance of external representations and the concept of function, reform approaches to teaching calculus (e.g., Hughes-Hallett et al., 2010) present students with multiple canonical representations of the same function (where canonical representations include graphs, tables, algebra-symbolic equations, or text descriptions). Despite an emphasis on this "'Rule of Four,' [in which] ideas are presented graphically, numerically, symbolically and verbally" (p. ix), research demonstrates that students have difficulty translating (or coordinating) between these representations (Bossé, Adu-Gyamfi & Cheetham, 2011; Van Dooren, De Bock & Verschaffel, 2012). While research has examined translation (or coordination) skills in the context of small-scale descriptive studies (Adu-Gyamfi & Bossé, 2014; Kendal & Stacey, 2003), to our knowledge, no study has examined the requirements for students to coordinate different representations of function in mathematics textbooks. In this study, we examine coordination of MR (CMR) of function in a reform calculus textbook through analyzing tasks that engage students in translating from one representation of function to another (we call these tasks CMR tasks or coordination tasks).

In the next section, we define CMR and CMR tasks. Following this, we review students' difficulties in coordinating different representations of function in different types of coordination tasks. Subsequently, we examine the literature on mathematics textbook analyses and the limited insights it provides as to the prevalence and nature of coordination tasks in curriculum texts. Insight as to how CMR tasks may be used differently within mathematics textbooks is then provided, given pedagogical concerns based on the topic covered, the perspectives promoted by the different tasks and their constituent representations, and whether or not students are asked to translate between representations via perception versus construction.

To shed light on (1) the prevalence of CMR tasks in a reform calculus textbook, (2) the distribution of CMR tasks by chapter/topic, and (3) the frequency with which students are asked to either perceive or construct CMR tasks, we analyzed 707 of these tasks in a widely used reform calculus textbook. Results suggest that different coordination tasks are used earlier versus later as text content progresses and for different topics. Finally, the implications of this research for teaching and learning calculus are discussed.

Coordination of Multiple Representations (CMR)

In this study, we limit our examination of CMR to functions in mathematics. Before explicitly defining CMR in and of itself, it is necessary to specify the terms *representation* and *coordination*. In this regard, Lithner's (2003) framework for defining components and properties of mathematical tasks is useful.

In examining mathematical tasks, Lithner (2003) delineates the object as the basic entity or thing that one interacts with or which results from such an interaction; this includes "numbers, variables, functions, graphs, diagrams, matrices, etc." (p. 33). Here, we focus our discussion on objects that represent functions, including tables, graphs, verbal descriptions, and algebra-symbolic formulas; thus, a "representation" refers to one of these four canonical representations of functions in mathematics. Second, Lithner (2003) describes a transformation as an action applied to one or several objects that, as a result, produces other objects. In this context, "coordination" of representations could be considered as a transformation where the objects being acted upon are one or more representations (the word "translation" is also commonly used). For example, one might begin with an equation for a function and produce a graph from it (algebra-symbolic→ graph). Conversely, one could examine a textual description of a function and then examine a provided graph to determine if the features of the graph match the description (text→graph; see Janvier, 1987 and Dick & Edwards, 2008 for a typology of skills required for translation to and from graphs, tables, formulas, and sentences in a 4 × 4 table). In this context, CMR is defined as the process of interacting with and translating between (transforming) two canonical representations (objects) of a function.

There is a large number of possible coordination skills given both the representations considered (i.e., formula, graph, table, or text) and the directionality of the coordination (e.g., from graph-to-table or vice versa). Below, we review the small literature on four categories of coordination skills: transitions to algebra-symbolic representations (A), transitions to graph representations (G), transitions to tabular representations (Ta), and transitions to text representations (Te), demonstrating the many difficulties students face in performing CMR. While we define CMR as the process of interacting with and translating between two canonical representations of a function, we necessarily define the "CMR task" as the specific requirement for students to engage in CMR between these two representational forms (e.g., to coordinate between the table and the graph of a function). In other words, a CMR task is one that requires students to coordinate two different canonical representations of function or to translate from one canonical representation of function to another (e.g., the table and the graph of a function). In proceeding we use the terms "CMR task" and "coordination task" interchangeably.

CMR Tasks in the Extant Literature

Coordination Tasks Involving Transitions to Algebra-Symbolic Representations

G→A coordination tasks involve "reading off" the slope or shape (e.g., cubic, logarithmic) and intercept from a graph and representing these appropriately in symbolic form. Often, students are not expected to identify the function precisely; rather, they could be asked to match a graph to one of a selection of equations (Elia, Panaoura, Eracleous & Gagatsis, 2007) or to roughly estimate the equation (Kendal & Stacey, 2003). Students might also connect patterns in the distribution of table values to characteristics of the corresponding equation (e.g., whether it is linear or exponential, Hughes-Hallett et al., 2010). Similar behaviors could be required in completing transitions from text representations. For example, students could construct an equation given a text description of a functional relationship (Gagatsis & Shiakalli, 2004).

Although students tend toward algebraic manipulation during problem solving (particularly due to difficulties with graph comprehension, Elia et al., 2009), students often cannot complete coordination tasks to algebra-symbolic representations accurately and efficiently. Scholars highlight the difficulty students have in producing equations from graphs specifically (G→A coordination tasks, Duval, 2006). Recently, Bossé, Adu-Gyamfi, and Chandler (2014) used a collective case study method to examine the steps students take during translation from the graph of a polynomial function to its equation. When 24 high school pre-calculus students were presented with the task, they found that just over half of the students were able to successfully construct the target symbolic representation given its graphical form.

Coordination Tasks Involving Transitions to Graph Representations

Many textbook examples provide a function in algebra-symbolic notation with pairs of values to plot; this encourages students to think in terms of plotting points rather than plotting functions (Elia et al., 2007, 2008; Monoyiou & Gagatsis, 2008). Undergraduates are also presented with A→G problems, but with more complicated functions. Students may also be asked to create a graph using values presented in a table. Graphing specific values is frequently practiced in middle school mathematics (Even,

1998) and entering data in tabular format and creating a graph is a very common scientific practice (Roth & Bowen, 2003). In transitions from text representations, students might be asked to sketch or estimate a graph given a verbal description of a functional relationship (Gagatsis & Shiakalli, 2004).

Students across grade levels have trouble with coordination tasks to graphical representations, for instance, falling into the "graph as picture" fallacy (Dugdale, 1993) or constructing inaccurate graphs (Geiger, Stradtmann, Vogel & Seufert, 2011). Van Dooren et al. (2012) found that undergraduates had the most difficulty completing A→G coordination tasks when compared to other transitions. They attribute this to students' general tendency to perceive functions locally (i.e., as a series of individual points) rather than globally, using concrete values to test for correspondence rather than assessing the order of the equation and its relation to the shape of a graph. In a more recent study, the same researchers presented 65 university students with a 24-item multiple choice test on functions; students were presented with either a table, equation, or graph and directed to identify its form in an alternate representation (i.e., a table, equation, or graph, depending on the source of representation). Similar to their previous study, pairwise comparisons indicated students scored least accurately on problems involving transitions between equations and graphs, particularly in A→G coordination tasks (De Bock et al., 2015). Leikin, Leikin, Waisman & Shaul (2013) also found that for high school students, matching corresponding equations and graphs, A→G coordination tasks posed more difficulty than their reverse form (i.e., G→A coordination tasks).

Coordination Tasks Involving Transitions to Tabular Representations

In transitions to tables, students are often asked to create a table of values from a function given in algebra-symbolic notation. G→Ta coordination tasks involve reading off approximate values from a graph and entering them appropriately into a table. Table entries are only expected to be rough estimates, but values that are off by more than reasonable estimation error signal a lack of coordination skills. Generating a table from a passage or verbal description of a function is possible as well, but the task could require much estimation if a symbolic representation were not included as an intermediate step. Rather, students may be presented with a function in verbal form and asked to relate the situation to a presented table of values.

In comparison to the other types of transitions, student management of coordination tasks to tables receives less attention in the literature. Still, researchers have noted that students may produce tables as an intermediate step linking graphs and equations and that producing a table from a given representation may be easier for students than producing the algebra-symbolic form (Geiger et al., 2011). In fact, Van Dooren et al. (2012) found undergraduate students were more accurate in completing coordination tasks to tables in comparison to CMR tasks to graphs or equations, suggesting that having concrete values to test for correspondence makes these transitions easier for students to complete.

Coordination Tasks Involving Transitions to Text Representations

In transitions to text, students given algebraic expressions are asked to interpret or verbalize features of the function (Geiger et al., 2011). Transitions from graphs or tables might require students to verbalize their understanding of the features of a functional relationship in plain text (Andrà et al., 2015; Geiger, et al., 2011) or might require students to produce/identify a qualitative interpretation where the covariation between x and y values is described in a real-world context (Gagatsis & Shiakalli, 2004; Mahir, 2010).The distinction between verbal descriptions of functions that are purely mathematical (e.g., defining the slope or shape of a function) versus text descriptions of natural or everyday situations that require mathematical modeling (e.g., in the form of a symbolic equation) should be noted as one layer of complexity that students must maneuver in working with text representations.

Given the scope and complexity of text representations, it is unsurprising that coordination tasks to text pose much difficulty for students. Geiger et al. (2011) demonstrated students' trouble with producing verbal representations that placed functions in a real-world context; students were more successful at using plain text—both describing the syntactical features of the representation and describing the function in a purely mathematical sense. Formulating text representations within a real-world context was particularly problematic for students presented with equations and tables. However, other research demonstrates that students have trouble completing G→Te coordination tasks specifically; Mahir (2010) found that undergraduates in calculus had trouble interpreting graphs absent cues providing a real-world context. Gagatsis and Shiakalli (2004) also found that undergraduates had low success rates in completing G→Te coordination tasks involving situational text and suggested that students fail to link the two representations conceptually. Instead, students in their study used equations as an intermediate linking graphs and text, failing to come to "an integrated and functional grasp" of the graph and verbal forms (Kilpatrick, Swafford & Findell, 2001, p. 118); students were more accurate in completing A→Te coordination tasks. By contrast, Andrà et al. (2013) used eye-tracking data to examine 46 university students' gaze behavior during completion of 43 multiple choice items pertaining to A→Te, Te→A, and G→Te coordination tasks specifically (i.e., students were presented with either a formula, plain text description, or graph and were asked to identify which of the four either plain text descriptions [when presented with formulas or graphs] or formulas [when presented with a plain text description] matched the given representation). In scoring student answers, they found that students were least accurate in completing A→Te coordination tasks.

In sum, despite the importance of coordination skills to students' mathematical understanding, research demonstrates students' difficulty with coordination, particularly when graphs are involved (Geiger et al., 2011; Mahir, 2010; Van Dooren et al., 2012). In this paper, we study how students' performance of CMR might be influenced by their educational context, particularly that of mathematics textbooks.

CMR in Mathematics Textbooks

Of course, students' ability to coordinate MR depends in part on their educational context. While emphasis could be rightly placed on classroom instruction, such instruction is often guided by curriculum materials—most often, the mathematics textbook. Often defined as part of the intended curriculum, research demonstrates that textbooks have great influence on the implemented curriculum (Johansson, 2005).

Content presented in math textbooks is more likely to be presented by teachers (Reys, Reys, Lapan, Holliday & Wasman, 2003) and the way topics are taught in textbooks influences teachers' pedagogy; in fact, teachers cite textbooks as their main reference in choosing pedagogical strategies (Schmidt et al., 2001). Aside from influencing instruction, the presentation of representations in textbooks can have a direct effect on student learning (McGee & Martinez-Planell, 2014), particularly as students not only structure their textbook use at the behest of their instructor but also use textbooks for self-directed learning (Rezat, 2009). Some scholars even suggest textbook characteristics are related to student performance on a national scale (e.g., that US texts containing more visual information vs. Chinese texts could be related to US students' higher performance on test items with visual representations; Zhu & Fan, 2006).

Extant research on mathematics textbooks offer limited insights when it comes to CMR. Studies that focus on teacher interactions with textbooks tend to examine how teachers rely on texts in designing mathematics lessons (Nicol & Crespo, 2006; Remillard, 2005). For example, Nicol and Crespo (2006) highlight how preservice teachers' lesson planning varied from more strict adherence to text content and presentation to more flexible creation alongside the text; adherence to the text was seen as more efficient, a stance that may be important given the current environment of high stakes testing in schools. Studies also demonstrate how math teachers gain content knowledge from textbooks (Davis, 2009; Remillard, 2005). Davis' (2009) study examining the influence of textbook use on high school mathematics teachers' comprehension of exponential functions demonstrated that textbook content mattered for both teachers' pedagogical content knowledge and mathematics skills. One text they examined promoted teachers' understanding of CMR tasks from tables to equations. However, data on how this was related to the content of the text (e.g., frequencies of text modeling Ta→A coordination tasks) are not provided.

Just as studies examining the influence of textbooks on teachers is limited in terms of their treatment of CMR, so too are studies that focus on student interactions with their mathematics textbooks. Studies often emphasize the failure of students to interact appropriately with their texts (Lithner, 2000, 2003, 2004; Shepherd, Selden & Selden, 2012; Weinberg & Wiesner, 2011). For example, Lithner (2000) asserts that students fail to grasp the intrinsic or central properties of mathematical ideas and practices as presented in textbooks, defaulting to superficial reasoning in solving text exercises (aided in part by the availability of these strategies given text content). Aside from a focus on problem solving, other scholars focus on student reading of mathematics textbooks, highlighting the complexities involved (Shepherd et al., 2012; Weinberg & Wiesner, 2011). For instance, Weinberg and Wiesner (2011) describe the diverse characteristics textbooks possess that readers must manage, including implicit and explicit directives, varied presentation formats, and symbols. In addition, readers must have the mathematical skills needed to meaningfully and accurately interact with the text; in this context, CMR may be considered both a behavior a text requires and a competency students must have in order to engage appropriately with their text.

Content analyses of mathematics textbooks are often limited to examining aspects of instruction within specific topics of the curriculum (e.g., Jones & Tarr, 2007 on cognitive demand in tasks on probability; Mesa, 2010, on verification strategies in initial value instruction; and Sood & Jitendra, 2007, on number sense instruction across traditional vs. reform texts). While these studies offer little insight into the nature of

CMR in textbooks, analyses examining problem solving in mathematics textbooks hint at the use of CMR tasks in texts. In their examination of the cognitive complexity of problems in best-selling mathematics textbooks in Australia, Vincent and Stacey (2008) found that textbooks generally do not present students with problems of high procedural complexity, which are most likely to include CMR tasks. Other studies suggest that US textbooks are more likely to include problems requiring CMR. For one, Fan and Zhu (2007) found that US texts had students make tables more frequently than mathematics textbooks in China or Singapore. In addition, Zhu and Fan (2006) compared the use of symbolic, verbal, and visual representations across Chinese and US mathematics textbooks and found that US texts were twice as likely to present students with problems that combined two of these representation types.

Thus, aside from these more general notions on the prevalence of CMR tasks in textbooks, the extant literature offers little guidance as to how often students are asked to perform each type of CMR task (e.g., Ta→A vs. G→A). Unlike prior content analyses of mathematics textbooks that tend to focus on only one topic within mathematics (Jones & Tarr, 2007; Mesa, 2010; Sood & Jitendra, 2007) or focus only on textbook exercises (Vincent & Stacey, 2008; Zhu and Fan, 2006), the current study will examine how often a reform calculus textbook require students to perform CMR tasks across four chapters, inclusive of expository text, worked examples, and text exercises. If CMR is intended as a core competency, our analysis will shed light on the specific coordination skills that are required from students as they progress through text content. Alternatively, CMR may be a cumulative skill that is built gradually as the text progresses; if this is the case, our analysis will provide a sense of how quickly CMR tasks build in frequency across chapters. In any case, since both teachers (Davis, 2009) and students (Lithner, 2003) have demonstrated an insufficient understanding of the unifying principles of the mathematics textbooks they use, this analysis should illuminate the centrality and nature of CMR within reform calculus.

Pedagogical Perspectives on the use of CMR Tasks

The prevalence of specific CMR tasks may vary based on differences in content that are inherent between topics or given pedagogical concerns where CMR tasks are used differently across chapters of the textbook (e.g., with more difficult tasks appearing in later chapters). Extant literature suggests that the topic addressed affects which CMR tasks are most often presented to students. For example, Davis (2009) suggests that CMR tasks involving tables are least common in content on exponential functions. In addition, Mesa (2010) suggests that in text content on initial value of derivatives, symbolic and verbal representations are most common with graphs being used less frequently. Still, we might expect to see all coordination tasks equally represented within a reform calculus textbook, since each is vital to conceptual understanding. Nitsch et al. (2015) argue that all possible translations are pertinent in assessing student proficiency such that "all types of translation [among graphs, tables, symbolic and verbal forms] should therefore be represented in textbooks, equally spread, and exercises should vary across all types of translation" (p. 674).

However, even a cursory review of a reform calculus text (e.g., Hughes-Hallett et al., 2010) suggests that CMR tasks are used differently by the authors depending on the topic under study. For example, when covering features of functions like maxima and minima, transitions to and from graphical representations may be important pedagogically since these features are easily illustrated using graphs. In addition, as Bell and Janvier (1981) posit, coordination tasks to tabular representations may be particularly helpful in concretizing behavior in certain areas of a function such as areas of slow versus fast growth and inflection points by having students monitor changes in adjacent ordered pairs. By contrast, when covering symbolic rules for determining derivatives, reform textbooks may rely less heavily on CMR tasks in general given the rules' emphasis on symbolic manipulation. In this way, topics under study may drive reliance on specific coordination tasks while deemphasizing others in illustrating concepts.

CMR tasks may also be used differently since representations can promote either a process or object perspective of functions (Moschkovich et al., 1993). Algebra-symbolic representations tend to be perceived by students from a process perspective where each x value is linked to a specific y value, promoting an understanding of the procedural characteristics of a function (Kölloffel, de Jong & Eysink, 2005). For example, students tend to plot ordered pairs when given an equation rather than assess the overall form of the equation and relate it to the shape of a graph (Monoyiou & Gagatsis, 2008). This process perspective also applies to tabular representations, which students tend to see as a collection of ordered pairs rather than values representing a functional relationship. Conversely, text or graph representations tend to promote an object perspective, where functions are understood as entities rather than as a series of points (Moschkovich et al., 1993). Graphs also provide distinctly qualitative information about, for example, the trajectory or direction of the functional relationship (Ainsworth, Bibby & Wood, 2002). In order to reinforce both perspectives, CMR tasks that include process-object representation pairs may be favored over coordination tasks that adhere to either the process or object perspective alone.

Perception Versus Construction in CMR Tasks

Researchers have pointed out a critical difference between mathematical tasks where learners are asked to construct a new representation from a given one versus being asked to perceive some relationship between two given representations (which may be termed *interpretation*). As Nitsch et al. (2015) explain, "construction requires an action of generating new parts that are not provided … [whereas] interpretation refers to all actions by which a student makes sense of or acquires a meaning from a specific form of a representation" (p. 664). Thus, translation (or coordination) tasks could require students' to construct a corresponding representation given another one (a construction task), recognize the same function in different representational forms, or identify the corresponding function in one representation given another one (interpretation tasks). As Leinhardt, Zaslavsky & Stein (1990) note, "whereas interpretation does not require any construction, construction often builds on some kind of interpretation" (p. 13). For example, if presented with a graph, students could be asked to either write the corresponding equation (a construction task) or choose the correct equation among a series of options (an interpretation or perception task). Given the differential learning advantages of the two tasks, it may be important to distinguish between tasks that require student construction versus perception. This distinction is often not explicitly defined in the current literature on CMR in mathematics (but see Nitsch et al., 2015).

Research Questions

Prior research demonstrates the importance of translation/coordination skills for students in understanding functions (Acevedo Nistal et al., 2009), the wide range of CMR tasks possible (Dick & Edwards, 2008), and the difficulty students have in performing CMR (e.g., Bossé et al., 2014; De Bock et al., 2015; Van Dooren et al., 2012). In addition, while the literature on mathematics textbooks establishes the significance of analyzing texts for insight into student learning, it offers little guidance as to the prevalence or nature of CMR tasks in textbooks. Given this, we catalogued the CMR tasks in a reform calculus textbook and compared which tasks were required as students move from introductory to later chapters. Documenting the prevalence of different CMR tasks this way can help situate prior studies of particular types of CMR tasks in the subtopics where these tasks are most often required and inform future analyses of students' coordination processes. In this study, we asked the following research questions: (1) What is the prevalence of different CMR tasks in a reform calculus textbook? (2) Does the distribution of CMR tasks vary by chapter/topic? And (3) does the distribution of CMR tasks vary by whether or not students are asked to perceive or construct the coordination?

Method

Reform Calculus Textbook

We analyzed CMR tasks from the reform calculus textbook, *Applied Calculus 4/E* by Hughes-Hallett et al. (2010). This text originates from the Harvard Calculus Consortium (HCC), which produced reform teaching materials that are among the most commonly used across college campuses (Ganter, 2001). As noted in the preface to *Applied Calculus*, this text takes a reform approach that foregrounds practical application of mathematical principles—an approach that is in danger of being lost in calculus courses when teaching is focused primarily on methods and formulas (Lithner, 2003). With contributions from educators and professionals across levels of education and applied fields, the text aims to emphasize equally both theoretical content and modeling for application, symbolic manipulation and technological use, and mathematical skills (e.g., through drills) and concepts. The text utilizes the "Rule of Four" in which "ideas are presented graphically, numerically, symbolically, and verbally" (Hughes-Hallett et al., 2010, p. ix).

The text consists of 11 content chapters ranging from a review of functions to geometric series designed to be covered either over a two-semester sequence or within a one-semester course with specific topics covered chosen by the instructor. In our analysis, we examined the first four chapters of *Applied Calculus*: (1) Functions and Change, (2) Rate of Change: The Derivative, (3) Short-cuts to Differentiation, and (4)

Using the Derivative. These chapters were chosen as they present foundational topics for understanding functions and their derivatives as well as emphasize the practical application of derivatives.

Coding

Below, we give our definition of the CMR task as the unit of analysis, describe the coding scheme as applied, and explain our procedure for coding and inter-rater reliability.

Defining the CMR Task as the Unit of Analysis. Given the complexity of coding entire chapters from an existing calculus reform textbook (rather than isolated problems), precise parameters for identifying CMR tasks needed to be specified. This study defined a CMR task as any instance where students are required to coordinate or translate between two different canonical representations of a function. Often, students are asked to engage in the use of a single representation rather than coordinate between two representations. For example, students might be asked to identify the value of a local maximum of a function from a table; this task requires engagement in table use, not coordination. By contrast, a problem asking students to identity a local maximum of a function from a table and then plot its graph do require engaging in CMR (Ta→G coordination). Alternatively, students may be presented with items that *could* be solved using CMR, but do not require it. For example, a problem asking students to find $f'(x) = 3\ln x - 7$ at some specified value of x could be solved using CMR—students could graph the function and locate the $f'(x)$ value on the graph constituting an A→G coordination, or students could forgo coordination and solve the problem numerically (i.e., by "plugging and chugging"). A problem explicitly directing students to use a graph of this function to determine the value would require students to solve via CMR. Given that students often default to calculation (Acevedo Nistal, Van Dooren & Verschaffel, 2012), we focused our attention on tasks *requiring* CMR in order to avoid collecting uninformative data. This excluded many tasks for which students were to provide numerical answers, unless the text specifically directed students to perform CMR (across the four chapters, this excluded 543 tasks from the analysis).

In addition, tasks were only coded if the student had to engage with two representations beyond the coordination of a single point of the function. For example, a problem asking students to simply interpret the meaning of the vertical intercept of a graph does not require CMR as it is defined here since students are only asked to determine the meaning of one particular point of the function. In this instance, students might engage in CMR by determining the meaning of the intercept *in relation to* the rest of the function, but since this type of thorough engagement in coordination was not specified given a directive to interpret the intercept only, we excluded such cases as not *specifically requiring* CMR of a function (although it does require coordination of one particular point on a function). By contrast, a problem presenting students with a graph of a function and directing them to describe the behavior of the function requires engaging in CMR (G→Te coordination) beyond the coordination of a single point.

Finally, it should be noted that we were only concerned with coordination tasks involving two different types of representations as opposed to coordination tasks between representations of the same type. For example, one could define an A→A coordination task via symbolic manipulation or Ta→Ta coordination tasks via data transformation. Since virtually none of the prior literature reviewed here is inclusive of these types of tasks (but see Dick & Edwards, 2008, who provide a theoretical framework for such tasks), we were comfortable excluding them from this analysis. In other words, since the extant literature on coordination tasks is almost entirely concerned with translations between two different types of representations, we chose to continue in this same vein. Thus, we examined the prevalence of 12 different CMR tasks (A→G/Ta/Te, G→A/Ta/Te, Ta→A/Ta/Te, and Te→A/G/Ta).

Coding Within Three Presentation Formats. Our coding of CMR tasks in the textbook was more complex than what might be suggested by studies analyzing exercise problems only. The CMR task as the unit of analysis needed to be distinguished across presentation formats in the textbook, given that content could be presented in one of three different formats: expository text, worked examples, and exercises.

These varied formats posed different challenges given the directionality inherent in translational tasks, where for instance, the cognitive processes required to perform G→A coordination are different from those required for A→G coordination. As Nitsch et al. (2015) note, it is difficult to assess directionality in this process for students since it often involves many translations between the two representations in question (i.e., it is not linear). However, our concern here was not the cognitive processes of students during CMR (where it may be inappropriate to distinguish directionality), but the presentation of CMR tasks in a textbook where consistent standards for directionality could be established.

To establish these standards the different formats of the text (i.e., expository, worked examples, and exercises) necessitated three slightly different coding procedures. In expository text, content was presented to students in paragraphs with the inclusion of equations and table/figure references to further illustrate concepts as needed. Given this format, the text was segmented by paragraph in order to isolate specific cases where CMR tasks could arise. CMR tasks were identified through a careful reading of each paragraph with attention paid to the order of presentation of representations (e.g., the presentation of a text representation followed by a reference to a provided table of the function, Te→Ta coordination) and the use of single representations as topics of entire paragraphs. As such, it was not uncommon for multiple CMR tasks to be identified within a single paragraph (e.g., see Fig. 1a).

The second presentation format, worked examples, were scattered throughout expository text to allow students to practice application alongside modeled solutions and explanations. For this format, CMR tasks were first assessed by identifying the coordination of two distinct modes of representations—the source, or initial representation, and the target, or final representation (Gagatsis & Shiakalli, 2004). Source representations were identified based on the information students were given in order to solve the problem whereas target representations were identified based on the information students were directed to construct or perceive as part of the solution. For instance, students might be provided with a text representation of a function (source) and be directed to identify which of the three provided graphs represent that function (target). Once CMR tasks were identified by assessing these source-target transitions, text-provided explanations of the solution were examined. This format required these explanations be examined separately since additional CMR tasks might be provided as explanation of the solution beyond what was required of students in the

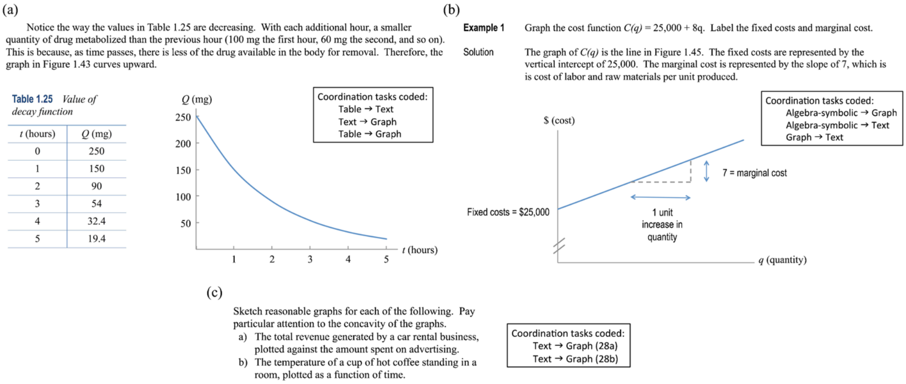

Fig. 1 Examples of coding across three presentation formats. (**a**) Coding within expository text. This example shows the directionality of coding within a single paragraph. This paragraph begins by introducing a table of values for a function followed by a text description of the behavior of the function. Following this text description, the paragraph then concludes with figure references directing the reader to a graph of the function. According to the coding scheme, this paragraph contains three CMR transitions: a Ta→Te coordination, a Te→G coordination, and a Ta→G coordination, linking the initial table representation introduced as the topic of the paragraph to the final graphical representation. All three tasks presented are "perceive" tasks. (**b**) Coding within a worked example. This example shows the directional of coding within a worked example. This problem direct students to sketch a graph of a provided algebra-symbolic equation. The equation provided is identified as the source representation coordinated with the target, or student-generated graph (A→G coordination). In the solution, the textbook then provides the graph as a means of modeling the answer followed by a text description of the behavior of the function. Two additional coordination tasks are identified as A→Te (the original source representation as tied to text in the solution) and G→Te (the original target representation as tied to text in the solution). The problem directing students to graph the function is a "construct" task. By contrast, both tasks involving the text provided in the solution are "perceive" tasks. (**c**) Coding textbook exercises. This example shows the directionality of coding within textbook exercises. Each sub-item is assessed as a potentially separate coordination task. For each sub-item, students are directed to construct a graph given a verbal description of a function (two Te→G tasks). Both of these are "construct" tasks

original problem. If this occurred, additional representations were coordinated with both the original source and target representations (e.g., see Fig. 1b).

Identifying CMR tasks in exercise portions of the text was more straightforward. These exercise sections were found at the end of each subchapter and following each chapter as a whole. Each item (e.g., item 1 or 2) or when applicable, each sub-item (item 1a or 1b), was considered separately to determine if the textbook directed students to perform CMR. Since these problems generally resembled those presented in worked examples (less the modeled solution), CMR tasks were similarly assessed considering source-target representations given the directives to students within each problem (e.g., see Fig. 1c).

Perceive Versus Construct. Once the type of CMR task was identified, the coordination was coded for whether students were asked to "perceive" or "construct" the corresponding target. Students were often asked to perceive coordination between representations within the expository text or worked example solutions. Alternatively, some exercises might ask students to match one representation to its equivalent given a set of options. Students were asked to construct target representations (e.g., make a graph, write an equation) in the context of worked examples and exercises. Here, students were given one representational form and were asked to create the second (see Fig. 1a–c for examples).

The Coding Process and Inter-rater Reliability. The first author coded 707 CMR tasks from the first four chapters. The second author was trained on data aside from those used to calculate inter-rater reliability and re-coded 35 % of the corpus (including approximately 250 CMR tasks). Two types of discrepancies were common. First were instances where one coder failed to identify a coordination task within a paragraph of expository text where multiple CMR tasks were present. For example, one coder might have passed over the Ta→G coordination task in Fig. 1a; however, as described previously, given that the first sentence of the paragraph identifies the table as the topic of the passage, this representation should be coded as tied to the graph at the end of the paragraph. Second were discrepancies in the directionality of the transition. For instance, in Fig. 1b, one coder might have mistakenly identified a Te→G transition (as opposed to a G→Te task) given the physical location of the text as above the graph on the page. However, as described above, since the text directs the reader to the figure before applying the figure to its real-world description, the graph is treated as the source representation. For both types, discrepancies were resolved by discussion and joint reading of the text passages pertaining to the coordination tasks under question while assessing whether or not a representation was used as a topic of an entire paragraph and logging the order of the presentation of different representations in the text. Inter-rater reliability was assessed using the kappa statistic; agreement was considered good to excellent (Cohen's kappa = .79; Fleiss, 1981).

Data Analysis

To compare the prevalence of CMR tasks across chapters, we used a series of chi squared tests (or Fisher's exact tests in the case where 0 cell counts were found; Ramsey & Schafer, 2002); separate tests were performed for each transition. Chi square tests were performed in SPSS version 21 and Fisher's exact tests were performed using SISA-Binomial, a web-based applet for performing common statistical analyses (Uitenbroek, 1997). Cramer's V statistic was used to estimate effect size where associations less than .10 are considered weak, .11 – .30 are moderate, and greater than .30 are strong (Healey, 2009). We also conducted an analysis of tasks where students must perceive the coordination versus those where students must construct a new representation. For this reason, we adjusted our alpha level to $p < .05/2 = .025$.

Results

Table 1 displays the cell counts and row/column percentages for each type of task, regardless of whether it required the student to perceive or construct the coordination. The most common tasks were A→G ($n = 232$), followed by Te→A ($n = 115$), Te→G ($n = 95$), and G→Te ($n = 93$). In comparison to these transitions, A→Te ($n = 39$), G→A ($n = 45$), Ta→A ($n = 31$), and Ta→Te ($n = 32$) tasks were much less frequent but still relatively common. A→Ta ($n = 4$), G→Ta ($n = 1$), Ta→G ($n = 13$), and Te→Ta ($n = 7$) tasks were relatively rare. The row percentages indicate that, for example, A→G transitions comprised 22 % of the total number of CMR tasks in chapter 1. The column percentages indicate that, for example, chapter 1 contained 41 % of the total number of A→G tasks across the four chapters.

Table 1 Overall distribution of CMR tasks among chapters

		Chapters																Total			
		1				2				3				4							
	From\to	-A	-G	-Ta	-Te	-A	-G	-Ta	-Te	-A	-G	-Ta	-Te	-A	-G	-Ta	-Te	-A	-G	-Ta	-Te
n			94	3	28		12	0	1		44	0	4		82	1	6	–	232	4	39
Row %	A-	–	22	1	7	–	27	0	2	–	64	0	6	–	51	1	4				
Column %			41	75	72		5	0	3		19	0	10		35	25	15				
n		37		1	58	0		0	13	6		0	4	2		0	18	45	–	1	93
Row %	G-	8	–	0	13	0	–	0	29	9	–	0	6	1	–	0	11				
Column %		82		100	62	0		0	14	13		0	4	4		0	19				
n		29	3		25	0	4		6	0	0		0	2	6		1	31	13	–	32
Row %	Ta-	7	1	–	6	0	9	–	13	0	0	–	0	1	4	–	1				
Column %		94	23		78	0	31		19	0	0		0	7	46		3				
n		101	50	4		0	8	1		6	5	0		8	32	2		115	95	7	–
Row %	Te-	23	11	1	–	0	18	2	–	9	7	0	–	5	20	1	–				
Column		88	53	57		0	8	14		5	5	0		7	34	29					
Total		433				45				69				160				707			

The table displays observed counts with row and columns percentages

A algebra-symbolic, *G* graph, *Ta* table, *Te* text

Chi square tests showed a significantly nonrandom distribution of transitions among chapters for 8 of the 12 possible CMR tasks. These results are summarized in Fig. 2 below.

Regarding transitions to algebra-symbolic representations, G→A, Ta→A, and Te→A tasks were nonrandomly distributed among chapters. As Fig. 2 shows, G→A transitions were overrepresented in chapter 1 and underrepresented in chapter 4 ($p < .001$; $\varphi_c = .142$), as were Ta→A ($p < .01$; $\varphi_c = .143$) and Te→A tasks, χ^2 (3, $n = 707$) = 42.40, $p < .001$; $\varphi_c = .245$. The size of these associations is moderate.

For transitions to graph representations, A→G, Ta→G, and Te→G tasks were nonrandomly distributed among chapters. As Fig. 2 illustrates, A→G transitions were overrepresented in chapter 4, χ^2 (3, $n = 716$) = 79.675 $p < .001$, with effect size estimates indicating a strong association ($\varphi_c = .338$). Ta→G tasks were overrepresented in chapter 2 and underrepresented in chapter 1 ($p < .01$; $\varphi_c = .168$) and Te→G tasks were overrepresented in chapter 4 and underrepresented in chapter 1, χ^2 (3, $n = 707$) = 10.26, $p < .025$; $\varphi_c = .120$, both moderate associations given effect size estimates.

Figure 2 also shows that all identified CMR tasks to table representations (i.e., A/G/Te→Ta) had observed counts that did not differ significantly from expected counts ($p > .025$).

Finally, for transitions to text representations, G→Te and Ta→Te tasks were nonrandomly distributed among the chapters. Whereas G→Te tasks were overrepresented in chapter 2 and underrepresented in chapter 3, χ^2 (3, $n = 707$) = 13.55, $p < .01$; $\varphi_c = .138$, Ta→Te tasks were overrepresented in chapter 1 and underrepresented in chapter 4 ($p < .01$; $\varphi_c = .162$). The size of both associations is considered moderate.

Appendix A displays the cell counts and row/column percentages for each type of CMR task, separated by those that required the student to perceive the coordination versus construct the target representation. As these data show, students were presented with CMR tasks in which they were directed to construct the target representation ($n = 454$) nearly two times more often than they were directed to perceive a coordination ($n = 253$). Among perceived CMR tasks, the most commonly required were A→G ($n = 57$), G→Te ($n = 42$), Te→G ($n = 41$), and A→Te ($n = 33$). Among constructed

	From \ To	-A		-G		-Ta	-Te	
Over	A-	--		**79.65*** **	4	ns	ns	
Under				**(.338)**	1			
Over	G-	**FET** **	1	--		ns	**13.55** **	2
Under		**(.142)**	4				**(.138)**	3
Over	Ta-	**FET** **	1	**FET** **	2	--	**FET** **	1
Under		**(.143)**	4	**(.168)**	1		**(.162)**	4
Over	Te-	**42.40*** **	1	**10.26***	4	ns	--	
Under		**(.245)**	4	**(.120)**	1			

Fig. 2 Results of chi square/Fisher's exact test for analysis of the overall distribution of CMR tasks among chapters. Chi square of Fisher's exact test results (FET) are reported for significant, nonrandom distribution. Effects size is reported in parentheses. For significant tests, chapters where transitions were over- or underrepresented are also shown. Notation *ns* not significant. *$p < .025$, **$p < .01$, ***$p < .001$

CMR tasks, the most commonly required were A→G (n = 175), Te→A (n = 87), Te→G (n = 54), and G→Te (n = 51).

The results for differences in distribution of transitions among chapters by perceive versus construct CMR tasks are summarized in Appendix B. Chi square tests showed nonrandom distribution of transitions among chapters for 4/16 of the perceived tasks. For student-constructed CMR tasks, significant nonrandom distribution of transitions among chapters was present for 6/16 of the coordination types. The pattern was similar when broken down by construct and perceive, as compared to analysis overall.

Discussion

The use of MR of functions and translation between them has been shown to be an important part of student learning in mathematics. Specifically, the ability to translate between symbolic, graphical, tabular, or verbal forms of functions are vital to students' ability to "identify the connecting elements of a functional dependency and to combine these" (Nitsch et al., 2015, p. 673). Students' ability to coordinate MR promotes a more complete understanding of abstract functional relationships by utilizing the advantages each representation affords (Duval, 2006). While prior literature demonstrates the importance of coordination skills to students' understanding of mathematics, it offers little guidance in regards to the prevalence or nature of CMR tasks presented to students in their educational contexts—particularly in mathematics textbooks. The purpose of this study was to catalog the different types of CMR tasks presented to students in a reform calculus textbook. The goal was to shed light on the specific coordination skills required from students as they progress through text content. At a most basic level, these results highlight not only the importance of the "Rule of Four" as suggested by the HCC, but also how text authors might use translations between canonical representations to promote understanding of functional relationships.

Our examination of CMR tasks in a reform calculus textbook shows the variability that exists in the presentation of these tasks among chapters. We found coding that appeared straightforward when student exercises were analyzed (e.g., Kendal & Stacey, 2003) is much more complex when coding across chapters of a text. Limiting our examination to tasks that *required* students to CMR (and excluding over 500 exercises in which students provided numerical answers only), we found that a very wide range of CMR tasks was used, although the majority were comprised of a few specific types. Furthermore, the types of transitions that predominate differed by chapter, suggesting an implicit sequencing of types of CMR tasks.

Differentiating these analyses across four chapters shows that discussion of the prevalence of CMR tasks in texts should recognize the potentially differential use of these tasks depending on text progression and content of chapters. In some prior literature, for example, as in Zhu and Fan's (2006) discussion of representational use in US and Chinese texts, the notion of the distribution of representation as dependent upon the topic content is wholly absent. We interpret the overrepresentation of Te→A transitions in chapter 1 as the authors' attempt to reacquaint students with known concepts (e.g., linear, exponential, and periodic functions) and introduce students to more complex and potentially unfamiliar concepts (e.g., financial applications of exponential functions). For example, chapter 1 situates functions in expository text in real-world situations and interpretations of the parameters and shape of functions. Students are also given situational descriptions of functions from which they are to model the symbolic form. Use of Te→A coordination tasks in an introductory chapter may be well placed given students' accuracy in completing these types of transitions relative to other types (Gagatsis & Shiakalli, 2004). Conversely, Te→A tasks are underrepresented in chapter 4 as concepts advance towards more complex functional forms that may be more difficult to symbolically model from text descriptions alone (e.g., logistic and surge functions). In addition, chapter 4 more characteristically features transitions to graphs, as both A→G and Te→G tasks are overrepresented. Both of these types of CMR tasks are typically required when covering concepts easily illustrated graphically like global/local maxima and minima and inflection points. Furthermore, G→Te tasks were overrepresented in Chapter 2, which introduces the derivative as a rate of change. This type of CMR task is presented as the text provides students with graphs of functions and either directs students to construct or perceive verbal descriptions of the their features (e.g., concavity) so that these features can be understood in relation to the derivative. G→Te tasks are underrepresented in chapter 3 given the chapter's focus on symbolic rules for obtaining the derivative given equations of functions and relatively low emphasis on text representations overall. In chapter 4, the heavy emphasis on CMR tasks requiring students to construct graphs (i.e., A→G, Te→G) may be appropriate as opposed to placement in earlier chapters given students' overall difficulty with such tasks (Geiger et al., 2011).

The prevalence of Te→A and A→G tasks in chapters 1 and 4, respectively, highlights the importance of coordination tasks that promote both process and object perspectives of functions (Moschkovich et al., 1993). As described previously, algebra-symbolic representations tend to be perceived from a process perspective where each x value is linked to a specific y value whereas text or graph representations promote the object perspective where functions are understood as entities rather than a series of points. Since use of both perspectives is vital to understanding the concept of a function, it is not surprising that CMR tasks linking these two perspectives were relatively common.

This differentiated use of CMR tasks could be considered in conflict with the work of Nitsch et al. (2015) who assert that all types of translation tasks should be equally represented across textbook content. While we understand this argument as emphasizing the importance of developing student skill in each type of CMR task, there are two distinct perspectives from which to approach the issue. The first perspective is that it may be appropriate to incorporate each type of CMR task equally across instructional content, although this is not, in fact, what was observed in the context of this corpus. Such an approach could help ensure students are coming to an understanding of functional relationships in the most holistic way possible. The second perspective is that it may be too simplistic to broadly assert that content within a mathematics text should rely equally on all translation tasks; it seems sensible that selection of CMR tasks could depend on the topic undertaken, where, for example, it may be inappropriate to use transitions involving tabular representations in a chapter focused on the symbolic rules for the formulation of the derivative. In educational environments where time for learning content is often scarce, the use of all CMR tasks across topics could put added strain on educators. Regardless of which perspective is taken, Nitsch et al.'s (2015) call for the specific consideration of translational tasks by curriculum and textbook designers is valid.

What is most certainly confirmed in this analysis is the complexity of the requirements students face in reading and learning from their mathematics textbooks. As echoed by Weinberg and Wiesner (2011) who describe the numerous textbook features that pose difficulty for students, we, too, found that even for researchers familiar with text content, navigating the various presentation formats of the text, the implicit and explicit directives, and language and symbols was no small task. These aspects in combination with text requests for students to perform CMR tasks highlight the many competencies students must have in order to use texts effectively. As Davis (2009) advocates, mathematics teachers should work through their course textbooks as they intend their students to. Not only would this practice deepen their own content knowledge but would also ensure they can appropriately model proper textbook use for students. Furthermore, if accurate coordination between different representational forms is a goal of the mathematics classroom, this goal should be made explicit to students by heightening their awareness of opportunities to coordinate between representations as provided in their instructional materials.

Another contribution of this study is our separate coding of perceive and construct CMR tasks. If students were asked to perform the A→G task in the Hughes-Hallett textbook, it was more likely that they were asked to construct it rather than perceive it. Perhaps requiring construction of the A→G coordination forces students to move beyond visually plotting points to use graphing calculators, reinforcing the transition in a more global sense (Leinhardt et al., 1990). Constructing Te→A coordination tasks was also required more frequently than perceiving, perhaps forcing students to move beyond their interpretations of the data to effectively model functional relationships symbolically. The opposite pattern held true for A→Te, Ta→Te, and Te→Ta tasks which students were more frequently asked to perceive rather than construct. The first two tasks were very common in Chapter 1, suggesting that the authors may be focusing on building conceptual understanding. Most of the analyses by chapter yielded the same results when broken down by perceive and construct. The prevalence of construct tasks versus perceive was unsurprising given the textbook's emphasis on worked example problems and student exercises. However, students were still required to perceive CMR tasks relatively frequently, particularly in expository text and worked example solutions. Since research shows students are not reading math textbooks effectively (Shepherd et al., 2012), it is likely that students miss out on practicing these crucial coordination skills.

This study has implications for both practice and future research. First, this study highlights the need for instructional designers and teachers to evaluate the role that coordinating MR of functions is to play in their own practice. If CMR is to be treated as a central theme of mathematics instruction as much research suggests it should, it must be a clear consideration in lesson and curriculum design. Given the difficulty students have with specific CMR tasks as demonstrated in prior literature (e.g., Bossé et al., 2014; De Bock et al., 2015; Van Dooren et al., 2012), textbook designers and math instructors would do well to flag these tasks as ones in which students might need additional scaffolding.

In terms of implications for research, an important contribution of this study is the detailed coding scheme for coordination tasks in expository text, worked examples, and end-of-chapter exercises. An important step in the development of the coding scheme was operationalizing what a CMR task is in a textbook differentiated by presentation formats within each chapter (i.e., expository text, worked examples, student exercises). Our refined coding scheme also provided a means of determining directionality of the transitions across multiple presentation forms (i.e., expository text, worked examples, and exercises), which is often a difficult task (Nitsch et al., 2015). Such distinctions could aid future researchers seeking to characterize CMR tasks across mathematics texts utilizing approaches distinct from that of the reform text analyzed here or across best-selling texts from various countries. It could also promote the analysis of textbook content beyond that of exercises alone which is often the focus of textbook content analyses (Fan & Zhu, 2007; Vincent & Stacey, 2008, Zhu & Fan, 2006).

As with any study, these findings and implications must be discussed in light of the limitations of the study. First, we limited these analyses to four chapters of one reform calculus textbook. Although the textbook selected is popularly used and the chapters covered represent foundational knowledge in the teaching of calculus, we did not analyze analogous chapters in other calculus texts. Still, this focus was needed to make the research feasible since our unit of analysis (i.e., the CMR task) was at the micro level. Second, although there is a distinction in the literature between situational or real-world text versus plain text descriptions in verbal forms of functions (e.g., Geiger et al., 2011), our goal was not to characterize CMR tasks as they differed within one representational form. Making this distinction was outside of the scope of this research; instead, we aimed to provide a broader analysis of CMR tasks as they exist given the four categories of canonical representational forms (see Dick & Edwards, 2008, who combine situational and plain text descriptions in defining translational tasks between text and other representational forms).

Future research on CMR could make this distinction by focusing on translational tasks involving text and distinguishing between how often students are provided with either plain text versus situational text. Future research on CMR tasks in the comprehension of functions might also further define which transitions are most difficult for students so that teachers or textbook authors might emphasize accurate coordination of these tasks. Finally, future studies might also do well to determine the prevalence and nature of CMR tasks in which students are able to make their own representational choices. While we focused on those CMR tasks that were prescribed by the text, as Acevedo Nistal et al. (2012) note, there is a need for research to focus on the contextual factors that promote "flexible representation choice" for students in mathematics. It is our hope that this study will help inform such future work on how textbooks might promote or hinder students' representational flexibility.

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